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Algebraic Coset Conformal Field Theories II

By

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Abstract

Some mathematical questions relating to coset conformal field theories (CFT) are considered in the framework of algebraic quantum field theory as developed previously by us. We consider the issue of fix point resolution in the diagonal cosets of type A. We show how to decompose certain reducible representations into irreducibles, and prove that the coset CFT gives rise to a unitary modular category and therefore may be used to construct 3-manifold invariants. We prove that if the coset inclusion satisfies certain conditions which can be checked in examples, the Kac-Wakimoto Hypothesis (KWH) is equivalent to the Kac-Wakimoto Conjecture (KWC), a result which seems to be hard to prove by purely representation considerations. Examples are also presented.

§1. Introduction

This paper is a sequel to [X4]. Let us first recall some definitions from [X4].

Let G be a simply connected compact Lie group and let $H \subset G$ be a connected Lie subgroup. Let π^i be an irreducible representations of LG with positive energy at level[1] k on Hilbert space H^i (cf. §2.1). Suppose when restricting to LH, H^i decomposes as:

$$H^i = \sum_\alpha H_{i,\alpha} \otimes H_\alpha,$$

and π_α are irreducible representations of LH on Hilbert space H_α. The set of (i, α) which appears in the above decompositions will be denoted by

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[1] When G is the direct product of simple groups, k is a multi-index, i.e., $k = (k_1, \ldots, k_n)$, where $k_i \in N$ corresponding to the level of the i-th simple group. The level of LH is determined by the Dynkin indices of $H \subset G$. To save some writing we write the coset as $H \subset G_k$.

exp.

We shall use π^1 (resp. π_1)[2] to denote the vacuum representation of LG (resp. LH). Let $\mathscr{A}$ be the vacuum sector of the coset G/H as defined in §2.1 of [X4]. The decompositions above naturally give rise to a class of covariant representations of $\mathscr{A}$, denoted by $\pi_{i,\alpha}$ or simply (i,α). By Th. 2.3 of [X4], $\pi_{1,1}$ is the vacuum representation of $\mathscr{A}$.

In §2.2 we consider the decompositions of certain reducible representations in the diagonal cosets of type A_{N-1} as considered in §4.3 of [X4] when the action of the Dynkin diagram automorphisms is not faithful (cf. (2) of Th. 4.3 of [X4]), which is part of the **fixed point resolution** problems known in physics literature (cf. [Gep], [LVW] and [SY]). Such problems have been known for some time, and there are no even clear mathematical formulations of such questions before. We will show that the results of [X4] provide the right mathematical framework for understanding such questions.

We first prove a general Lemma 2.1 which we believe will play an important role in all fixed point resolution problems. Using Lemma 2.1 and Lemma 2.2, we prove (cf. (1) of Th. 2.3) that certain S-matrices are non-degenerate. It follows from Th. 2.3 (Cor. 2.4) that the diagonal cosets of type A_{N-1} give rise to a unitary modular category in the sense of Turaev (cf. P. 74 and P. 113 of [Tu]), and may be used to construct 3-manifold invariants (cf. P. 160 of [Tu]). We also calculate S matrices when N is prime. The results agree with some of the results of [FSS1], [SY] from different considerations based on physics.

To describe the results in §3, let us denote by S_{ij} (resp. $\dot{S}_{\alpha\beta}$) the S matrices of LG (resp. LH) at level k (resp. certain level of LH determined by the inclusion $H \subset G_k$). Define[3]

$$b(i,\alpha) = \sum_{(j,\beta)} S_{ij}\overline{\dot{S}_{\alpha\beta}}\langle (j,\beta),(1.1)\rangle \tag{1}$$

where $\langle\, , \rangle$ is defined in §2.1. Note the above summation is effectively over those (j,β) such that $(j,\beta) \in exp$. The Kac-Wakimoto Conjecture (KWC) states

[2] This is slightly different from the notation π^0 (resp. π_0) in [X4]: it seems to be more appropriate since these representations correspond to identity sectors.

[3] Our (j,β) corresponds to (M,μ) on P. 186 of [KW], and it follows from the definitions that $\langle (j,\beta),(1,1)\rangle$ is then equal to $mult_M(\mu,p)$ which appears in 2.5.4 of [KW]. Our formula (1) is then identical to 2.5.4 of [KW].

that if $(i,\alpha)\in exp$, then $b(i,\alpha)>0$.

The Kac-Wakimoto Hypothesis (KWH) states that if $\langle(j,\beta),(1,1)\rangle>0$ and $(i,\alpha)\in exp$, then $S_{ij}\overline{\dot{S}_{\alpha\beta}}\geq 0$.

Note that since $S_{i1}>0$, $\dot{S}_{\alpha 1}>0$, $(1,1)\in exp$, KWH implies KWC.

KWC has proved to be true in all known examples. In fact, in §2 of [X4] an even stronger conjecture, Conjecture 2 (C2) is formulated (also cf. [L4]).

Unfortunately KWH is not true. In [X2] counter examples were found by using subfactors associated with conformal inclusions. However, KWH has been checked to be true in so many examples, and it seems that it should be true or equivalent to KWC under some general conditions. The first main result in §3.1 is to describe such a condition (cf. Th. 3.3). The condition[4] is that:

$$\text{if } \langle(1,\beta),(1,1)\rangle>0, \text{ then } \beta=1. \tag{2}$$

Th. 3.3 states that if $H\subset G_k$ satisfies (2), and certain assumptions in §3.1 which are expected to be true in general, then KWH is equivalent to KWC for the inclusion $H\subset G_k$.

Condition (2) can be shown to be equivalent to the normality of certain inclusions, but we will not discuss this in this paper.

In §3.1 we also give an example which does not satisfy (2), and verifies KWC but not KWH. This is also the first example of non-conformal inclusion which does not verify KWH.

It is interesting to note that Th. 3.3 can be thought as a statement about representations of affine Kac-Moody algebras without even mentioning von Neumann algebras, yet it seems to be hard to obtain such results without using subfactor theory (cf. [J]). We give another example of this nature in Prop. 3.2. For more such statements, see inequality on P. 11 of [X2] and in particular (2) of Th. 4.3 of [X4].

In §3.2 we prove a property (Prop. 3.4) of Conjecture 2 (C2) in [X4]. It states that if $H_1\subset H_2$ and $H_2\subset G$ verify C2, then $H_1\subset G$ also verifies C2, thus reducing C2 to maximal inclusions which are classified in [Dyn1], [Dyn2]. We also give an example related to $N=2$ superconformal theories.

[4] If we identify $(1,\beta)$ with (M,μ), where M is the vacuum representation, as on P. 186 of [KW], then condition (2) is the statement that if $(M,\mu)\in S_m$, with S_m defined on P. 186 of [KW], then μ must be the vacuum representation of the subalgebra.

§2. Fixed Point Resolutions in the Diagonal Cosets of Type A_{N-1}

§2.1 Preliminaries. Let us first recall some definitions from [X2]. Let M be a properly infinite factor and End(M) the semigroup of unit preserving endomorphisms of M. In this paper M will always be the unique hyperfinite III_1 factors. Let Sect(M) denote the quotient of End(M) modulo unitary equivalence in M. We denote by $[\rho]$ the image of $\rho \in \mathrm{End}(M)$ in Sect(M).

It follows from [L3] and [L4] that Sect(M), with M a properly infinite von Neumann algebra, is endowed with a natural involution $\theta \to \bar{\theta}$; moreover, Sect(M) is a semiring with identity denoted by *id*.

If given a normal faithful conditional expectation $\epsilon: M \to \rho(M)$, we define a number d_ϵ (possibly ∞) by:

$$d_\epsilon^{-2} := \mathrm{Max}\{\lambda \in [0, +\infty) \,|\, \epsilon(m_+) \geq \lambda m_+,\ \forall m_+ \in M_+\}$$

(cf. [PP]).

We define

$$d = \mathrm{Min}_\epsilon \{d_\epsilon \,|\, d_\epsilon < \infty\}.$$

d is called the statistical dimension of ρ. It is clear from the definition that the statistical dimension of ρ depends only on the unitary equivalence classes of ρ. The properties of the statistical dimension can be found in [L1], [L2] and [L3]. We will denote the statistical dimension of ρ by d_ρ in the following. d_ρ^2 is called the minimal index of ρ.

Recall from [X2] that we denote by $\mathrm{Sect}_0(M)$ those elements of Sect(M) with finite statistical dimensions. For $\lambda, \mu \in \mathrm{Sect}(M)$, let $\mathrm{Hom}(\lambda, \mu) \subset M$ denote the space of intertwiners from λ to μ, i.e. $a \in \mathrm{Hom}(\lambda, \mu)$ iff $a\lambda(x) = \mu(x)a$ for any $x \in M$. $\mathrm{Hom}(\lambda, \mu)$ is a finite dimensional vector space and we use $\langle \lambda, \mu \rangle$ to denote the dimension of this space. $\langle \lambda, \mu \rangle$ depends only on $[\lambda]$ and $[\mu]$. Moreover we have $\langle \nu\lambda, \mu \rangle = \langle \lambda, \bar{\nu}\mu \rangle$, $\langle \nu\lambda, \mu \rangle = \langle \nu, \mu\bar{\lambda} \rangle$ which follows from Frobenius duality (See [L2]). We will also use the following notation: if μ is a subsector of μ, we will write as $\mu \prec \lambda$ or $\lambda \succ \mu$. A sector is said to be irreducible if it has only one subsector.

