

Hilbert C^* -Module Representation on Haagerup Tensor Products and Group Systems

By

Jaeseong HEO*

Abstract

Using the Hilbert C^* -module representation associated with completely multi-positive linear maps [Heo], we give another representation on Haagerup tensor product without the bridging maps. We also construct covariant representations of covariant group systems on Hilbert C^* -modules.

§1. Introduction and Preliminaries

In the theory of C^* -algebras, we can see many results with various kinds of dilations. The two fundamental results are the Stinespring dilation theorem [St] and the Hilbert C^* -module representation given by Paschke [Pas]. Moreover, there are dilations associated with a multi-state [Kap] and a completely multi-positive map [Heo] enhanced more than existing dilation theorems.

Christensen and Sinclair [CS] formulated the notion of completely bounded (respectively, completely positive) multilinear operators from a C^* -algebra into $\mathcal{B}(\mathcal{H})$ and gave representations for completely bounded multilinear operators. Paulsen and Smith [PS] extended a representation of completely bounded multilinear maps to the case of subspaces of C^* -algebras using the correspondence between completely bounded multilinear maps and completely bounded linear maps on Haagerup tensor products.

In this paper, we will give another representation on the Haagerup tensor

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* Department of Mathematics, Seoul National University, Seoul 151–742, Korea.
e-mail : hjs@math.snu.ac.kr

product using the Hilbert C^* -module representation associated with completely multi-positive linear maps. In the quantum field theory and statistical mechanics, the representations of covariant algebras play an important role. Takesaki [Tak] made a study of covariant algebras and their representations. We will construct covariant representations of covariant group systems on Hilbert C^* -modules. This implies that every covariant completely positive linear map from covariant group systems into C^* -algebras arises from a dilation to a covariant unitary representation on Hilbert C^* -modules.

A *pre-Hilbert B -module* over a C^* -algebra B is a right B -module X equipped with a B -valued inner product $\langle \cdot, \cdot \rangle : X \times X \rightarrow B$ which is B -linear in the second variable and has the properties;

$$\langle x, y \rangle = \langle y, x \rangle^*, \quad \langle x, x \rangle \geq 0 \text{ with the equality iff } x = 0.$$

If, in addition, X is complete with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$, then X is called a *Hilbert B -module* or *Hilbert C^* -module over B* .

Let X and Y be Hilbert B -modules. We denote by $\mathcal{B}_B(X, Y)$ the space of all bounded B -linear operators of X into Y . We write $\mathcal{B}_B(X)$ for $\mathcal{B}_B(X, X)$. With the operator norm, $\mathcal{B}_B(X)$ is a Banach algebra. We denote by $\mathcal{L}_B(X, Y)$ the set of all B -module maps $T : X \rightarrow Y$ for which there is an operator $T^* : Y \rightarrow X$, called the *adjoint* of T , such that

$$\langle Tx, y \rangle = \langle x, T^*y \rangle, \quad x \in X, y \in Y.$$

By the Banach-Steinhaus Theorem, $T \in \mathcal{L}_B(X, Y)$ is bounded. We write $\mathcal{L}_B(X)$ for $\mathcal{L}_B(X, X)$, which becomes a C^* -algebra with the operator norm [JT, Lemma 1.1.7]. By a *representation* of a C^* -algebra A on a Hilbert B -module X , we mean a $*$ -homomorphism $\pi : A \rightarrow \mathcal{L}_B(X)$.

The following theorem [Heo] generalizes the Stinespring representation for a completely bounded linear map whose proof depends on the fact that any closed subspace of a Hilbert space is complemented [Pau]. Since Hilbert C^* -modules are, in general, not complemented, the decomposition property of the Hilbert C^* -module can not be used in the proof of Theorem 1.1. To get a representation on a Hilbert C^* -module associated with completely bounded maps, we have used the representation associated with a completely multi-positive linear map.

Theorem 1.1. *Let A and B be C^* -algebras with B injective. If $\phi : A \rightarrow B$ is a completely bounded linear map, then there exist a Hilbert B -module X ,*

a representation π of A on X and vectors $x_1, x_2 \in X$ with the properties:

- (i) $\phi(a) = \langle x_1, \pi(a)x_2 \rangle$ for each $a \in A$,
- (ii) the set $\{\pi(a)(x_i \cdot b) : a \in A, b \in B, i = 1, 2\}$ spans a dense subspace of X .

