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On the Spherically Symmetric Solution to the Mixed Problem for a Weakly Hyperbolic Equation of Second Order

Dedicated to Professor T. Kakita on his 70th birthday

By

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Introduction

We consider the following mixed problem in the domain $(0, T) \times \Omega$:

$$P[v(t,x)] = f(t,|x|), \tag{0.1}$$

$$v = 0 \quad \text{on } (0,T) \times \partial\Omega, \tag{0.2}$$

$$v = 0, \qquad v_t = 0 \qquad \text{at } t = 0, \tag{0.3}$$

where $\Omega = \{x \in \boldsymbol{R}^n; |x| < 1\}$,

$$P[v] = v_{tt} - \sum_{i=1}^{n} (a(t,|x|)v_{x_i})_{x_i} + \sum_{i=1}^{n} b_i(t,x)v_{x_i} + b_0(t,|x|)v_t + c(t,|x|)v,$$

all the coefficients of P are real valued and $a(t,|x|) \geq 0$ for all $(t,x) \in [0,T] \times \bar{\Omega}$. In order to obtain the spherically symmetric solution of (0.1)–(0.3), P is reduced to the following operator for $r \in (0,1) = I$, $r = (x_1^2 + \cdots + x_n^2)^{1/2}$ and $V(t,|x|) = v(t,x)$.

$$P_r[V] = V_{tt} - (a(t,r)V_r)_r + \sum_{i=1}^{n} \frac{1}{r}(x_i b_i(t,x) - (n-1)a(t,r))V_r + b_0(t,r)V_t + c(t,r)V,$$

where $a(t,r)$, $b_0(t,r)$, $c(t,r)$ are in $\mathscr{B}^\infty([0,T] \times \bar{I})$. We impose the assumption on P_r:

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(A-i) $r^{-1}(\sum_{i=1}^{n} x_i b_i(t,x) - (n-1)a(t,|x|))$ is a radial function in x and belongs to $\mathscr{B}^\infty([0,T]\times\bar{I})$ as a function in (t,r).

(0.1)–(0.3) is reduced to the following problem:

$$(0.1)' \qquad P_r[V(t,r)] = f(t,r) \qquad \text{in } (0,T)\times I,$$

$$(0.2)' \qquad V_r(t,0) = V(t,1) = 0,$$

$$(0.3)' \qquad V(0,r) = 0, \qquad V_t(0,r) = 0.$$

Therefore in order to obtain the solution of $(0.1)'$–$(0.3)'$ it is enough to solve the following mixed problem:

$$(0.4) \qquad L_r[u(t,r)] = f(t,r) \qquad \text{in } (0,T)\times I,$$

$$(0.5) \qquad u_r(t,0) = u(t,1) = 0,$$

$$(0.6) \qquad u(0,r) = 0, \qquad u_t(0,r) = 0,$$

where $L_r[u] = u_{tt} - (a(t,r)u_r(t,r))_r + b(t,r)u_r + b_0(t,r)u_t + c(t,r)u$ and $b(t,r) \in \mathscr{B}^\infty([0,T]\times\bar{I})$. We further impose the following assumptions.

(A-ii) (Oleinik [14]) For a positive constant $t_0 \le T$ it holds that

$$(0.7) \qquad \alpha t b^2(t,r) \le A a(t,r) + a_t(t,r),$$

where $\alpha > (2p+6)^{-1}$ (p being an integer ≥ 0), A is some constant if $0 \le t \le t_0$, and α and A are some positive constants if $t_0 \le t \le T$.

(A-iii) 1) $a(t,0) = a_r(t,0) = 0$.

2) There exists an extension function $\tilde{a}(t,r) \in \mathscr{B}^2([0,T]\times R)$ such that $\tilde{a} \ge 0$ and $\tilde{a}(t,r) = a(t,r)$ in $[0,T]\times\bar{I}$.

(A-iv) It holds that $\partial_t^j f(t,r) = 0$ at $t=0$, $j = 0,1,\dots,p$ and $\partial_r^i f(t,r) = 0$, $i = 0,\dots,s-1$ on $r=0$ for an even number s.

Remark 1. If $b_i(t,x) \equiv 0$, $i = 0,\dots,n$, (A-i) is satisfied because of (A-iii)-1). An example of the case where $a(t,r)$ and $b_i(t,x) \not\equiv 0$, $i = 1,\dots,n$ satisfy (A-i), (A-ii) and (A-iii) will be discussed in the last section in details.

Remark 2. If (A-ii) and (A-iii)-1) are satisfied, the assumption (A-iv) is seemed to be natural in the following sence. It is well known that the mixed problem for L_r with the Dirichlet condition is reduced to the following type of problem by the usual argument, if initial data, boundary data and the forcing term are sufficiently smooth in $[0,T], \bar{I}$ and $[0,T]\times\bar{I}$ respectively and appropriate compatibility conditions are satisfied (cf. [5], [7], [14]).

$$(M)\begin{cases} L_r[u] = F_0(t,r), & \\ u = 0 & \text{on } (0,T)\times\partial I, \\ u = 0, u_t = 0 & \text{at } t = 0, \end{cases}$$

where $F_0(t,r)$ is sufficiently smooth in $[0,T]\times\bar{I}$ and satisfies the former part of (A-iv). Then we can find an appropriate function $U(t,r)$ so that $L_r[u-U] = F_0(t,r)-L_r[U]=F_1(t,r)$ satisfies (A-iv) and $u-U$ satisfies (0.5) and (0.6). This argument will be discussed in Appendix in details.

We give simple examples of the equation (0.1).

$$\text{i)}\qquad v_{tt}-|x|^{2k}t^{2l}\Delta v(t,x)+d(|x|)|x|^{k-1}t^{l-1}\sum_{i=1}^{n}x_iv_{x_i},\qquad (t,x)\in(0,T)\times\Omega,$$

$$d(r)\in\mathscr{B}^{\infty}(\bar{I}),\qquad k\in\boldsymbol{N}=\{1,2,\dots\},\qquad l\in\boldsymbol{Z}_+=\{0,1,\dots\},$$

where especially we put $t^{l-1}\equiv 1$ for $l=0$ in the third term. Ebihara-Kawashima-Levine [2] obtained the spherically symmetric solution of the mixed problem for $v_{tt}-|x|^{2k}\Delta v+|v|^{\alpha}v=0$ for $\alpha>0$. This type of equation is the wave operator describing a model of wave phenomenon, on or through inhomogenous medium, especially which is extremely dense near the center $(x=0)$ and then the speed of the wave vanishes near the center.

Also $a(t,|x|)$ further admits the following degeneracy on the boundary.

$$\text{ii)}\qquad a(t,|x|)=|x|^{2k}t^{2l}(|x|-1)^{2\kappa}(1-|x|+t)^{2\theta},\qquad (t,x)\in(0,T)\times\Omega$$

$$\text{for } k\in\boldsymbol{N},\quad \kappa,l \text{ and } \theta\in\boldsymbol{Z}_+.$$

i) is reduced to the following equation:

$$\text{i)}'\qquad V_{tt}-r^{2k}t^{2l}V_{rr}+(r^kt^{l-1}d(r)-(n-1)r^{2k-1}t^{2l})V_r.$$

Also examples of L_r corresponding to ii) are given as follows.

$$\text{ii)}'\quad a(t,r)=r^{2k}t^{2l}(r-1)^{2\kappa}(1-r+t)^{2\theta},\qquad b(t,r)=r^kt^{l-1}(r-1)^{\kappa}(1-r+t)^{\theta}$$

where especially we put $t^{l-1}\equiv 1$ for $l=0$ in $b(t,r)$.

