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Sierpiński Gasket as a Martin Boundary II
(*The Intrinsic Metric*)

By

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Abstract

It is shown in [DS] that the Sierpiński gasket $\mathscr{S} \subset \boldsymbol{R}^N$ can be represented as the Martin boundary of a certain Markov chain and hence carries a canonical metric ρ_M induced by the embedding into an associated Martin space M. It is a natural question to compare this metric ρ_M with the Euclidean metric. We show first that the harmonic measure coincides with the normalized $H = (\log(N+1)/\log 2)$-dimensional Hausdorff measure with respect to the Euclidean metric. Secondly, we define an intrinsic metric ρ which is Lipschitz equivalent to ρ_M and then show that ρ is not Lipschitz equivalent to the Euclidean metric, but the Hausdorff dimension remains unchanged and the Hausdorff measure in ρ is infinite. Finally, using the metric ρ, we prove that the harmonic extension of a continuous boundary function converges to the boundary value at every boundary point.

§1. Introduction

The Sierpiński gasket in $\boldsymbol{R}^{N-1}$ (see Sierpiński's work (1915) in [S] and Mandelbrot [M]) is a fundamental example of fractal sets. Its Hausdorff dimension equals $H = \dfrac{\log N}{\log 2}$ in the Euclidean metric $\|\cdot\|$ and the H-dimensional Hausdorff measure μ is positive and finite ([Ma] and [F]). The harmonic analysis of the Sierpiński gasket has been investigated by many authors. For example, Barlow and Perkins [BP] defined a Brownian motion on the Sierpiński gasket and Kigami [K] established a harmonic analysis from an analytical viewpoint. On the other hand, the authors [DS] represented the Sierpiński

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gasket as the Martin boundary of a certain Markov chain. This note is a continuation of the investigations in [DS] intending to establish a harmonic analysis of the Sierpiński gasket from this point of view.

It is known (see [DS]) that the Sierpiński gasket $(\mathscr{S}, \|\cdot\|)$ in $\mathbb{R}^{N-1}$ is homeomorphic to the Martin boundary $(\partial M, \rho_M)$ of some Markov chain $\mathbb{X}=\{X_n\}$, where ρ_M denotes the Martin metric (see (2) below). The state space $\mathscr{W}$ is the word space over the alphabet $\mathscr{A}=\{1,\cdots,N\}$, and the associated Markov operator is denoted by P. Dynkin's theorem says that every bounded harmonic function f for P has an integral representation

$$f(\mathbf{w})=\int_{\mathscr{S}} k(\mathbf{w},\xi)\phi(\xi)\mu_1(d\xi)$$

for a function $\phi\in L_\infty(\mu_1)$, where μ_1 is the harmonic measure on $\mathscr{S}$ and $k=k(\mathbf{w},\xi)$ $(\mathbf{w}\in\mathscr{W},\ \xi\in\Xi)$ denotes the Martin kernel extended to $\mathscr{S}$. Our first result shows that the harmonic measure μ_1 equals the normalized canonical Hausdorff measure on $\mathscr{S}$.

The transition probabilities are defined by

$$\mathbb{P}(X_2=\mathbf{w}a\,|\,X_1=\mathbf{w})=\begin{cases}\dfrac{1}{2N} & \text{if } \mathbf{w}\neq\mathbf{w}^{\#},\quad a\in\mathscr{A}\\[2ex] \dfrac{1}{N} & \text{if } \mathbf{w}=\mathbf{w}^{\#},\quad a\in\mathscr{A}\end{cases}$$

for some involution $\#$ and where $\mathbf{w}=a_1^{l_1}\cdots a_s^{l_s}$, $(1\le a_i\le N, l_i\ge 1, 1\le i\le s)$ is a finite word over the alphabet $\mathscr{A}$ (see Section 2 below for details). This Markov chain has the state space $\mathscr{W}$ consisting of all finite words over $\mathscr{A}$ and has long range dependence with respect to the natural metric on the tree $\mathscr{W}$. It follows that $M=\mathscr{W}\cup\mathscr{S}$ is a model of the Martin space of $\mathbb{X}$ equipped with the coarsest topology for which the functions $\mathscr{S}\ni\xi\mapsto k(\mathbf{w},\xi)$ $(\mathbf{w}\in\mathscr{W})$ are continuous. This topology is determined by the extension of the metric

$$\text{(1)}\qquad \rho(\mathbf{w},\mathbf{v})=|2^{-d(\mathbf{w})}-2^{-d(\mathbf{v})}|+\sum_{n=0}^{\infty}\frac{1}{(2N)^n}\sup_{\substack{\mathbf{u}\in\mathscr{W}\\ d(u)=n}}|k(\mathbf{u},\mathbf{w})-k(\mathbf{u},\mathbf{v})|$$

to M, where $d(\mathrm{W})$ denotes the length of the word $\mathbf{w}\in\mathscr{W}$. It should be noted that (1) is Lipschitz equivalent to the Martin metric

$$(2) \qquad \rho_M(\mathbf{w},\mathbf{v}) = |2^{-d(\mathbf{w})} - 2^{-d(\mathbf{v})}| + \sum_{\mathbf{u}\in\mathscr{W}} \frac{1}{(2N)^{d(\mathbf{u})}} |k(\mathbf{u},\mathbf{w} - k(\mathbf{u},\mathbf{v})|$$

as can be easily deduced from Theorem 3.4 in [DS] (see also Lemma 2.1 below). (2) is the standard metric for the Martin space introduced by Dynkin in [Dy]. Hence (1) provides the canonical metric structure on M, and therefore we call ρ the *intrinsic metric.* Likewise we call $\rho|_{\mathscr{S}\times\mathscr{S}}$, the restriction of ρ to $\mathscr{S}\times\mathscr{S}$, the intrinsic metric on the Sierpiński gasket and denote it also by ρ.

Let Ξ denote the space of one-sided infinite sequences $\mathbf{x}=(x_k)\in\mathscr{A}^{\mathbf{N}}$, and define an equivalence relation $\mathbf{x}\sim\mathbf{y}$ iff $\mathbf{x}=\mathbf{y}$ or

$$\exists n\geq 1 \text{ such that } x_k=y_k \ \forall k<n \text{ and } x_n=y_{n+k},\ y_n=x_{n+k}\ (\forall k\geq 1).$$

It is known that $\mathscr{S}$ is bi-Lipschitz equivalent to the quotient space $\Xi/\sim$, where $\mathscr{S}$ carries the Euclidean metric $\|\xi-\eta\|$ $(\xi,\eta\in\mathscr{S})$ and $\Xi/\sim$ a metric derived from the word space metric $\Sigma_{n\geq 1}2^{-n}1_{\{x_n\neq y_n\}}$ (for $\mathbf{x}=(x_n)$, $\mathbf{y}=(y_n)\in\Xi$). Our second result is to show that (see Section 3)

$$(3) \qquad \frac{1}{32}\|\xi-\eta\|\log_2\frac{1}{\|\xi-\eta\|}\leq\rho(\xi,\eta)\leq\frac{12}{A}\|\xi-\eta\|\log_2\frac{A}{\|\xi-\eta\|}$$

where $\xi,\eta\in\mathscr{S}$, and where A is some constant depending only on N. A particular consequence of (3) is that the intrinsic metric ρ is not equivalent to the Euclidean metric on $\mathscr{S}$. It also follows from this inequality that the Hausdorff dimension H under the intrinsic metric does not change and that the H-dimensional Hausdorff measure with respect to ρ is infinite.

A harmonic function h on $\mathscr{W}$ is an eigenfunction for the eigenvalue 1 of the Markov operator P of $\mathbf{X}$. It is known that the algebra of bounded harmonic functions is isomorphic to $L_\infty(\mu_1)$. In Section 4 we estimate the modulus of continuity for harmonic functions in terms of its representing bounded measurable function on $\mathscr{S}$. It turns out that the modulus of continuity (over cylinder sets) of harmonic functions h is uniformly bounded by the variation of its representing function in the space $C(\mathscr{S})$ of continuous functions on $\mathscr{S}$ over cylinders. Another consequence of (3) is that uniformly continuous functions in the ρ-metric are uniformly continuous in the word space metric and vice versa. Hence we can define the space $\mathscr{H}_C$ of uniformly continuous harmonic functions independently of the metric, and it follows that the algebra $\mathscr{H}_C$ is isomorphic to $C(\mathscr{S})$.

§2. Harmonic Measure on the Sierpiński Gasket

Let $\Delta=\Delta(p_1,\cdots,p_N)$ denote the non-degenerate regular simplex generated by N points $p_1,\cdots,p_N\in\boldsymbol{R}^{N-1}$ $(N\geq 2)$. For every fixed $i_0\in\{1,\cdots,N\}$, the midpoints $p_{j,i_0}=\dfrac{p_{i_0}+p_j}{2}$, $(j=1,\cdots,N)$ define a corresponding simplex

$$\Delta(i_0)=\Delta(p_{1,i_0},\cdots,p_{N,i_0})\subset\Delta$$

and an affine map

$$f_{i_0}:\Delta(p_1,\cdots,p_N)\to\Delta(p_{1,i_0},\cdots,p_{N,i_0})$$

satisfying $f_{i_0}(p_i)=p_{i,i_0}$. We denote the diameter of a subset $B\subset\boldsymbol{R}^{N-1}$ by $|B|$ and, for simplicity, assume $|\Delta|=1$. It follows from [Ha] and [Hu] that the iterated function system $\{f_i;\ 1\leq i\leq N\}$ has a unique nonempty compact set $\mathscr{S}$, called the *Sirpiński gasket*, satisfying

$$\mathscr{S}=\bigcup_{i=1}^{N}f_i(\mathscr{S}).$$

Let $\mathscr{A}=\{1,2,3,\cdots,N\}$ be the *alphabet* of N letters $(N\geq 2)$ and

$$\mathscr{W}=\{w_1w_2w_3\cdots w_n;\ w_k\in\mathscr{A},\ n\geq 0\}$$

be the space of *finite words*, where we also allow $n=0$ to denote the *empty word* $\emptyset$. If $\mathbf{v}=v_1v_2v_3\cdots v_n$ and $\mathbf{w}=w_1w_2w_3\cdots w_{n'}$ are two words their *product* is defined by

$$\mathbf{vw}=v_1v_2v_3\cdots v_nw_1w_2w_3\cdots w_{n'},$$

and the *length of* $\mathbf{v}$ is denoted by $d(\mathbf{v})=n$. Let $\mathscr{W}_n$ denote the set of words of length $n\,(n\geq 0)$, $\mathscr{W}_+=\bigcup_{n=1}^{\infty}\mathscr{W}_n$ and Ξ the set of all $\mathscr{A}$-valued sequences. We define $d(\mathbf{x})=\infty$ for $\mathbf{x}\in\Xi$.

If a finite word $\mathbf{w}$ includes at least two different letters, then $\mathbf{w}$ has a representation $\mathbf{w}=\mathbf{u}ab^k$, where $\mathbf{u}\in\mathscr{W}$, $a,b\in\mathscr{A}$, $(a\neq b)$, and $k\geq 1$, and we define the *conjugate* of $\mathbf{w}$ by $\mathbf{w}^{\#}=\mathbf{u}ba^k$. If $\mathbf{w}$ contains at most one letter, then $\mathbf{w}=a^k$ for $a\in\mathscr{A}$ and $k\geq 0$ (where $a^0=\emptyset$) and we define the conjugate of $\mathbf{w}$ to be $\mathbf{w}^{\#}=a^k=\mathbf{w}$. Let $\mathscr{W}^{a}$ denote the set of all finite words $\mathbf{w}$ for which $\mathbf{w}\neq\mathbf{w}^{\#}$.

