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Determinant Formula for Solutions of the Quantum Knizhnik-Zamolodchikov Equation Associated with $U_q(sl_n)$ at $|q| = 1$

By

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Abstract

We construct the hypergeometric solutions for the quantized Knizhnik-Zamolodchikov equation with values in a tensor product of vector representations of $U_q(sl_n)$ at $|q| = 1$ and give an explicit formula for the corresponding determinant in terms of the double sine function.

Introduction

In this paper we study the hypergeometric solutions of the quantized Knizhnik-Zamolodchikov (qKZ) equation with values in a tensor product of vector representations of $U_q(sl_n)$, see Section 1 for the precise formulation of the problem. It is known that the qKZ equation respects the weight decomposition of the tensor product. For each weight subspace we construct a fundamental matrix solution of the qKZ equation and explicitly calculate the corresponding determinant, see Theorem 3.1.

Formal integral representations for solutions of the qKZ equation in the sl_n case, both in the rational and trigonometric situation, were constructed in [TV1]. Though to write down the phase function explicitly in the trigonometric situation it had been assumed in [TV1] that the multiplicative step p of the qKZ equation is inside the unit circle: $0 < |p| < 1$, all the construction in [TV1] used only difference equations for the phase function and after obvious modifications remained valid for an arbitrary step $p \neq 0, 1$. However, the problem of integrating the formal integral representations suitably and getting in this way

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actual solutions of the qKZ equation is much more analytically involved; one can see this looking at the sl_2 case.

In the last four years the hypergeometric solutions of the qKZ equation in the sl_2 case were studied quite well. The generic situation was considered in [TV2] (the rational case) and in [TV3] (the trigonometric case for $0 < |p| < 1$). The construction was generalized to the trigonometric case for $|p| = 1$ in [MT1] and to the elliptic case of the quantized Knizhnik-Zamolodchikov-Bernard ($qKZB$) equation in [FTV1], [FTV2].

If some of the representations are finite-dimensional, the situation is no more generic. Rather detailed study of this case has been done in [MV1]; see also [S], [JKMQ], [NPT], [T1] for some important particular cases.

Less is known for the case of $n > 2$. Some integral formulae for solutions of the qKZ equation were obtained in [S], [KQ], [N], [MT2], but in all the considered cases solutions takes values in a tensor product of vector representations. Recently Varchenko and the third author have managed to extend the construction of [TV2], [TV3] to the higher rank case and get solutions taking values in a tensor product of arbitrary highest weight representations [TV4]. Let us also mention a paper [M], where integral formulae for solutions of another type of the qKZ equation were suggested.

In this paper we evaluate a determinant of a certain matrix whose entries are given by multidimensional integrals of q-hypergeometric type. In the case of ordinary multidimensional hypergeometric integrals a problem of evaluating similar determinants appears, say, in studying arrangements of hyperplanes, and several results have been obtained in this direction, see for instance [V1], [L], [LS], [DT], [MTV], [MV2]. In some particular cases these determinant formulae have another meaning; namely they imply that under certain assumptions the hypergeometric solutions of the differential Knizhnik-Zamolodchikov equation form a basis of solutions [SV], [V2]. There are similar determinant formulae for solutions of the qKZ equation in the sl_2 case. They have been obtained for the rational case in [TV2], [T1], and for the trigonometric case in [TV3] for $0 < |p| < 1$ and in [MT1] for $|p| = 1$. It turns out that there is a nice connection of constructions given in [TV3] and [MT1], which in particularly allows to derive the determinant formula for $|p| = 1$ from the determinant formula for $0 < |p| < 1$. This subject will be addressed elsewhere [T2].

The paper is organized as follows. The first section contains preliminaries and precise definitions on the qKZ equation. In Section 2 we construct the hypergeometric pairing and give integral formulae for solutions of the qKZ equation. The main result of the paper is formulated in Section 3, see Theorem 3.1. We show that both the left hand side and the right hand side of formula (3.2) satisfy the same system of difference equations and have to be proportional. To compute the proportionality coefficient we study suitable asymptotics of the hypergeometric solutions. We see that the proportionality coefficient splits into a product of contributions of each tensor factor, which are calculated in Section 5. In the last Section we complete the proof of Theorem 3.1. A short Appendix contains the necessary information of the double sine function for the convenience of the reader.