O. A. Oleinik [14] considered the Cauchy problem to weakly hyperbolic equation of second order admitting general degeneracy in $(t,x)\in[0,T]\times\boldsymbol{R}^n$:

$$(0.8)\quad v_{tt}-\sum_{i,j=1}^{n}(A_{ij}(t,x)v_{x_i})_{x_j}+\sum_{i=1}^{n}B_i(t,x)v_{x_i}+B_0(t,x)v_t+C(t,x)v=F(t,x),$$

assuming that the following inequalities hold

$$(0.9)\qquad \alpha t\left(\sum_{i=1}^{n}B_i(t,x)\xi_i\right)^2\le A\left(\sum_{i,j=1}^{n}A_{ij}(t,x)\xi_i\xi_j+\sum_{i,j=1}^{n}A_{ijt}(t,x)\xi_i\xi_j\right),$$

$$0\le\sum_{i,j=1}^{n}A_{ij}(t,x)\xi_i\xi_j,$$

for any $\xi \in \mathbb{R}^n$ where all the coefficients of (0.8) are sufficiently smooth and bounded. Note that (0.7) is the special case of (0.9). Then she proved that there exists the smooth solution. Her result was extended to the equation of higher order by Menikoff [10] and Ohya [11].

The well-posedness of the mixed problem for regularly hyperbolic equations of second order was proved by Ikawa [3], [4]. But nothing is known about the mixed problem corresponding to Oleinik's result except for the simple degeneracy case. In fact, Kimura [5] restricted degeneracy of (0.8) to the case where A_{ij} and B_i degenerate in t only of polynomial order (cf. Chi Min-you [1]) and impose the null Dirichlet boundary condition in a bounded domain in $\mathbb{R}^n$ with compact smooth boundary. Then she proved the well-posedness of the problem (cf. [7]). The mixed problems of a weakly hyperbolic equations of second order with other kind of degeneracy was studied by [8] and [9]. On the other hand, Krasnov [6] and Oleinik [12] showed the existence theorems and uniqueness theorems in the sense of a generalized solution to the mixed problem for weakly hyperbolic equations under some conditions on the coefficients and data.

Our purpose is to obtain the smooth spherically symmetric solution of the mixed problem (0.1)–(0.3) corresponding to the Cauchy problem considered in Oleinik [14]. Let us introduce some notations:

$$(h,g)_{I(\tau)} = \int_0^\tau \int_I h(t,r)g(t,r)drdt, \qquad (h,g)_I(t) = \int_I h(t,r)g(t,r)dr,$$

$$\|h\|_{I,k}(t) = \left\{ \sum_{i+j \le k} (\partial_r^j \partial_t^i h, \partial_r^j \partial_t^i h)_I(t) \right\}^{1/2}, \qquad \|h\|_{I(\tau),k} = \left\{ \int_0^\tau \|h\|_{I,k}^2(t)dt \right\}^{1/2},$$

$$\|h\|_{I;q,s,k}(t) = \left\{ \sum_{i \le q, j \le s, q+s \le k} (\partial_r^j \partial_t^i h, \partial_r^j \partial_t^i h)_I \right\}^{1/2},$$

$$\|h\|_{I(\tau);q,s,k} = \left\{ \sum_{i \le q, j \le s, q+s \le k} (\partial_r^j \partial_t^i h, \partial_r^j \partial_t^i h)_{I(\tau)} \right\}^{1/2},$$

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_r = \frac{\partial}{\partial r}, \quad \partial_{x_i} = \frac{\partial}{\partial x_i}, \quad i = 1, \dots, n, \quad D_x^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n},$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index. Let λ_j, $j = 1, 2, \dots$, be eigen values of $-\partial_r^2$ with the null Dirichlet condition such that $0 < \lambda_1 < \lambda_2 < \cdots$ and corresponding eigen functions $\varphi_1(r), \varphi_2(r), \dots$. We define a functional space

$$V^{(k)}(I) = \left\{ v(r) = \sum_{i=1}^\infty g_i \varphi_i(r); \sum_{i=1}^\infty \lambda_i^{2k} g_i^2 < \infty, r \in I \right\}.$$

It is well known that $V^{(k)}(I) \subset \mathring{H}^1(I) \cap H^k(I)$ and that the following inequality holds. For some positive constants c_i, $i = 1, 2$ and $v(r) \in V^{(k)}(I)$ it holds that

$$c_1 \|v\|_{I,k}^2 \leq \sum_{i=1}^{\infty} g_i^2 \lambda_i^{2k} \leq c_2 \|v\|_{I,k}^2. \tag{0.10}$$

Theorem 1. *Let* $f(t,r)$ *be in* $\bigcap_{i=0}^{s+p+1} C^i([0,T]; V^{(s+p+1-i)}(I))$ *for any even number* s. *Assume that* (A-ii)–(A-iv) *hold. Then there eixists a unique solution* $u(t,r) \in \bigcap_{i=0}^{s} C^i([0,T]; V^{(s-i)}(I))$ *of the problem* (0.4)–(0.6) *and satisfies the following estimate for some constant* $M > 0$ *and* $\tau \in [0,T]$:

$$\|u\|_{I,s}^2(\tau) \leq M(\|f\|_{I(T);0,s,s}^2 + \|f\|_{I,s-2}^2(\tau) + \max_{0 \leq \sigma \leq t_0} \|f\|_{I;p+1+s,s,p+1+s}^2(\sigma)). \tag{0.11}$$

Replacing $b(t,r)$ by $\sum_{i=1}^{n} r^{-1}(x_i b_i(t,x) - (n-1)a(t,r))$ in (0.7) and going back to the original problem through $(0.1)'$–$(0.3)'$, we have the following result.

Theorem 2. *Let* $f(t,r) \in \bigcap_{i=0}^{s+1+p} C^i([0,T]; V^{(s+p+1-i)}(I))$ *for any even number* s. *Assume that* (A-i), (A-iii), (A-iv) *and* (A-ii) *replaced by* $\sum_{i=1}^{n} r^{-1}(x_i b_i(t,x) - (n-1)a(t,r))$ *instead of* $b(t,r)$ *in* (0.7) *hold. Then there exists a solution* $v(t,x) \in C^{s-1}([0,T] \times \bar{\Omega})$ *of* (0.1)–(0.3).

Remark 3. (A-iii)-1) is required to prove (0.5) in Theorem 1 and to show the regularity of the solution in Theorem 2. However even if (A-iii)-1) is not satisfied, the proof of Theorem 1 guarantees the existence of a smooth solution satisfying the mixed problem (0.4), (0.6) and $u(t,0) = u(t,1) = 0$.