Similarly we define the conjugate of $\mathbf{x}\in\Xi$ by

$$\mathbf{x}^{\#}=\begin{cases} x_1\cdots x_n ba^{\infty}, & \text{if } \exists a, b (a\neq b)\in\mathscr{A} \text{ such that } \mathbf{x}=x_1\cdots x_n ab^{\infty} \\ \mathbf{x}, & \text{otherwise,}\end{cases}$$

where $x_1\cdots x_n ba^{\infty}$ denotes $x_1\cdots x_n baaa\cdots$. The conjugation defines an equivalence relation $\sim$ on Ξ by

$$\mathbf{x}\sim\mathbf{y} \rightleftharpoons \mathbf{x}=\mathbf{y} \text{ or } \mathbf{y}^{\#}.$$

It is known that the Sierpiński gasket $\mathscr{S}$ can be identified with the quotient space $\Xi/\sim$ and in the sequel we do not distinguish between them. Let Π: $\Xi\mapsto\Xi/\sim$ denote the canonical projection. We extend Π to a map defined on $\mathscr{W}\cup\Xi$ taking the identity operator on $\mathscr{W}$ and define Π_n: $\mathscr{W}\cup\Xi\mapsto\mathscr{W}, n\in\boldsymbol{N}$ by

$$\Pi_n(\mathbf{x}):=\begin{cases} x_1x_2\cdots x_n\in\mathscr{W}_n, & d(\mathbf{x})\geq n, \\ \mathbf{x}, & d(\mathbf{x})<n.\end{cases}$$

for $\mathbf{x}=x_1x_2x_3\cdots\in\mathscr{W}\cup\Xi$.

Let ν be the Bernoulli measure on Ξ, that is, the product measure $\nu=\Pi_{k=1}^{\infty}\nu_k$, where each ν_k $k\geq 1$, is the uniform probability measure on $\mathscr{A}$. It is known that $\mu=\nu\circ\Pi^{-1}$ is the normalized Hausdorff measure on $\mathscr{S}$.

In [DS] we considered the Markov chain $(X_n)_{n\geq 1}$ with state space $\mathscr{W}$ defined by the following transition probabilities $p(\mathbf{v},\mathbf{w})$, $\mathbf{v},\mathbf{w}\in\mathscr{W}$:

(a) For $\mathbf{v}=a^k$ where $a\in\mathscr{A}$ and $k\geq 0$

$$p(a^k,\mathbf{w})=\begin{cases} \dfrac{1}{N}, & \text{if } \exists c\in\mathscr{A} \text{ such that } \mathbf{w}=a^k c \\ 0, & \text{otherwise.}\end{cases}$$

(b) For $\mathbf{v}=\mathbf{u}ab^k$ where $\mathbf{u}\in\mathscr{W}$, $a,b\in\mathscr{A}$, $(a\neq b)$, and $k\geq 1$

$$p(\mathbf{u}ab^k,\mathbf{w})=\begin{cases} \dfrac{1}{2N}, & \text{if } \exists c\in\mathscr{A} \text{ such that } \mathbf{w}=\mathbf{v}c \text{ or } \mathbf{v}^{\#}c \\ 0, & \text{otherwise.}\end{cases}$$

The associated Markov operator P is defined by

$$Pf(\mathbf{v})=\sum_{\mathbf{w}\in\mathscr{W}} p(\mathbf{v},\mathbf{w})f(\mathbf{w}),$$

and a function f: $\mathscr{W}\to\boldsymbol{R}$ is called *harmonic* if $Pf=f$. Every harmonic function

h satisfies

$$h(\mathbf{v}) = h(\mathbf{v}^{\#}) \quad \textit{for } \mathbf{v} \in \mathscr{W}. \tag{2.1}$$

We call a function f on $\mathscr{W}$ *symmetric* if f satisfies (2.1).

The n-step transition probabilities are given by

$$p(0, \mathbf{v}, \mathbf{w}) = \delta_{\mathbf{v},\mathbf{w}}$$

$$p(n, \mathbf{v}, \mathbf{w}) = \sum_{\mathbf{z} \in \mathscr{W}} p(\mathbf{v}, \mathbf{z}) p(n-1, \mathbf{z}, \mathbf{w}), \quad n \geq 1, \quad \mathbf{v}, \mathbf{w} \in \mathscr{W},$$

the Green function $g(\mathbf{v}, \mathbf{w})$ by

$$g(\mathbf{v}, \mathbf{w}) = \sum_{n=0}^{\infty} p(n, \mathbf{v}, \mathbf{w}), \quad \mathbf{v}, \mathbf{w} \in \mathscr{W}.$$

and the Martin kernel by

$$k(\mathbf{v}, \mathbf{w}) = \frac{g(\mathbf{v}, \mathbf{w})}{g(\emptyset, \mathbf{w})}, \quad \mathbf{v}, \mathbf{w} \in \mathscr{W}.$$

It is shown in [DS] that $g(\emptyset, \mathbf{w}) = N^{-d(\mathbf{w})} > 0$,

$$g(\mathbf{v}, \mathbf{w}) = p(d(\mathbf{w}) - d(\mathbf{v}), \mathbf{v}, \mathbf{w})$$

and hence

$$k(\mathbf{v}, \mathbf{w}) = N^{d(\mathbf{w})} p(d(\mathbf{w}) - d(\mathbf{v}), \mathbf{v}, \mathbf{w}) \tag{2.2}$$

for $\mathbf{v}, \mathbf{w} \in \mathscr{W}$ such that $d(\mathbf{v}) \leq d(\mathbf{w})$.

For a finite word $\mathbf{w} = w_1 w_2 w_3 \cdots w_n \in \mathscr{W}_+$ define

$$\mathbf{w}^- = \begin{cases} w_1 w_2 w_3 \cdots w_{n-1}, & \text{if } n \geq 2, \\ \emptyset & \text{if } n = 1, \end{cases}$$

and define the cylinder set $\langle\!\langle \mathbf{w} \rangle\!\rangle$ in $\mathscr{W} \cup \Xi$ by

$$\langle\!\langle \mathbf{w} \rangle\!\rangle = \{\mathbf{u} = (u_k) \in \mathscr{W} \cup \Xi;\ d(\mathbf{u}) \geq d(\mathbf{w}) \text{ and } u_k = w_k,\ \forall 1 \leq k \leq d(\mathbf{w})\}.$$

We also use the notations $\langle \mathbf{w} \rangle = \langle\!\langle \mathbf{w} \rangle\!\rangle \cap \mathscr{W}_+$ and $[\mathbf{w}] = \langle\!\langle \mathbf{w} \rangle\!\rangle \cap \Xi$.

An explicit formula of the Martin kernel is derived in [DS], Theorem 3.4.

Lemma 2.1. *Let* $k(\mathbf{v},\mathbf{w})>0$. *Then either* $\mathbf{v}=\mathbf{w}$ *and* $k(\mathbf{v},\mathbf{w})=N^{d(\mathbf{v})}$, *or* $d(\mathbf{v})+1\leq d(\mathbf{w})$ *and* $\mathbf{w}$ *has the form*

$$\mathbf{w}=\mathbf{v}^{-}w_0w_1w_2\cdots w_nc \quad or \quad (\mathbf{v}^{\#})^{-}w_0w_1w_2\cdots w_nc.$$

In case $\mathbf{w}=\mathbf{v}^{-}w_0w_1w_2\cdots w_nc$ *we have*

$$k(\mathbf{v},\mathbf{w})=k(\mathbf{v}^{\#},\mathbf{w})=\frac{\theta(\mathbf{v})}{4}N^{d(\mathbf{v})}\left(\sum_{k=0}^{n}\frac{\mathbf{I}_{\tau(\mathbf{v})}(w_k)}{2^k}+\frac{\mathbf{I}_{\tau(\mathbf{v})}(w_n)}{2^n}\right), \tag{2.3}$$

where $\tau(\mathbf{v})$ *denotes the last letter of* $\mathbf{v}$, *and* $\theta(\mathbf{v})$ *is defined by*

$$\theta(\mathbf{v})=\begin{cases}1, & if \quad \mathbf{v}\neq\mathbf{v}^{\#},\\ 2, & if \quad \mathbf{v}=\mathbf{v}^{\#}.\end{cases}$$

In view of this, we define a metric ρ_M on $\mathscr{W}$ by

$$\rho_M(\mathbf{v},\mathbf{w})=|2^{-d(\mathbf{v})}-2^{-d(\mathbf{w})}|+\sum_{\mathbf{z}\in\mathscr{W}}(2N)^{-d(\mathbf{z})}|\,k(\mathbf{z},\mathbf{v})-k(\mathbf{z},\mathbf{w})|,$$

for $\mathbf{v},\mathbf{w}\in\mathscr{W}$ (cf. the introduction). Let $M=\overline{\mathscr{W}}$ be the ρ_M-completion of $\mathscr{W}$. Then (M,ρ_M) is a compact metric space and the functions $\mathbf{w}\mapsto k(\mathbf{v},\mathbf{w})$, $(\mathbf{v}\in\mathscr{W})$, are extended to M continuously. These extensions are also denoted by $k(\mathbf{v},\xi)$, $\xi\in\overline{\mathscr{W}},\mathbf{v}\in\mathscr{W}$. The boundary $\partial M=\overline{\mathscr{W}}\setminus\mathscr{W}$ is called the *Martin boundary* and can be identified with the Sierpiński gasket $\mathscr{S}$ (see [DS]). In fact, combined with [Dy] this result leads to

Theorem 2.2. [DS]

(1) *The function* $\mathbf{v}\rightarrow k_\xi(\mathbf{v})=k(\mathbf{v},\xi)$ *is harmonic in* $\mathbf{v}$ *for every* $\xi\in\mathscr{S}$.

(2) $\mathscr{S}$ *is the* space of exits *as defined in* [Dy].

(3) *For every harmonic function* $h\geq 0$ *there exists a unique finite measure* μ_h *on* $\mathscr{S}$ *such that*

$$h(\mathbf{v})=\int_{\mathscr{S}}k(\mathbf{v},\xi)\mu_h(d\xi).$$

(4) *For every bounded harmonic function h, there exists a unique bounded measurable function* φ *on* $\mathscr{S}$ *such that*

$$h(\mathbf{v})=\int_{\mathscr{S}} k(\mathbf{v},\xi)\varphi(\xi)\mu_1(d\xi), \tag{2.4}$$

$$\lim_{n\to\infty} h(X_n)=\varphi(X_\infty) \quad \mathbf{P}_{\mathbf{v}}-a.s., \ \forall \mathbf{v}\in\mathscr{W},$$

$$\exists X_\infty \text{ such that } h(\mathbf{v})=\mathbf{E}_{\mathbf{v}}[\varphi(X_\infty)], \quad \forall \mathbf{v}\in\mathscr{W}.$$

(5) *Conversely for every bounded measurable function* φ *on* $\mathscr{S}$

$$h_\varphi(\mathbf{v})=\int_{\mathscr{S}} k(\mathbf{v},\xi)\varphi(\xi)\mu_1(d\xi), \quad \mathbf{v}\in\mathscr{W} \tag{2.5}$$

defines a harmonic function on $\mathscr{W}$.

We shall denote the map sending a bounded measurable function φ on $\mathscr{S}$ to h_φ by $\mathscr{I}$, that is,

$$\mathscr{I}(\varphi)=h_\varphi,$$

and we call h_φ *the harmonic extension of* φ.

Theorem 2.3. *The harmonic measure* μ_1 *on* $\mathscr{S}$ *in Theorem* 2.2, *coincides with the canonical normalized Hausdorff measure* $\mu=\nu\circ\Pi^{-1}$.