Recall (cf. [L2]) of each $\rho \in \mathrm{End}(M)$ and its conjugate $\bar{\rho}$ with finite minimal index, there exists $R_\rho \in \mathrm{Hom}(id, \bar{\rho}\rho)$ and $\bar{R}_\rho \in \mathrm{Hom}(id, \rho\bar{\rho})$ such that

$$\bar{R}_\rho^* \rho(R_\rho) = 1,\ R_\rho^* \bar{\rho}(\bar{R}_\rho) = 1$$

and $\|R_\rho\| = \|\bar{R}_\rho\| = \sqrt{d_\rho}$. The minimal left inverse ϕ_ρ of ρ is defined by

$$\phi_\rho(m) = R_\rho^* \bar{\rho}(m) R_\rho .$$

The following lemma plays a fundamental role in §2.2.

Lemma 2.1. *Let $a, b, c \in End(M)$, $[c] = [ab]$ and a, b have finite statistical dimensions. Suppose $\tau \in End(M)$ has order t in $Sect(M)$, i.e., t is the least positive integer such that $[\tau^t] = [id]$, and $[a\tau] = [a]$, $[\tau b] = [b]$. If*

$$\langle c, c \rangle = t,$$

then $\mathrm{Hom}(c, c)$ *is an abelian algebra with dimension t and hence there exist irreducible sectors $c_1, \ldots, c_t$ such that*

$$c = \sum_{1 \le k \le t} c_k .$$

Moreover, $d_{c_k} = \frac{1}{t} d_c$, $k = 1, \ldots, t$.

Proof. From $[a\tau] = [a]$ we conclude by using Frobenius duality that $\langle \bar{a}a, \tau^i \rangle \ge 1$, and since τ has order t in $Sect(M)$, we must have

$$\bar{a}a \succ \sum_{0 \le i \le t-1} \tau^i .$$

Similarly

$$b\bar{b} \succ \sum_{0 \le i \le t-1} \tau^i .$$

Since

$$t = \langle c, c \rangle = \langle ab, ab \rangle = \langle \bar{a}a, b\bar{b} \rangle$$

$$\ge \sum_{0 \le i \le t-1} \langle \bar{a}a, \tau^i \rangle \langle \tau^i, b\bar{b} \rangle \ge t,$$

it follows that all the $\ge$ are $=$ and in particular $\langle \bar{a}a, id \rangle = \langle \bar{b}b, id \rangle = 1$, i.e., both a and b are irreducible. It is enough to prove the case $c = ab$. Since $[\tau b] = [b]$, there exist unitary elements $v \in M$ such that

$$v \in \mathrm{Hom}(b, \tau b).$$

Define $\tau_v = v^* \tau v$, then

$$b=\tau_v b$$

and so

$$b=\tau_v^t b.$$

Since $[\tau^t]=[id]$, there exists a unitary v_1 such that $\tau_v^t=Ad_{v_1}$, where $Ad_{v_1}(m):=v_1 m v_1^*$, $\forall m\in M$. So $v_1\in\mathrm{Hom}(b,b)$. Since b is irreducible, v_1 is equal to identity up to a complex number whose absolute value is 1, so

$$\tau_v^t=id.$$

From $[a\tau]=[a\tau_v]=[a]$ there exists a unitary u such that $u\in\mathrm{Hom}(a\tau_v,a)$. It follows that

$$u^t a(m)=u^t a(\tau_v^t(m))=a(m)u^t,\ \forall m\in M,$$

so $u^t=x.1$ with $x\in\mathbf{C}$, x has absolute value one, since a is irreducible. Define $w=x^{\frac{1}{t}}u$ so that $w^t=1$. Note $w\in\mathrm{Hom}(ab,ab)$ since $b=\tau_v b$. Denote by ϕ_a, ϕ_b the minimal left inverses of a, b respectively. We claim that $\phi_b\phi_a(w^i)=d_a d_b\delta_{i0}$ for $0\le i\le t-1$. It is enough to show $\phi_b\phi_a(w^i)=0$ for $0<i\le t-1$. Since b is irreducible, $\phi_a(w^i)\in\mathrm{Hom}(b,b)\equiv\mathbf{C}.1$, and so $\phi_b\phi_a(w^i)=R_{\bar b}^*\phi_a(w^i)R_{\bar b}$. But

$$\begin{aligned}R_{\bar b}^*\phi_a(w^i)R_{\bar b}&=R_{\bar b}^*R_a^*\bar a(w^i)R_aR_{\bar b}=R_a^*\bar a a(R_{\bar b}^*)\bar a(w^i)R_aR_{\bar b}\\&=R_a^*\bar a(a(R_{\bar b}^*)w^i)R_aR_{\bar b}=R_a^*\bar a(w^i a\tau_v^i(R_{\bar b}^*))R_aR_{\bar b}\\&=R_a^*\bar a(w^i)\bar a a(\tau_v^i(R_{\bar b}^*))R_aR_{\bar b}\\&=R_a^*\bar a(w^i)R_a\tau_v^i(R_{\bar b}^*)R_{\bar b},\end{aligned}$$

and use the fact $\tau_v b=b$ we have $\tau_v^i(R_{\bar b}^*)R_{\bar b}\in\mathrm{Hom}(b,b)(id,\tau_v^i)$. Since $\langle\tau^i,id\rangle=0$, $0<i\le t-1$, it follows that $\tau_v^i(R_{\bar b}^*)R_{\bar b}=0$ and so $\phi_b\phi_a(w^i)=d_a d_b\delta_{i0}$ for $0\le i\le t-1$.

If $\Sigma_{0\le i\le t-1}x_i w^i=0$, $x_i\in\mathbf{C}$, multiply both sides by w^{t-i} and apply $\phi_b\phi_a$, we get $x_i=0$. So $id,w,w^2,\cdots,w^{t-1}$ are linearily independent in $\mathrm{Hom}(c,c)$. And since $\mathrm{Hom}(c,c)$ has dimension t, $\mathrm{Hom}(c,c)$ is therefore an abelian algebra with basis $id,w,w^2,\cdots,w^{t-1}$. Note $w^t=1$, so the minimal projections $P_k,k=1,\cdots,t$ in $\mathrm{Hom}(c,c)$ are given by $P_k=\Sigma_{0\le j\le t-1}\frac{1}{t}(\exp(\frac{2\pi ik}{t})w)^j$. Let $c_k\prec c$ be the irreducible sector corresponding to P_k, then by Th. 5.5 of [L1]

$$d_{c_k}=\phi_a\phi_b(P_k)=\frac{1}{t}d_a d_b=\frac{1}{t}d_c,\ k=1,\cdots,t.\qquad\text{Q.E.D.}$$

Next we will recall some of the results of [Reh] (also cf. [FRS]) and introduce notations.

Let $\{[\rho_i], i \in I\}$ be a finite set of equivalence classes of irreducible superselection sectors (cf. [GL]). Suppose this set is closed under conjugation and composition. We will denote the conjugate of $[\rho_i]$ by $[\rho_{\bar{i}}]$ and identity sector by [1] if no confusion arises, and let $N_{ij}^k = \langle [\rho_i][\rho_j], [\rho_k] \rangle$. We will denote by $\{T_e\}$ a basis of isometries in $\mathrm{Hom}(\rho_k, \rho_i\rho_j)$. The univalence of ρ_i (cf. P. 12 of [GL]) will be denoted by ω_{ρ_i}.

Let ϕ_i be the unique minimal left inverse of ρ_i, define:

$$Y_{ij} := d_{\rho_i} d_{\rho_j} \phi_j(\epsilon(\rho_j, \rho_i)\epsilon(\rho_i, \rho_j))^*, \tag{0}$$

where $\epsilon(\rho_j, \rho_i)$ is the unitary braiding operator (cf. [GL]).

We list two properties of Y_{ij} (cf. (5.13), (5.14) of [Reh]) which will be used in §2.2:

$$Y_{ij} = Y_{ji} = Y_{ij}^* = Y_{\bar{i}\bar{j}}. \tag{1}$$

$$Y_{ij} = \sum_k N_{ij}^k \frac{\omega_i \omega_j}{\omega_k} d_{\rho_k}. \tag{2}$$

Let us explain the proof of (2) since similar but different proof appears in the proof of lemma 2.2.

We have:

$$\phi_j\left(\sum_e T_e T_e^* \epsilon(\rho_i, \rho_j)^* \epsilon(\rho_j, \rho_i)^* T_e T_e^*\right) = \sum_e \frac{\omega_i \omega_j}{\omega_k} \phi_j(T_e T_e^*)$$

$$= \sum_e \frac{\omega_i \omega_j}{\omega_k} N_{ij}^k \frac{d_{\rho_k}}{d_{\rho_i} d_{\rho_j}},$$

where in the first = we used the monodromy equation (cf. [FRS] or P. 359 [X1]), and the second = follows from [L3]. (2) now follows immediately.

Define $\tilde{\sigma} := \Sigma_i d_{\rho_i}^2 \omega_{\rho_i}^{-1}$. If the matrix (Y_{ij}) is invertible, by the proposition on P. 351 of [Reh] $\tilde{\sigma}$ satisfies $|\tilde{\sigma}|^2 = \Sigma_i d_{\rho_i}^2$. Suppose $\tilde{\sigma} = |\tilde{\sigma}| \exp(ix)$, $x \in \boldsymbol{R}$. Define matrices

$$S := |\tilde{\sigma}|^{-1} Y, T := \exp\left(i\frac{x}{3}\right) Diag(\omega_{\rho_i}). \tag{3}$$

Then these matrices satisfy the algebra:

$$SS^{\dagger}=TT^{\dagger}=id, \tag{4}$$

$$TSTST=S, \tag{5}$$

$$S^2=C,\ TC=CT=T, \tag{6}$$

where $C_{ij}=\delta_{i\bar{j}}$ is the conjugation matrix. Moreover

$$N_{ij}^{k}=\sum_m \frac{S_{im}S_{jm}S_{km}^{*}}{S_{1m}}. \tag{7}$$

(7) is known as Verlinde formula.