Now let us explain our argument. In Oleinik [14] the energy estimate played an essential role. If we try to derive the energy estimate for (0.8), (0.2) and (0.3) in $(0,T) \times \Omega$ following to her method in the usual Sobolev space, integrating by parts we have many remainder terms of inner products defined on the boundary. They are in very complicated forms and so harmful to deriving the energy inequality of higher order. Hence crutial point in our problem lies in treatment with them. In the first step, to simplify the form of them we restrict our attention to seeking the spherically symmetric solution of (0.4)–(0.6). For this purpose P is reduced to P_r by $r = (x_1^2 + \cdots + x_n^2)^{1/2}$. Since $r^{-1}(\sum_{i=1}^{n} x_i b_i(t,x) - (n-1)a(t,|x|))$ may be regarded as a radial function under the assumption (A-i), it is enough to consider L_r instead of P_r.

In the second step, we introduce the functional space $V^{(2k)}(I)$ spanned by eigen functions of $-\partial_r^2$ satisfying null Dirichlet boundary condition. It holds that $\partial_r^{2l} \varphi_j(r) = 0$ at $r = 0, 1, l = 0, 1, \ldots, j \in N$. By the use of this property most of such simplified remainder terms become harmless and we obtain the desired estimate. Consequently our main purpose in this paper will be carried out through $(0.1)'$–$(0.3)'$ by seeking the spherically symmetric solution of (0.4)–(0.6) by Galerkin's method. The energy inequality of (0.4)–(0.6) plays an important role and it is derived according to Oleinik [14].

This paper is organized as follows. In the first section, we derive the basic energy inequality for (0.4)–(0.6). In section 2, in the first subsection we obtain the energy inequality of higher order with respect to r. Most crutial point in this paper lies here. Next, using it, we have the energy inequality of higher order with respect to (t, r). By this estimate we obtain the existence of smooth solution of (0.4)–(0.6). Finally, from this result the existence of solution of the original problem (0.1)–(0.3) follows. We also give an example of $a(t, |x|)$ and $b_i(t, x) \not\equiv 0$, $i = 1, \ldots, n$, satisfying (A-i)–(A-iii).

§ 1. Basic Energy Inequality

1.1. Construction of Solution

$f(t, r)$ is written in the form of $\sum_{i=0}^{\infty} f_i(t)\varphi_i(r)$ where $f_i(t) = (f(t, r), \varphi_i(r))_I$. Put $f_J(t, r) = \sum_{i=0}^{J} f_i(t)\varphi_i(r)$ and set $u_J(t, r) = \sum_{i=0}^{J} g_{Ji}(t)\varphi_i(r)$ for $J = 1, 2, \ldots$. Then we construct $u_J(t, r)$ so that $u_\infty(t, r)$ satisfies the problem (0.4)–(0.6). For this purpose, we consider $(L_r[u_J], \varphi_j)_I = (f_J, \varphi_j)_I$. The left hand side of this equality is written in the form:

$$(L_r[u_J], \varphi_j)_I = \partial_t^2 g_{Jj}(t) + P_{Jj}(t; g_{J1}, \ldots, g_{JJ}, \partial_t g_{J1}, \ldots, \partial_t g_{JJ}),$$

where P_{Jj} is a linear operator in $g_{J1}, \ldots, g_{JJ}$, $\partial_t g_{J1}, \ldots$ and $\partial_t g_{JJ}$. We determine $g_{Jj}(t)$, $j = 1, \ldots, J$ so that $g_{Jj}(t)$ satisfies the following problem:

$$\begin{cases} \partial_t^2 g_{Jj}(t) + P_{Jj}(t; g_{J1}, \ldots, g_{JJ}, \partial_t g_{J1}, \ldots, \partial_t g_{JJ}) = f_j, \\ g_{Jj} = 0, \qquad g_{Jjt} = 0 \qquad \text{at } t = 0, j = 1, \ldots, J. \end{cases}$$

This problem is reduced to the following ordinary differential system in g_{Jj} *and* $G_{Jj} = \partial_t g_{Jj}$, $j = 1, \ldots, J$:

$$\begin{cases} \partial_t g_{Jj}(t) = G_{Jj}(t), \\ \partial_t G_{Jj}(t) + P_j(t; g_{J1}, \ldots, g_{JJ}, G_{J1}, \ldots, G_{JJ}) = f_j, \\ g_{Jj}(0) = 0, \qquad G_{Jj}(0) = 0, \qquad j = 1, \ldots, J. \end{cases}$$

It is well known that there is a smooth solution in $[0, T]$ of this problem. In fact coefficients of P_{Jj} are sufficiently smooth in t. Since $f_j(t) \in C^{p+1+s}([0, T])$, we have $g_{Jj}(t) \in C^{p+3+s}([0, T])$.

1.2. Basic Energy Inequality for $0 \le t \le t_0$

For simplicity, put $u_J(t,r) = u(t,r)$ and put $f_J(t,r) = f(t,r)$ again through this section and next section. Then we have

$$\begin{cases} (L_r[u], u)_I = (f, u)_I, \\ u(t,0) = u(t,1) = 0, \\ u(0,r) = u_t(0,r) = 0. \end{cases}$$

In this subsection our desired estimate will be derived by the manner due to Oleinik [14]. We have the following basic energy estimate of u.

Lemma 1. *It holds that for* $0 \le \tau \le t_0 \le T$ *we have for a constant* M_1

$$\|u\|^2_{I,0}(\tau) \le M_1 \max_{0 \le \sigma \le t_0} \|f\|^2_{I;p+1,0,p+1}(\sigma), \tag{1.0}$$

supposed that (A-ii) *and* (A-iv) *hold.*

Proof. Denote $w(t,x) = \int_t^\tau u(\sigma,x)d\sigma$ for $0 \le t \le \tau$ according to Oleinik [14]. It is easily seen that it holds for $\theta > 0$

$$(L[u], e^{\theta t}w)_{I(\tau)} = (f, e^{\theta t}w)_{I(\tau)}. \tag{1.1}$$

First we have

$$\begin{aligned}(u_{tt}, we^{\theta t})_{I(\tau)} &= -(u_t, w\theta e^{\theta t})_{I(\tau)} + (u_t, ue^{\theta t})_{I(\tau)} \\ &= (u, -u\theta e^{\theta t} + w\theta^2 e^{\theta t})_{I(\tau)} + (u_t, ue^{\theta t})_{I(\tau)}.\end{aligned}$$

On the other hand it holds

$$(u_t, ue^{\theta t})_{I(\tau)} = \frac{-(u, u\theta e^{\theta t})_{I(\tau)} + (u, ue^{\theta t})_I(\tau)}{2}.$$

Therefore we have

$$(u_{tt}, we^{\theta t})_{I(\tau)} = -\frac{3}{2}(u, u\theta e^{\theta t})_{I(\tau)} + (u, w\theta^2 e^{\theta t})_{I(\tau)} + \frac{1}{2}(u, ue^{\theta t})_I(\tau). \tag{1.2}$$

Next we estimate the elliptic part.

$$\begin{aligned}((a(t,r)u_r)_r, we^{\theta t})_{I(\tau)} &= -(au_r, w_r e^{\theta t})_{I(\tau)} \\ &= [(aw_r, w_r e^{\theta t})_I]_0^1 - (aw_r, \partial_t(w_r e^{\theta t}))_{I(\tau)} - (a_t w_r, w_r e^{\theta t})_{I(\tau)}.\end{aligned}$$