Two Hilbert B -modules $(X_1 \langle \cdot, \cdot \rangle_1), (X_2 \langle \cdot, \cdot \rangle_2)$ over a fixed C^* -algebra B are isomorphic as Hilbert C^* -modules if and only if there exists a bijective bounded B -linear mapping $S : X_1 \rightarrow X_2$ such that the identity $\langle x, y \rangle_1 = \langle S(x), S(y) \rangle_2$ holds for every $x, y \in X$.

Examples 1.2. (1) Let \mathcal{O}_∞ be the C^* -algebra generated by these isometries s_1, s_2, \dots such that

$$s_i^* s_j = \delta_{ij} 1 \quad \text{and} \quad \sum_{i=1}^\infty s_i s_i^* = 1.$$

Denote by $\mathcal{H}_{\mathcal{O}_\infty}$ the space of all sequences $\{a_i\}$ in \mathcal{O}_∞ that are square summable in the sense that $\sum_{i=1}^\infty a_i^* a_i$ converges in \mathcal{O}_∞ with the inner product

$$\langle \{a_i\}, \{b_i\} \rangle = \sum_{i=1}^\infty a_i^* b_i.$$

Define $T : \mathcal{O}_\infty \rightarrow \mathcal{H}_{\mathcal{O}_\infty}$ and $S : \mathcal{H}_{\mathcal{O}_\infty} \rightarrow \mathcal{O}_\infty$ by

$$T(a) = \{s_i^*\} a = \{s_i^* a\} \quad \text{and} \quad S(\{a_i\}) = \sum_{i=1}^\infty s_i a_i.$$

Clearly T is an injective \mathcal{O}_∞ -module map. We easily see that $TS(\{a_i\}) = \{a_i\}$, $ST(a) = a$ and $\langle T(a), T(b) \rangle = \langle a, b \rangle$. Hence \mathcal{O}_∞ and $\mathcal{H}_{\mathcal{O}_\infty}$ are isomorphic as Hilbert C^* -modules.

(2) Let $\mathcal{O}_n, (n \geq 2)$ be the C^* -algebra generated by isometries s_1, \dots, s_n such that $\sum_{i=1}^n s_i s_i^* = 1$. Let l_k^2 be the space of all complex sequences $\{z_i\}_{i=1}^k$ and $l_k^2 \otimes \mathcal{O}_n$ the space of all sequences $\{a_i\}_{i=1}^k$ with $a_i \in \mathcal{O}_n$.

We see that if $k = l(n - 1) + 1$ for each $l = 1, 2, \dots$, then there exist orthogonal projections p_1, \dots, p_k in \mathcal{O}_n such that

$$p_1 + \dots + p_k = 1, \quad s_i s_i^* = p_i \quad \text{and} \quad s_i^* s_i = 1 \quad (i = 1, \dots, n).$$

Then \mathcal{O}_n and $l_k^2 \otimes \mathcal{O}_n, (k = l(n - 1) + 1)$ are isomorphic as Hilbert C^* -modules. For let s_1, \dots, s_k be as in above argument. Define $T : \mathcal{O}_n \rightarrow l_k^2 \otimes \mathcal{O}_n$

and $S: l_k^2 \otimes \mathcal{O}_n \rightarrow \mathcal{O}_n$ by

$$T(a) = \{s_i^* a\}_{i=1}^k \quad \text{and} \quad S(\{a_i\}_{i=1}^k) = \sum_{i=1}^k s_i a_i.$$

It is easy to check that $TS(\{a_i\}) = \{a_i\}$, $ST(a) = a$ and $\langle T(a), T(b) \rangle = \langle a, b \rangle$. In particular, \mathcal{O}_n and $\mathcal{H}_{\mathcal{O}_n}$ are isomorphic as Hilbert C^* -modules.

(3) Let A be a unital C^* -algebra with orthogonal projections p_1, \dots, p_k such that $p_1 + \dots + p_k = 1_A$, $u_i u_i^* = p_i$ and $u_i^* u_i = 1_A$ for each $1 \leq i \leq k$. By the same argument in (2), we have that $A \simeq l_k^2 \otimes A$ as Hilbert C^* -modules.