Now let us consider an example which verifies (1) to (7) above as in §1 of [X1]. Let $G=SU(N)$. We denote LG the group of smooth maps $f:S^1\mapsto G$ under pointwise multiplication. The diffeomorphism group of the circle DiffS^1 is naturally a subgroup of Aut(LG) with the action given by reparametrization. In particular the group of rotations Rot$S^1\simeq U(1)$ acts on LG. We will be interested in the projective unitary representation $\pi: LG\to U(H)$ that are both irreducible and have positive energy. This means that π should extend to $LG \ltimes \mathrm{Rot}\, S^1$ so that $H=\oplus_{n\geq 0}H(n)$, where the $H(n)$ are the eigenspace for the action of RotS^1, i.e., $r_\theta\xi=\exp(in\theta)\xi$ for $\xi\in H(n)$ and dim $H(n)<\infty$ with $H(0)\neq 0$. It follows from [PS] that for fixed level K which is positive integer, there are only finite number of such irreducible representations indexed by the finite set

$$P_{++}^{h}=\left\{\lambda\in P\mid \lambda=\sum_{i=1,\cdots,N-1}\lambda_i\Lambda_i,\ \lambda_i\geq 1,\ \sum_{i=1,\cdots,N-1}\lambda_i<h\right\}$$

where P is the weight lattice of $SU(N)$ and Λ_i are the fundamental weights and $h=N+K$. We will use 1 to denote the trivial representation of $SU(N)$. For $\lambda,\mu,\nu\in P_{++}^{h}$, define

$$N_{\lambda\mu}^{\nu}=\sum_{\delta\in P_{++}^{h}}\frac{S_{\lambda\delta}S_{\mu\delta}S_{\nu\delta}^{*}}{S_{1\delta}} \tag{8}$$

where $S_{\lambda\delta}$ is given by the Kac-Peterson formula:

$$S_{\lambda\delta}=c\sum_{w\in S_N}\varepsilon_w\exp(iw(\delta)\cdot\lambda 2\pi/n). \tag{9}$$

Here $\varepsilon_w=\det(w)$ and c is a normalization constant fixed by the requirement that $(S_{\lambda\delta})$ is an orthonormal system. It is shown in [Kac] P. 288 that $N^{\nu}_{\lambda\mu}$ are nonnegative integers. Moreover, define $Gr(C_K)$ to be the ring whose basis are elements of P^h_{++} with structure constants $N^{\nu}_{\lambda\mu}$. The natural involution $*$ on P^h_{++} is defined by $\lambda\mapsto\lambda^*$ = the conjugate of λ as representation of $SU(N)$. All the irreducible representations of $Gr(C_K)$ are given by $\lambda\to\frac{S_{\lambda\mu}}{S_{1\mu}}$ for some μ.

The irreducible positive energy representations of $LSU(N)$ at level K give rise to an irreducible conformal precosheaf $\mathscr{A}$ and its covariant representations (cf. P. 362 of [X1]). The unitary equivalent classes of such representations are the superselection sectors. We will use λ to denote such representations.

For λ irreducible, the univalence ω_λ is given by an explicit formula. Let us first define

$$\Delta_\lambda=\frac{c_2(\lambda)}{K+N} \tag{10}$$

where $c_2(\lambda)$ is the value of Casimir operator on representation of $SU(N)$ labeled by dominant weight λ (cf. 1.4.1 of [KW]). Δ_λ is usually called the conformal dimension, and $\omega_\lambda=\exp(2\pi i\Delta_\lambda)$.

Define the central charge (cf. 1.4.2 of [KW])

$$C_G:=\frac{K\dim(G)}{K+N} \tag{11}$$

and T matrix as

$$T=diag(\dot{\omega}_\lambda) \tag{12}$$

where $\dot{\omega}_\lambda=\omega_\lambda exp\left(\frac{-2\pi iC_G}{24}\right)$. By Th. 13.8 of [Kac] S matrix as defined in (9) and T matrix in (12) satisfy relation (4), (5) and (6).

By Cor. 1 in §34 of [W], the fusion ring generated by all $\lambda\in P^h_{++}$ is isomorphic to $Gr(C_K)$, with structure constants $N^{\nu}_{\lambda\mu}$ as defined in (8). One may therefore ask what are the Y matrix (cf. (0)) in this case. By using (2) and the formula for $N^{\nu}_{\lambda\mu}$, a simple calculation shows:

$$Y_{\lambda_\mu}=\frac{S_{\lambda_\mu}}{S_{1_\mu}},$$

and it follows that $Y_{\lambda\mu}$ is nondegenerate, and S, T matrices as defined in (3) are indeed the same S, T matrix defined in (8) and (11), which is a surprising fact. If the analogue of Cor. 1 in §34 of [W] is established for other types of simple and simply connected Lie groups, then this fact is also true for other types of groups by the same argument.

In §2.2 we will also consider the case when G is the direct product of two type A groups. In that case the S, T matrices are just the tensor product of the S, T matrices corresponding to each subgroup.

§2.2 **Fixed point resolutions.** We preserve the set up of §4.3 of [X4]. We consider the coset $G:=SU(N)_{m'}\times SU(N)_{m''}/H:=SU(N)_{m'+m''}$, where the embedding $H\subset G$ is diagonal. Let $\Lambda_1,\dots,\Lambda_{N-1}$ be the fundamental weights of $SL(N)$. Let $k\in N$. Recall that the set of integrable weights of the affine algebra $\widehat{SL(N)}$ at level k is the following subset of the weight lattice of $SL(N)$:

$$P^h_{++}=\{\lambda=\lambda_1\Lambda_1+\cdots+\lambda_{N-1}\Lambda_{N-1}\mid\lambda_i\in N,\ \lambda_1+\cdots+\lambda_{N-1}<h\}$$

where $h=k+N$. This set admits a Z_N automorphism generated by

$$\sigma_1:\lambda=(\lambda_1,\lambda_2,\dots,\lambda_{N-1})\to\sigma_1(\lambda)=(h-\sum_{j=1}^{N-1}\lambda_j,\lambda_1,\dots,\lambda_{N-2}).$$

We define the color $\tau(\lambda):\equiv\Sigma_i(\lambda_i-1)i\,\mathrm{mod}(N)$ and Q to be the root lattice of $\widehat{SL(N)}$ (cf. §1.3 of [KW]). Note that $\lambda\in Q$ iff $\frac{1}{N}\tau(\lambda)\in Z$.

We use i (resp. α) to denote the irreducible positive energy representations of LG (resp. LH). To compare our notations with that §2.7 od [KW], note that our i is (Λ',Λ'') of [KW], and our α is Λ of [KW]. We will identify $i=(\Lambda',\Lambda'')$ and $\alpha=\Lambda$ where Λ',Λ'', Λ are the weights of $SL(N)$ at levels m', m'', $m'+m''$ respectively. Suppose

$$i=(\Lambda'_1,\Lambda''_1), j=(\Lambda'_2,\Lambda''_2),\ k=(\Lambda'_3,\Lambda''_3),\ \alpha=\Lambda_1,\ \beta=\Lambda_2,\ \delta=\Lambda_3.$$

Then the fusion coefficients $N^k_{ij}:=N^{\Lambda'_3}_{\Lambda'_1\Lambda'_2}N^{\Lambda''_3}_{\Lambda''_1\Lambda''_2}$ (resp. $N^\delta_{\alpha\beta}:=N^{\Lambda_3}_{\Lambda_1\Lambda_2}$) of LG (resp. LH) are given by Verlinde formula (cf. §2.1). Recall from §4.3 of [X4] that $\pi_{i,\alpha}$ are the covariant representations of the coset G/H. The set of all $(i,\alpha):=(\Lambda',\Lambda'',\Lambda)$ which appears in the decompositions of π^i of LG with respect to LH is denoted by *exp*. This set is determined on P. 194 of [KW] to be $(\Lambda',\Lambda'',\Lambda)\in exp$ iff $\Lambda'+\Lambda''-\Lambda\in Q$. The Z_N action on (i,α), $\forall i$, $\forall\alpha$ is denoted by $\sigma(i,\alpha):=(\sigma(\Lambda'),\sigma(\Lambda''),\sigma(\Lambda))$, $\sigma\in \boldsymbol{Z}_N$. This is also known as diagram automorphisms since they corresponds to the automorphisms of Dykin diagrams. Note that this $\boldsymbol{Z}_N$ action preserves *exp* and therefore induces a $\boldsymbol{Z}_N$ action on *exp*.

We define a vector space W over $\boldsymbol{C}$ whose orthonormal basis are denoted by $i\otimes\alpha$ with $i=(\Lambda',\Lambda'')$, $\alpha=\Lambda$. W is also a commutative ring with structure constants given by $N_{ij}^k N_{\alpha\beta}^\delta$. Let V be the vector space over $\boldsymbol{C}$ whose basis are given by the irreducible components of $\sigma_i a_{1\otimes\bar{\alpha}}$ (cf. §4.3 of [X4]). Then $V=V_0\oplus V_1$, where V_0 is a subspace of V whose basis are given by the irreducible components of $\sigma_i a_{1\otimes\bar{\alpha}}$ with $(i,\alpha)\in exp$, and V_1 is the orthogonal complement of V_0 in V. The composition of sectors gives V a ring structure. By (1) of Th. 4.3 of [X4], the irreducible subrepresentations of (i,α) of the coset are in one-to-one correspondence with the basis of V_0 and this map is a ring isomorphism by (1) of Prop. 4.2 of [X4], and we will identify the irreducible subrepresentations of (i,α) of the coset with the basis of V_0 in the following when no confusion arises. Note that V_0 is a subring of V and $V_0 . V_1\subset V_1$.