Therefore we have

$$((a(t,r)u_r)_r, we^{\theta t})_{I(\tau)} = -\frac{((\theta a + a_t)w_r, w_r e^{\theta t})_{I(\tau)}}{2} - \frac{(aw_r, w_r)_I(0)}{2}. \tag{1.3}$$

Lower order terms of $L_r[u]$ are estimated as follows.

$$(1.4)\qquad (b_0(t,r)u_t, we^{\theta t})_{I(\tau)} = (u, e^{\theta t}b_0(t,r)u - (b_0(t,r)e^{\theta t})_t w)_{I(\tau)}.$$

$$\begin{aligned}(1.5)\qquad (b(t,r)u_r, we^{\theta t})_{I(\tau)} &= -(b_r(t,r)u, we^{\theta t})_{I(\tau)} - (b(t,r)w_r, ue^{\theta t})_{I(\tau)} \\ &\le M_2\tau^2(u, t^{-1}ue^{\theta t})_{I(\tau)} + \frac{1}{2}(\alpha t b^2(t,r)w_r, w_r e^{\theta t})_{I(\tau)} \\ &\quad + (2\alpha)^{-1}(u, t^{-1}ue^{\theta t})_{I(\tau)}.\end{aligned}$$

Finally we estimate the right hand side of (1.1). By integrating by parts we have

$$(f, we^{\theta t})_{I(\tau)} = (-1)^{p+1}(\partial_t^{p+1} f, w_{p+1})_{I(\tau)},$$

where $w_{p+1}(t,x) = \int_t^\tau w_p(\sigma,x)d\sigma$ and $w_0 = we^{\theta t}$. Then we have

$$\begin{aligned}|w_{p+1}|^2 &\le \left|\int_{t_{p+1}}^\tau \cdots \int_{t_1}^\tau \int_{t_0}^\tau we^{\theta t} dt dt_0 \dots dt_p\right|^2 \\ &\le \left|\tau^{p+1}e^{\theta\tau}\int_0^\tau u(t,x)dt\right|^2 \le \tau^{2p+2}e^{2\theta\tau}\left(\int_0^\tau u(t,x)dt\right)^2 \\ &\le \tau^{2p+3}e^{2\theta\tau}\int_0^\tau u^2(t,x)dt.\end{aligned}$$

Hence we have

$$\int_t^\tau |w_{p+1}|^2 dt \le \tau^{2p+4}e^{2\theta\tau}\int_0^\tau u^2 dt \le \tau^{2p+5}e^{2\theta\tau}\int_0^\tau t^{-1}u^2 dt.$$

Therefore we obtain for a constant $\delta > 0$

$$|(f, we^{\theta t})_{I(\tau)}| \le \delta(u, t^{-1}ue^{\theta t})_{I(\tau)} + \frac{e^{\theta t_0}\tau^{2p+6}}{4\delta}\max_{0\le\sigma\le t_0}\|f\|^2_{I;p+1,0,p+1}(\sigma).$$

Put $\theta = A$, for $\tau \le t_0$ and set $y(\tau) = (u, t^{-1}ue^{\theta t})_{I(\tau)}$. Then taking account of (A-ii) we have

$$\tau y'(\tau) \le (\alpha^{-1} + 2\delta)y(\tau) + M_3\tau y(\tau) + M_4\tau^{2p+6}\max_{0\le\sigma\le t_0}\|f\|^2_{I;p+1,0,p+1}(\sigma).$$

Term by term multiplying by $\tau^{-\alpha^{-1}-2\delta}e^{-M_3\tau}$

$$\begin{aligned}(1.6)\qquad y(\tau) &\le \tau^{\alpha^{-1}+2\delta}e^{M_3\tau}\int_0^\tau\left(M_4 e^{-M_3\sigma}\max_{0\le\theta\le t_0}\|f\|^2_{I;p+1,0,p+1}(\theta)\sigma^{2p+6-\alpha^{-1}-2\delta}\right)d\sigma \\ &\le M_4\tau^{2p+6}\max_{0\le\sigma\le t_0}\|f\|^2_{I;p+1,0,p+1}(\sigma).\end{aligned}$$

This completes Lemma 1. ∎

§ 2. Energy Inequality of Higher Order

In this section we derive the energy inequality of higher order for $L_r[u]$, provided that (A-ii), (A-iii)-2) and (A-iv) hold.

2.1. Energy Inequality of Higher Order in r-Derivatives for $0 \leq t \leq t_0$

In this subsection, we derive the energy estimate to r-derivatives of u of higher order. For this purpose we prepare the following two lemmas.

Lemma 2.1. *For* $k = 0, 1, 2, \ldots$, *we have*

$$(\partial_r^{2k+1}u, \partial_r^{2k+1}u)_{I,0} \leq \frac{1}{2}(\|\partial_r^{2k}u\|_{I,0}^2 + \|\partial_r^{2k+2}u\|_{I,0}^2). \tag{2.0}$$

Proof. Since $\partial_r^{2k}u = 0$ at $r = 0, 1$, we have

$$(\partial_r^{2k+1}u, \partial_r^{2k+1}u)_{I,0} = [\partial_r^{2k}u\partial_r^{2k+1}u]_0^1 - (\partial_r^{2k}u, \partial_r^{2k+2}u)_{I,0} = -(\partial_r^{2k}u, \partial_r^{2k+2}u)_{I,0}.$$

By Cauchy-Schwarz's inequality we have the desired result. ■

Lemma 2.2. *Suppose that* $g(r) \geq 0$, $r \in \boldsymbol{R}$ *and* $g(r) \in \mathscr{B}^2(\boldsymbol{R})$. *Then we have*

$$(\partial_r g(r))^2 \leq 2\{\sup_{r \in R} |g_{rr}|\} g(r), \qquad r \in \boldsymbol{R}. \tag{2.1}$$

Proof. The proof is done according to Oleinik [13, Lemma 4]. Suppose that (2.1) is not satisfied at a point $r_0 \in \boldsymbol{R}$, i.e., $g_r^2(r_0) > 2\{\sup_{r \in R}|g_{rr}|\}g(r_0)$. Consider a point $\tilde{r}_0 = r_0 - 2\dfrac{g(r_0)}{g_r(r_0)}$. By Taylor's formula for a point $\bar{r}$ between $\tilde{r}_0$ and r_0 we have

$$g(\tilde{r}_0) = g(r_0) - 2\frac{g(r_0)}{g_r(r_0)}g_r(r_0) + 2\frac{g^2(r_0)}{g_r^2(r_0)}g_{rr}(\bar{r}) = -g(r_0)\left(1 - 2\frac{g(r_0)}{g_r^2(r_0)}g_{rr}(\bar{r})\right).$$

In view of the hypotheses $1 - 2\dfrac{g(r_0)}{g_r^2(r_0)}g_{rr}(\bar{r}) > 0$ holds. If $g(r_0) > 0$, then it holds $g(\tilde{r}_0) < 0$, which contradicts $g(r) \geq 0$. In case $g(r_0) = 0$, $g_r^2(r_0) > 0$ holds, which contradicts $g(r) \geq 0$ in a neighborhood of r_0. Hence we proved that (2.1) holds for $r \in \boldsymbol{R}$. ■

Lemma 2.1 implies that to carry out our aim in this subsection it is enough to derive the inequality for derivatives of even order of u with respect to r. Put $y_\nu(\tau) = (\partial_r^\nu u, e^{\theta_1 t}t^{-1}\partial_r^\nu u)_{I(\tau)}$. We have the following result.