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§1. The Quantized Knizhnik-Zamolodchikov Equation

Consider the vector representation V of sl_n:

$$V = \bigoplus_{j=0}^{n-1} \mathbf{C} v_j.$$

Let $\varepsilon_1, \dots, \varepsilon_n$ be the fundamental weights of sl_n. We consider the basis vectors $v_0, \dots, v_{n-1}$ as weight vectors with respect to the Cartan subalgebra of sl_n with weights $\varepsilon_1, \dots, \varepsilon_n$, respectively.

Fix a complex number ρ and a weight $\bar{\mu} = \sum_{j=1}^{n} \mu_j \varepsilon_j$, $(\mu_j \in \mathbf{R}, j = 1, \dots, n)$. Consider a diagonal matrix

$$D(\bar{\mu}) = \operatorname{diag}(e^{2\pi i \mu_1}, \dots, e^{2\pi i \mu_n})$$

and a matrix $R^{(\rho)}(\beta) \in \operatorname{End}(V \otimes V)$ with the following entries: $R^{(\rho)}(\beta)_{jj}^{jj} = 1$,

$$R^{(\rho)}(\beta)_{jk}^{jk} = \frac{\operatorname{sh} \frac{\pi}{\rho} \beta}{\operatorname{sh} \frac{\pi}{\rho} \left(\beta - \frac{2\pi i}{n} \right)} \tag{1.1}$$

for $j \neq k$,

$$R^{(\rho)}(\beta)_{kj}^{jk} = -\frac{e^{(\pi/\rho)\beta} \operatorname{sh} \frac{2\pi^2 i}{\rho n}}{\operatorname{sh} \frac{\pi}{\rho} \left(\beta - \frac{2\pi i}{n} \right)}, \qquad R^{(\rho)}(\beta)_{jk}^{kj} = -\frac{e^{-(\pi/\rho)\beta} \operatorname{sh} \frac{2\pi^2 i}{\rho n}}{\operatorname{sh} \frac{\pi}{\rho} \left(\beta - \frac{2\pi i}{n} \right)} \tag{1.2}$$

for $j < k$, and $R^{(\rho)}(\beta)_{lm}^{jk} = 0$, otherwise. We have

$$R^{(\rho)}(\beta) v_l \otimes v_m = \sum_{j,k=0}^{n-1} R^{(\rho)}(\beta)_{jk}^{lm} v_j \otimes v_k.$$

Fix a complex number λ and define the *qKZ* operators $K_1^{(p)}, \dots K_N^{(p)}$ acting in the tensor product $V^{\otimes N}$:

$$(1.3) \quad K_m^{(p)}(\beta_1, \dots, \beta_N) = R_{m,m-1}^{(p)}(\beta_m - \beta_{m-1} - \lambda i) \dots R_{m1}^{(p)}(\beta_m - \beta_1 - \lambda i)$$
$$\times D_m(-\bar{\mu}/p) R_{mN}^{(p)}(\beta_m - \beta_N) \dots R_{m,m+1}^{(p)}(\beta_m - \beta_{m+1}).$$

In this paper we consider the *qKZ* equation for a function $f(\beta_1, \dots, \beta_N)$ taking values in $V^{\otimes N}$, which is the following system of difference equations:

$$(1.4) \quad f(\beta_1, \dots \beta_m - \lambda i, \dots \beta_N) = K_m^{(p)}(\beta_1, \dots, \beta_N) f(\beta_1, \dots, \beta_N), \qquad m = 1, \dots N.$$

We also consider the mirror *qKZ* equation for a similar function $g(\beta_1, \dots, \beta_N)$:

$$(1.5) \quad g(\beta_1, \dots \beta_m - p i, \dots \beta_N) = K_m^{(\lambda)}(\beta_1, \dots, \beta_N) g(\beta_1, \dots, \beta_N), \qquad m = 1, \dots N.$$

The *qKZ* operators respects the sl_n weight decomposition of the tensor product. Therefore, one can consider solutions of the *qKZ* and mirror *qKZ* equations taking values in a weight subspace $(V^{\otimes N})_\xi$ for any given weight ξ.