Define a linear map $P: W\to V$ such that $P(i\otimes\alpha)=\sigma_i a_{1\otimes\bar{\alpha}}$. By Th. 4.3 of [X4]

$$P(i\otimes\alpha)=\sigma_i a_{1\otimes\bar{\alpha}}=P(i'\otimes\bar{\alpha}')=\sigma_{i'}a_{1\otimes\alpha'}$$

iff $\sigma^s(i)=i'$, $\sigma^s(\alpha)=\alpha'$ for some $s\in\boldsymbol{Z}$. Also $\langle P(i\otimes\alpha), P(j\otimes\beta)\rangle=0$ if $P(i\otimes\alpha)\neq P(j\otimes\beta)$ by (*) of §4.3 of [X4].

Note $P(\sigma(1)\otimes\sigma(1))=1$ and P is a ring homomorphism from W to V. Define $W_0:=P^{-1}(V_0)$, $W_1:=P^{-1}(V_1)$, then $W=W_0\oplus W_1$ since *exp* is σ invariant. Note that $i\otimes\alpha\in W_0$ iff $i-\alpha\in Q$. Define the action of Z_N on W as $\sigma(i\otimes\alpha)=\sigma(i)\otimes\sigma(\alpha)$. Much of the following depends on the relation between W and V.

Assume $\sigma^s(i\otimes\alpha)=i\otimes\alpha$ for some $i\otimes\alpha\in W$, and $0<s\leq N$ is the least positive integer with this property. Let $t=\frac{N}{s}$. By equation (*) in §4.3 of [X4] we have:

$$\langle P(i\otimes\alpha),\ P(i\otimes\alpha)\rangle=t.$$

Our first question is to decompose $P(i\otimes\alpha)$ when $t>1$.

Apply Lemma 2.1 to the present case with

$$P(i\otimes 1)=a,\ P(1\otimes\alpha)=b,\ P(i\otimes\alpha)=c,\ P(\sigma^s\otimes 1)=\tau,$$

we conclude that there exists $c_1,\cdots,c_t\in V$ such that

$$P(i\otimes\alpha)=\sum_{1\le k\le t} c_k$$

and $d_{c_k}=\frac{1}{t}d_i d_\alpha$, $k=1,\dots,t$. Note that if $P^{-1}(P(i\otimes\alpha))=\{i_1\otimes\alpha_1,\dots,i_s\otimes\alpha_s\}$, then $st=N$.

Note we identify the covariant representations of the coset with the basis of $P(W_0)=V_0$. The univalence of $A:=P(i\otimes\alpha)$, $i\otimes\alpha\in W_0$ are given by: $\omega_A=\exp(2\pi i(\Delta_i-\Delta_\alpha))$, where Δ_i, Δ_α are the conformal dimensions (cf. §2.1, and if $i=(\Lambda',\Lambda'')$, $\Delta_i:=\Delta_{\Lambda'}+\Delta_{\Lambda''}$). Note if $A\succ a$, then $\omega_a=\omega_A$. The univalence is only defined for covariant sectors which correspond to elements of V_0. However, for convenience let us define $\omega_{i\otimes\alpha}:=\omega_i\omega_\alpha^{-1}$ for $i\otimes\alpha\in W$. Then if $i\otimes\alpha\in W_0$, $\omega_i\omega_\alpha^{-1}$ which is the univalence of $P(i\otimes\alpha)$ depends only on $P(i\otimes\alpha)$, i.e,. $\omega_i\omega_\alpha^{-1}$ depends only on the orbit of $i\otimes\alpha$ under the $\boldsymbol{Z}_N$ action.

Suppose $A:=P(i\otimes\alpha)$, $B:=P(j\otimes\beta)$, $i\otimes\alpha$, $j\otimes\beta\in W_0$. Let ϕ_A, ϕ_B be the unique minimal left inverses of A, B. Define

$$Y_{AB}:=d_A d_B\phi_B(\phi_A(\epsilon(B,A)\epsilon(A,B))^*). \tag{1}$$

Note (1) is similar to (0) of §2.1, the difference here is that our A, B may be reducible and hence we need to include ϕ_B in the definition since $\phi_A(\epsilon(B,A)\epsilon(A,B))^*$ may not be a scalar.

To avoid confusions we will denote S matrices associated to indices i (Recall from §2.1 this is the tensor product of S matrices associated to two type A subgroup of G) by S_{ij} and the S matrices associated to indices α by $\dot{S}_{\alpha\beta}$.

Lemma 2.2. *Suppose $A:=P(i\otimes\alpha)$, $B:=P(j\otimes\beta)$, $i\otimes\alpha$, $j\otimes\beta\in W_0$, and $A=\Sigma_{1\le i\le t}c_i$ with $d_{c_i}=\frac{1}{t}d_A$. Then:*

(1) $\langle c_iB, P(k\otimes\delta)\rangle=\frac{1}{t}\langle AB, P(k\otimes\delta)\rangle$, $\forall k\otimes\delta\in W$;

(2) $Y_{AB}=\dfrac{S_{ij}}{S_{11}}\dfrac{\overline{\dot{S}_{\alpha\beta}}}{\dot{S}_{11}}$;

(3) $Y_{c_iB}=\frac{1}{t}Y_{AB}$;

(4) *If* $B=\Sigma_j b_j$, *then* $\Sigma_j Y_{c_ib_j}=Y_{c_iB}$.

Proof. Ad(1): Denote by $C:=P(k\otimes\delta)$. Then by Frobenius duality

$$\langle c_iB, C\rangle=\langle c_i, C\bar{B}\rangle.$$

By the definitions of B, C,

$$C\bar{B}=lA+\sum D$$

where $l\geq 0$ is an integer, and D are elements of the form $P(k'\otimes\delta')$ which are different from A, and by (*) in §4.3 of [X4] $\langle D,A\rangle=0$, and so $\langle D,c_i\rangle=0$. It follows that

$$\langle c_iB,C\rangle=\langle c_i,lA\rangle=l$$

which is independent of i. Since $A=\Sigma_{1\leq i\leq t}c_i$, (1) follows.

Ad (2): The main point of the proof is that even though A, B may be reducible, their univalence are complex numbers, so the monodromy equation (cf. [FRS] or P. 359 of [X1]) holds, and we have:

$$\begin{aligned}\phi_B(\phi_A(\epsilon(B,A)\epsilon(A,B))^*)&=\phi_B(\phi_A(\sum_{e\in V}T_eT_e^*\epsilon(A,B)^*\epsilon(B,A)^*T_eT_e^*))\\ &=\sum_{e\in V}\frac{\omega_A\omega_B}{\omega_e}N_{AB}^e\frac{d_e}{d_Ad_B}\\ &=\sum_{e\in V}\frac{\omega_i\omega_j\omega_\alpha^{-1}\omega_\beta^{-1}}{\omega_e}\langle AB,e\rangle\frac{d_e}{d_Ad_B}\end{aligned}$$

where $e\in V$ means we sum over the basis of V. Note that the summation above is effectively over V_0 since $A\in V_0$, $B\in V_0$ and V_0 is a subring. Suppose $P(k\otimes\delta)=e_1+e_2+\cdots+e_m$, $k\otimes\delta\in W_0$, with $d_{e_i}=\frac{1}{m}d_kd_\delta$, $\omega_{e_i}=\omega_k\omega_\delta^{-1}$. Assume $P^{-1}(P(k\otimes\delta))=k_1\otimes\delta_1,\cdots,k_n\otimes\delta_n$, with $mn=N$. Then

$$\sum_{e_i}\frac{\omega_i\omega_j\omega_\alpha^{-1}\omega_\beta^{-1}}{\omega_{e_i}}\langle AB,e_i\rangle\frac{d_{e_i}}{d_Ad_B}=\frac{1}{m}\frac{\omega_i\omega_j\omega_\alpha^{-1}\omega_\beta^{-1}}{\omega_k\omega_\delta^{-1}}\langle AB,P(k\otimes\delta)\rangle\frac{d_kd_\delta}{d_Ad_B}.$$

By equation (*) in §4.3 of [X4],

$$\begin{aligned}\langle AB,P(k\otimes\delta)\rangle&=\langle\sigma_ia_{1\otimes\bar{\alpha}}\sigma_ja_{1\otimes\bar{\beta}},\sigma_ka_{1\otimes\bar{\delta}}\rangle=N_{ij}^{k'}N_{\bar{\alpha}\bar{\beta}}^{\bar{\delta}'}\langle\sigma_{k'}a_{1\otimes\bar{\delta}'},\sigma_ka_{1\otimes\bar{\delta}}\rangle\\ &=m(N_{ij}^{k_1}N_{\bar{\alpha}\bar{\beta}}^{\bar{\delta}_1}+\cdots+N_{ij}^{k_n}N_{\bar{\alpha}\bar{\beta}}^{\bar{\delta}_n})\end{aligned}$$