Proof. By Theorem 2.2, the harmonic measure μ_1 is uniquely determine by

$$1=\int_{\mathscr{S}} k(\mathbf{v},\xi)\mu_1(d\xi), \quad \forall \mathbf{v}\in\mathscr{W}.$$

On the other hand for every $\mathbf{v}\in\mathscr{W}$ we have

$$k(\mathbf{v},\Pi(\mathbf{x}))=\lim_n k(\mathbf{v},\Pi_n(\mathbf{x})) \quad \text{and} \quad \sup_{n\geq 1, x\in\Xi} k(\mathbf{v},\Pi_n(\mathbf{x}))\leq N^{d(\mathbf{v})}.$$

Therefore, by the bounded convergence theorem, (2.2) and the definition of ν, for every $\mathbf{v}\in\mathscr{W}$ we have

$$\begin{aligned}&\int_{\Xi/\sim} k(\mathbf{v},\xi)\nu\circ\Pi^{-1}(d\xi)\\ &=\int_{\Xi} k(\mathbf{v},\Pi(\mathbf{x}))\nu(d\mathbf{x})\\ &=\int_{\Xi} \lim_n k(\mathbf{v},\Pi_n(\mathbf{x}))\nu(d\mathbf{x})\end{aligned}$$

$$=\lim_n \int_\Xi k(\mathbf{v}, \Pi_n(\mathbf{x}))\nu(d\mathbf{x})$$

$$=\lim_n \sum_{\mathbf{w}\in\mathscr{W}_n} k(\mathbf{v}, \mathbf{w})\nu([\mathbf{w}])$$

$$=\lim_n \sum_{\mathbf{w}\in\mathscr{W}_n} N^n p(n-d(\mathbf{v}), \mathbf{v}, \mathbf{w}) \frac{1}{N^n}=1.$$

Remark 2.4. The measure μ is *full* on $\mathscr{S}$, that is, every non-empty open subset of $\mathscr{S}$ has a positive measure.

Proof. Since ν is full on Ξ with respect to the product topology, and since the map Π is surjective and continuous, it is evident that $\mu=\nu\circ\Pi^{-1}$ is also full.

§3. The Intrinsic Metric

There is a natural metric on the Sierpiński gasket induced by the Euclidean norm $\|\xi-\eta\|$, $\xi, \eta\in\mathscr{S}$. In [DS] we defined another metric $\tilde{d}$ which is Lipschitz equivalent to $\|\xi-\eta\|$, but only defined on $\mathscr{S}(=\partial M=\overline{\mathscr{W}}\setminus\mathscr{W})$ and not on the word space $\mathscr{W}$. In this section, using the Martin kernel, we define a new metric ρ on $\mathscr{W}_+\cup\mathscr{S}$. The metric ρ is Lipschitz equivalent to the metric ρ_M of the Martin space M (by Lemma 2.1) and, when restricted to $\mathscr{S}$, is 'almost' Lipschitz equivalent to $\|\xi-\eta\|$.

This metric ρ on $\mathscr{W}\cup\mathscr{S}$ is defined by

$$\rho(\xi, \eta)=|2^{-d(\xi)}-2^{-d(\eta)}|+\sum_{n=1}^{\infty}\frac{1}{(2N)^n}\sup_{\mathbf{u}\in\mathscr{W}_n}|k(\mathbf{u}, \xi)-k(\mathbf{u}, \eta)|,$$

for $\xi, \eta\in\mathscr{W}_+\cup\mathscr{S}$, where $d(\xi)=+\infty$ for $\xi\in\mathscr{S}$, $\rho(\xi, \emptyset)=\rho(\emptyset, \xi)=1$ and $\rho(\emptyset, \emptyset)=0$.

ρ is Lipschitz equivalent to ρ_M. In fact, by definition, it is evident that $\rho\le\rho_M$. On the other hand by Lemma 2.1 $k(\mathbf{u}, \mathbf{w})$ does not vanish only if $\mathbf{u}=\Pi_{d(\mathbf{u})}(\mathbf{w})^- a$ or $\mathbf{u}=(\Pi_{d(\mathbf{u})}(\mathbf{w})^{\#})^- a$ for some $a\in\mathscr{A}$ so that

$$\sum_{\mathbf{u}\in\mathscr{W}_n}|k(\mathbf{u}, \mathbf{w})-k(\mathbf{u}, \mathbf{v})|\le 4N\sup_{\mathbf{u}\in\mathscr{W}_n}|k(\mathbf{u}, \mathbf{w})-k(\mathbf{u}, \mathbf{v})|,$$

which implies $\rho_M\le 4N\rho$.

Let $\mathbf{x},\mathbf{y}\in\mathscr{W}\cup\Xi$, define $\alpha(\mathbf{x},\mathbf{y})$ by

$$\alpha(\mathbf{x},\mathbf{y}):=\begin{cases}\min\{k\geq 1;\ \Pi_k(\mathbf{x})\neq\Pi_k(\mathbf{y})\}, & \text{if } \mathbf{x}\neq\mathbf{y}\\ +\infty, & \text{if } \mathbf{x}=\mathbf{y},\end{cases}$$

and $\beta(\mathbf{x},\mathbf{y})$ by

$$\beta(\mathbf{x},\mathbf{y}):=\begin{cases}\alpha(\mathbf{x},\mathbf{y}), & \text{if } \Pi_k(\mathbf{x})\notin\{\Pi_k(\mathbf{y}),\Pi_k(\mathbf{y})^{\#}\} \text{ for } \forall k>\alpha(\mathbf{x},\mathbf{y}),\\ \min\{l>\alpha(\mathbf{x},\mathbf{y}); \Pi_l(\mathbf{x})\neq\Pi_l(\mathbf{y}),\Pi_l(\mathbf{y})^{\#}\} & \\ \qquad \text{if } \Pi_k(\mathbf{x})\in\{\Pi_k(\mathbf{y}),\Pi_k(\mathbf{y})^{\#}\} \text{ for some } k>\alpha(\mathbf{x},\mathbf{y}), & \\ +\infty, & \text{if } \mathbf{x}=\mathbf{y} \text{ or } \mathbf{y}^{\#}\end{cases}$$

Note that $\alpha(\mathbf{x},\mathbf{y})=d(\mathbf{x})+1$ if $\mathbf{y}=\mathbf{x}\mathbf{z}$ $(\mathbf{x}\in\mathscr{W},\mathbf{z}\in\mathscr{W}_+\cup\Xi)$, and that $\Pi_k(\mathbf{x})=\Pi_k(\mathbf{y})$ for $k<\alpha(\mathbf{x},\mathbf{y})$. Obviously we have $\alpha(\mathbf{x},\mathbf{y})\leq\beta(\mathbf{x},\mathbf{y})$.

First we prove the following lemma.

Lemma 3.1. $\frac{1}{8}\cdot 2^{-\beta(\mathbf{x},\mathbf{y})}\leq\rho(\Pi(\mathbf{x}),\Pi(\mathbf{y}))$ *for every* $\mathbf{x},\mathbf{y}\in\mathscr{W}_+\cup\Xi$.

Proof. Let $\mathbf{x},\mathbf{y}\in\mathscr{W}_+\cup\Xi$. Then if $\mathbf{y}=\mathbf{x}$ or $\mathbf{x}^{\#}$, we have $\beta(\mathbf{x},\mathbf{y})=\infty$ so that the assertion is trivial.

Next, consider the case where $\mathbf{y}\neq\mathbf{x},\mathbf{x}^{\#},\mathbf{y}\in\mathscr{W}_+$ and $\mathbf{x}$ is an extension of $\mathbf{y}$, *i.e.* $\mathbf{x}$ has the form $\mathbf{x}=\mathbf{y}x_m x_{m+1}x_{m+2}\cdots$, where $m=d(\mathbf{y})+1$, whence $\alpha(\mathbf{x},\mathbf{y})=\beta(\mathbf{x},\mathbf{y})=m$. Therefore for $\mathbf{u}=\Pi_m(\mathbf{x})=\mathbf{y}x_m$ we have $k(\mathbf{u},\mathbf{y})=0$ and $k(\mathbf{u},\mathbf{x})\geq\frac{1}{4}N^m$ so that

$$\rho(\Pi(\mathbf{x}),\Pi(\mathbf{y}))\geq\frac{1}{(2N)^m}\frac{N^m}{4}=2^{-m-2}=\frac{1}{4}2^{-\beta(\mathbf{x},\mathbf{y})}.$$

Now consider the general case when $\mathbf{x}$ and $\mathbf{y}$ are neither dual nor an extension of the other sequence. Then we may write

$$\mathbf{x}=\mathbf{v}ab^p x_{m+p+1}x_{m+p+2}\cdots$$

$$\mathbf{y}=\mathbf{v}ba^q y_{m+q+1}y_{m+q+2}\cdots,$$

where $\mathbf{v}=x_1x_2\cdots x_{m-1}\in\mathscr{W}$, $m=\alpha(\mathbf{x},\mathbf{y})$, $a,b(a\neq b)\in\mathscr{A}$, $x_{m+p+1}(\neq b)$, x_{m+p+2}, $x_{m+p+3},\cdots,y_{m+q+1}(\neq a),y_{m+q+2},y_{m+q+3},\cdots\in\mathscr{A}\cup\{\emptyset\}$, and $0\leq p\leq q\leq+\infty$. Then, sicne $\mathbf{y}\neq\mathbf{x},\mathbf{x}^{\#}$, we have $m+p<\infty$ and

$$\beta(\mathbf{x},\mathbf{y})=\begin{cases} m, & \text{if } p=0,\\ m+p+1, & \text{if } p\geq 1.\end{cases}$$

Case 1. $d(\mathbf{x})\geq m+p+1$. In this case put $\mathbf{u}=\Pi_{m+p+1}(\mathbf{x})=\mathbf{v}ab^p x_{m+p+1}$. Then we have

$$\mathbf{u}^{\#}=\begin{cases} \mathbf{v}ab^{p-1}x_{m+p+1}b, & \text{if } 1\leq p\leq q,\\ \mathbf{v}x_{m+1}a, & \text{if } 0=p\leq q \text{ and } x_{m+1}\neq a,\\ \mathbf{v}^{-}a\tau(\mathbf{v})^2, & \text{if } 0=p\leq q \text{ and } x_{m+1}=a\neq\tau(\mathbf{v}),\\ \mathbf{v}^{\#}\tau(\mathbf{v}^{\#})^2, & \text{if } 0=p\leq q \text{ and } x_{m+1}=a=\tau(\mathbf{v}).\end{cases}$$

It follows that $\mathbf{u}^{-}$, $(\mathbf{u}^{\#})^{-}\neq\Pi_{m+p}(\mathbf{y})$ so that (by Lemma 2.1) $k(\mathbf{u},\Pi(\mathbf{y}))=0$ and $k(\mathbf{u},\Pi(\mathbf{x}))\geq\frac{\theta(\mathbf{u})}{4}N^{m+p+1}$. Consequently we obtain

$$\rho(\Pi(\mathbf{x}),\Pi(\mathbf{y}))\geq\frac{1}{(2N)^{m+p+1}}\,\frac{\theta(\mathbf{u})}{4}N^{m+p+1}\geq\frac{1}{4}\cdot 2^{-(m+p+1)}\geq\frac{1}{8}\cdot 2^{-\beta(\mathbf{x},\mathbf{y})}.$$

Case 2. $d(\mathbf{x})=m+p$, $p\geq 1$. Since $\mathbf{y}\neq\mathbf{x}^{\#}=\mathbf{v}ab^p$, we get that $\mathbf{y}=\mathbf{x}^{\#}a^{q-p}y_{m+p+1}\cdots$. For $\mathbf{u}=\Pi_{m+p+1}(\mathbf{y})$ it follows that $k(\mathbf{u},\Pi(\mathbf{x}))=0$ and $k(\mathbf{u},\Pi(\mathbf{y}))\geq\frac{\theta(\mathbf{u})}{2}N^{m+p+1}$ hence

$$\rho(\Pi(\mathbf{x}),\Pi(\mathbf{y}))\geq\frac{1}{(2N)^{m+p+1}}\,\frac{\theta(\mathbf{u})}{2}N^{m+p+1}\geq\frac{1}{2}\cdot 2^{-\beta(\mathbf{x},\mathbf{y})}.$$