(4) If A is a unital C^* -algebra with orthogonal projections p_1, p_2, \dots such that $\sum_{i=1}^\infty p_i = 1_A$, $u_i u_i^* = p_i$ and $u_i^* u_i = 1_A$ for each $1 \leq i \leq k$, then we see that $A \simeq \mathcal{H}_A$ as Hilbert C^* -modules. Let X be a countably generated Hilbert A -module. By the Kasparov's stabilization theorem, we get $A \simeq X \oplus A$ as Hilbert C^* -modules.

Let X be a Hilbert A -module and Y a Hilbert B -module. Let $A \odot B$ be the algebraic tensor product and $A \otimes B$ the spatial tensor product of A and B . We define an $A \odot B$ -valued inner product

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle x_1, x_2 \rangle \otimes \langle y_1, y_2 \rangle$$

for $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. We can extend the $A \odot B$ -module structure by continuity in two steps to get a right $A \otimes B$ -module structure on $X \hat{\otimes} Y$. $X \hat{\otimes} Y$ is called the exterior tensor product [JT].

Let A and B be C^* -algebras. We regard A and B as a Hilbert A -module and a Hilbert B -module, respectively. Consider the exterior tensor product $A \hat{\otimes} B$ as a Hilbert $A \otimes B$ -module where $A \otimes B$ is the spatial tensor product. Then $A \hat{\otimes} B$ is exactly $A \otimes B$ as a Hilbert C^* -module with the $A \otimes B$ -valued inner product

$$\langle a_1 \otimes b_1, a_2 \otimes b_2 \rangle = a_1^* a_2 \otimes b_1^* b_2.$$

Proposition 1.3. *Let A and B be unital C^* -algebras. Let $\phi: A \rightarrow B$ be a unital completely positive map. Then the followings are equivalent:*

- (i) ϕ is a $*$ -homomorphism;
- (ii) $\ker \Phi$ is a Hilbert $A \otimes B$ -submodule of $A \hat{\otimes} B$ where a linear map $\Phi: A \hat{\otimes} B \rightarrow B$ is defined by $\Phi(a \otimes b) = \phi(a)^* b$.

Proof. (i) \Rightarrow (ii) Suppose that ϕ is a $*$ -homomorphism. Clearly, $\ker \Phi$ is a linear space. To show that $\ker \Phi$ is a Hilbert $A \otimes B$ -submodule of $A \widehat{\otimes} B$, take any $x = \sum a_i \otimes b_i \in \ker \Phi$ and $a \otimes b \in A \otimes B$. Then we have

$$\begin{aligned} \Phi(x \cdot (a \otimes b)) &= \Phi\left(\sum_i a_i a \otimes b_i b\right) = \sum_i \phi(a_i a) * b_i b \\ &= \sum_i \phi(a) * \phi(a_i) * b_i b = \phi(a) * \left(\sum_i \phi(a_i) * b_i\right) b \\ &= \phi(a) * \Phi(x) b = 0. \end{aligned}$$

(ii) \Rightarrow (i) Conversely, suppose that $\ker \Phi$ is a Hilbert $A \otimes B$ -submodule of $A \widehat{\otimes} B$. For any $a_1 \in A$, we have $a_1 \otimes 1_B - 1_A \otimes \phi(a_1)^* \in \ker \Phi$. Since we have

$$(a_1 \otimes 1_B - 1_A \otimes \phi(a_1)^*) (a_2 \otimes 1_B) = a_1 a_2 \otimes 1_B - a_2 \otimes \phi(a_1)^* \in \ker \Phi$$

for any $a_1, a_2 \in A$, we get $\phi(a_1 a_2)^* = \phi(a_2)^* \phi(a_1)^*$. Hence ϕ is a homomorphism. By the representation theorem in [Pas, Theorem 5.2], we get ϕ is a $*$ -homomorphism. \square

§2. Representations of Haagerup Tensor Products

In recent years, the Haagerup tensor product has played an important role in various aspects of the operator algebra theory and is closely tied to the advances in the theory of operator spaces, quantum groups and cohomology theory of C^* -algebras. Completely bounded multilinear maps can be linearized by considering the Haagerup tensor product.