We conclude that

$$\sum_{e_i}\frac{\omega_i\omega_j\omega_\alpha^{-1}\omega_\beta^{-1}}{\omega_e}\langle AB,e_i\rangle\frac{d_{e_i}}{d_Ad_B}=\frac{\omega_i\omega_j\omega_\alpha^{-1}\omega_\beta^{-1}}{\omega_k\omega_\delta^{-1}}(N_{ij}^{k_1}N_{\bar{\alpha}\bar{\beta}}^{\bar{\delta}_1}+\cdots+N_{ij}^{k_n}N_{\bar{\alpha}\bar{\beta}}^{\bar{\delta}_n})\frac{d_kd_\delta}{d_Ad_B},$$

and so

$$\phi_B(\phi_A(\epsilon(B,A)\epsilon A,B))^*)=\sum_{k\otimes\delta\in W}\frac{\omega_i\omega_j\omega_{\bar{\alpha}}^{-1}\omega_{\bar{\beta}}^{-1}}{\omega_k\omega_{\bar{\delta}}^{-1}}N_{ij}^k N_{\bar{\alpha}\bar{\beta}}^{\bar{\delta}}\frac{d_k d_\delta}{d_i d_\alpha d_j d_\beta}$$

$$=\sum_k\frac{\omega_i\omega_j}{\omega_k}N_{ij}^k\frac{d_k}{d_i d_j}\times\sum_\delta\frac{\omega_\alpha^{-1}\omega_\beta^{-1}}{\omega_\delta^{-1}}N_{\alpha\beta}^\delta\frac{d_\delta}{d_\alpha d_\beta}$$

$$=\frac{S_{ij}}{S_{11}}\frac{\overline{\dot{S}_{\alpha\beta}}}{\dot{S}_{11}}$$

where in the last $=$ we have used (2) and the comment after (12) in §2.1.

Ad (3): As in the proof of (2) let $P(k\otimes\delta)$, $k\otimes\delta\in W_0, P(k\otimes\delta)=e_1+\cdots+e_m$, $P^{-1}(P(k\otimes\delta))=\{k_1\otimes\delta_1,\ldots,k_n\otimes\delta_n\}$, with $mn=N$. Then

$$\sum_{e_j}\langle c_iB,e_j\rangle\frac{\omega_{c_i}\omega_B}{\omega_{e_j}}d_{e_j}=\frac{1}{m}\langle c_iB,P(k\otimes\delta)\rangle\frac{\omega_A\omega_B}{\omega_{k\otimes\delta}}d_k d_\delta$$

$$=\frac{1}{t}\frac{1}{m}\langle AB,P(k\otimes\delta)\rangle\frac{\omega_A\omega_B}{\omega_{k\otimes\delta}}d_k d_\delta$$

$$=\frac{1}{t}\sum_{e_j}\langle AB,e_j\rangle\frac{\omega_A\omega_B}{\omega_{e_j}}d_{e_j}$$

where on the second $=$ we used (1), and (3) follows immediately.

Ad (4): First note $\omega_{b_j}=\omega_B$. By (2) of §2.1, we have:

$$\sum_j Y_{c_ib_j}=\sum_j\sum_e\langle c_ib_j,e\rangle\frac{\omega_{c_i}\omega_{b_j}}{\omega_e}d_e=\sum_e\sum_j\langle c_ib_j,e\rangle\frac{\omega_{c_i}\omega_B}{\omega_e}d_e$$

$$=\sum_e\langle c_iB,e\rangle\frac{\omega_{c_i}\omega_B}{\omega_e}d_e=Y_{c_iB}.$$

Q.E.D.

Recall the basis of V_0 corresponds to a finite set of irreducible covariant sectors of the coset: it is closed under composition, conjugation and contains identity by Th. 4.3 of [X4]. Define the Y matrix as in (0) of §2.1. Then we have:

Theorem 2.3. (1) *The matrix Y is invertible.*

(2) *The number* $\tilde{\sigma} := \Sigma_{e \in V} d_e^2 \omega_e^{-1}$ *is given by*

$$\tilde{\sigma} = \frac{1}{N} S_{11}^{-1} \dot{S}_{11}^{-1} \exp\left(\frac{-6\pi i}{24}(C_G - C_H)\right)$$

where

$$\frac{1}{24}(C_G - C_H) = \frac{N^2-1}{24} - \frac{N(N^2-1)}{24}\left(\frac{1}{m'+N} + \frac{1}{m'+N} - \frac{1}{m'+m''+N}\right).$$

Proof. By (i) of Prop. on P. 351 of [Reh] it is enough to show that if e is such that

$$Y_{eg} = d_e d_g, \quad \forall g,$$

then $e=1$ (Remember if a sector is denoted by 1, then it is the identity or equivalently vacuum sector). Suppose $B := P(j \otimes \beta) = \Sigma_{1 \le i \le t_B} g_i \succ g$, $B \in V_0$. By (4) of Lemma 2.2

$$Y_{eB} = \sum_i Y_{eg_i} = \sum_i d_e d_{g_i} = d_e d_B.$$

Suppose $A = P(i \otimes \alpha) = \Sigma_{1 \le i \le t} c_i \succ e$, $d_{c_i} = \frac{1}{t} d_c$. Then by (3) of Lemma 2.2 we have:

$$Y_{AB} = d_A d_B, \ \forall B \in V_0.$$

By (2) of Lemma 2.2 we have

$$\frac{S_{ij}}{S_{11}} \frac{\overline{\dot{S}_{\alpha\beta}}}{\dot{S}_{11}} = d_i d_j d_\alpha d_\beta,$$

and it follows that

$$\frac{S_{ij}}{S_{1j}} \frac{\overline{\dot{S}_{\alpha\beta}}}{\dot{S}_{1\beta}} = d_i d_\alpha$$

for any $j \otimes \beta \in W_0$. Note

$$\left|\frac{S_{ij}}{S_{1j}}\right| \le d_i, \quad \left|\frac{\overline{\dot{S}_{\alpha\beta}}}{\dot{S}_{1\beta}}\right| \le d_\alpha,$$

there must exist $a_{ij}\in \boldsymbol{R}$ such that

$$\frac{S_{ij}}{S_{1j}}=\exp(ia_{ij})d_i,\ \frac{\dot{S}_{\alpha\beta}}{\dot{S}_{1\beta}}=\exp(ia_{ij})d_\alpha$$

for any $j\otimes\beta\in W_0$. Suppose $(j,\beta)=(\Lambda',\Lambda'';\Lambda)$, $(i,\alpha)=(M',M'';M)$, then $\Lambda'+\Lambda''-\Lambda\in Q$, $M'+M''-M\in Q$. From

$$\frac{\dot{S}_{\alpha\beta}}{\dot{S}_{1\beta}}=\exp(ia_{ij})d_\alpha$$

we get

$$\left|\frac{S_{M\Lambda}}{S_{1\Lambda}}\right|=d_M$$

for any $\Lambda\in P_+^{(K'+K'')}$, and so

$$\sum_\Lambda|S_{M\Lambda}|^2=d_M^2\sum_\Lambda|S_{1\Lambda}|^2,$$

i.e., $d_M^2=1$. It follows that $M\bar{M}$ is the identity sector, and so $M\Lambda$ is always irreducible. Choose Λ corresponding to the defining representation of $SU(N)$, it follows from the fusion rules that M must be of the form $\sigma^s(1)$ for some $s\in \boldsymbol{Z}$. Since $P(i\otimes\alpha)=P(\sigma^{-s}(i\otimes\alpha))=A$, replacing $(i,\alpha)=(M',M'';\sigma^s(1))$ by $\sigma^{-s}(i,\alpha)=(\sigma^{-s}(M'),\ \sigma^{-s}(M'');1)$ if necessary, we may assume $s=0$, i.e., $\alpha=1$.

Similarly $(M',M'')=(\sigma^{s_1}(1),\sigma^{s_2}(1))$, and using $\alpha=1$ we have

$$\frac{S_{M'\Lambda'}}{S_{1\Lambda'}}\frac{S_{M''\Lambda''}}{S_{1\Lambda''}}=d_{M'}d_{M''}=1,\ \forall(\Lambda',\Lambda'').$$

Choose Λ' to be the vaccum representation, from the above equation we have

$$S_{M''\Lambda''}=S_{1\Lambda''},\ \forall\Lambda''.$$

Since S matrix is unitary by (9) in §2.1, we conclude M'' is the vacuum representation. By the same argument M' is also the vacuum representation.

So we have proved $P(i\otimes\alpha)\succ e$ is the vacuum sector, and therefore e must be the vacuum sector.