Case 3. $d(\mathbf{x})=m<d(\mathbf{y})$ $(p=0)$. If $\mathbf{u}=\Pi_{m+1}(\mathbf{y})$, then $k(\mathbf{u},\Pi(\mathbf{x}))=0$ and $k(\mathbf{u},\Pi(\mathbf{y}))\ \geq\frac{1}{4}N^{m+1}$ whence

$$\rho(\Pi(\mathbf{x}),\Pi(\mathbf{y}))\geq\frac{1}{4(2N)^{m+1}}\ N^{m+1}=2^{-m-3}\geq\frac{1}{8}\cdot 2^{-\beta(\mathbf{x},\mathbf{y})}.$$

Case 4. $d(\mathbf{x})=m=d(\mathbf{y})$, $p=0$. Since $\mathbf{x}\neq\mathbf{y}$, for $\mathbf{u}=\mathbf{x}$, we have $k(\mathbf{u},\mathbf{y})=0$ and $k(\mathbf{u},\mathbf{x})=N^m$ so that

$$\rho(\Pi(\mathbf{x}),\Pi(\mathbf{y}))\geq\frac{1}{(2N)^m}\ N^m=2^{-m}=2^{-\beta(\mathbf{x},\mathbf{y})}.$$

The estimate in the previous lemma is sharp as the following example shows. However, restricting the metric on $\mathscr{S}$ we are able to improve the estimate as shown in Lemma 3.3 below.

Example 3.2. *Let* $a \in \mathscr{A}$. *Then* $\alpha(a^\infty, a^m)=\beta(a^\infty, a^m)=m+1$ *and* $\rho(\Pi(a^\infty),a^m)=4\cdot 2^{-\beta(a^\infty,a^m)}$ *for every* $m\geq 2$.

A direct calculation using (2.3) shows that

$$k(\mathbf{u}, a^m)=\begin{cases} N^n, & \text{if } 1\leq n\leq m, \quad \mathbf{u}=a^n \\ 0, & \text{otherwise} \end{cases}$$

and

$$k(\mathbf{u}, a^\infty)=\begin{cases} N^n, & \text{if } \mathbf{u}=a^n \text{ for some } n\geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

It follows that

$$\rho(\Pi(a^\infty), a^m)=2^{-m}+\sum_{n=1}^{m}\frac{0}{(2N)^n}+\sum_{n=m+1}^{\infty}\frac{1}{(2N)^n}N^n=\frac{1}{2^{m-1}}=4\cdot 2^{-\beta(a^\infty,a^m)}.$$

Lemma 3.3. $\rho(\Pi(\mathbf{x}),\Pi(\mathbf{y}))\geq\frac{1}{8}\beta(\mathbf{x},\mathbf{y})\ 2^{-\beta(\mathbf{x},\mathbf{y})}$ *for any* $\mathbf{x},\mathbf{y}\in\Xi$.

Proof. We may assume that $\mathbf{y}\neq\mathbf{x},\mathbf{x}^\#$. Let $s(\mathbf{u})=|k(\mathbf{u},\Pi(\mathbf{x}))-k(\mathbf{u},\Pi(\mathbf{y}))|$ for $\mathbf{u}\in\mathscr{W}_+$. Then, as before, we may write

$$\mathbf{x}=\mathbf{v}ab^pcx_{m+p+2}x_{m+p+3}\cdots$$
$$\mathbf{y}=\mathbf{v}ba^qy_{m+q+1}y_{m+q+2}\cdots,$$

where $\mathbf{v}=v_1v_2\cdots v_{m-1}\in\mathscr{W}$, $m=\alpha(\mathbf{x},\mathbf{y})$, $a,b(a\neq b)\in\mathscr{A}$, $c(\neq b)$, $x_{m+p+2},x_{m+p+3},\cdots$, $y_{m+q+1}(\neq a)$, y_{m+q+2}, $y_{m+q+3},\cdots\in\mathscr{A}$, and $0\leq p\leq q\leq+\infty$. Then, since $\mathbf{y}\neq\mathbf{x},\mathbf{x}^\#$, we have $m+p<\infty$ and

$$\beta(\mathbf{x},\mathbf{y})=\begin{cases} m, & \text{if } p=0, \\ m+p+1, & \text{if } p\geq 1. \end{cases}$$

Fix $1\leq n\leq m$. Then for $\mathbf{u}=v_1v_2\cdots v_{n-1}b$ it follows from (2.3) again that

$k(\mathbf{u},\Pi(\mathbf{x}))$

$$=\frac{\theta(\mathbf{u})}{4}N^n\left(\sum_{k=0}^{m-1-n}\frac{\mathbb{I}_b(v_{n+k})}{2^k}+\sum_{k=m-n+1}^{m+p-n}\frac{1}{2^k}+\sum_{k=m+p-n+1}^{\infty}\frac{\mathbb{I}_b(x_{n+k})}{2^k}\right)$$

$$\leq\frac{\theta(\mathbf{u})}{4}N^n\left(\sum_{k=0}^{m-1-n}\frac{\mathbb{I}_b(v_{n+k})}{2^k}+\sum_{k=m-n+1}^{m-n+p}\frac{1}{2^k}+\sum_{k=m+p-n+2}^{\infty}\frac{1}{2^k}\right)$$

(since $c\neq b$) and

$$k(\mathbf{u},\Pi(\mathbf{y}))\geq\frac{\theta(\mathbf{u})}{4}N^n\left(\sum_{k=0}^{m-1-n}\frac{\mathbb{I}_b(v_{n+k})}{2^k}+\frac{1}{2^{m-n}}\right).$$

Notice that $k(\mathbf{u},\Pi(\mathbf{y}))\geq k(\mathbf{u},\Pi(\mathbf{x}))$, hence

$$s(\mathbf{u})\geq k(\mathbf{u},\Pi(\mathbf{y}))-k(\mathbf{u},\Pi(\mathbf{x}))\geq\frac{\theta(\mathbf{u})}{4}N^n\frac{1}{2^{m-n+p+1}}.$$

Fix $m<n\leq m+p$. Then for $\mathbf{u}=\mathbf{v}ab^{n-m-1}c$

$$k(\mathbf{u},\Pi(\mathbf{x}))\geq\frac{\theta(\mathbf{u})}{4}N^n\frac{1}{2^{m+p-n+1}}$$

and it is not difficult to show using Lemma 2.1 that $\Pi_{n-1}(\mathbf{y})\neq\mathbf{u}^-$, $(\mathbf{u}^\#)^-$, hence $k(\mathbf{u},\Pi(\mathbf{y}))=0$.

As a result wer get

$$s(\mathbf{u})\geq k(\mathbf{u},\Pi(\mathbf{x}))-k(\mathbf{u},\Pi(\mathbf{y}))\geq\frac{\theta(\mathbf{u})}{4}N^n\frac{1}{2^{m-n+p+1}}$$

and finally

$$\rho(\Pi(\mathbf{x}),\Pi(\mathbf{y}))\geq\sum_{n=1}^{m+p}\frac{1}{(2N)^n}\frac{\theta(\mathbf{u})}{4}N^n\frac{1}{2^{m-n+p+1}}.$$

$$=\frac{m+p}{4}\frac{1}{2^{m+p+1}}\geq\frac{1}{8}\beta(\mathbf{x},\mathbf{y})2^{-\beta(\mathbf{x},\mathbf{y})}.$$

Example 3.4. *Let $N\geq 3$ and let a,b,c be different letters in $\mathscr{A}$. Define $\mathbf{x}=ab^pc^\infty$ and $\mathbf{y}=ba^{p+1}c^\infty$ for some $p\geq 1$. It can be calculated in a similar way as above that $\alpha(\mathbf{x},\mathbf{y})=1$, $\beta(\mathbf{x},\mathbf{y})=p+2$ and*

$$\rho(\Pi(\mathbf{x},\Pi(\mathbf{y}))=\frac{\beta(\mathbf{x},\mathbf{y})+5}{4}2^{-\beta(\mathbf{x},\mathbf{y})}.$$

The details are left to the reader.

Next we give an upper estimate for $\rho(\Pi(\mathbf{x},\Pi(\mathbf{y}))$ for $\mathbf{x}\in\Xi$ and $\mathbf{y}\in\mathscr{W}_+\cup\Xi$.

Lemma 3.5. $\rho(\Pi(\mathbf{x}),\Pi(\mathbf{y}))\le 6\beta(\mathbf{x},\mathbf{y})2^{-\beta(\mathbf{x},\mathbf{y})}$
for any $\mathbf{x}\in\Xi$ *and* $\mathbf{y}\in\mathscr{W}_+\cup\Xi$.

Proof. If $\Pi(\mathbf{x})=\Pi(\mathbf{y})$, then $\mathbf{x}=\mathbf{y}$ or $\mathbf{y}^\#$. It follows that $\rho(\Pi(\mathbf{x}),\Pi(\mathbf{y}))=0$ and $\beta(\mathbf{x},\mathbf{y})=\infty$, proving the lemma in this case.

Let $\mathbf{x},\mathbf{y}\in\Xi$ and assume $\Pi(\mathbf{x})\ne\Pi(\mathbf{y})$. Then without loss of generality $\mathbf{x}$ and $\mathbf{y}$ have representations

$$\mathbf{x}=(x_k)=\mathbf{v}ab^p x_{m+p+1}x_{m+p+2}\cdots \text{ and } \mathbf{y}=(y_k)=\mathbf{v}ba^q y_{m+q+1}y_{m+q+2}\cdots,$$

where $\mathbf{v}=v_1v_2\cdots v_{m-1}$, $a\ne b$, $a\ne y_{m+q+1}$, $b\ne x_{m+p+1}$, and $0\le p\le q\le+\infty$. We have $\alpha(\mathbf{x},\mathbf{y})=m$, and $\beta(\mathbf{x},\mathbf{y})=m$ if $p=0$ and $\beta(\mathbf{x},\mathbf{y})=m+p+1$ if $p\ge 1$. Since $\Pi(\mathbf{x})\ne\Pi(\mathbf{y})$, $m+p<+\infty$.

Let $\mathbf{u}\in\mathscr{W}$ be arbitrary.