In this paper for any given weight ξ such that $(V^{\otimes N})_\xi$ is nontrivial we will construct a function $\Psi_\xi(\beta_1, \dots, \beta_N)$ taking values in $(V^{\otimes N})_\xi \otimes (V^{\otimes N})_\xi$, which solves the *qKZ* equation (1.4) in the first tensor factor and solves the mirror *qKZ* equation (1.5) in the second tensor factor. Also we will compute the determinant $\det \Psi_\xi(\beta_1, \dots, \beta_N)$.

The matrix $R^{(p)}(\beta)$ is the R-matrix associated with the tensor product of the evaluation vector representations of $U_q(\widehat{sl_n})$ for

$$q = e^{-2\pi^2 i/pn}.$$

Similarly, $R^{(\lambda)}(\beta)$ is associated with $U_{q'}(\widehat{sl_n})$ for

$$q' = e^{-2\pi^2 i/\lambda n}.$$

All over this paper we assume that p and λ are real positive. Thus, under our assumptions we have that

$$|q| = |q'| = 1.$$

However, it is clear from the consideration that all our construction remain valid if p and λ have small enough imaginary parts of arbitrary sign. Therefore, q and q' can deviate from the unit circle and vary in a narrow annulus.

In addition to the reality and positivity of p and λ we assume that both of them and their ratio are not rational. Sometimes we take p and λ to be sufficiently large.

§ 2. Integral Formulae for Solutions

For non-negative integers $\nu_1, \dots, \nu_{n-1}$ satisfying

$$N = \nu_0 \geqslant \nu_1 \geqslant \cdots \geqslant \nu_{n-1} \geqslant \nu_n = 0, \tag{2.1}$$

we denote by $\mathscr{Z}_{\nu_1,\dots,\nu_{n-1}}$ the set of all N-tuples $J = (J_1, \dots, J_N) \in (\mathbf{Z}_{\geqslant 0})^N$ such that

$$\#\{r; J_r \geqslant j\} = \nu_j. \tag{2.2}$$

For $J = (J_1, \dots, J_N) \in \mathscr{Z}_{\nu_1,\dots,\nu_{n-1}}$, we set

$$\mathscr{N}_j^J = \{r; J_r \geqslant j\} \tag{2.3}$$

and define integers $r_{j,m}^J$, $(0 \leqslant j \leqslant n-1, 1 \leqslant m \leqslant \nu_j)$ as follows:

$$\mathscr{N}_j^J = \{r_{j,1}^J, \dots, r_{j,\nu_j}^J\}, \qquad r_{j,1}^J < \cdots < r_{j,\nu_j}^J. \tag{2.4}$$

We have, in particular, $r_{0,m}^J = m$.

Now we set

$$w_J^{(\rho)}(\{\gamma_{j,m}\}; \beta_1, \dots, \beta_N) = \mathrm{Skew}_{n-1} \circ \cdots \circ \mathrm{Skew}_1\, g_J^{(\rho)}(\{\gamma_{j,m}\}; \beta_1, \dots, \beta_N), \tag{2.5}$$

$$\begin{aligned} &g_J^{(\rho)}(\{\gamma_{j,m}\}; \beta_1, \dots, \beta_N) \\ &= \prod_{j=1}^{n-1} \left\{ \prod_{1 \leqslant m < m' \leqslant \nu_j} \mathrm{sh}\frac{\pi}{\rho}\left(\gamma_{j,m'} - \gamma_{j,m} - \frac{2\pi i}{n}\right) \right. \\ &\quad \times \prod_{m=1}^{\nu_j} \left\{ e^{-(\pi/\rho)(\gamma_{j,m} - \gamma_{j-1,m^*(J,j,m)})} \prod_{\substack{m' \\ r_{j-1,m'}^J < r_{j\,m}^J}} \mathrm{sh}\frac{\pi}{\rho}\left(\gamma_{j,m} - \gamma_{j-1,m'} + \frac{\pi i}{n}\right) \right. \\ &\quad \left. \left. \times \prod_{\substack{m' \\ r_{j,m}^J < r_{j-1,m'}^J}} \mathrm{sh}\frac{\pi}{\rho}\left(\gamma_{j,m} - \gamma_{j-1,m'} - \frac{\pi i}{n}\right) \right\} \right\}. \end{aligned} \tag{2.6}$$