Lemma 2.3. *There exists a constant* M_5 *and a positive constant* θ_1 *such that we have for* $\nu \leq s$ *and* $\tau \in [0, t_0]$

$$y_\nu(\tau) \leq M_5 \tau^{2p+6} \max_{0 \leq \sigma \leq t_0} \|f\|_{I;p+1,s,p+1+s}^2(\sigma). \tag{2.2}$$

Proof. Recall that s is any even number. Suppose that the following inequality holds for $v \leq s-2$ for a constant E_v.

$$(2.3) \qquad y_v(\tau) \leq E_v \tau^{2p+6} \sum_{q \leq v} \max_{0 \leq \sigma \leq t_0} \|\partial_r^q f\|^2_{I;p+1,0,p+1}(\sigma).$$

In the same way as derived (1.2) we have for a positive constant θ_1

$$(\partial_r^s u_{tt}, \partial_r^s w e^{\theta_1 t})_{I(\tau)} = \frac{1}{2}(\partial_r^s u, \partial_r^s u e^{\theta_1 t})_I(\tau) + \left(\partial_r^s u e^{\theta_1 t}, \theta_1^2 \partial_r^s w - \frac{3\theta_1}{2}\partial_r^s u\right)_{I(\tau)}.$$

Estimating the elliptic part we have

$$\begin{aligned}(\partial_r^s(a(t,r)u_r)_r, \partial_r^s w e^{\theta_1 t})_{I(\tau)} &= ((a(t,r)\partial_r^s u_r)_r, \partial_r^s w e^{\theta_1 t})_{I(\tau)} \\ &\quad + \sum_{v \leq s} C_v(\partial_r^v a \partial_r^{s-v} u_r, e^{\theta_1 t}\partial_r^s w_r)_{I(\tau)} = I + II,\end{aligned}$$

where C_v is a constant. Then we have in the same way as derived (1.3)

$$I = -\frac{1}{2}((\theta_1 a + a_t)\partial_r^s w_r, \partial_r^s w_r e^{\theta_1 t})_{I(\tau)} - \frac{1}{2}(a\partial_r^s w_r, \partial_r^s w_r)_I(0).$$

In the case of $v = 1$ in II, we have

$$\begin{aligned}&((\partial_r a)\partial_r^{s-1} u_r, e^{\theta_1 t}\partial_r^s w_r)_{I(\tau)} \\ &\quad = ((\partial_r a)\partial_r^{s-1} w_r, e^{\theta_1 t}\partial_r^s w_r)_I(0) + ((\partial_t \partial_r a)\partial_r^{s-1} w_r, e^{\theta_1 t}\partial_r^s w_r)_{I(\tau)} \\ &\qquad + ((\partial_r a)\partial_r^{s-1} w_r, \theta_1 e^{\theta_1 t}\partial_r^s w_r)_{I(\tau)} - ((\partial_r a)\partial_r^{s-1} w_r, e^{\theta_1 t}\partial_r^s u_r)_{I(\tau)}\end{aligned}$$

considering that s is an even number

$$\begin{aligned}&= -\frac{1}{2}((\partial_r^2 a)\partial_r^{s-1} w_r, e^{\theta_1 t}\partial_r^s w)_I(0) - \frac{1}{2}((\partial_t \partial_r^2 a)\partial_r^{s-1} w_r, e^{\theta_1 t}\partial_r^s w)_{I(\tau)} \\ &\quad - \frac{1}{2}((\partial_r^2 a)\partial_r^{s-1} w_r, \theta_1 e^{\theta_1 t}\partial_r^s w)_{I(\tau)} + ((\partial_r a)\partial_r^{s-1} w_{rr}, e^{\theta_1 t}\partial_r^s u)_{I(\tau)} \\ &\quad + ((\partial_r^2 a)\partial_r^{s-1} w_r, e^{\theta_1 t}\partial_r^s u)_{I(\tau)} \\ &\leq -\frac{((\partial_r^2 a)\partial_r^{s-1} w_r, e^{\theta_1 t}\partial_r^s w)_I(0)}{2} + ((\partial_r a)\partial_r^{s-1} w_{rr}, e^{\theta_1 t}\partial_r^s u)_{I(\tau)} + |A_1|.\end{aligned}$$

Here as well as below we denote by A_j the integrals which can be estimated in the following way: $|A_j| \leq N_j \sum_{v \leq s} \tau(\partial_r^v u, t^{-1}\partial_r^v u e^{\theta_1 t})_{I(\tau)}$ for some constants $N_j > 0$. By Lemma 2.2 and (A-iii)-2) we have

$$\begin{aligned}&((\partial_r a)\partial_r^{s-1} w_{rr}, e^{\theta_1 t}\partial_r^s u)_{I(\tau)} \\ &\quad \leq (a(\partial_r^{s-1} w)_{rr}, e^{\theta_1 t}(\partial_r^{s-1} w)_{rr})_{I(\tau)} + const.(e^{\theta_1 t}\partial_r^s u, \partial_r^s u)_{I(\tau)}.\end{aligned}$$

On the other hand

$$
\begin{aligned}
&-\frac{((\partial_r^2 a)\partial_r^{s-1} w_r, e^{\theta_1 t}\partial_r^s w)_I(0)}{2} \\
&\quad = \int_0^\tau \partial_t \frac{((\partial_r^2 a)\partial_r^{s-1} w_r, e^{\theta_1 t}\partial_r^s w)_I(t)}{2}\, dt \\
&\quad = \frac{((\partial_r^2 \partial_t a)\partial_r^{s-1} w_r, e^{\theta_1 t}\partial_r^s w)_{I(\tau)}}{2} - \frac{((\partial_r^2 a)\partial_r^{s-1} u_r, e^{\theta_1 t}\partial_r^s w)_{I(\tau)}}{2} \\
&\qquad - \frac{((\partial_r^2 a)\partial_r^{s-1} w_r, e^{\theta_1 t}\partial_r^s u)_{I(\tau)}}{2} + \frac{((\partial_r^2 a)\partial_r^{s-1} w_r, \theta_1 e^{\theta_1 t}\partial_r^s w)_{I(\tau)}}{2}.
\end{aligned}
$$

Hence we have

$$
(a_r \partial_r^{s-1} u_r, e^{\theta_1 t}\partial_r^s w_r)_{I(\tau)} \le (a(\partial_r^{s-1} w)_{rr}, e^{\theta_1 t}(\partial_r^{s-1} w)_{rr})_{I(\tau)} + |A_2|.
$$

Now let us estimate the case of $v \ge 2$ in II

$$
\begin{aligned}
&\sum_{s \ge v \ge 2} C_v((\partial_r^v a)\partial_r^{s-v} u_r, e^{\theta_1 t}\partial_r^s w_r)_{I(\tau)} \\
&\qquad = -\sum_{s \ge v \ge 2} C_v((\partial_r^v a)\partial_r^{s-v} u_{rr}, e^{\theta_1 t}\partial_r^s w)_{I(\tau)} \le |A_3|.
\end{aligned}
$$

Combining above these inequalities we have

$$
II \le (a(\partial_r^{s-1} w)_{rr}, e^{\theta_1 t}(\partial_r^{s-1} w)_{rr})_{I(\tau)} + |A_4|.
$$