Let A and B be unital C^* -algebras. The *Haagerup norm* on the algebraic tensor product $A \otimes B$ is defined by

$$(2.1) \quad \|u\|_h = \inf \left\{ \left\| \sum_{i=1}^n a_i a_i^* \right\|^{\frac{1}{2}} \left\| \sum_{i=1}^n b_i^* b_i \right\|^{\frac{1}{2}} \right\}$$

where this infimum is taken over all representations of u as a finite sum $\sum a_i \otimes b_i$. This is equivalent to

$$\|u\|_h = \inf \{ \|E\| \|F\| : u = E \otimes F, \quad E \in M_{1n}(A), \quad F \in M_{n1}(B) \}.$$

The infimum in the definition is attained. Moreover E and F may be chosen to have linearly independent sets of components. The resulting normed space

is written $A \otimes_h B$ (without completion). The Haagerup norm is injective. This is surprising since the definition closely resembles that of the projective tensor product norm. For a fuller account of the theory of operator spaces and Haagerup tensor products, see [PS] and [SS].

Christensen and Sinclair showed that each completely bounded multilinear operator is representable in terms of representations of the C^* -algebra using Wittstock's theorem of matrix-sublinear functionals. Paulsen and Smith gave the representation for completely bounded maps on subspaces of C^* -algebras using the correspondence between completely bounded multilinear maps and completely bounded linear maps on Haagerup tensor products. In [CS] and [PS], the bridging operators were used to connect Hilbert spaces and another Hilbert spaces. Using representations for completely multi-positive linear maps we will give a representation without bridging operators.

Let $V: \mathcal{V} \times \mathcal{W} \rightarrow \mathcal{B}(\mathcal{H})$ be a bilinear map where \mathcal{V} and \mathcal{W} are operator spaces. With each bilinear map $V: \mathcal{V} \times \mathcal{W} \rightarrow \mathcal{B}(\mathcal{H})$ we may associate a linear map $\phi: \mathcal{V} \otimes_h \mathcal{W} \rightarrow \mathcal{B}(\mathcal{H})$ on the Haagerup tensor product by $\phi(v \otimes w) = V(v, w)$ for $v \in \mathcal{V}$, $w \in \mathcal{W}$. The following lemma was proved by Paulsen and Smith [PS].

Lemma 2.1. *Let $V: \mathcal{V} \times \mathcal{W} \rightarrow \mathcal{B}(\mathcal{H})$ be a bilinear map and let $\phi: \mathcal{V} \otimes_h \mathcal{W} \rightarrow \mathcal{B}(\mathcal{H})$ be the associated linear map. Then V is completely bounded if and only if ϕ is completely bounded and $\|V\|_{cb} = \|\phi\|_{cb}$.*

We call an element v of a $*$ -algebra A *quasi-unitary* if $vv^* = v^*v = v + v^*$, and say that A is a U^* -algebra if it is the linear span of its quasi-unitary elements. All Banach $*$ -algebras are U^* -algebras. Notice that if A is unital, then $u \in A$ is unitary if and only if $1 - u$ is a quasi-unitary, so in this case A is a U^* -algebra if and only if it is spanned by unitaries.

Even though the completion of the Haagerup tensor product of C^* -algebras is not a C^* -algebras, it becomes a Banach algebra [ASS]. But Paschke gave a representation on a Hilbert C^* -module for a completely positive linear map from a unital U^* -algebra to a unital C^* -algebra. Note that we can give a $*$ -operation on $A \otimes_h B$ with $(a \otimes b)^* = a^* \otimes b^*$, which is, in general, not isometric. Since A and B are unital and $A \otimes_h B$ is generated by unitaries, we may regard $A \otimes_h B$ as a U^* -algebra. We can see that Theorem 5.2 in [Pas] can be applied to a Banach algebra which is an operator space. Hence we can use Theorem 1.1 to prove the following Proposition.

Proposition 2.2. *Let C be an injective C^* -algebra. If $\phi: A \otimes_{\hbar} B \rightarrow C$ is a completely bounded linear map, then there exist a Hilbert C -module X , representations π_1 of A , π_2 of B on X and vectors $x_1, x_2 \in X$ such that on elementary tensors*

$$\phi(a \otimes b) = \langle x_1, \pi_1(a)\pi_2(b)x_2 \rangle, \quad a \in A, b \in B.$$

Proof. By Theorem 1.1, there exist a Hilbert C -module X , a representation π of $A \otimes_{\hbar} B$ on X and vectors $x_1, x_2 \in X$ such that

$$\phi(u) = \langle x_1, \pi(u)x_2 \rangle, \quad u \in A \otimes_{\hbar} B.$$

We define $\pi_1(a) = \pi(a \otimes 1)$, $a \in A$ and $\pi_2(b) = \pi(1 \otimes b)$, $b \in B$. Hence we have $\phi(a \otimes b) = \langle x_1, \pi_1(a)\pi_2(b)x_2 \rangle$ on elementary tensors $a \otimes b$ with $a \in A, b \in B$. This completes the proof. \square