The notation in the above formulae is as follows. The operator Skew_j is the skew-symmetrization with respect to the variables $\{\gamma_{j,m}\}_{m=1,\dots,\nu_j}$:

$$\mathrm{Skew}_j\, X(\gamma_{j,1}, \dots, \gamma_{j,\nu_j}) = \sum_{\sigma \in S_{\nu_j}} (\mathrm{sgn}\,\sigma) X(\gamma_{j,\sigma(1)}, \dots, \gamma_{j,\sigma(\nu_j)}). \tag{2.7}$$

The integer $m^*(J, j, m)$ is uniquely determined by the condition

$$r^J_{j,m} = r^J_{j-1,m^*(J,j,m)}, \tag{2.8}$$

and $\gamma_{0,m} = \beta_m$. We abbreviate $w^{(\rho)}_J(\{\gamma_{j,m}\};\beta_1,\ldots,\beta_N)$ to $w^{(\rho)}_J(\beta_1,\ldots,\beta_N)$ when the dependence on the abbreviated variables is irrelavant.

Set

$$\mathscr{F}^{(\rho)}_{\nu_1,\ldots,\nu_{n-1}}(\beta_1,\ldots,\beta_N) = \sum_{J\in\mathscr{Z}_{\nu_1,\ ,\nu_{n-1}}} \mathbb{C}w^{(\rho)}_J(\beta_1,\ldots,\beta_N). \tag{2.9}$$

In the following, we define a pairing between

$$w^{(\rho)}_J \in \mathscr{F}^{(\rho)}_{\nu_1,\ldots,\nu_{n-1}}(\beta_1,\ldots,\beta_N) \qquad \text{and} \qquad w^{(\lambda)}_J \in \mathscr{F}^{(\lambda)}_{\nu_1,\ldots,\nu_{n-1}}(\beta_1,\ldots,\beta_N). \tag{2.10}$$

We use

$$\varphi(x) = \frac{1}{S_2\left(ix-\frac{\pi}{n}\right)S_2\left(-ix-\frac{\pi}{n}\right)}, \qquad \psi(x) = \frac{1}{S_2\left(ix+\frac{2\pi}{n}\right)S_2\left(-ix+\frac{2\pi}{n}\right)}, \tag{2.11}$$

where $S_2(x) = S_2(x|\rho,\lambda)$ is the double sine function with periods ρ and λ.

For $J, J' \in \mathscr{Z}_{\nu_1,\ldots,\nu_{n-1}}$, we set

$$\begin{aligned} I(w^{(\rho)}_J, w^{(\lambda)}_{J'}) = {} & \left(\prod_{j=1}^{n-1}\prod_{m=1}^{\nu_j}\int_{C_j} d\gamma_{j,m}\right) K(\{\gamma_{j,m}\};\beta_1,\ldots,\beta_N;\mu_1,\ldots,\mu_n) \\ & \times w^{(\rho)}_J(\{\gamma_{j,m}\};\beta_1,\ldots,\beta_N)w^{(\lambda)}_{J'}(\{\gamma_{j,m}\};\beta_1,\ldots,\beta_N). \end{aligned} \tag{2.12}$$