We first consider the case when $d(\mathbf{u})\le m$. Then $k(\mathbf{u},\Pi(\mathbf{x}))$ and $k(\mathbf{u},\Pi(\mathbf{y}))>0$ only if either $\mathbf{u}^-$ or $(\mathbf{u}^\#)^-=\Pi_{d(\mathbf{u})-1}(\mathbf{x})=\Pi_{d(\mathbf{u})-1}(\mathbf{y})$ and we have

$$k(\mathbf{u},\Pi(\mathbf{x}))-k(\mathbf{u},\Pi(\mathbf{y}))=\frac{\theta(\mathbf{u})}{4}N^{d(\mathbf{u})}\sum_{k=d(\mathbf{u})}^{\infty}\left(\frac{\mathbf{I}_{\tau(\mathbf{u})}(x_k)}{2^{k-d(\mathbf{u})}}-\frac{\mathbf{I}_{\tau(\mathbf{u})}(y_k)}{2^{k-d(\mathbf{u})}}\right)$$

or

$$=\frac{\theta(\mathbf{u})}{4}N^{d(\mathbf{u})}\sum_{k=d(\mathbf{u})}^{\infty}\left(\frac{\mathbf{I}_{\tau(\mathbf{u}^\#)}(x_k)}{2^{k-d(\mathbf{u})}}-\frac{\mathbf{I}_{\tau(\mathbf{u}^\#)}(y_k)}{2^{k-d(\mathbf{u})}}\right).$$

Consider the first case, i.e. $\mathbf{u}^-=\Pi_{d(\mathbf{u})-1}(\mathbf{x})=\Pi_{d(\mathbf{u})-1}(\mathbf{y})$. If $\tau(\mathbf{u})\ne a,b$, then

$$|k(\mathbf{u},\Pi(\mathbf{x}))-k(\mathbf{u},\Pi(\mathbf{y}))|\le\frac{\theta(\mathbf{u})}{4}N^{d(\mathbf{u})}\sum_{k=m+p-d(\mathbf{u})+1}^{\infty}\frac{1}{2^k}\le\frac{N^{d(\mathbf{u})}}{2^{m+p-d(\mathbf{u})}}.$$

If $\tau(\mathbf{u})=a$, then

$$\sum_{k=m-d(\mathbf{u})}^{m+p-d(\mathbf{u})}\left(\frac{\mathbf{I}_a(x_{d(\mathbf{u})+k})}{2^k}-\frac{\mathbf{I}_a(y_{d(\mathbf{u})+k})}{2^k}\right)=\frac{1}{2^{m-d(\mathbf{u})}}-\sum_{k=m-d(\mathbf{u})+1}^{m+p-d(\mathbf{u})}\frac{1}{2^k}=\frac{1}{2^{m+p-d(\mathbf{u})}}.$$

Similarly if $\tau(\mathbf{u})=b$, then we have

$$\sum_{k=m-d(\mathbf{u})}^{m+p-d(\mathbf{u})}\left(\frac{\mathbf{I}_b(x_{d(\mathbf{u})+k})}{2^k}-\frac{\mathbf{I}_b(y_{d(\mathbf{u})+k})}{2^k}\right)=\sum_{k=m-d(\mathbf{u})+1}^{m+p-d(\mathbf{u})}\frac{1}{2^k}-\frac{1}{2^{m-d(\mathbf{u})}}=\frac{-1}{2^{m+p-d(\mathbf{u})}}.$$

The case where $(\mathbf{u}^\#)^- = \Pi_{d(\mathbf{u})-1}(\mathbf{x}) = \Pi_{d(\mathbf{u})-1}(\mathbf{y})$ is similar. Consequently we have

$$|k(\mathbf{u},\Pi(\mathbf{x}))-k(\mathbf{u},\Pi(\mathbf{y}))|\leq 2\frac{\theta(\mathbf{u})}{4}\,N^{d(\mathbf{u})}\frac{1}{2^{m+p-d(\mathbf{u})}}\leq\frac{N^{d(\mathbf{u})}}{2^{m+p-d(\mathbf{u})}}.$$

Now consider the case where $m<d(\mathbf{u})\leq m+p$. Then $k(\mathbf{u},\Pi(\mathbf{x}))>0$ if and only if $\mathbf{u}^-$ or $(\mathbf{u}^\#)^- = \Pi_{d(\mathbf{u})-1}(\mathbf{x})$.

Assume first that $\mathbf{u}^- = \Pi_{d(\mathbf{u})-1}(\mathbf{x})$. Then $k(\mathbf{u},\Pi(\mathbf{y}))>0$ if and only if $\tau(\mathbf{u})=b$ where $\mathbf{u}^\# = \Pi_{d(\mathbf{u})}(\mathbf{y})$. Therefore, if $\tau(\mathbf{u})=b$, we have

$$|k(\mathbf{u},\Pi(\mathbf{x}))-k(\mathbf{u},\Pi(\mathbf{y}))|\leq\frac{1}{2}\,N^{d(\mathbf{u})}\sum_{k=m+p-d(\mathbf{u})+1}^{\infty}\frac{1}{2^k}\leq\frac{N^{d(\mathbf{u})}}{2^{m+p-d(\mathbf{u})+1}}.$$

If $\tau(\mathbf{u})\neq b$ we have

$$\begin{aligned}|k(\mathbf{u},\Pi(\mathbf{x}))-k(\mathbf{u},\Pi(\mathbf{y}))|=k(\mathbf{u},\Pi(\mathbf{x}))&\leq\frac{1}{2}\,N^{d(\mathbf{u})}\sum_{k=m+p-d(\mathbf{u})+1}^{\infty}\frac{1}{2^k}\\&\leq\frac{N^{d(\mathbf{u})}}{2^{m+p-d(\mathbf{u})+1}}.\end{aligned}$$

Replacing $\mathbf{u}^-$ by $(u\,\sharp)^-$, i.e. $(\mathbf{u}^\#)^- = \Pi_{d(\mathbf{u})-1}(\mathbf{x})$ and $\mathbf{u}^-$ or $(\mathbf{u}^\#)^- = \Pi_{d(\mathbf{u})-1}(\mathbf{y})$, we obtain the analogous estimate

$$|k(\mathbf{u},\Pi(\mathbf{x}))-k(\mathbf{u},\Pi(\mathbf{y}))|=k(\mathbf{u},\Pi(\mathbf{x}))\leq\frac{N^{d(\mathbf{u})}}{2^{m+p-d(\mathbf{u})+1}}.$$

Last, consider the case where $d(\mathbf{u})>m+p$. Then we have

$$|k(\mathbf{u},\Pi(\mathbf{x}))-k(\mathbf{u},\Pi(\mathbf{y}))|\leq\frac{\theta(\mathbf{u})}{4}\,N^{d(\mathbf{u})}\sum_{k=0}^{\infty}\frac{1}{2^k}\leq N^{d(\mathbf{u})}.$$

The above estimations show that

$$\rho(\Pi(\mathbf{x}), \Pi(\mathbf{y}))$$

$$\leq \sum_{n=1}^{m} \frac{1}{(2N)^n} \frac{N^n}{2^{m+p-n}} + \sum_{n=m+1}^{m+p} \frac{1}{(2N)^n} \frac{N^n}{2^{m+p-n+1}} + \sum_{n=m+p+1}^{\infty} \frac{N^n}{(2N)^n}$$

$$= 2^{-(m+p)}m + p2^{-(m+p+1)} + 2^{-(m+p)} \leq 2\beta(\mathbf{x},\mathbf{y})2^{-\beta(\mathbf{x},\mathbf{y})}.$$

It is left to consider the case where $\mathbf{x} \in \Xi$ and $\mathbf{y} \in \mathscr{W}_+$.
Define $s(\mathbf{u}) = |k(\mathbf{u}, \Pi(\mathbf{x})) - k(\mathbf{u}, \mathbf{y})|$ for $\mathbf{u} \in \mathscr{W}_+$. We have $\beta(\mathbf{x},\mathbf{y}) \leq d(\mathbf{y}) + 1 < +\infty$.

Assume first that $\mathbf{x}$ is an extension of $\mathbf{y}$. Then $\mathbf{x}$ has the form $\mathbf{x} = \mathbf{y}x_m x_{m+1} x_{m+2} \cdots$ where $m = d(\mathbf{y}) + 1 (\geq 2)$, hence $m = \alpha(\mathbf{x},\mathbf{y}) = \beta(\mathbf{x},\mathbf{y})$.

We consider three cases. First, let $d(\mathbf{u}) \geq m$. Since $d(\mathbf{u}) > d(\mathbf{y})$, we have $k(\mathbf{u}, \mathbf{y}) = 0$ and

$$s(\mathbf{u}) = k(\mathbf{u}, \Pi(\mathbf{x})) \leq \frac{\theta(\mathbf{u})}{4} N^{d(\mathbf{u})} \sum_{k=0}^{\infty} \frac{1}{2^k} \leq N^{d(\mathbf{u})}.$$

Next, let $d(\mathbf{u}) = m-1$. Then, if $\mathbf{u} = \mathbf{y}$ we have

$$s(\mathbf{u}) = \left| N^{m-1} - \frac{\theta(\mathbf{u})}{4} N^{m-1} \sum_{k=0}^{\infty} \frac{\mathbb{I}_{\tau(\mathbf{y})}(x_{m-1+k})}{2^k} \right| \leq \frac{3}{4} N^{m-1},$$

and if $\mathbf{u} \neq \mathbf{y}$ we have $k(\mathbf{u}, \mathbf{y}) = 0$ so that $s(\mathbf{u}) \leq \frac{1}{2} N^{m-1}$.

Finally, let $1 \leq d(\mathbf{u}) \leq m-2$, which implies $m \geq 3$. Then $k(\mathbf{u}, \Pi(\mathbf{x})) = k(\mathbf{u}, \mathbf{y}) = 0$ unless $\mathbf{u}^-$ or $(\mathbf{u}^\#)^- = \Pi_{d(\mathbf{u})-1}(\mathbf{y})$. If $\mathbf{u}^- = \Pi_{d(\mathbf{u})-1}(\mathbf{y})$ then we have

$$s(\mathbf{u})$$

$$\leq \frac{\theta(\mathbf{u})}{4} N^{d(\mathbf{u})} \left| \sum_{k=0}^{\infty} \frac{\mathbb{I}_{\tau(\mathbf{y})}(x_{d(\mathbf{u})+k})}{2^k} - \sum_{k=0}^{d(\mathbf{y})-1-d(\mathbf{u})} \frac{\mathbb{I}_{\tau(\mathbf{u})}(y_{d(\mathbf{u})+k})}{2^k} - \frac{\mathbb{I}_{\tau(\mathbf{u})}(y_{d(\mathbf{y})-1})}{2^{d(\mathbf{y})-1-d(\mathbf{u})}} \right|$$

$$\leq \frac{\theta(\mathbf{u})}{4} N^{d(\mathbf{u})} \left(\sum_{k=d(\mathbf{y})+1-d(\mathbf{u})}^{\infty} \frac{1}{2^k} + \frac{1}{2^{d(\mathbf{y})-d(\mathbf{u})}} \right)$$

$$\leq \frac{\theta(\mathbf{u})}{4} N^{d(\mathbf{u})} \frac{2}{2^{d(\mathbf{y})-d(\mathbf{u})}} \leq \frac{1}{2^{m-d(\mathbf{u})}} N^{d(\mathbf{u})}.$$

We have the same estimate when $(\mathbf{u}^\#)^- = \Pi_{d(\mathbf{u})-1}(\mathbf{y})$.

Combining the above estimations and observing that $m \geq 2$ we get

$$\rho(\Pi(\mathbf{x}),\Pi(\mathbf{y}))=\rho(\Pi(\mathbf{x}),\mathbf{y})$$

$$\leq 2^{-(m-1)}+\sum_{n=1}^{m-2}\frac{1}{(2N)^n}\frac{N^n}{2^{m-n}}+\frac{3}{4}\cdot\frac{1}{(2N)^{m-1}}N^{m-1}+\sum_{n=m}^{\infty}\frac{N^n}{(2N)^n}$$

$$=2^{-m+1}+(m-2)2^{-m}+\frac{3}{2}\cdot 2^{-m}+2\cdot 2^{-m}$$

$$\leq 3m2^{-m}\leq 3\beta(\mathbf{x},\mathbf{y})2^{-\beta(\mathbf{x},\mathbf{y})}.$$

Finally, consider the case where $\mathbf{x}$ is not an extension of $\mathbf{y}$. Then without loss of generality $\mathbf{x}$ and $\mathbf{y}$ are expressed in the form

$$\mathbf{x}=\mathbf{v}ab^p x_{m+p+1}x_{m+p+2}\cdots \quad\text{and}\quad \mathbf{y}=\mathbf{v}ba^p y_{m+p+1}y_{m+p+2}\cdots y_{d(\mathbf{y})},$$

where $\mathbf{v}=v_1v_2\cdots v_{m-1}$, $a\neq b$, and $1\leq m\leq m+p<+\infty$ and either $a\neq y_{m+q+1}$ or $b\neq x_{m+p+1}$. We have $\alpha(\mathbf{x},\mathbf{y})=m$, and $\beta(\mathbf{x},\mathbf{y})=m$ if $p=0$ and $\beta(\mathbf{x},\mathbf{y})=m+p+1$ if $p\geq 1$.