Corollary 2.3. *Let C be an injective C^* -algebra. If $V: A \times B \rightarrow C$ be a completely bounded bilinear map, then there exist a Hilbert C -module X , representations $\pi_1: A \rightarrow \mathcal{L}_C(X)$, $\pi_2: B \rightarrow \mathcal{L}_C(X)$ and vectors $x_1, x_2 \in X$ such that for each $a \in A$ and $b \in B$*

$$V(a, b) = \langle x_1, \pi_1(a)\pi_2(b)x_2 \rangle.$$

Proof. By Lemma 2.1, the associated linear map $\phi: A \otimes_{\hbar} B \rightarrow C$ is completely bounded. By Proposition 2.2, there exist a Hilbert C -module X , representations $\pi_1: A \rightarrow \mathcal{L}_C(X)$, $\pi_2: B \rightarrow \mathcal{L}_C(X)$ and vectors $x_1, x_2 \in X$ such that on elementary tensors

$$\phi(a \otimes b) = \langle x_1, \pi_1(a)\pi_2(b)x_2 \rangle.$$

Hence we have $V(a, b) = \langle x_1, \pi_1(a)\pi_2(b)x_2 \rangle$ for each $a \in A, b \in B$. \square

§3. Covariant Completely Positive Operators on Group Systems

Let G and H be topological groups and let $\text{Aut}(G)$ be the group of all automorphisms of G endowed with the pointwise convergence topology. The action τ of H on G is a continuous homomorphism of H into $\text{Aut}(G)$. We will refer to this as a *group system*. Regarding the group G as a subset of the group algebra $C[G]$, we may consider a completely positive linear map $\phi: G \rightarrow B$ of a group G into a C^* -algebra B . Let X be a Hilbert B -module. We

also call a $*$ -homomorphism $\pi: G \rightarrow \mathcal{L}_B(X)$ a representation π of G on X . If $\phi: G \rightarrow B$ is a completely positive linear map, we see that ϕ is a positive definite B -valued function on G , that is, for each $g_1, \dots, g_n \in G$ and $b_1, \dots, b_n \in B$,

$$(3.1) \quad \sum_{i,j=1}^n b_i^* \phi(g_i^{-1} g_j) b_j \geq 0.$$

Let G be a topological group, B a C^* -algebra and X a Hilbert B -module. Given a $*$ -homomorphism $\pi: G \rightarrow \mathcal{L}_B(X)$ and an element $\xi \in X$, we define the linear map $\phi: G \rightarrow B$ by

$$(3.2) \quad \phi(g) = \langle \xi, \pi(g)\xi \rangle \quad \text{for each } g \in G.$$

Then the linear map ϕ is completely positive. In the rest of this section, we will assume that G and H are topological groups and B is a unital C^* -algebra. The following proposition and theorem are motivated by [JQ].

Proposition 3.1. *Let $\phi: G \rightarrow B$ be a completely positive linear map. Then there exist a Hilbert B -module X , a representation $\pi: G \rightarrow \mathcal{L}_B(X)$ and a vector $\xi \in X$ such that*

- (i) $\phi(g) = \langle \xi, \pi(g)\xi \rangle$ for all $g \in G$,
- (ii) the set $\{\pi(g)(\xi \cdot b) : g \in G, b \in B\}$ spans a dense subspace of X .

Proof. We only sketch the construction since the details are routine and similar to the proof of Theorem 2.1 in [Heo]. Consider the vector space $\mathcal{F}_s(G, B)$ of all finitely supported functions from G into B . For each $x = \sum_{i=1}^n a_i \cdot \chi_{g_i} \in \mathcal{F}_s(G, B)$ and $b \in B$, we define

$$x \cdot b = \sum_{i=1}^n a_i b \cdot \chi_{g_i} \quad \text{where } \chi_g(h) = \begin{cases} 1_B, & \text{if } g = h \\ 0, & \text{if } g \neq h. \end{cases}$$