Here the function K is defined by

$$\begin{aligned} & K(\{\gamma_{j,m}\};\beta_1,\ldots,\beta_N;\mu_1,\ldots,\mu_n) \\ & = e^{(2\pi/\rho\lambda)\sum_{j=0}^{n-1}\sum_{m=1}^{\nu_j}\gamma_{j,m}(\mu_{j+1}-\mu_j)} \\ & \quad \times \prod_{j=1}^{n-1}\left\{\prod_{m=1}^{\nu_j}\prod_{m'=1}^{\nu_{j-1}}\varphi(\gamma_{j,m}-\gamma_{j-1,m'})\prod_{1\leqslant m<m'\leqslant\nu_j}\psi(\gamma_{j,m}-\gamma_{j,m'})\right\}, \end{aligned} \tag{2.13}$$

where $\mu_0 = 0$. We say the variable $\gamma_{j,m}$ belongs to the point $r_{j,m}$. Then, J_r is the number of integral variables which belong to the point r.

The contour C_j for $\gamma_{j,m}$, $(m = 1,\ldots,\nu_j)$ is a deformation of the real line $(-\infty,\infty)$ such that the poles at

$$\gamma_{j-1,m'} - \frac{\pi i}{n} + \rho i\mathbb{Z}_{\geqslant 0} + \lambda i\mathbb{Z}_{\geqslant 0}, \qquad \gamma_{j,m'} + \frac{2\pi i}{n} + \rho i\mathbb{Z}_{\geqslant 0} + \lambda i\mathbb{Z}_{\geqslant 0} \tag{2.14}$$

are above C_j and the poles at

$$(2.15)\qquad \gamma_{j-1,m'}+\frac{\pi i}{n}-\rho i\mathbf{Z}_{\geqslant 0}-\lambda i\mathbf{Z}_{\geqslant 0},\qquad \gamma_{j,m'}-\frac{2\pi i}{n}-\rho i\mathbf{Z}_{\geqslant 0}-\lambda i\mathbf{Z}_{\geqslant 0}$$

are below C_j, where $\gamma_{0,m}=\beta_m$.

These conditions are not compatible if all the poles really exist. Pinching of the integration contours by poles occurs for each triple of variables γ_{j,m_1}, $\gamma_{j,m_2}, \gamma_{j-1,m}$. However, we can improve the definition (2.12) as follows. We have that

$$I(w_J^{(\rho)},w_{J'}^{(\lambda)})=\sum_{\sigma,\sigma'\in\mathbf{S}_{\nu_1}\times\cdots\times\mathbf{S}_{\nu_{n-1}}}(\operatorname{sgn}\sigma)(\operatorname{sgn}\sigma')\left(\prod_{j=1}^{n-1}\prod_{m=1}^{\nu_j}\int_{C_j}d\gamma_{j,m}\right)F_{J,J',\sigma,\sigma'}(\{\gamma_{j,m}\})$$

where $\sigma=(\sigma_1,\dots,\sigma_{n-1})\in\mathbf{S}_{\nu_1}\times\cdots\times\mathbf{S}_{\nu_{n-1}}$, $\operatorname{sgn}\sigma=\prod_{j=1}^{n-1}(\operatorname{sgn}\sigma_j)$ and

$$\begin{aligned}F_{J,J',\sigma,\sigma'}(\{\gamma_{j,m}\})&=K(\{\gamma_{j,m}\};\beta_1,\dots,\beta_N;\mu_1,\dots,\mu_n)\\&\quad\times g_J^{(\rho)}(\{\gamma_{j,\sigma_j(m)}\};\beta_1,\dots,\beta_N)g_{J'}^{(\lambda)}(\{\gamma_{j,\sigma'_j(m)}\};\beta_1,\dots,\beta_N).\end{aligned}$$

A partial integrand $F_{J,J',\sigma,\sigma'}(\{\gamma_{j,m}\})$ does not have poles at some points (2.14) and (2.15) because of the zeros of $g_J^{(\rho)}(\{\gamma_{j,\sigma(m)}\};\beta_1,\dots,\beta_N)$. So, given J and σ there is a choice of integration contours $C_j^{(J,\sigma)}$ satisfying the required conditions for the actual poles of $F_{J,J',\sigma,\sigma'}(\{\gamma_{j,m}\})$ for arbitrary J',σ'. Similarly, given J' and σ' there is a choice of integration contours $C_{j,(J',\sigma')}$ satisfying the required conditions for the actual poles of $F_{J,J',\sigma,\sigma'}(\{\gamma_{j,m}\})$ for arbitrary J,σ. Finally, one can easily check that the integrals of the term $F_{J,J',\sigma,\sigma'}(\{\gamma_{j,m}\})$ over the contours $C_j^{(J,\sigma)}$ and over the contours $C_{j,(J',\sigma')}$ are equal.