Ad (2): First we claim:

$$\sum_{e\in V_0}d_e^2\omega_e^{-1}=\frac{1}{N}\sum_{g\in W_0}d_g^2\omega_g^{-1}.$$

Suppose $P(g)=f$, $P^{-1}(P(g))=\{g_1,\cdots,g_n\}$, and $P(g)=f=e_1+\cdots+e_m$, with $mn=N$. Since $\omega_{e_i}=\omega_g$, $d_{e_i}=\frac{d_g}{m}$, $d_{g_j}=d_g$, $\omega_{g_j}=\omega_g$, we have

$$\sum_{e_i} d_{e_i}^2\omega_{e_i}^{-1}=\sum_{e_i}\frac{d_g^2}{m^2}\omega_g^{-1}=\frac{d_g^2}{m}\omega_g^{-1}$$

and so

$$\sum_{e_i} d_{e_i}^2\omega_{e_i}^{-1}=\frac{1}{N}nd_g^2\omega_g^{-1}=\frac{1}{N}\sum_{g_j} d_{g_j}^2\omega_{g_j}^{-1}.$$

It follows that

$$\sum_{e\in V_0} d_e^2\omega_e^{-1}=\frac{1}{N}\sum_{g\in W_0} d_g^2\omega_g^{-1}.$$

Next let us show

$$\sum_{g\in W_1} d_g^2\omega_g^{-1}=0.$$

Again Suppose $g=i\otimes\alpha$, $P(g)=f$, $P^{-1}(P(g))=\{g_1,\cdots,g_n\}$, and $P(g)=f=e_1+\cdots+e_m$, with $mn=N$. So we have $\sigma^n(i,\alpha)=(i,\alpha)$, $g_k=\sigma^{k-1}g_1$, $1\le k\le n$. Note that by definitions

$$\omega_{\sigma(i)}\omega_{\sigma(\alpha)}^{-1}=\exp\left(\frac{-2\pi i(\tau(i)-\tau(\alpha))}{N}\right)\omega_i\omega_\alpha^{-1}.$$

Denote by $z:=\exp\left(\frac{2\pi i(\tau(i)-\tau(\alpha))}{N}\right)$, then $z^n=1$ since $\sigma^n(i,\alpha)=(i,\alpha)$, but $z\neq 1$ since $i\otimes\alpha$ is not in W_0, i.e., $\frac{(\tau(i)-\tau(\alpha))}{N}\notin \mathbf{Z}$. So

$$\sum_{g_k} d_{g_k}^2\omega_{g_k}^{-1}=\sum_{1\le k\le n} d_g^2\omega_{g_1}^{-1}z^{k-1}=0.$$

We have

$$\sum_{e\in V_0} d_e^2\omega_e^{-1}=\frac{1}{N}\sum_{g\in W_0} d_g^2\omega_g^{-1}=\frac{1}{N}\sum_{g\in W} d_g^2\omega_g^{-1}$$
$$=\frac{1}{N}\sum_k d_k^2\omega_k^{-1}\sum_\delta d_\delta^2\omega_\delta$$
$$=\frac{1}{N}\exp\left(\frac{-6\pi i(C_G-C_H)}{24}\right)\frac{1}{S_{11}}\frac{1}{S^{\cdot}_{11}}$$

where in the last = we have used (3), (12) and comments after (12) in §2.1, and C_G and C_H are central charges given by (1.4.2) of [KW], i.e.,

$$C_G=\frac{(N^2-1)m'}{m'+N}+\frac{(N^2-1)m''}{m''+N},$$
$$C_H=\frac{(N^2-1)(m'+m'')}{m'+m''+N}.$$

(2) now follows by a simple calculation. Q.E.D.

Corollary 2.4. *The irreducible covariant sectors of the diagonal coset of type A corresponding to the basis of V_0, with its braiding and S, T matrices as defined in* §2.1, *is a unitary modular category* (*cf.* [*Tu*]).

Proof. By the definition of unitary modular category as on P. 74 and P. 113 of [Tu], it is enough to show that the Y matrix is invertible, which follows from (1) of Th. 2.3. Q.E.D.

When the Z action on *exp* is faithful, i.e., for any $(i,\alpha)\in exp$, $\sigma^s(i,\alpha)=(i,\alpha)$ iff $N|s$, by (2) of Th. 2.3 we have that $S_{AB}=NS_{ij}\overline{S^{\cdot}_{\alpha\beta}}$ where $A:=P(i\otimes\alpha)$, $B:=P(j\otimes\beta)$ and both A, B are irreducible.

Let us calculate S-matrices in the case N is a prime and there exists a (necessarily unique) fixed point $F:=P(i_0\otimes\alpha_0)\in V_0$ under the action of σ. By Lemma 2.1, $F=\Sigma_{1\le i\le N}F_i$ with irreducible and $d_{F_i}=\frac{1}{N}d_F$. Recall $S_{ab}:=|\tilde{\sigma}|^{-1}Y_{ab}$, where

$$|\tilde{\sigma}|^2=\frac{1}{N^2}\frac{1}{S_{11}^2}\frac{1}{S^{\cdot 2}_{11}}$$

which follows from (2) of Th. 2.3.

So $S_{ab}=NS_{11}S^{\cdot}_{11}Y_{ab}$, and by (3) of Lemma 2.2 S_{ab} for $a=P(i\otimes\alpha)\neq F_i$ is determined as follows:

$$S_{ab}=NS_{ij}\overline{S^{\cdot}_{\alpha\beta}} \tag{2}$$

if $b:=P(j\otimes\beta)\neq F_k$, and

$$S_{aF_k}=S_{ii_0}\overline{S^{\cdot}_{\alpha\alpha_0}}. \tag{3}$$

It remains to determine $S_{F_iF_j}$. Note

$$STS=T^{-1}ST^{-1}$$

and $T_{ab}=\delta_{ab}\dot{\omega}_a$, where $\dot{\omega}_a=\omega_a\omega$, and ω is determined by (2) of Th. 2.3 and (3) of §2.1 as:

$$\omega:=\exp\left(\frac{-2\pi i}{24}(C_G-C_H)\right).$$

We have

$$\begin{aligned}
\dot{\omega}_{F_i}{}^{-1}S_{F_iF_k}\dot{\omega}_{F_k}{}^{-1}&=\sum_a S_{F_iA}\dot{\omega}_A S_{AF_k}\\
&=\sum_l S_{F_iF_l}\dot{\omega}_{F_l}S_{aF_k}+\sum_{A\neq F}S_{F_iA}\dot{\omega}_A S_{AF_k}\\
&=\dot{\omega}_F\sum_l S_{F_iF_l}S_{F_lF_k}+\sum_{A\neq F}S_{F_iA}\dot{\omega}_A S_{AF_k}\\
&=\dot{\omega}_F(\delta_{F_i\bar{F}_k}-\sum_{B\neq F}S_{F_iB}S_{BF_k})+\sum_{A\neq F}S_{F_iA}\dot{\omega}_A S_{AF_k}\\
&=\dot{\omega}_F\delta_{F_i\bar{F}_k}+\sum_{A\neq F}S_{F_iA}(\dot{\omega}_A-\dot{\omega}_F)S_{AF_k}\\
&=\dot{\omega}_F\delta_{F_i\bar{F}_k}+\frac{1}{N^2}\sum_{A\neq F}S_{FA}(\dot{\omega}_A-\dot{\omega}_F)S_{AF}\\
&=\dot{\omega}_F\delta_{F_i\bar{F}_k}+\frac{1}{N^2}\sum_A S_{FA}(\dot{\omega}_A-\dot{\omega}_F)S_{AF}
\end{aligned}$$

where we used S^2 is equal to conjugate matrix in the fourth $=$, and (3) of Lemma 2.2 in the sixth $=$. So we have

$$S_{F_iF_k} = \omega_F^{\,3} + \omega_F^{\,2}\frac{1}{N^2}\sum_{A\in V_0} S_{FA}(\omega_A - \omega_F)S_{AF}.$$

But

$$\begin{aligned}\sum_{A\in V_0} S_{FA}(\omega_A - \omega_F)S_{AF} &= \sum_{A\in V_0, 1\le i,j\le N} S_{F_iA}(\omega_A - \omega_F)S_{AF_j} \\ &= \sum_{A\in V_0, 1\le i,j\le N} S_{F_iA}\omega_A S_{AF_j} - \sum_{A\in V_0, 1\le i,j\le N} S_{F_iA}\omega_F S_{AF_j} \\ &= \sum_{1\le i,j\le N} \omega_F^{\,-2} S_{F_iF_j} - \sum_{A\in V_0, 1\le i,j\le N} S_{F_iA}\omega_F S_{AF_j} \\ &= \omega_F^{\,-2} S_{FF} - \omega_F \sum_{1\le i,j\le N} \delta_{F_i\bar{F}_j} \\ &= \omega_F^{\,-2} S_{FF} - \omega_F N\end{aligned}$$

where $S_{FF} = |\tilde{\sigma}|^{-1} Y_{FF}$, and in the fourth $=$ we used (4) of Lemma 2.2, and in the last step we used $\bar{F} = F$ and

$$\sum_{1\le i,j\le N} \delta_{F_i\bar{F}_j} = \langle F, \bar{F}\rangle = \langle F, F\rangle = N.$$

So we have:

$$S_{F_iF_k} = \omega_F^{\,3}\delta_{F_i\bar{F}_k} + \frac{1}{N^2}(S_{FF} - \omega_F^{\,3}N).$$

Let us show

$$\omega_F^{\,3} = \omega_F^{\,3}\omega^3 = 1.$$

Recall $\omega_F = \exp(2\pi i \Delta_F)$, and since F is the unique fixed point, by a simple calculation using (10) of §2.1 we get

$$\Delta_F = \frac{N^2-1}{24} - \frac{N(N^2-1)}{24}\left(\frac{1}{m'+N} + \frac{1}{m'+N} - \frac{1}{m'+m''+N}\right),$$

and it follows that

$$\omega = \exp\left(-2\pi i \frac{1}{24}(C_G - C_H)\right) = \omega_F^{-1}$$

by (2) of Th. 2.3, so $\omega_F = 1$.

Therefore we have

$$S_{F_iF_k} = \delta_{F_i\bar{F}_k} + \frac{1}{N^2}(S_{FF} - N).$$

Since $\bar{F} = F$, S_{FA} is real for all A, so S_{F_ka} is real for any irreducible a, and we must have $\bar{F}_k = F_k$ since S matrix is invertible. So:

$$S_{F_iF_k} = \delta_{ik} + \frac{1}{N^2}(S_{FF} - N) \tag{4}$$

The formula (2), (3) and (4) above agree wiht formula (4.40) of [FSS1] (Note our $S_{FF} = NS_{i_0i_0}\overline{S^{\cdot}_{\alpha_0\alpha_0}}$, where $F = P(i_0 \otimes \alpha_0)$). However one should notice that our definition of S matrices are very different from those of [FSS1].