Applying (2.3) to $k(\mathbf{u},\mathbf{y})$, then by similar arguments as those in the case $\mathbf{x},\mathbf{y}\in\Xi$ (observe that an additional term $2^{-d(\mathbf{y})+d(\mathbf{u})}$ appears due to the fact that $\mathbf{y}$ is a finite word) we have

$$s(\mathbf{u})\leq\begin{cases}(2^{-(m+p-d(\mathbf{u}))}+2^{-(d(\mathbf{y})-d(\mathbf{u}))})N^{d(\mathbf{u})}, & \text{if } 1\leq d(\mathbf{u})\leq m,\\ (2^{-(m+p-d(\mathbf{u})+1)}+2^{-(d(\mathbf{y})-d(\mathbf{u}))})N^{d(\mathbf{u})}, & \text{if } m<d(\mathbf{u})\leq m+p,\\ 3N^{d(\mathbf{u})}, & \text{if } d(\mathbf{u})>m+p,\end{cases}$$

so that

$$\rho(\Pi(\mathbf{x}),\mathbf{y})\leq 2^{-d(\mathbf{y})}+\sum_{n=1}^{\infty}\frac{1}{(2N)^n}\sup_{\mathbf{u}\in\mathscr{W}_n}s(\mathbf{u})$$

$$\leq\frac{3(m+p+1)}{2^{m+p}}\leq 6\beta(\Pi(\mathbf{x}),\mathbf{y})2^{-\beta(\Pi(\mathbf{x}),\mathbf{y})}.$$

Lemma 3.6. *There exists a positive constant $A=A_N$ (depending only on N) such that*

$$A2^{-\beta(\mathbf{x},\mathbf{y})-1}\leq\|\Pi(\mathbf{x})-\Pi(\mathbf{y})\|\leq 4\cdot 2^{-\beta(\mathbf{x},\mathbf{y})}$$

for all $\mathbf{x},\mathbf{y}\in\Xi$.

Note that under our assumption of $|\Delta|=1$ we have $A\leq 1$, and in the case of a symmetric simplex Δ of diameter 1, it follows from elementary calculations that $A_2=\frac{1}{4}$ and $16A_N^2=1-(8A_{N-1})^{-2}$.

Proof. Let $\mathbf{x},\mathbf{y}\in\Xi$.

If $\Pi(\mathbf{x})=\Pi(\mathbf{y})$, then we have $\beta(\mathbf{x},\mathbf{y})=\infty$ and nothing has to be shown.

Therefore, we may assume that $\Pi(\mathbf{x})\neq\Pi(\mathbf{y})$. Then $\beta=\beta(\mathbf{x},\mathbf{y})<+\infty$, and $\Pi_{\beta-1}(\mathbf{x})=\Pi_{\beta-1}(\mathbf{y})$ or $=\Pi_{\beta-1}(\mathbf{y})^{\#}$.

If $\mathbf{w}=w_1\cdots w_l$ is a finite word, we denote by Δ (w) the image of Δ under the composition of naps $f_{w_l}\circ\cdots\circ f_{w_1}$.

In the first case we have $\Pi(\mathbf{x})$, $\Pi(\mathbf{y})\in\Delta(\Pi_{\beta-1}(\mathbf{x}))$ so that

$$\|\xi-\eta\|\leq|\Delta(\Pi_{\beta-1}(\mathbf{x}))|=2^{1-\beta}.$$

In the second case $\Delta(\Pi_{\beta-1}(\mathbf{x}))$ and $\Delta(\Pi_{\beta-1}(\mathbf{x})^{\#})$ have a point $\zeta=\Pi(\mathbf{z})=\Pi(\mathbf{z}^{\#})$ in common, where $\mathbf{z}=\Pi_{\beta-1}(\mathbf{x})\tau(\Pi_{\beta-1}(\mathbf{x}))^{\infty}$. Therefore

$$\|\Pi(\mathbf{x})-\Pi(\mathbf{y})\|\leq\|\Pi(\mathbf{x})-\zeta\|+\|\zeta-\Pi(\mathbf{y})\|\leq 2|\Delta(\Pi_{\beta-1}(\mathbf{x}))|=2^{2-\beta}.$$

In order to derive the lower bound, notice that $\Delta(\Pi_{\beta+1}(\mathbf{x}))$ and $\Delta(\Pi_{\beta+1}(\mathbf{y}))$ do not have a point in common, hence

$$\|\Pi(\mathbf{x})-\Pi(\mathbf{y})\|\geq A\cdot|\Delta(\Pi_{\beta+1}(\mathbf{x}))|=A\cdot 2^{-\beta-1},$$

where A is the minimum distance of two disjoint triangles in $\{\Delta(\mathbf{w});\mathbf{w}\in\mathscr{W}_2\}$.

Combining Lemmas 3.3, 3.5 and 3.6, we can compare $\rho(\xi,\eta)$ and $\|\xi-\eta\|$ on $\mathscr{S}$.

Theorem 3.7. *For any $\xi,\eta\in\mathscr{S}$, we have*

$$\frac{1}{32}\|\xi-\eta\|\log_2\frac{1}{\|\xi-\eta\|}\leq\rho(\xi,\eta)\leq\frac{12}{A}\|\xi-\eta\|\log_2\frac{A}{\|\xi-\eta\|}.$$

Proof. This follows from a direct calculation observing that the function $n\mapsto ne^{-n}$ is decreasing in n.

The above theorem shows that the Hausdorff dimensions H of $\mathscr{S}$ with respect to the metrics $\|\xi-\eta\|$ and $\rho(\xi,\eta)$ coincide. It also follows from standard considerations that the H-dimensional Hausdorff measure with respect to the metric ρ is infinite.

Our next aim is to characterize the cylinder sets by the metric ρ.

Lemma 3.8. $\rho(\mathbf{v}, \mathbf{w}) \le 6\alpha(\mathbf{v}, \mathbf{w}) 2^{-\alpha(\mathbf{v},\mathbf{w})}$ *for* $\mathbf{v}, \mathbf{w} \in \mathscr{W}_+$.

Proof. Without loss of generality we may assume $\mathbf{v} \neq \mathbf{w}$ and

$$1 \le d(\mathbf{v}) \le d(\mathbf{w}) \quad \text{and} \quad m = \alpha(\mathbf{v}, \mathbf{w}) \le d(\mathbf{v}) + 1.$$

Let $m=1$. Since $k(\mathbf{u}, \mathbf{w}) \le N^{d(\mathbf{u})}$ for any $\mathbf{u} \in \mathscr{W}_+$ and $\mathbf{w} \in \mathscr{W}_+$, we have

$$\rho(\mathbf{v}, \mathbf{w}) 2^{-d(\mathbf{v})} + \sum_{n=1}^{\infty} \frac{2N^n}{(2N)^n} \le 2^{-1} + 2 = 5 \cdot 2^{-\alpha(\mathbf{v},\mathbf{w})}.$$

Assume $m \ge 2$. Then we have $v_k = w_k$, $1 \le k \le m-1$.

For $\mathbf{u} \in \mathscr{W}_+$ such that $d(\mathbf{u}) \le m-1$ we have $w_1 w_2 \cdots w_{m-1} = v_1 v_2 \cdots v_{m-1}$ and $k(\mathbf{u}, \mathbf{v})$ and $k(\mathbf{u}, \mathbf{w})$ do not vanish if and only if $\mathbf{u}^-$ or $(\mathbf{u}^{\#})^- = w^1 w^2 \cdots w_{d(\mathbf{u})-1}$, where $w_0 = \emptyset$. In each of the cases we get

$$|k(\mathbf{u}, \mathbf{v}) - k(\mathbf{u}, \mathbf{w})| \le \frac{\theta(\mathbf{u})}{4} N^{d(\mathbf{u})} \left(\sum_{k \ge m - d(\mathbf{u})} \frac{1}{2^k} + \frac{1}{2^{d(\mathbf{v})-1}} + \frac{1}{2^{d(\mathbf{w})-1}} \right)$$

$$\le \frac{3}{2} N^{d(\mathbf{u})} 2^{-(m-d(\mathbf{u}))}.$$

For $\mathbf{u} \in \mathscr{W}_+$ such that $d(\mathbf{u}) \ge m$ we have

$$|k(\mathbf{u}, \mathbf{v}) - k(\mathbf{u}, \mathbf{w})| \le \frac{\theta(\mathbf{u})}{4} N^{d(\mathbf{u})} \left(\sum_{k=0}^{\infty} \frac{1}{2^k} + \frac{1}{2^{d(\mathbf{v})-1}} + \frac{1}{2^{d(\mathbf{w})-1}} \right) \le 2 N^{d(\mathbf{u})}$$

so that

$$\rho(\mathbf{v}, \mathbf{w}) \le 2^{-d(\mathbf{v})} + \frac{3}{2} \sum_{n=1}^{m-1} \frac{2^{n-m} N^n}{(2N)^n} + 2 \sum_{n=m}^{\infty} \frac{2N^n}{(2N)^n}$$

$$\le \frac{2}{2^m} + \frac{3}{2} \frac{m-1}{2^m} + 2 \cdot \frac{4}{2^m} \le \frac{6m}{2^m} \le 6\alpha(\mathbf{v}, \mathbf{w}) 2^{-\alpha(\mathbf{v},\mathbf{w})}.$$

As a corollary of Lemma 3.5 and 3.8 we derive

Lemm 3.9. *For any* $\mathbf{w} \in \mathscr{W}_+$ *we have*

$$\rho(\mathbf{w}, \Pi(\mathbf{x})) \le 6\alpha(\mathbf{w}, \mathbf{x}) 2^{-\alpha(\mathbf{w},\mathbf{x})} \le 6 d(\mathbf{w}) 2^{-d(\mathbf{w})} \quad \text{for } \forall \mathbf{x} \in \langle\!\langle \mathbf{w} \rangle\!\rangle.$$

Proof. Let $\mathbf{w} \in \mathscr{W}_+$. If $\mathbf{x} \in \langle \mathbf{w} \rangle$ then $\alpha(\mathbf{w}, \mathbf{x}) = d(\mathbf{w}) + 1$ and by Lemma 3.8

$$\rho(\mathbf{w}, \mathbf{x}) \le 6\alpha(\mathbf{w}, \mathbf{x}) 2^{-\alpha(\mathbf{w},\mathbf{x})} = 6(d(\mathbf{w}) + 1) 2^{-(d(\mathbf{w})+1)} \le 6 d(\mathbf{w}) 2^{-d(\mathbf{w})}.$$

If $x \in [w]$, then by Lemma 3.5

$$\rho(w, \Pi(x)) \leq 6\beta(w, x)2^{-\beta(w,x)} \leq 6\alpha(w, x)2^{-\alpha(w,x)} = (3d(w)+3)2^{-d(w)},$$

since $\alpha(w, x) \leq \beta(w, x)$.

Lemma 3.10. *Let* $w \in \mathscr{W}_+$ *and* $x \in \mathscr{W}_+ \cup \Xi$. *Then* $\rho(w, \Pi(x)) < \frac{1}{8} 2^{-d(w)}$ *implies* $x \in \langle\langle w \rangle\rangle$ *or* $\langle\langle w^\# \rangle\rangle$.