In this paper we assume that ρ an λ are large positive. Then, as we will see in the next section, there is a region of the parameters $\mu_1,\dots,\mu_n$ where the integral (2.12) is absolutely convergent (see (4.5)).

Consider the vector representation V of sl_n:

$$(2.16)\qquad V=\bigoplus_{j=0}^{n-1}\mathbf{C}v_j.$$

Theorem 2.1. *For $J\in\mathscr{Z}_{\nu_1,\dots,\nu_{n-1}}$, we set*

$$(2.17)\qquad f_J=\sum_{J'\in\mathscr{Z}_{\nu_1,\dots,\nu_{n-1}}}I(w_{J'}^{(\rho)},w_J^{(\lambda)})v_{J'},\qquad \text{where } v_{J'}=v_{J'_1}\otimes\cdots\otimes v_{J'_N}.$$

Then f_J is a solution to (1.4).

Proof. For $J=(J_1,\ldots,J_N)\in\mathscr{Z}_{\nu_1,\ldots,\nu_{n-1}}$, we set $w^{(\rho)}_{J_1,\ldots,J_N}=w^{(\rho)}_J$. In the same way as the proof of Lemma 1 and Lemma 3 in [MT2], we can show the following formulae:

$$
\begin{aligned}
(2.18)\qquad & w^{(\rho)}_{J_1,\ldots,J_{k+1},J_k,\ldots,J_N}(\beta_1,\ldots,\beta_{k+1},\beta_k,\ldots,\beta_N)\\
&=\sum_{J'_k,J'_{k+1}} R^{(\rho)}(\beta_k-\beta_{k+1})^{J'_k,J'_{k+1}}_{J_k,J_{k+1}}\\
&\quad\times w^{(\rho)}_{J_1,\ldots,J'_k,J'_{k+1},\ldots,J_N}(\beta_1,\ldots,\beta_k,\beta_{k+1},\ldots,\beta_N),
\end{aligned}
$$

$$
\begin{aligned}
(2.19)\qquad & I(g^{(\rho)}_{J_N,J_1,\ldots,J_{N-1}}(\beta_N,\beta_1,\ldots,\beta_{N-1}),w^{(\lambda)}(\beta_1,\ldots,\beta_N))|_{\beta_N\to\beta_N-\lambda i}\\
&=e^{-(2\pi i/\rho)\mu_{J_N+1}}I(g^{(\rho)}_{J_1,\ldots,J_N}(\beta_1,\ldots,\beta_{N-1},\beta_N),w^{(\lambda)}(\beta_1,\ldots,\beta_N)),
\end{aligned}
$$

where $w^{(\lambda)}\in\mathscr{F}^{(\lambda)}_{\nu_1,\ldots,\nu_{n-1}}(\beta_1,\ldots,\beta_N)$ and the left hand side of (2.19) is understood as the analytic continuation of the integral. It is easy to prove Theorem 2.1 from (2.18) and (2.19). □

We note that the weight of the solution ψ_J is given by

$$
(2.20)\qquad \sum_{j=1}^{n}\lambda_j\varepsilon_j,\qquad\text{where }\lambda_j=\nu_{j-1}-\nu_j.
$$

Now we set

$$
(2.21)\qquad \Psi_\xi(\beta_1,\ldots,\beta_N)=\sum_{J,J'\in\mathscr{Z}_{\nu_1,\ldots,\nu_{n-1}}} I(w^{(\rho)}_J,w^{(\lambda)}_{J'})v^{(\rho)}_J\otimes v^{(\lambda)}_{J'},
$$

where

$$
(2.22)\qquad \xi=\sum_{j=1}^{n}\lambda_j\varepsilon_j.
$$

Then Ψ_ξ is the fundamental matrix solution mentioned in Section 1.