On the other hand we have

$$
(\partial_r^s(b(t,r)u_r), e^{\theta_1 t}\partial_r^s w)_{I(\tau)} \le |(b(t,r)\partial_r^s u, e^{\theta_1 t}\partial_r^s w_r)_{I(\tau)} + A_5|
$$

in the same way as derived (1.5)

$$
\le \frac{\alpha}{2} t(b^2(t,r)\partial_r^s w_r, e^{\theta_1 t}\partial_r^s w_r)_{I(\tau)} + (2\alpha)^{-1}(\partial_r^s u, t^{-1} e^{\theta_1 t}\partial_r^s u)_{I(\tau)} + |A_5|.
$$

Other lower order terms are estimated as follows.

$$
\begin{aligned}
&(\partial_r^s(b_0(t,r)u_t), \partial_r^s w e^{\theta_1 t})_{I(\tau)} \\
&\qquad = (\partial_r^s(b_0 u), e^{\theta_1 t}\partial_r^s(u - \theta_1 w))_{I(\tau)} - (\partial_r^s(b_{0t} u), e^{\theta_1 t}\partial_r^s w)_{I(\tau)} = A_6.
\end{aligned}
$$

Finally, choosing θ_1 sufficiently large we have by using Lemma 2.1 and (2.3) for a positive constant C

$$
\tau y_s'(\tau) \le (\alpha^{-1} + 2\delta) y_s(\tau) + C\tau y_s(\tau) + M_6 \tau^{2p+6} \sum_{v \le s} \max_{0 \le \sigma \le t_0} \|\partial_r^v f\|^2_{I;p+1,0,p+1}(\sigma).
$$

Thus we have in the same way as derived (1.6)

$$y_s(\tau) \le M_7 \tau^{2p+6} \sum_{\nu \le s} \max_{0 \le \sigma \le t_0} \|\partial_r^\nu f\|^2_{I;p+1,0,p+1}(\sigma). \tag{2.4}$$

Considering into Lemma 2.1 we see that (2.2) holds. ∎

2.2. Energy Inequality of Higher Order for $0 \le t \le t_0$

In this subsection, we derive the energy estimate of higher order derivatives of $u(t, r)$ in t and r.

Lemma 2.4. *There exists a constant $M_8 > 0$ such that it holds for $\tau \in [0, t_0]$*

$$\|u\|^2_{I,s}(\tau) \le M_8(\|f\|^2_{I(t_0);0,s-1,s-1} + \|f\|^2_{I,s-2}(\tau) + \max_{0 \le \sigma \le t_0} \|f\|^2_{I;p+1,s,p+s+1}(\sigma)). \tag{2.5}$$

Proof. In the case of $\nu \le s-1$, we have for a positive constant θ_2

$$(\partial_r^\nu u_{tt}, e^{-\theta_2 t}\partial_r^\nu u_t)_{I(\tau)} = \frac{1}{2}(\partial_r^\nu u_t, e^{-\theta_2 t}\partial_r^\nu u_t)_I(\tau) + \frac{1}{2}(\partial_r^\nu u_t, \theta_2 e^{-\theta_2 t}\partial_r^\nu u_t)_{I(\tau)}.$$

Next we have

$$\begin{aligned}
&(\partial_r^\nu (a(t,r)u_r)_r, \partial_r^\nu u_t e^{-\theta_2 t})_{I(\tau)} \\
&\quad = \frac{1}{2}((-\theta_2 a + a_t)\partial_r^\nu u_r, \partial_r^\nu u_r e^{-\theta_2 t})_{I(\tau)} - \frac{1}{2}(a\partial_r^\nu u_r, \partial_r^\nu u_r)(\tau) \\
&\qquad + \sum_{1 \le k \le \nu} C_k((\partial_r^k a \partial_r^{\nu-k} u_r)_r, e^{-\theta_2 t}\partial_r^\nu u_t)_{I(\tau)}.
\end{aligned}$$

On the other hand we have

$$(\partial_r^\nu (b(t,r)u_r), e^{-\theta_2 t}\partial_r^\nu u_t)_{I(\tau)} \le |(\partial_r^\nu u_t, e^{-\theta_2 t}\partial_r^\nu u_t)_{I(\tau)}| + M_9 \sum_{\nu+1 \ge k} (\partial_r^k u, e^{-\theta_2 t}\partial_r^k u)_{I(\tau)},$$

$$(\partial_r^\nu (b_0(t,r)u_t), e^{-\theta_2 t}\partial_r^\nu u_t)_{I(\tau)} \le M_{10} \sum_{\nu \ge k} (\partial_r^k u_t, e^{-\theta_2 t}\partial_r^k u_t)_{I(\tau)}.$$

Taking Lemma 2.3 into account and choosing θ_2 sufficiently large we have

$$\begin{aligned}
&\sum_{\nu \le s-1} (\partial_r^\nu u_t, \partial_r^\nu u_t)_I(\tau) \\
&\quad \le M_{11}\left(\tau^{2p+6} \max_{0 \le \sigma \le t_0} \sum_{\nu \le s} \|\partial_r^\nu f\|^2_{I;p+1,0,p+1}(\sigma) + \sum_{\nu \le s-1} (\partial_r^\nu f, \partial_r^\nu f)_{I(t_0)}\right).
\end{aligned} \tag{2.6}$$

In order to estimate the derivatives of the form: $\partial_r^\nu \partial_t^{\rho+2} u$, $\rho \ge 0$, $\nu + \rho \le s-2$, we apply the operator $\partial_r^\nu \partial_t^\rho$ to the equation (0.4) and we obtain

$$\partial_r^\nu \partial_t^{\rho+2} u = \partial_r^\nu \partial_t^\rho (L_r[u] - \partial_t^2 u). \tag{2.7}$$

Since in the right hand side of (2.7) derivatives of u are in the form: $\partial_r^i \partial_t^j u$, $i+j \leq s$, $j \leq \rho+1$, by using (2.7) we arrive at (2.5). ■

2.3. Energy Inequality of Higher Order for $0 \leq t \leq T$

We may obtain the desired estimate for $\tau \in [t_0, T]$ in a similar way as in the above. But the estimate for $(b(t,r)u_r, we^{\theta t})_{I(\tau)}$ should be derived as in the following.

$$\begin{aligned}(b(t,r)u_r, we^{\theta t})_{I(\tau)} &= -(b_r(t,r)u, we^{\theta t})_{I(\tau)} - (b(t,r)u, w_r e^{\theta t})_{I(\tau)} \\ &\leq M_{12}(u, e^{\theta t}u)_{I(\tau)} + \frac{\alpha}{2}(tb^2(t,r)w_r, w_r e^{\theta t})_{I(t_0)} \\ &\quad + (2\alpha)^{-1}(u, t^{-1}e^{\theta t}u)_{I(t_0)} + \frac{\alpha_1}{2}(b^2(t,r)w_r, w_r e^{\theta t})_{I(t_0,\tau)} \\ &\quad + (2\alpha_1)^{-1}(u, ue^{\theta t})_{I(t_0,\tau)}\end{aligned}$$

where α_1 is a constant and $\alpha_1 \leq \alpha t$ for $t \geq t_0$ and $I(t_0, \tau) = (t_0, \tau) \times I$.