Proof. Without loss of generality we may assume that $x \neq w, w^\#$. Lemma 3.1 implies $\beta(w, x) > d(w)$ and, by definition of $\beta(w, x)$, we have $\beta(w, x) \leq d(w)+1$. Therefore $\beta(w, x) = d(w)+1$ and we have $x \in \langle\langle w \rangle\rangle$ or $\langle\langle w^\# \rangle\rangle$.

Combining Lemmas 3.1, 3.3, 3.5 and 3.9, we derive the following theorem.

Theorem 3.11. *If* $v, w \in \mathscr{W}_+$ *then*

$$\frac{1}{8} \cdot 2^{-\beta(v,w)} \leq \rho(v, w) \leq 6\alpha(v, w)2^{-\alpha(v,w)}.$$

If $w \in \mathscr{W}_+$ and $\xi \in \mathscr{S}$ *then*

$$\frac{1}{8} 2^{-\beta(w,x)} \leq \rho(w, \xi) \leq 6\beta(w, x)2^{-\beta(w,x)}, \quad \textit{for } \forall x \in \Pi^{-1}(\xi).$$

If $\xi, \eta \in \mathscr{S}$ *then*

$$\frac{1}{8}\beta(x, y)2^{-\beta(x,y)} \leq \rho(\xi, \eta) \leq 6\beta(x, y)2^{-\beta(x,y)},$$

for every $x \in \Pi^{-1}(\xi)$ *and* $y \in \Pi^{-1}(\eta)$.

§4. Variation of Harmonic Functions

In this section we estimate the variation of harmonic functions for the Markov chain by that of the boundary functions on the Sierpiński gasket, which is identified with the quotient space $\Xi/\sim$. The following lemma is basic in the sequel.

Lemma 4.1. *For every bounded measurable function* $\varphi: \mathscr{S} \to \mathbb{R}$ *the harmonic function*

$$h(\mathbf{w})=\int_{\mathscr{S}} k(\mathbf{w},\xi)\varphi(\xi)\mu(d\xi)$$

satisfies

$$|h(\mathbf{v})-h(\mathbf{w})|\leq\frac{1}{2}\sup_{\mathbf{x}\in[\mathbf{v}^-],\mathbf{y}\in[\mathbf{w}^-]}|\varphi(\Pi(\mathbf{x}))-\varphi(\Pi(\mathbf{y}))|$$

$$+\frac{1}{2}\sup_{\mathbf{x}\in[(\mathbf{v}^\#)^-],\mathbf{y}\in[(\mathbf{w}^\#)^-]}|\varphi(\Pi(\mathbf{x}))-\varphi(\Pi(\mathbf{y}))|.$$

for every $\mathbf{v},\mathbf{w}\in\mathscr{W}_+$.

Note that the case when $\mathbf{v}=\mathbf{v}^\#$ (resp. $\mathbf{w}=\mathbf{w}^\#$)is formally included in the statement, since in this case $\theta(\mathbf{v})=2$ (resp. $\theta(\mathbf{w})=2$) and the proof below covers also this case.

Proof. Let $\mathbf{v}=v_1v_2\cdots v_m$ and $\mathbf{w}=w_1w_2w_3\cdots w_n$ be words in $\mathscr{W}_+$. Then by Lemma 2.1 and Theorem 2.3, we have

$$\begin{aligned}
h(\mathbf{v}) &= \int_{\mathscr{S}} k(\mathbf{v},\xi)\varphi(\xi)\mu(d\xi)\\
&= \int_{\Xi} k(\mathbf{v},\Pi(\mathbf{x}))\varphi(\Pi(\mathbf{x}))\nu(d\mathbf{x})\\
&= \int_{\Xi} \lim_n k(\mathbf{v},\Pi_n(\mathbf{x}))\varphi(\Pi(\mathbf{x}))\nu(d\mathbf{x})\\
&= \frac{N^{d(\mathbf{v})}}{4}\left\{\int_{[\mathbf{v}^-]}\sum_{k=0}^{\infty}2^{-k}\mathbf{I}_{\tau(\mathbf{v})}(x_{d(\mathbf{v})+k})\varphi(\Pi(\mathbf{x}))\nu(d\mathbf{x})\right.\\
&\quad\left.+\int_{[(\mathbf{v}^\#)^-]}\sum_{k=0}^{\infty}2^{-k}\mathbf{I}_{\tau(\mathbf{v}^\#)}(x_{d(\mathbf{v})+k})\varphi(\Pi(\mathbf{x}))\nu(d\mathbf{x})\right\}\\
h(\mathbf{w}) &= \frac{N^{d(\mathbf{w})}}{4}\left\{\int_{[\mathbf{w}^-]}\sum_{k=0}^{\infty}2^{-k}\mathbf{I}_{\tau(\mathbf{w})}(x_{d(\mathbf{w})+k})\varphi(\Pi(\mathbf{x}))\nu(d\mathbf{x})\right.\\
&\quad\left.+\int_{[(\mathbf{w}^\#)^-]}\sum_{k=0}^{\infty}2^{-k}\mathbf{I}_{\tau(\mathbf{w}^\#)}(x_{d(\mathbf{w})+k})\varphi(\Pi(\mathbf{x}))\nu(d\mathbf{x})\right\}
\end{aligned}$$

where x_k denotes the k-th coordinate of $\mathbf{x}\in\Xi$.

Define a bijective map $t:[\mathbf{v}^-]\to[\mathbf{w}^-]$ by

$$t(\mathbf{x})_k = \begin{cases} \tau(\mathbf{w}), & \text{if } x_{k-d(\mathbf{w})+d(\mathbf{v})} = \tau(\mathbf{v}) \text{ and } k \geq d(\mathbf{w}) \\ \tau(\mathbf{v}), & \text{if } x_{k-d(\mathbf{w})+d(\mathbf{v})} = \tau(\mathbf{w}) \text{ and } k \geq d(\mathbf{w}) \\ x_{k-d(\mathbf{w})+d(\mathbf{v})}, & \text{if } x_{k-d(\mathbf{w})+d(\mathbf{v})} \neq \tau(\mathbf{v}),\ \tau(\mathbf{w}) \text{ and } k \geq d(\mathbf{w}) \\ w_k, & \text{if } 1 \leq k \leq d(\mathbf{w}) - 1. \end{cases}$$

For example we have

$$\begin{aligned} &\mathbf{x} = v_1 v_2 \cdots v_{m-1} x_m x_{m+1} \cdots x_{m+j} v_m x_{m+j+2} \cdots x_{m+l} w_n x_{m+l+2} \cdots \\ \to\ &t(\mathbf{x}) = w_1 w_2 \cdots w_{n-1} x_m x_{m+1} \cdots x_{m+j} w_n x_{m+j+2} \cdots x_{m+l} v_m x_{m+l+2} \cdots. \end{aligned}$$

Then it follows that for every non-negative measurable function f on Ξ

$$\int_{[\mathbf{v}^-]} f(t(\mathbf{x}))\nu(d\mathbf{x}) = N^{d(\mathbf{w})-d(\mathbf{v})} \int_{[\mathbf{w}^-]} f(\mathbf{y})\nu(d\mathbf{y})$$

and that

$$\sum_{k=0}^{\infty} 2^{-k} \mathbb{I}_{\tau(\mathbf{w})}(t(\mathbf{x})_{d(\mathbf{w})+k}) = \sum_{k=0}^{\infty} 2^{-k} \mathbb{I}_{\tau(\mathbf{v})}(x_{d(\mathbf{v})+k}).$$

Therefore we conclude that

$$\begin{aligned} &\left| N^{d(\mathbf{v})} \int_{[\mathbf{v}^-]} \sum_{k=0}^{\infty} 2^{-k} \mathbb{I}_{\tau(\mathbf{v})}(x_{d(\mathbf{v})+k}) \varphi(\Pi(\mathbf{x}))\nu(d\mathbf{x}) \right. \\ &\left. - N^{d(\mathbf{w})} \int_{[\mathbf{w}^-]} \sum_{k=0}^{\infty} 2^{-k} \mathbb{I}_{\tau(\mathbf{w})}(x_{d(\mathbf{w})+k}) \varphi(\Pi(\mathbf{x}))\nu(d\mathbf{x}) \right| \\ &= \left| N^{d(\mathbf{v})} \int_{[\mathbf{v}^-]} \sum_{k=0}^{\infty} 2^{-k} \mathbb{I}_{\tau(\mathbf{v})}(x_{d(\mathbf{v})+k}) \varphi(\Pi(\mathbf{x}))\nu(d\mathbf{x}) \right. \\ &\left. - N^{d(\mathbf{v})} \int_{[\mathbf{v}^-]} \sum_{k=0}^{\infty} 2^{-k} \mathbb{I}_{\tau(\mathbf{w})}(t(\mathbf{x})_{d(\mathbf{w})+k}) \varphi(\Pi(t(\mathbf{x})))\nu(d\mathbf{x}) \right| \\ &= \left| N^{d(\mathbf{v})} \int_{[\mathbf{v}^-]} \sum_{k=0}^{\infty} 2^{-k} \mathbb{I}_{\tau(\mathbf{v})}(x_{d(\mathbf{v})+k}) \varphi(\Pi(\mathbf{x}))\nu(d\mathbf{x}) \right. \\ &\left. - N^{d(\mathbf{v})} \int_{[\mathbf{v}^-]} \sum_{k=0}^{\infty} 2^{-k} \mathbb{I}_{\tau(\mathbf{v})}(x_{d(\mathbf{v})+k}) \varphi(\Pi(t(\mathbf{x})))\nu(d\mathbf{x}) \right| \\ &\leq N^{d(\mathbf{v})} \int_{[\mathbf{v}^-]} \sum_{k=0}^{\infty} 2^{-k} \mathbb{I}_{\tau(\mathbf{v})}(x_{d(\mathbf{v})+k}) |\varphi(\Pi(\mathbf{x})) - \varphi(\Pi(t(\mathbf{x})))| \nu(d\mathbf{x}) \end{aligned}$$

$$\leq 2 \sup_{\mathbf{x}\in[\mathbf{v}^-],\mathbf{y}\in[\mathbf{w}^-]} |\varphi(\Pi(\mathbf{x}))-\varphi(\Pi(\mathbf{y}))|.$$

Applying the same reasoning to $\mathbf{v}^\#$ and $\mathbf{w}^\#$, the lemma follows immediately.

Lemma 4.2. *For every* $\mathbf{w}\in\mathscr{W}_+$, $\mathbf{v}\in\langle\mathbf{w}\rangle$ *implies* $(\mathbf{v}^\#)^-\in\langle\mathbf{w}^-\rangle\cup\langle(\mathbf{w}^\#)^-\rangle$.

Proof. For every $\mathbf{v}\in\langle\mathbf{w}\rangle$, there exists $\mathbf{u}\in\mathscr{W}$ such that $\mathbf{v}=\mathbf{wu}$. If $\mathbf{u}\neq\mathbf{u}^\#$, then $\mathbf{v}^\#=\mathbf{wu}^\#$ so that $(\mathbf{v}^\#)^-\in\langle\mathbf{w}^-\rangle$. If $\mathbf{u}=\mathbf{u}^\#$, then there exists $a\in\mathscr{A}$ and $k\geq 0$ such that $\mathbf{u}=a^k$. Moreover, if $a\neq\tau(\mathbf{w})$, then we have $\mathbf{v}^\#=\mathbf{w}^-a\tau(\mathbf{w})^{d(\mathbf{u})}$, hence $(\mathbf{v}^\#)^-\in\langle\mathbf{w}^-\rangle$. Finally, if $a=\tau(\mathbf{w})$, then $\mathbf{v}^\#=\mathbf{w}^\#\tau(\mathbf{w}^\#)^{d(\mathbf{u})}$ implies that $(\mathbf{v}^\#)^-\in\langle(\mathbf{w}^\#)^-\rangle$.