§ 3. Determinant Formula for the Solutions

In the following sections we calculate the determinant

$$
(3.1)\qquad D_{\lambda_1,\ldots,\lambda_n}(\beta_1,\ldots,\beta_N)=\det(I(w^{(\rho)}_J,w^{(\lambda)}_{J'}))_{J,J'\in\mathscr{Z}_{\nu_1,\ldots,\nu_{n-1}}}.
$$

The result is as follows.

Theorem 3.1.

$$
\begin{aligned}
\text{(3.2)}\quad & \det(I(w_J^{(\rho)}, w_{J'}^{(\lambda)}))_{J,J' \in \mathscr{Z}_{\nu_1,\ldots,\nu_{n-1}}} \\
&= 2^{-d_{\nu_1,\ldots,\nu_{n-1}} \Lambda^{(0)}_{\lambda_1,\ldots,\lambda_n}} \exp\left((N^2 - N - 2)\left(\frac{1}{\rho} + \frac{1}{\lambda}\right) \frac{\pi^2 i}{n} \Lambda^{(2)}_{\lambda_1,\ldots,\lambda_n} \right) \\
&\times \left(\frac{\sqrt{\rho\lambda}}{S_2\left(-\frac{2\pi}{n}\right)} \right)^{\Lambda^{(0)}_{\lambda_1,\ldots,\lambda_n} \sum_{j=1}^{n} (j-1)\lambda_j} \prod_{1 \leqslant r' < r \leqslant n} \left\{ \prod_{a=0}^{\min\{\lambda_r - 1, \lambda_{r'}\}} \right. \\
&\times \left. \left(\frac{S_2\left(\mu_r - \mu_{r'} - \frac{\pi}{n}(\lambda_r + \lambda_{r'} - 2a)\right)}{S_2\left(\mu_r - \mu_{r'} + \frac{\pi}{n}(\lambda_r + \lambda_{r'} - 2a)\right)} \right)^{\binom{\lambda_r + \lambda_{r'}}{a}} \right\}^{\Lambda^{(0)}_{\lambda_1,\ldots,\lambda_n} / \binom{\lambda_r + \lambda_{r'}}{\lambda_r}} \\
&\times \exp\left(\frac{2\pi}{\rho\lambda} \left(\sum_{j=1}^{n} \binom{N-1}{\lambda_1, \ldots, \lambda_j - 1, \ldots, \lambda_n} \mu_j \right) \sum_{m=1}^{N} \beta_m \right) \\
&\times \left(\prod_{1 \leqslant r < s \leqslant N} \frac{S_2\left(i(\beta_r - \beta_s) + \frac{2\pi}{n}\right)}{S_2\left(i(\beta_r - \beta_s) - \frac{2\pi}{n}\right)} \right)^{\Lambda^{(2)}_{\lambda_1,\ldots,\lambda_n}},
\end{aligned}
$$

where

$$\lambda_j = \nu_{j-1} - \nu_j, \qquad d_{\nu_1,\ldots,\nu_{n-1}} = \sum_{j=1}^{n-1} (2\nu_j \nu_{j-1} + \nu_j^2 - 3\nu_j),$$

$$\Lambda^{(0)}_{\lambda_1,\ldots,\lambda_n} = \binom{N}{\lambda_1, \ldots, \lambda_n}, \qquad \Lambda^{(2)}_{\lambda_1,\ldots,\lambda_n} = \sum_{1 \leqslant j < k \leqslant n} \binom{N-2}{\lambda_1, \ldots, \lambda_j - 1, \ldots, \lambda_k - 1, \ldots, \lambda_n}.$$

Note that

$$\text{(3.3)}\quad \Lambda^{(0)}_{\lambda_1,\ldots,\lambda_n} \Big/ \binom{\lambda_r + \lambda_{r'}}{\lambda_r} = \binom{N}{\lambda_1, \ldots, \lambda_{r'-1}, \lambda_{r'+1}, \ldots, \lambda_{r-1}, \lambda_{r+1}, \ldots, \lambda_n, \lambda_{r'} + \lambda_r}$$

is a positive integer.