Then $\mathcal{F}_s(G, B)$ becomes a right B -module. Define a B -valued positive semi-definite form on $\mathcal{F}_s(G, B)$ by

$$\left\langle \sum_{i=1}^m a_i \cdot \chi_{g_i}, \sum_{j=1}^n b_j \cdot \chi_{h_j} \right\rangle = \sum_{i=1}^m \sum_{j=1}^n a_i^* \phi(g_i^{-1} h_j) b_j$$

for $g_i, h_j \in G$ and $a_i, b_j \in B$. By the Cauchy-Schwarz inequality, the set

$$N = \{x \in \mathcal{F}_s(G, B) : \langle x, x \rangle = 0\}$$

is a subspace of $\mathcal{F}_s(G, B)$. Furthermore, it becomes a B -submodule of $\mathcal{F}_s(G, B)$. The induced B -valued inner product on the quotient B -module $\mathcal{F}_s(G, B)/N$ given by

$$\langle x + N, y + N \rangle = \langle x, y \rangle, \quad x, y \in \mathcal{F}_s(G, B)$$

is well-defined. Let X be the completion of $\mathcal{F}_s(G, B)/N$ with respect to the induced inner product.

The representation π on X_0 is given by

$$\pi(g) \left(\sum_{i=1}^n a_i \cdot \chi_{g_i} + N \right) = \sum_{i=1}^n a_i \cdot \chi_{gg_i} + N.$$

By a simple calculation, we see that π is a unitary representation of G on X_0 . Putting $\xi = \chi_e + N$ where e is the identity of G , we have $\pi(g)(\chi_e \cdot b + N) = b \cdot \chi_g + N$, so the linear span of $\{\pi(g)(\xi \cdot b) : g \in G, b \in B\}$ is precisely X_0 , which is dense in X . Further, we have

$$\phi(g) = \langle \chi_e, \chi_g \rangle = \langle \chi_e + N, \pi(g)(\chi_e + N) \rangle = \langle \xi, \pi(g)\xi \rangle$$

for all $g \in G$. This completes the proof. \square

Theorem 3.2. *Let $\tau : H \rightarrow \text{Aut}(G)$ be an action of H on G and $u : H \rightarrow \mathcal{U}(B)$ a strongly continuous unitary representation. If $\phi : G \rightarrow B$ is a u -covariant completely positive linear map, then there exist*

- (i) a Hilbert B -module X with a generating vector ξ ,
- (ii) a strongly continuous unitary representation $\tilde{\tau} : H \rightarrow \mathcal{L}_B(X)$,
- (iii) a $*$ -representation $\pi : G \rightarrow \mathcal{L}_B(X)$,
- (iv) an element $v \in \mathcal{L}_B(B, X)$,

such that

- (1) $\phi(g) = \langle \xi, \pi(g)\xi \rangle$ for each $g \in G$,
- (2) the linear span of $\{\pi(g)(\xi \cdot b) : g \in G, b \in B\}$ is a dense subspace of X ,
- (3) $v^* \pi(g)v = m_{\phi(g)}$ for each $g \in G$,
- (4) $\tilde{\tau}_h v = v m_{u_h}$ for each $h \in H$,

where m is a left multiplication operator on B .

Proof. We will follow the notations of the proof of Proposition 3.1. The statements (i) and (iii) are the part of Proposition 3.1, so it will be sufficient to construct a unitary representation $\tilde{\tau}$ of H on X and a B -module map v

satisfying (3) and (4). Since the details are similar to the proof of Theorem 3.1 in [Heo], we will only sketch the proof.

For each $h \in H$, we define a linear map $\tilde{\tau}_h: \mathcal{F}_s(G, B) \rightarrow \mathcal{F}_s(G, B)$ by

$$\tilde{\tau}_h \left(\sum_{i=1}^n b_i \cdot \chi_{g_i} \right) = \sum_{i=1}^n u_h b_i \cdot \chi_{\tau_h(g_i)}$$

for each $b_i \in B$ and $g_i \in G$. Let $\sum_{i=1}^n a_i \cdot \chi_{g_i}, \sum_{j=1}^m b_j \cdot \chi_{h_j} \in \mathcal{F}_s(G, B)$. By the u -covariance property of ϕ , we have

$$\begin{aligned} \left\langle \tilde{\tau}_h \left(\sum_{i=1}^n a_i \cdot \chi_{g_i} \right), \tilde{\tau}_h \left(\sum_{j=1}^m b_j \cdot \chi_{h_j} \right) \right\rangle &= \left\langle \sum_{i=1}^n u_h a_i \cdot \chi_{\tau_h(g_i)}, \sum_{j=1}^m u_h b_j \cdot \chi_{\tau_h(h_j)} \right\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^m a_i^* u_h^* \phi(\tau_h(g_i^{-1} h_j)) u_h b_j \\ &= \sum_{i=1}^n \sum_{j=1}^m a_i^* \phi(g_i^{-1} h_j) b_j \\ &= \left\langle \sum_{i=1}^n a_i \cdot \chi_{g_i}, \sum_{j=1}^m b_j \cdot \chi_{h_j} \right\rangle . \end{aligned}$$