Set $z = (u, ue^{\theta t})_{I(\tau)}$ for $t_0 \leq \tau \leq T$. By using the estimate (1.6) for $y(t_0) = (u, t^{-1}ue^{\theta t})_{I(t_0)}$ we have for sufficiently large θ and $\tau > t_0$

$$\begin{aligned}z'(\tau) &\leq M_{13}(z(\tau) + y(t_0) + (f,f)_{I(T)}) \\ &\leq M_{13}(z(\tau) + (f,f)_{I(T)}) + M_{14} \max_{0 \leq \sigma \leq t_0} \|f\|^2_{I;p+1,0,p+1}(\sigma).\end{aligned}$$

Therefore we have

$$(u,u)_{I(\tau)} \leq M_{15}\left(\max_{0 \leq \sigma \leq t_0} \|f\|^2_{I;p+1,0,p+1}(\sigma) + (f,f)_{I(T)}\right).$$

Using the estimate (2.2), in the same way we estimate $(\partial_r^\nu u, \partial_r^\nu u)(\tau)$ for $\nu \leq s$ in case $\tau > t_0$. The derivatives of the form $\partial_r^\nu \partial_t$, $\nu \leq s-1$ and $\partial_r^\nu \partial_t^{\rho+2}$, $\rho \geq 0$, $\nu + \rho \leq s-2$ for $\tau > t_0$ are estimated in the same way as for $\tau \leq t_0$. Then we have the following result.

Lemma 2.5. *We have for some constant $M_{16} > 0$ and $0 \leq \tau \leq T$*

$$\|u\|^2_{I,s}(\tau) \leq M_{16}(\|f\|^2_{I(T);0,s,s} + \|f\|^2_{I,s-2}(\tau) + \max_{0 \leq \sigma \leq t_0} \|f\|^2_{I;p+1,s,p+1+s}(\sigma)).$$

§ 3. The Proof of Theorems

3.1. The Proof of Theorems

First we discuss the convergence of formal series $\{f_J\}_{J=1}^{\infty}$.

Lemma 3.1. *It holds that*

i) $f_J(t,r) \to f(t,r)$ *strongly in* $\bigcap_{i=0}^{p+1+s} C^i([0,T]; V^{p+1+s-i}(I))$ as $J \to \infty$,

ii) $f_J(t,r) \to f(t,r)$ *strongly in* $H^{s-1}((0,T) \times I)$ as $J \to \infty$.

Proof. i) Taking (0.10) into account, it holds that for $t \in [0,T]$

$$\|f_J\|^2_{I,p+1+s}(t) = \sum_{i+j \le p+1+s} \sum_{l=1}^{J} (\partial_t^i f_l)^2 \lambda_l^{2j}$$

$$\to \sum_{i+j \le p+1+s} \sum_{l=1}^{\infty} (\partial_t^i f_l)^2 \lambda_l^{2j} = \|f\|^2_{I,p+1+s}(t) \qquad \text{as } J \to \infty.$$

Since $\max_{0 \le t \le T} \|f\|^2_{I,p+1+s}(t) < +\infty$, we have

$$\max_{0 \le t \le T} \|f - f_J\|^2_{I,p+1+s}(t) = \max_{0 \le t \le T} \sum_{i+j \le p+1+s} \sum_{l=J}^{\infty} (\partial_t^i f_l)^2 \lambda_l^{2j} \to 0 \qquad \text{as } J \to \infty.$$

ii) Considering into the proof of i) we have by Lebesgue's convergence theorem

$$\int_0^T \|f - f_J\|^2_{I,s-1}(\sigma) d\sigma \to 0.$$

Hence we proved ii). ■

Proof of Theorem 1. For any $J_1, J_2 \in N$ with $J_1 \ge J_2$, from Lemma 2.5 it follows that $u_{J_1} - u_{J_2}$ satisfies

$$\begin{aligned} (3.1) \qquad \|u_{J_1} - u_{J_2}\|_{I,s}(\tau) \le M_{17}(&\|f_{J_1} - f_{J_2}\|^2_{I(T);0,s,s} + \|f_{J_1} - f_{J_2}\|^2_{I,s-1}(\tau) \\ &+ \max_{0 \le \sigma \le t_0} \|f_{J_1} - f_{J_2}\|^2_{I;p+1,s,p+1+s}(\sigma)). \end{aligned}$$

Lemma 3.1 yields that the right hand side of (3.1) tends to zero as $J_1, J_2 \to \infty$. Hence there exists a limiting function $u(t,r)$ such that

$$u_J \to u \text{ strongly in } \bigcap_{i=0}^{s} C^i([0,T]; V^{s-i}(I)) \qquad \text{as } J \to \infty.$$

Since $f_J \to f$ in $L^2((0,T) \times I)$, for any $w(t,x) \in \mathscr{D}((0,T) \times I)$ we have

$$(L_r[u_J], w)_{(0,T)\times I} = (f_J, w)_{(0,T)\times I}$$

$$\to (L_r[u], w)_{(0,T)\times I} = (f, w)_{(0,T)\times I} \qquad \text{as } J \to \infty.$$

It is easily seen that

$$(3.2) \qquad u(t,r) = 0 \qquad \text{on } r = 0, 1.$$

From (A-ii) and (A-iii)-1) $b(t,0)=0$ follows for $t\in[0,T]$. Put $v_j(t)=\partial_r^j u(t,0)$, $j=0,1,2,\dots$. Suppose that $v_j=0$, $j=0,1,\dots,k-1$ for $s-1\ge k$. Since $(\partial_r^i f)(t,0)=0$, $i=0,1,\dots,s-1$, differentiating the both sides of (0.4) in r of order k, we have for $t\in[0,T]$, taking (A-iii)-1) into account,

$$\begin{cases} \partial_t^2 v_k+\sum_{i=0}^k {}_kC_i\{(\partial_r^i b)(t,0)v_{k+1-i}+(\partial_r^i b_0)(t,0)\partial_t v_{k-i}+(\partial_r^i c)(t,0)v_{k-i}\} \\ \qquad =\partial_t^2 v_k+kb_r(t,0)v_k+b_0(t,0)\partial_t v_k+c(t,0)v_k=0, \\ v_k=v_{kt}=0 \qquad \text{at } t=0. \end{cases}$$

Then we have $v_k(t)\equiv 0$. Therefore we obtain

$$\partial_r^i u=0, \qquad i=0,1,\dots,s-1 \qquad \text{on } r=0. \tag{3.3}$$

Also the estimate (3.1) implies that (0.11) holds for $u(t,r)$. Thus the function $u(t,r)$ is the desired solution of (0.4)–(0.6). ■

The proof of Theorem 2. By Theorem 1 we obtain the solution $V(t,r)$ of (0.1)′–(0.3)′ in $\bigcap_{i=0}^s C^i([0,T];V^{s-i}(I))\cap C^{s-1}([0,T]\times\bar{I})$. Repeating the same procedure from (3.2) to (3.3) we have

$$\partial_r^i\partial_t^j V(t,r)=0, \qquad i+j\le s-1 \qquad \text{on } r=0. \tag{3.4}$$