In [FSS1] and [FSS2], certain formula about S matrices were derived from other considerations in the case when N is not prime and other types of simple simply connected Lie groups, and it will be interesting to extend our calculations above to these cases and to see if the results agree with [FSS1] and [FSS2].

By Cor. 2.4, one may calculate 3-manifold invariants using S matrices obtained above as in [Tu]. These and related questions will be addressed in another publication.

§3. Miscellaneous Results

§3.1 KWH and KWC. Let $H \subset G_k$ be as in the introduction. Through out this section, we will assume the following: H and G verifies the statements as in Cor. 1 in §34 of [W] (cf: comments after (12) in §2.1), and $H \subset G_k$ is cofinite as defined in §3 of [X4].

Note the assumption is satisfied by many examples (cf. Cor. 4.2 of [X4]) and is expected to be true in general.

We also assume that $H \subset G_k$ is not conformal, so the coset theory is non-trival.

We will use the notations of §4.2 of [X4] and ideas of [X2]. We denote the set of irreducible sectors of $\sigma_i a_{1\otimes\lambda}$ by V. Notice $\sigma_i \in V$, and these are referred to as special nodes in §3.4 of [X1]. Let:

$$a_{1\otimes\lambda} a = \sum_{b \in V} V^{\lambda}_{ab} b,$$

where V^λ_{ab} are nonnegative integers. Denote by V^λ the matrix such that $(V^\lambda)^b_a = V^\lambda_{ab}$. Define matrix N_c by $N^b_{ca} = \langle ca, b\rangle$ for $a, b, c \in V$. Then $V^\lambda = \Sigma_c V^\lambda_{1c} N_c$. Since $[a_{1\otimes\bar{\lambda}}] = [\bar{a}_{1\otimes\lambda}]$, $[\sigma_j a_{1\otimes\lambda}] = [a_{1\otimes\lambda}\sigma_j]$, V^λ, N_{σ_j} are commuting normal matrices, so they can be simultaneously diagonalized. Recall the irreducible representations of the ring $Gr(C_k)$ generated by λ's are given by

$$\lambda \to \frac{S_{\lambda\mu}}{S_{1\mu}}.$$

Assume

$$V^\lambda_{ab} = \sum_{i,\mu,s\in(\mathrm{Exp})} \frac{S_{\lambda\mu}}{S_{1\mu}} \psi_a^{(i,\mu,s)} \psi_b^{(i,\mu,s)*}$$

where $\psi_a^{(i,\mu,s)}$ are normalized orthogonal eigenvectors of V^λ (resp. N_{σ_i}) with eigenvalue $\frac{S_{\lambda\mu}}{S_{1\mu}}$ (resp. $\frac{S_{ij}}{S_{1j}}$). (Exp) is a set of i, μ, s's and s is an index indicating the multiplicity of i, μ. We denote by Exp the set of (i, μ) such that $(i, \mu, s) \in (Exp)$ for some s. Recall if a representation is denoted by 1, it will always be the vacuum representation. The Perron-Frobenius eigenvector $\psi^{(1,1)}$ is given by $\Sigma_a d_a a$, up to a positive constant. Note all the entries of $\psi^{(1,1)}$ are positive.

Proposition 3.1. *$(i, \alpha) \in Exp$ if and only if $b(i, \alpha) > 0$.*

Proof. Recall (1) of §1:

$$b(i, \alpha) = \sum_{(j,\beta)} S_{ij} \overline{\check{S}_{\alpha\beta}} \langle (i, \alpha), (1, 1)\rangle.$$

By the proof of (2) of Prop. 4.2 of [X4] and (2) of Cor. 3.5 of [X1] we have

$$\langle (i, \alpha), (1, 1)\rangle = \langle \sigma_i a_{1\otimes\bar{\alpha}}, 1\rangle = \langle \sigma_{\bar{i}} a_{1\otimes\alpha}, 1\rangle,$$

so:

$$b(i, \alpha) = \sum_{(j,\beta)} S_{ij} \overline{\check{S}_{\alpha\beta}} \langle \sigma_{\bar{j}} a_{1\otimes\beta}, 1\rangle = \sum_{(j,\beta)} S_{ij} \overline{\check{S}_{\alpha\beta}} (N_{\bar{j}} V^\beta)_{11}$$

$$= \sum_{(k,\delta,s)\in(Exp)} S_{ij}\overline{\dot{S}_{\alpha\beta}} \frac{\dot{S}_{\beta\delta}}{\dot{S}_{1\delta}} \frac{\overline{S_{jk}}}{S_{1k}} |\psi_1^{(k,\delta,s)}|^2$$

$$= \sum_s \frac{1}{\dot{S}_{1\alpha}} \frac{1}{S_{1i}} |\psi_1^{(i,\alpha,s)}|^2.$$

Note the equality above is similar to (1) on P. 12 of [X2], and the rest of the proof is the same as the proof on P. 12 of [X2]. Q.E.D.

Note if $b(i,\alpha)>0$, then $(i,\alpha)\in exp$, so by Prop. 3.1 $Exp \subset exp$, and KWC is equivalent to the statement that $Exp=exp$.

Proposition 3.2. *If* $b(i,1)>0$, *then* $\frac{b(i,1)}{b(1,1)}=\frac{S_{i1}}{S_{11}}$.

Proof. If $b(i,1)>0$, by Prop. 3.1, $(i,1)\in Exp$. Suppose $(j,\beta)\succ(1,1)$. Then

$$\langle (j,\beta),(1,1)\rangle = \langle \sigma_j a_{1\otimes\bar{\beta}}, 1\rangle = \langle \sigma_j, a_{1\otimes\beta}\rangle > 0$$

since σ_j is irreducible. It follows that

$$a_{1\otimes\beta} \succ \sigma_j,$$

and if $b=a_{1\otimes\beta}-\sigma_j$, then $V^b:=V^\beta - N_j$ is a normal matrix with non-negative entries, with a Perron-Frobenius eigenvalue $\frac{\dot{S}_{\beta 1}}{\dot{S}_{11}}-\frac{S_{j1}}{S_{11}}$. It follows that

$$\frac{\dot{S}_{\beta 1}}{\dot{S}_{11}}-\frac{S_{j1}}{S_{11}} \geq |\langle V^b\psi^{(i,1,s)}, \psi^{(i,1,s)}\rangle| = \left|\frac{\dot{S}_{\beta 1}}{\dot{S}_{11}}-\frac{S_{ji}}{S_{1i}}\right|$$

$$\geq \frac{\dot{S}_{\beta 1}}{\dot{S}_{11}} - \left|\frac{S_{ji}}{S_{1i}}\right|$$

$$\geq \frac{\dot{S}_{\beta 1}}{\dot{S}_{11}} - \frac{S_{j1}}{S_{11}}.$$

So all the $\geq$ are $=$, which happens only if

$$\frac{S_{ji}}{S_{1i}} = \frac{S_{j1}}{S_{11}}.$$

Prop. 3.1 now follows from the definitions. Q.E.D.

Theorem 3.3. *If* $H \subset G_k$ *satisfies* (2) *in the introduction, then KWH is* *equivalent to KWC.*

Proof. We just have to show that KWC implies KWH.

By (2) of the introduction we have if $\langle(1,\delta),(1,1)\rangle>0$, then $\delta=1$. For any α, β, we have (cf. Prop. 4.2 of [X4]):

$$\begin{aligned}\langle a_{1\otimes\alpha}, a_{1\otimes\beta}\rangle=\langle a_{1\otimes\alpha}a_{1\otimes\bar{\beta}}, 1\rangle=N_{\alpha\bar{\beta}}^{\delta}\langle a_{1\otimes\delta}, 1\rangle &\\ =N_{\alpha\bar{\beta}}^{\delta}\langle(1,\delta),(1,1)\rangle &\\ =N_{\alpha\bar{\beta}}^{1} &\\ =\langle\alpha,\beta\rangle. &\end{aligned}$$

In particular $a_{1\otimes\beta}$ is irreducible if β is irreducible.

Suppose $\langle(j,\beta),(1,1)\rangle>0$, then $\langle\sigma_j a_{1\otimes\bar{\beta}}, 1\rangle>0$, and so $\langle\sigma_j, a_{1\otimes\beta}\rangle>0$. Since both σ_j and $a_{1\otimes\beta}$ are irreducible, it follows that

$$\sigma_j=a_{1\otimes\beta}.$$

Now suppose $(i,\alpha)\in exp$. By KWC, $b(i,\alpha)>0$, so by Prop. 3.1, $(i,\alpha)\in Exp$, and from $\sigma_j=a_{1\otimes\beta}$ we must have:

$$\frac{S_{ji}}{S_{1i}}=\frac{\dot{S}_{\beta\alpha}}{\dot{S}_{1\alpha}}.$$

Therefore

$$S_{ji}\overline{\dot{S}_{\beta\alpha}}=S_{1i}\frac{|\dot{S}_{\beta\alpha}|^2}{\dot{S}_{1\alpha}}\geq 0,$$

which proves KWH. Q.E.D.

