Publ. RIMS, Kyoto Univ.
35 (1999), 847–852

Note on the Paper "Strong Unique Continuation Property for the Dirac Equation" by De Carli and Ōkaji

By

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Our aim in this brief note is to bring out the identity, equation (7) below, that underlies a Carleman estimate proved by De Carli and Ōkaji for the n-dimensional Dirac expression $\alpha \cdot p$. Here $n \geq 2$, $N := 2^{[(n+1)/2]}$, $p := -i\nabla$ and $\alpha := (\alpha_1, \ldots, \alpha_n)$, the α_j being anti-commuting Hermitian $N \times N$ matrices with square one. We formulate their Theorem 4.4 as follows. ($\|\cdot\|$ is the norm of $L^2(\mathbf{R}^n)^N$.)

Theorem 1. *There exists a sequence of positive numbers m_j with $m_j \to \infty$ as $j \to \infty$ and*

$$\|r^{-(m_j+1)}\varphi\| \leq 2\|r^{-m_j}\alpha \cdot p\varphi\| \qquad (\varphi \in C_0^\infty(\mathbf{R}^n \backslash \{0\})^N,\ j \in \mathbf{N}).$$

(*More precisely, the m_j can be chosen in* $\mathbf{N}$ *or* $\mathbf{N} + \frac{1}{2}$ *if n is odd or even, respectively.*)

Once this Carleman estimate is established, routine arguments (see, for example, Pan [8, p. 958 f.]) yield the following result De Carli and Ōkaji [DO, Theorem 2.2]. (Letters in square brackets refer to our list of references, numbers in square brackets to the list in the paper of De Carli and Ōkaji.)

Theorem 2. *Let $\Omega \subset \mathbf{R}^n$ be a domain that contains the origin. Then any function $u \in W_{\mathrm{loc}}^{1,2}(\Omega)^N$ which satisfies*

Communicated by Y. Takahashi, May 31, 1999.
1991 Mathematics Subject Classification(s): 35B05, 35J45, 35R45
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$$|\alpha \cdot pu| \leq \frac{C}{r}|u|$$

a.e. on Ω *for some* $C \in \left(0, \frac{1}{2}\right)$ *has the strong unique continuation property.*

For brevity we write $D_0 := C_0^\infty(\mathbf{R}^n \backslash \{0\})$ and

$$\alpha_r := \alpha \cdot \frac{x}{r} \qquad (x \in \mathbf{R}^n \backslash \{0\}).$$

For $f \in C^\infty((0, \infty))$ we define a multiplication operator in $L^2(\mathbf{R}^n)$ or $L^2(\mathbf{R}^n)^N$ by

$$u \mapsto f(|\cdot|)u \qquad (u \in D_0 \quad \text{or} \quad u \in D_0^N)$$

and denote it again by f. To approach Theorem 1, we note

$$e^g \alpha \cdot p\varphi = (\alpha \cdot p + ig'\alpha_r)e^g\varphi$$

for all $\varphi \in D_0^N$ and $g \in C^\infty((0, \infty))$. The radial momentum operator in $L^2(\mathbf{R}^n)$ defined by

$$p_r\varphi := -ir^{(1-n)/2}\frac{\partial}{\partial r}(r^{(n-1)/2}\varphi) \qquad (\varphi \in D_0)$$

is a symmetric operator which commutes with α_r and which satisfies the commutation relation

$$p_r f - f p_r = -if' \tag{1}$$

on D_0 if $f \in C^\infty((0, \infty))$. With p_r and the Laplace-Beltrami operator Δ_S we have the familiar decomposition

$$(\alpha \cdot p)(\alpha \cdot p) = p^2 = p_r^2 + \frac{(n-1)(n-3)}{4r^2} + \frac{1}{r^2}(-\Delta_S). \tag{2}$$

The operator

$$G\varphi := -\sum_{1 \leq j < k \leq n} i\alpha_j\alpha_k(x_j p_k - x_k p_j)\varphi \qquad (\varphi \in D_0^N),$$

also employed by De Carli and Ōkaji, is a symmetric operator in $L^2(\mathbf{R}^n)^N$ (or in $L^2(S^{n-1})^N$) and with

$$S := \frac{n-1}{2} + G$$

we find a decomposition analogous to (2), namely

$$\alpha \cdot p = \alpha_r\left(p_r + \frac{i}{r}S\right). \tag{3}$$

S is called the spin-orbit coupling operator. It was introduced, for $n=3$, by Dirac, who used the notation βK, a notation retained by physicists, sometimes also in arbitrary dimensions (Blanchard, Combe, Sirugue and Sirugue-Collin [BCSC]). With

$$\sigma_1 := -i\alpha_2\alpha_3, \qquad \sigma_2 := -i\alpha_3\alpha_1, \qquad \sigma_3 := -i\alpha_1\alpha_2,$$

$\sigma := (\sigma_1, \sigma_2, \sigma_3)$ and the orbital angular momentum operator $L := x \times p$ we have

$$S = 1 + \sigma \cdot L \tag{4}$$

in three dimensions. In the plane there is the simple representation

$$S = \frac{1}{2} + \sigma_3 L_3 \tag{5}$$

where $L_3 := x_1 p_2 - x_2 p_1$ and σ_3 is the third Pauli matrix.