Therefore, the set $N = \{x \in \mathcal{F}_s(G, B) : \langle x, x \rangle = 0\}$ is invariant under the action of $\tilde{\tau}_h$. By passing to the quotient, we get an isometric linear map, again denoted by $\tilde{\tau}$. It is straightforward to check that $\tilde{\tau}_h$ is a B -module map. To show that $\tilde{\tau}_h \in \mathcal{L}_B(X)$, take $x = \sum_{i=1}^n a_i \cdot \chi_{g_i} + N$ and $y = \sum_{j=1}^m b_j \cdot \chi_{h_j} + N$ in X . Then we have

$$\begin{aligned} \langle \tilde{\tau}_h(x), y \rangle &= \left\langle \sum_{i=1}^n u_h a_i \cdot \chi_{\tau_h(g_i)} + N, \sum_{j=1}^m b_j \cdot \chi_{h_j} + N \right\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^m (u_h a_i)^* \phi(\tau_h(g_i)^{-1} h_j) b_j \\ &= \sum_{i=1}^n \sum_{j=1}^m a_i^* \phi(g_i^{-1} \tau_{h^{-1}}(h_j)) u_h^* b_j \\ &= \left\langle \sum_{i=1}^n a_i \cdot \chi_{g_i} + N, \sum_{j=1}^m u_{h^{-1}} b_j \cdot \chi_{\tau_{h^{-1}}(h_j)} + N \right\rangle \\ &= \langle x, \tilde{\tau}_{h^{-1}}(y) \rangle . \end{aligned}$$

Letting $x = \sum_{i=1}^n a_i \cdot \chi_{g_i} + N \in X, g \in G$ and $h \in H$, it follows that

$$\begin{aligned} \tilde{\tau}_h \pi(g) \tilde{\tau}_h^*(x) &= \tilde{\tau}_h \pi(g) \left(\sum_{i=1}^n u_{h^{-1}} a_i \cdot \chi_{\tau_{h^{-1}}(g_i)} + N \right) \\ &= \tilde{\tau}_h \left(\sum_{i=1}^n u_{h^{-1}} a_i \cdot \chi_{g \tau_{h^{-1}}(g_i)} + N \right) \\ &= \sum_{i=1}^n a_i \cdot \chi_{\tau_h(g)g_i} + N \\ &= \pi(\tau_h(g))(x) \end{aligned}$$

Define the linear map $v : B \rightarrow X$ by

$$v(b) = \xi \cdot b = b \cdot \chi_e + N \quad \text{for each } b \in B.$$

It follows immediately from the definition that v is a B -module map from B into X . For each $b \in B$ and $x = \sum_{i=1}^n a_i \cdot \chi_{g_i} \in X$, we have

$$\langle v(b), x \rangle = \sum_{i=1}^n b^* \phi(g_i) a_i = \left\langle b, \sum_{i=1}^n \phi(g_i) a_i \right\rangle,$$

which implies that $v^*(\sum_{i=1}^n a_i \cdot \chi_{g_i}) = \sum_{i=1}^n \phi(g_i) a_i$, and so $v \in \mathcal{L}_B(B, X)$. For each $g \in G$ and $b \in B$, we have

$$v^* \pi(g) v(b) = v^* \pi(g) (b \cdot \chi_e + N) = \phi(g) b = m_{\phi(g)}(b).$$

It is easy to check that $\sigma_h v = v m_{u_h}$, which completes the proof. \square

If G is a locally compact group, we can use Paschke’s result [Pas] about the corresponding group C^* -algebra to give another proof. But the method in Theorem 3.2 applies to arbitrary topological groups [JQ]. Note that for (covariant) completely multi-positive linear maps from G into B we can give the associated (covariant) representations on Hilbert B -modules similar to Theorem 2.1 (Theorem 3.1) in [Heo].

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