Let us give an example which does not satisfy the assumption of our theorem, and verifies KWC but not KWH. This is the coset $SU(2)_8\subset SU(3)_2$ discussed in §4.4 of [X4] (also cf. [DJ]) and we will use the notations there. The vacuum representation space H of $LSU(3)$ at level 2 decomposes as:

$$H=(00,0)\otimes 0+(00,4)\otimes 4+(00,8)\otimes 8,$$

and since

$$(00,0)=(00,8)$$

as representations of the coset, our assumption is not satisfied. In §4.4 of [X4] we checked C2 is satisfied, which implies that KWC is true. However, since

$$(11,4)=(00,0)$$

and $(10,4)\in exp$, KWH implies that

$$S_{11,10}\dot{S}_{4,4}\geq 0.$$

But a direct calculation using (9) of §2.1 gives

$$\frac{S_{11,10}}{S_{00,10}}=\frac{1-\sqrt{5}}{2}$$

and

$$\dot{S}_{4,4}=\sqrt{\frac{1}{5}},$$

which shows that

$$S_{11,10}\dot{S}_{4,4}<0$$

since $S_{00,10}>0$. In fact this was discovered when we verified C2 in this example.

Note that all diagonal inclusions of type A satisfies the assumption of Th. 3.3 by 2.7.12 of [KW]. To give a slightly different example, let us consider the following inclusions

$$SU(2)_{11k}\subset SU(2)_{8k}\times SU(2)_{3k}\subset SU(3)_{2k}\times SU(2)_{3k}\subset SU(6)_k$$

with $k\in N$, where the first inclusion is diagonal, the second inclusion comes from the conformal inclusion $SU(2)_4\subset SU(3)_1$, and the third inclusion comes from the conformal inclusion $SU(3)_2\times SU(2)_3\subset SU(6)$ (cf. [X3]).

By (2) of Prop. 3.1 and (2) of Cor. 3.1 of [X4], the inclusion $SU(2)_{11k}\subset SU(6)_k$ is cofinite and verifies the assumptions at the beginning of this section. Note if $(1,\alpha)\succ(1,1)$, then $\Delta_\alpha\in \boldsymbol{Z}$ by definition. Here $1\leq\alpha\in \boldsymbol{Z}\leq 11k+1$ and $\Delta_\alpha=\frac{\alpha(\alpha-1)}{11k+2}$. So if $11k+2$ is prime, then $(1,\alpha)\succ(1,1)$ iff $\alpha=1$, and it follows that the conditions of Th. 2.3 are satisfied. So the conclusions of Th. 2.3 hold for the inclusion

$$SU(2)_{11k}\subset SU(6)_k$$

if $11k+2$ is prime, and by Dirichlet's theorem, there are infinitely many such k's.

§3.2. A property of C2.

Proposition 3.4. *If $H_1 \subset H_2$ (resp. $H_2 \subset G_k$) verifies C2 or is a conformal inclusion, and assume $H_1 \subset G_k$ is not a conformal inclusion to avoid trivality, then $H_1 \subset G_k$ verifies C2.*

Proof. For simplicity we will use π_x, π_y, π_z to denote the irreducible representations of LH_1, LH_2 and LG respectively, and $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ to denote the vacuum sector of cosets $H_1 \subset H_2$, $H_2 \subset G$, $H_1 \subset G$ respectively. Note we have natural inclusions $\mathscr{A}(I) \otimes \mathscr{B}(I) \subset \mathscr{C}(I)$, corresponding to the natural inclusions

$$(\pi(L_1 H_1)' \cap \pi(L_I H_2)'') \otimes (\pi(L_I H_2)' \cap \pi(L_I G)'') \subset \pi(L_I H_1)' \cap \pi(L_I G)'',$$

where I is a proper open interval of S^1. From the decompositions:

$$\pi_z \simeq \sum_y \pi_{(z,y)} \otimes \pi_y \simeq \sum_{y,x} \pi_{(z,y)} \otimes \pi_{(y,x)} \otimes \pi_x \simeq \sum_x \pi_{(z,x)} \otimes \pi_x$$

we conclude that

$$\pi_{(z,x)} \simeq \sum_y \pi_{(z,y)} \otimes \pi_{(y,x)}$$

which is understood as the decomposition of representation $\pi_{(z,x)}$ of C when restricted to $A \otimes B \subset C$. By local equivalence (cf. P. 502 of [W]), the minimal index of

$$\pi_{(z,x)}(A(I) \otimes B(I)) \subset \pi_{(z,x)}(C(I))$$

is the same as that of

$$\pi_{(1,1)}(A(I) \otimes B(I)) \subset \pi_{(1,1)}(C(I))$$

(I is a proper interval of the circle), which by Haag duality (cf. Prop. 1.1 of [GL]) and Th. 5.5 of [L1] is given by

$$\sum_y d_{(1,y)} d_{(y,1)}.$$

Here when $H_2 \subset G$ (resp. $H_1 \subset H_2$) is a conformal inclusion, $d_{(z,y)}$ (resp. $d_{(y,x)}$) is defined to be the multiplicity of irreducible representation y (resp. x) which appears in z (resp. y) when restricting to LH_2 (resp. LH_1). Otherwise $d_{(z,y)}$ (resp. $d_{(y,x)}$) is the statistical dimension (cf. §2.1) of sector (z,y) (resp. (y,x)).

By Cor. 2.2 of [L3], the statistical dimension of the inclusion: $\pi_{(z,x)}(A(I)\otimes B(I)) \subset \pi_{(z,x)}(C(I)) \subset \pi_{(z,x)}(C(I'))' \subset \pi_{(z,x)}(A(I')\otimes B(I'))'$ (I' is the complement of I in S^1 as defined on P. 14 of [GL]) is

$$d_{(z,x)}\sum_y d_{(1,y)}d_{(y,1)}.$$

On the other hand, by Th. 5.5 of [L1], the statistical dimension of the above inclusion is also given by:

$$\sum_y d_{(z,y)}d_{(y,x)},$$

and so:

$$d_{(z,x)}\sum_y d_{(1,y)}d_{(y,1)} = \sum_y d_{(z,y)}d_{(y,x)}.$$

Therefore

$$d_{(z,x)} = \frac{\Sigma_y d_{(z,y)}d_{(y,x)}}{\Sigma_y d_{(1,y)}d_{(y,1)}}.$$

The proof now follows from the assumptions and

$$b(z,x) = \sum_y b(z,y)b(y,x)$$

which follows from Th. B of [KW]. Q.E.D.

Let us give one application of the above proposition. Consider the superconformal coset models (cf. [Gep], [LVW] or [NS]):

$$G(m,n,k) := \frac{SU(m+n)_k \times SO(2mn)_1}{SU(m)_{n+k}\times SU(n)_{m+k}\times U(1)_{mn(m+n)(m+n+k)}}.$$

In our setting, when $2mn > 2$, the inclusion is given by $H\subset G$ with $H = SU(m)_{n+k}\times SU(n)_{m+k}\times U(1)_{mn(m+n)(m+n+k)}$ and $G = SU(m+n)_k \times Spin(2mn)_1$. The inclusion $H\subset G$ is constructed by the composition of two inclusions:

$$H \subset SU(m)_n \times SU(m)_k \times SU(n)_m \times SU(n)_k \\ \times U(1)_{mn(m+n)(m+n)} \times U(1)_{mn(m+n)(k)} \tag{1}$$

and

$$(SU(m)_n \times SU(n)_m \times U(1)_{mn(m+n)(m+n)}) \times (SU(m)_k \times SU(n)_k \\ \times U(1)_{mn(m+n)(k)}) \subset Spin(2mn)_1 \times SU(m+n)_k . \tag{2}$$

The tangent space of the Grassmanian

$$\frac{SU(m+n)}{SU(m) \times SU(n) \times U(1)}$$

at the point corresponding to the identity of $SU(m+n)$ is isomorphic to $\mathbf{C}^m \otimes \mathbf{C}^n$, which is a fundamental representation of $Spin(2mn)$. The natural action of $SU(m) \times SU(n) \times U(1)$ on the tangent space gives the conformal inclusion (cf. §4.2 of [KW])

$$SU(m)_n \times SU(n)_m \times U(1)_{mn(m+n)(m+n)} \subset Spin(2mn)_1 .$$

The inclusion

$$SU(m)_k \times SU(n)_k \times U(1)_{mn(m+n)(k)} \subset SU(m+n)_k$$

comes from the conformal inclusion (cf. Prop. 4.2 of [KW])

$$SU(m)_1 \times SU(n)_1 \times U(1)_{mn(m+n)} \subset SU(m+n)_1 .$$

The inclusion in (1) is diagonal, and the SU part of the inclusion verifies C2 by (4) of Th. 4.3 in [X4]. For the $U(1)$ part, we consider the following inclusions:

$$U(1)_{2a} \times U(1)_{2b} \subset SU(2)_a \times SU(2)_b ,$$

and

$$U(1)_{2a+2b} \subset SU(2)_{a+b} \subset SU(2)_a \times SU(2)_b ,$$

with $a := \frac{1}{2}mn(m+n)^2$, $b := \frac{1}{2}mn(m+n)k$. It follows from (3) of Cor. 3.1 of [X4] and the proof of (1) of Prop. 3.1 that

$$U(1)_{2a+2b} \subset U(1)_{2a} \times U(1)_{2b}$$

is cofinite, and since all the endomorphisms involved are automorphisms as in the paragraph after lemma 3.2 of [X4], C2 is immediately verified in this case.

It follows from Prop. 3.1 and Th. 4.2 of [X4], [W] and [B] that $G(m,n,k)$ coset verifies Conj. 1 of [X4], and so is indeed a "rational" conformal field theory.

By Proposition 3.2, we see that when $k=1$ and $mn>1$, the above coset verifies C2.

The fixed point resolution problems for $G(m,n,k)$ are discussed in [Gep], [LVW] (also cf. [NS]). It will be interesting to work out this problem along the lines of §2.

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