In any dimension S and α_r anti-commute and S commutes of course with the operator of multiplication generated by a function $f \in C^\infty((0,\infty))$. Now it is easy to verify the following identity, which was first noted by Schmincke [S, Lemma 4] and used by him in connection with the selfadjointness of Dirac operators (cf. also Kalf [K]). (These papers have $n=3$, but the extension to general n is straightforward.)

Lemma. *Let $A := \alpha \cdot p + if\alpha_r$, where $f \in C^\infty((0,\infty))$ is a real-valued function. Then*

$$A^*A = p_r^2 + \frac{1}{r^2}(S^2 - S) + \frac{2f}{r}S + f' + f^2$$

on D_0^N.

From (1) and (3) we immediately get

$$(\alpha \cdot p)(\alpha \cdot p) = p_r^2 + \frac{1}{r^2}(S^2 - S),$$

and comparing with (2), we find

$$\left(S - \frac{1}{2}\right)^2 = -\Delta_S + \left(\frac{n-2}{2}\right)^2.$$

Since the right-hand side is an essentially self-adjoint operator in $L^2(S^{n-1})^N$, so is $S - \frac{1}{2}$ (Reed and Simon [RS, p. 341]) and therefore S. A closer look at the relationship between S and Δ_S reveals that the spectrum of the closed extension $\bar{S}$ is

$$\sigma(\bar{S}) = -\left(\mathbb{N}_0 + \frac{n-1}{2}\right) \cup \left(\mathbb{N}_0 + \frac{n-1}{2}\right) \tag{6}$$

where $\mathbf{N}_0 = \{0, 1, 2, \dots\}$ (see, e.g., Jerison [6, p. 121 f.] or Delanghe, Sommen and Souček, [DSS, p. 161 ff.]; note that $\sigma(\bar{S})$ is equal to $\mathbf{Z} + \frac{1}{2}$ if $n = 2$ and $\mathbf{Z} \setminus \{0\}$ if $n = 3$, which follows rather directly from (4) or (5), respectively).

Proof of Theorem 1. We use the Lemma in the form

$$A^*A = p_r^2 - \frac{1}{4r^2} + \frac{1}{r^2}\left(S + rf - \frac{1}{2}\right)^2 + f' + \frac{f}{r} \tag{7}$$

with

$$g = m(-\log r), \qquad f = g' = -\frac{m}{r} \quad (m > 0)$$

to obtain

$$\|r^{-m}\alpha \cdot p\varphi\|^2 = \left\langle \left(p_r^2 - \frac{1}{4r^2}\right) r^{-m}\varphi, r^{-m}\varphi \right\rangle$$
$$+ \left\langle \left(S - m - \frac{1}{2}\right)^2 r^{-(m+1)}\varphi, r^{-(m+1)}\varphi \right\rangle \qquad (\varphi \in D_0^N).$$

From (6) and the spectral theorem it is clear that

$$\left(S - m_j - \frac{1}{2}\right)^2 \geq \frac{1}{4}$$

for a suitable sequence (m_j), and Hardy's inequality $p_r^2 \geq \dfrac{1}{4r^2}$ yields the desired estimate. □

We note in passing that Vogelsang's strong unique continuation theorem, which is mentioned in De Carli and Ōkaji [DO, Corollary 2.3], follows from (7) on setting

$$f = g' \qquad \text{where } g = m(r^{\varepsilon/2} - \log r) \quad (m, \varepsilon > 0),$$

since

$$g'' + \frac{g'}{r} = m\left(\frac{\varepsilon}{2}\right)^2 r^{-2+(\varepsilon/2)}$$

and

$$\left(S - rf - \frac{1}{2}\right)^2 \geq 0,$$

not using any spectral information about S at all.

The analogue of Theorem 2 for the Laplace operator, namely that solutions of the inequality

$$|\Delta u| \leq \frac{C}{r^2}|u| \tag{8}$$

(no restriction on the size of C) have the strong unique continuation property, was first proved by Heyn [H] in 1956. It is a pity that his beautiful paper, which has in fact a slightly more general result, was largely overlooked in the literature. Heyn estimated the Fourier coefficients (with respect to an orthonormal system in $L^2(S^{n-1})$) of a solution of (8) directly, while Pan [8], unaware of Heyn's paper, emplaced the heavier gun of a Carleman estimate of Amrein-Berthier-Georgescu. In the context of Theorem 2, however, Heyn's method would lead to a stronger restriction on C.

We conclude with the remark that one faces a simpler situation than the one of Theorem 2.1 in De Carli and Ōkaji [DO] with solutions of

$$(\alpha \cdot p + V(r))u = 0$$

when V is a scalar function with the property that $rV(r)$ is bounded. In this case, each Fourier coefficient $U(r)$ of u satisfies an inequality of the form

$$|U'(r)| \leq \frac{C}{r}|U(r)|,$$

where C is a positive constant. Then, if $U(r_0) \neq 0$ for some $r_0 > 0$, we would have

$$|U(r_0)|\left(\frac{r}{r_0}\right)^C \leq |U(r)| \qquad (0 < r \leq r_0),$$

which implies strong unique continuation.

Acknowledgements

We gratefully acknowledge the generosity with which Professors L. De Carli and T. Ōkaji informed us about the results of [DO] at a very early stage.

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