Also (3.4) implies that for the solution $V(t,r)$ of (0.1)′–(0.3)′ the derivatives of $V(t,|x|)$ in x up to the order $s-1$ are continuous at $x=0$. In fact, we have

$$\sum_{|\alpha|\le s-1}|D_x^\alpha V(t,|x|)|\le C\sum_{i+j\le s-2} r^{-j}|(\partial_r^{i+1}V)(t,|x|)|$$

by Ohya [11, Lemma 14.1]

$$< C\max_{0\le r\le 1}|(\partial_r^{s-1}V)(t,r)|<+\infty.$$

Similarly we have $\max_{x\in\bar{\Omega}}\sum_{|\alpha|+j\le s-1}|\partial_t^j D_x^\alpha V(t,|x|)|<+\infty$. Therefore $V(t,|x|)$ is our desired solution (0.1)–(0.3). ■

3.2. Example of $a(t,|x|)$ and $b_i(t,x)$

Recall that in Theorem 2 (A-ii) is assumed to hold for $\sum_{i=1}^n r^{-1}(x_i b_i(t,x)-(n-1)a(t,r))$ instead of $b(t,r)$ in (0.7). In this subsection we give an example of $b_i(t,x)$ and $a(t,r)$ satisfying (A-i), (A-iii)-1) and (0.7) replaced by $\sum_{i=1}^n r^{-1}(x_i b_i(t,x)-(n-1)a(t,r))$ instead of $b(t,r)$.

Assume that $a(t,r)=r^2\eta(t,r)$ for a function $\eta(t,r)(\ge 0)\in\mathscr{B}^\infty([0,T]\times\bar{I})$, which satisfies (A-iii)-1). Let $\beta(t,r)$ belong to $\mathscr{B}^\infty([0,T]\times\bar{I})$. Define $b_i(t,x)=x_i\beta/r$, $i=1,\dots,n$. Then we have $\sum_{i=1}^n b_i(t,x)\partial_{x_i}=\beta(t,r)\partial_r$. Therefore (A-i) is satisfied and P_r is written in the form:

$$P_r[V] = V_{tt} - (a(t,r)V_r)_r + (\beta(t,r) - r(n-1)\eta(t,r))V_r + b_0(t,r)V_t + c(t,r)V.$$

For a constant $A' > 0$ we see that $a(t,r)$ satisfies $(r^{-1}(n-1)a)^2 \leq A'a$. We assume that $a(t,r)$ and $\beta(t,r)$ satisfy

$$2\alpha t\beta^2 \leq (A - 2A')a + a_t \tag{3.5}$$

where $\alpha > (2p+6)^{-1}$ (p being an integer ≥ 0), $A - 2A'$ is some constant if $0 \leq t \leq t_0$, and α and $A - 2A'$ are some positive constant if $t > t_0$. It is easily seen that $a(t,|x|)$ and $b_i(t,x)$ defined in above are our desired ones.

Appendix

Considering into (A-ii) and (A-iii)-1) we may assume that $a(t,r)$ and $b(t,r)$ are denoted by $r^2A(t,r)$ and $rB(t,r)$ respectively for $A(t,r)$ and $B(t,r) \in \mathscr{B}^\infty([0,T]\times\bar{I})$. Then we have the following result:

Proposition. *Assume that $F_0(t,r) \in \mathscr{B}^\infty([0,T]\times\bar{I})$ and that $(\partial_t^i F_0)(0,r) = 0$, $i = 0,\dots,l$ for a positive integer $l \geq 2$. For any positive integer $k \leq l/2$, there exists a smooth function $U(t,r)$ such that for $F_1(t,r) = F_0(t,r) - L_r[U]$*

$$(\partial_r^i F_1)(t,0) = 0, \qquad i = 0,\dots,k,$$

$$(\partial_t^i F_1)(0,r) = 0, \qquad i = 0,\dots,l-2k.$$

Proof. Note that

$$L_r = \partial_t^2 - r^2A(t,r)\partial_r^2 - r^2A_r(t,r)\partial_r + r(B(t,r) - 2A(t,r))\partial_r + b_0(t,r)\partial_t + c(t,r).$$

Define operators $L_r^{1,i}$ and $L_r^{2,i}$, $i = 0,\dots,k$ as in the following.

$$\begin{cases} L_r^{1,0} = \partial_t^2 + b_0(t,r)\partial_t + c(t,r), \\ L_r^{1,1} = \partial_t^2 + b_0(t,r)\partial_t + c(t,r) + (B(t,r) - 2A(t,r)), \\ L_r^{1,i} = \partial_t^2 + b_0(t,r)\partial_t + c(t,r) - i(i-1)A(t,r) + i(B(t,r) - 2A(t,r)), \qquad i \geq 2, \end{cases}$$

$$r^{i+1}L_r^{2,i}[\cdot] = L_r[r^i\cdot] - r^iL_r^{1,i}[\cdot], \qquad i \geq 0.$$

Then we consider the Cauchy problem of the following ordinary differential equation for $i = 0,\dots,k$:

$$(Li)\begin{cases} L_r^{1,i}[w_i(t,r)] = (\partial_r^i F_0)(t,0) - iL_r^{2,i-1}[w_{i-1}], \\ w_i = w_{it} = 0 \qquad \text{at } t = 0, \end{cases}$$

where $L_r^{2,-1} \equiv 0$. We see that there exists a unique smooth solution $w_i(t,r)$ of (Li) (see Kubo [8, Proposition 1.1]). It is seen that $(\partial_t^j w_i)(0,r) = 0$, $j = 0,\dots,l+2-2i$. Put $U(t,r) = \sum_{j=0}^k w_j(t,r)\rho(r)r^j/j!$ for a non-negative function $\rho(r) \in \mathscr{B}^\infty(\bar{I})$ satisfying $\rho(r) = 1$ near $r = 0$ and $\rho(r) = 0$ near $r = 1$. Then we have near $r = 0$

$$F_0(t,r) - L_r[U] = F_0(t,r) - \sum_{j=0}^{k}\left((\partial_r^j F)(t,0)\frac{r^j}{j!} - r^j L_r^{2,j-1}\left[\frac{w_{j-1}}{(j-1)!}\right] + r^{j+1}L_r^{2,j}\left[\frac{w_j}{j!}\right]\right)$$

$$= F_0(t,r) - \sum_{j=0}^{k}(\partial_r^j F)(t,0)\frac{r^j}{j!} - r^{k+1}L_r^{2,k}\left[\frac{w_k}{k!}\right].$$

Therefore we have $F_1(t,r) = O(r^{k+1})$ and $(\partial_t^i F_1)(0,r) = 0$, $j = 0, \ldots, l - 2k$ hold. Hence the proof is complete. ∎

Let u be a smooth solution of (M) in *Remark 2*. Proposition implies that $L_r[u - U] = F_1$ holds. Since it holds that $L_r[u]|_{r=0} = L_r^{1,0}[u]|_{r=0}$, considering into $(L0)$ on $r = 0$, from the uniqueness of the solution of $(L0)$ restricted on $r = 0$ $w_0(t,0) = u(t,0)$ follows. Since it holds $L_r[u - U] = F_1$, repeating the procedure from (3.2) to (3.3) for $u - U$, we have $\partial_r^j(u - U) = 0$, $j = 0, \ldots, k$ on $r = 0$. Thus we see that $u - U$ satisfies (0.5) and (0.6).

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