For every $\mathbf{w}\in\mathscr{W}_+$, define the *variation at* $\mathbf{w}$ of a function h on $\mathscr{W}$ by

$$\mathrm{Var}_h(\mathbf{w})=\sup_{\mathbf{u},\mathbf{v}\in\langle\mathbf{w}\rangle}|h(\mathbf{u})-h(\mathbf{v})|,$$

and the *variation at* $\mathbf{w}$ of a function φ on $\mathscr{S}$ by

$$Var_\varphi(\mathbf{w})=\sup_{\mathbf{x},\mathbf{y}\in[\mathbf{w}]}|\varphi(\Pi(\mathbf{x}))-\varphi(\Pi(\mathbf{y}))|.$$

From Lemmas 4.1 and 4.2, we obtain the following proposition.

Proposition 4.3. *For every bounded measurable function* $\varphi:\mathscr{S}\to\boldsymbol{R}$ *the harmonic function*

$$h(\mathbf{w})=\int k(\mathbf{w},\xi)\varphi(\xi)\nu(d\xi)$$

satisfies

$$\mathrm{Var}_h((\mathbf{w})\leq Var_\varphi(\mathbf{w}^-)+Var_\varphi((\mathbf{w}^\#)^-),\quad \forall\mathbf{w}\in\mathscr{W}_+.$$

Proof. It is evident that $\mathbf{v}\in\langle\mathbf{w}\rangle$ implies $[\mathbf{v}^-]\subset[\mathbf{w}^-]$ and that Lemma 4.2 implies $[(\mathbf{v}^\#)^-]\subset[\mathbf{w}^-]\cup[(\mathbf{w}^\#)^-]$. Therefore by Lemma 4.1 it follows that for any $\mathbf{u},\mathbf{v}\in\langle\mathbf{w}\rangle$

$$|h(\mathbf{u})-h(\mathbf{v})|\leq \sup_{\mathbf{x},\mathbf{y}\in[\mathbf{w}^-]\cup[(\mathbf{w}^\#)^-]}|\varphi(\Pi(\mathbf{x}))-\varphi(\Pi(\mathbf{y}))|.$$

Define $\mathbf{z}=\mathbf{w}\tau(\mathbf{w})^\infty$ and conclude that $\mathbf{z}^\#=\mathbf{w}^\#\tau(\mathbf{w}^\#)^\infty$, hence $\mathbf{z}^\#\in[\mathbf{w}^-]$ and $\mathbf{z}^\#\in[(\mathbf{w}^\#)^-]$. Since $\Pi(\mathbf{z})=\Pi(\mathbf{z}^\#)$ the proposition follows from

$$\begin{aligned}
&\mathrm{Var}_h(\mathbf{w})\\
&\leq \max\left\{\sup_{\mathbf{x}\in[\mathbf{w}^-];\mathbf{y}\in[(\mathbf{w}^\sharp)^-]} |\varphi(\Pi(\mathbf{x}))-\varphi(\Pi(\mathbf{y}))|,\ Var_\varphi(\mathbf{w}^-),\ Var_\varphi((\mathbf{w}^\sharp)^-)\right\}\\
&\leq \max\left\{\sup_{\mathbf{x}\in[\mathbf{w}^-];\mathbf{y}\in[(\mathbf{w}^\sharp)^-]} (|\varphi(\Pi(\mathbf{x}))-\varphi(\Pi(\mathbf{z}))|+|\varphi(\Pi(\mathbf{z}^\sharp))-\varphi(\Pi(\mathbf{y}))|),\right.\\
&\qquad \left. Var_\varphi(\mathbf{w}^-),\ Var_\varphi((\mathbf{w}^\sharp)^-)\right\}\\
&\leq Var_\varphi(\mathbf{w}^-)+Var_\varphi((\mathbf{w}^\sharp)^-).
\end{aligned}$$

Corollary 4.4. *Let φ be an s-Hölder continuous function on $(\mathscr{S},\rho)$ with Hölder constant C_φ. Then for every $\mathbf{w}\in\mathscr{W}$ we have*

$$\mathrm{Var}_{h_\varphi}(\mathbf{w})\leq 2C_\varphi(12d(\mathbf{w}))^s 2^{-sd(\mathbf{w})}.$$

Proof. Let $\mathbf{w}\in\mathscr{W}_+$. Then for every $\mathbf{x}\in[\mathbf{w}^-]$ we have $\alpha(\mathbf{w}^-,\mathbf{x})=\beta(\mathbf{w}^-,\mathbf{x})=d(\mathbf{w})$ and by Lemma 3.9 $\rho(\mathbf{w},\Pi(\mathbf{x}))\leq 6d(\mathbf{w})2^{-d(\mathbf{w})}$ whence $\rho(\Pi(\mathbf{x}),\Pi(\mathbf{y}))\leq 12d(\mathbf{w})2^{-d(\mathbf{w})}$ for every $\mathbf{x},\mathbf{y}\in[\mathbf{w}^-]$. This proves the corollary in view of Proposition 4.3.

Lemma 4.5. *For every bounded measurable function $\varphi:\mathscr{S}\to\boldsymbol{R}$ the harmonic function*

$$h(\mathbf{w})=\int_{\mathscr{S}} k(\mathbf{w},\xi)\varphi(\xi)\mu(d\xi)$$

satisfies

$$\begin{aligned}
&\sup\{|h(\mathbf{v})-h(\mathbf{w})|;\ \rho(\mathbf{v},\mathbf{w})<2^{-(n+3)},\ \mathbf{v},\mathbf{w}\in\mathscr{W}_+\}\\
&\leq 4\sup\{|\varphi(\xi)-\varphi(\eta)|;\ \rho(\xi,\eta)<12n2^{-n},\ \xi,\eta\in\mathscr{S}\}.
\end{aligned}$$

Proof. Fix any $\mathbf{v},\mathbf{w}\in\mathscr{W}_+$ such that $\rho(\mathbf{v},\mathbf{w})<2^{-(n+3)}$. Without loss of generality we may assume that $d(\mathbf{w})\leq d(\mathbf{v})$. By Lemma 3.1 we have

$$2^{-\beta(\mathbf{v},\mathbf{w})}\leq 8\rho(\mathbf{v},\mathbf{w})<2^{-n}$$

so that $\beta(\mathbf{v},\mathbf{w})>n$, which implies $\mathbf{v},\mathbf{w}\in\langle\mathbf{z}\rangle\cup\langle\mathbf{z}^\sharp\rangle$ where $\mathbf{z}=w_1w_2\cdots w_n$. Consequently by Proposition 4.3, even in the case where $\mathbf{v}\in\langle\mathbf{z}\rangle$ and $\mathbf{w}\in\langle\mathbf{z}^\sharp\rangle$, we have

$$|h(\mathbf{v})-h(\mathbf{w})| \leq |h(\mathbf{v})-h(\mathbf{z})|+|h(\mathbf{z}^{\#})-h(\mathbf{w})|$$
$$\leq \mathrm{Var}_h(\mathbf{z})+\mathrm{Var}_h(\mathbf{z}^{\#}) \leq 2(Var_{\varphi}(\mathbf{z}^-)+Var_{\varphi}((\mathbf{z}^{\#})^-)).$$

On the other hand by Lemma 3.9 $\mathbf{x}\in[\mathbf{z}^-]$ implies $\rho(\Pi(\mathbf{x}),\Pi(\mathbf{z}^-)) \leq 6d(\mathbf{z})2^{-d(\mathbf{z})}$, since $\alpha(\mathbf{z},\mathbf{x})=d(\mathbf{z})$. Hence $\mathbf{x},\mathbf{y}\in[\mathbf{z}^-]$ implies $\rho(\Pi(\mathbf{x}),\Pi(\mathbf{y})) \leq 12d(\mathbf{z})2^{-d(\mathbf{z})} \leq 12n2^{-n}$.

Theorem 4.6. *Let φ be a continuous function on $\mathcal{S}$. Then h_φ is extended to a continuous function on $\mathcal{W}\cup\mathcal{S}$, which coincides with φ on $\mathcal{S}$. In particular we have*

$$\lim_{\mathbf{w}\to\xi} h_\varphi(\mathbf{w})=\varphi(\xi)$$

for every $\xi\in\mathcal{S}$.

Proof. Since by Lemma 4.5 h_φ is uniformly continuous on $\mathcal{W}_+$ and $\emptyset$ is an isolated point, h_φ is uniformly continuous on the dense subset $\mathcal{W}$ of a compact metric space $\mathcal{W}\cup\mathcal{S}$ and extends to a continuous function $\bar{h}_\varphi$ on $\mathcal{W}\cup\mathcal{S}$.

On the other hand by Theorem 2.2(4) we have

$$\lim_n h_\varphi(X_n)=\varphi(X_\infty), \quad a.s.(\boldsymbol{P}_\emptyset)$$

and since $\mu=\boldsymbol{P}_\emptyset\circ X_\infty^{-1}$ we have

$$\bar{h}_\varphi(\xi)=\varphi(\xi), \quad a.s.(\mu).$$

Since μ is a Radon measure on $\mathcal{S}$ and full by Remark 2.4, we have $\bar{h}=\varphi$.

Denote the set of all continuous function on $\mathcal{S}$ by $C(\mathcal{S})$ and that of all bounded uniformly continuous harmonic functions on $(\mathcal{W},\rho)$ by $\mathcal{H}_C$.

Corollary 4.7. $\mathcal{I}(C(\mathcal{S}))=\mathcal{H}_C$.

Lemma 4.8. *If a function f on $\mathcal{W}$ is uniformly continuous, then we have*

$$\lim_{n\to\infty}\sup_{\mathbf{w}\in\mathcal{W}_n}\mathrm{Var}_f(\mathbf{w})=0. \tag{4.1}$$

Conversely, if a function f is symmetric, (4.1) *implies the uniform continuity of f.*

Proof. Assume that f is uniformly continuous on $\mathcal{W}$ and let $\mathbf{w}\in\mathcal{W}_+$.

Then, since $\alpha(\mathbf{u},\mathbf{w})=d(\mathbf{w})+1$ for every $\mathbf{u}\in\langle\mathbf{w}\rangle$ and by Theorem 3.9, we have $\rho(\mathbf{u},\mathbf{v})\leq 6d(\mathbf{w})2^{-d(\mathbf{w})}$ for any $\mathbf{u},\mathbf{v}\in\langle\mathbf{w}\rangle$. Therefore, the uniform continuity implies

$$\begin{aligned}&\lim_n \sup_{\mathbf{w}\in\mathscr{W}_n} \mathrm{Var}_f(\mathbf{w})\\ &\leq \lim_n \sup\{|f\mathbf{u})-f(\mathbf{v})|;\ \rho(\mathbf{u},\mathbf{v})\leq 6n2^{-n},\ \mathbf{u},\mathbf{v}\in\mathscr{W}_+\}=0.\end{aligned}$$

Conversely, assume that $\mathbf{u},\mathbf{v}\in\mathscr{W}_+$ and $\rho(\mathbf{u},\mathbf{v})<2^{-(n+3)}$ for some $n\geq 1$. Then without loss of generality we may assume $d(\mathbf{u})\leq d(\mathbf{v})$ and as in the proof of Lemma 4.5 we conclude that $\mathbf{u},\mathbf{v}\in\langle\mathbf{z}\rangle\cup\langle\mathbf{z}^{\#}\rangle$ where $\mathbf{z}=u_1u_2u_3\cdots u_n$. This yields

$$|f(\mathbf{u})-f(\mathbf{v})|\leq \mathrm{Var}_f(\mathbf{z})+\mathrm{Var}_f(\mathbf{z}^{\#})\leq 2\sup_{\mathbf{w}\in\mathscr{W}_n}\mathrm{Var}_f(\mathbf{w})\to 0$$

as $n\to\infty$.

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