First, we determine the dependence on $\beta_1, \ldots, \beta_N$ of $D_{\lambda_1,\ldots,\lambda_n}$. From Theorem 2.1, we find that

$$\text{(3.4)}\quad \frac{D_{\lambda_1,\ldots,\lambda_n}(\beta_1, \ldots, \beta_m - \lambda i, \ldots, \beta_N)}{D_{\lambda_1,\ldots,\lambda_n}(\beta_1, \ldots, \beta_m, \ldots, \beta_N)} = \det_{\lambda_1,\ldots,\lambda_n} K_m^{(\rho)}(\beta_1, \ldots, \beta_N),$$

$$\text{(3.5)}\quad \frac{D_{\lambda_1,\ldots,\lambda_n}(\beta_1, \ldots, \beta_m - \rho i, \ldots, \beta_N)}{D_{\lambda_1,\ldots,\lambda_n}(\beta_1, \ldots, \beta_m, \ldots, \beta_N)} = \det_{\lambda_1,\ldots,\lambda_n} K_m^{(\lambda)}(\beta_1, \ldots, \beta_N).$$

Here $\det_{\lambda_1,\dots,\lambda_n} K_m$ stands for the determinant of the operator which is a restriction of the operator K_m to the weight subspace of the weight $\sum_{j=1}^{n} \lambda_j \varepsilon_j$.

Using formulae (1.1) and (1.2), we have

$$
\det_{\lambda_1,\dots,\lambda_n} K_m^{(p)}(\beta_1,\dots,\beta_N) = \exp\left(-\frac{2\pi i}{p}\sum_{j=1}^{n}\binom{N-1}{\lambda_1,\dots,\lambda_j-1,\dots,\lambda_n}\mu_j\right)
\times\left(\prod_{m'=1}^{m-1}\frac{\operatorname{sh}\frac{\pi}{p}\left(\beta_m-\beta_{m'}-\lambda_i+\frac{2\pi i}{n}\right)}{\operatorname{sh}\frac{\pi}{p}\left(\beta_m-\beta_{m'}-\lambda_i-\frac{2\pi i}{n}\right)}
\times\prod_{m'=m+1}^{N}\frac{\operatorname{sh}\frac{\pi}{p}\left(\beta_m-\beta_{m'}+\frac{2\pi i}{n}\right)}{\operatorname{sh}\frac{\pi}{p}\left(\beta_m-\beta_{m'}-\frac{2\pi i}{n}\right)}\right)^{\Lambda^{(2)}_{\lambda_1,\dots,\lambda_n}}. \tag{3.6}
$$

Now we set

$$
E_{\lambda_1,\dots,\lambda_n}(\beta_1,\dots,\beta_N) = \exp\left(\frac{2\pi}{p\lambda}\left(\sum_{j=1}^{n}\binom{N-1}{\lambda_1,\dots,\lambda_j-1,\dots,\lambda_n}\mu_j\right)\sum_{m=1}^{N}\beta_m\right)
\times\left(\prod_{1\leqslant r<s\leqslant N}\frac{S_2\left(i(\beta_r-\beta_s)+\frac{2\pi}{n}\right)}{S_2\left(i(\beta_r-\beta_s)-\frac{2\pi}{n}\right)}\right)^{\Lambda^{(2)}_{\lambda_1,\dots,\lambda_n}}. \tag{3.7}
$$

Then by using (6.3) we can check that $E_{\lambda_1,\dots,\lambda_n}(\beta_1,\dots,\beta_N)$ satisfies (3.4) and (3.5). Therefore, we have

Proposition 3.2.

$$
D_{\lambda_1,\dots,\lambda_n}(\beta_1,\dots,\beta_N) = c_{\lambda_1,\dots,\lambda_n}(\mu_1,\dots,\mu_n;p,\lambda)E_{\lambda_1,\dots,\lambda_n}(\beta_1,\dots,\beta_N), \tag{3.8}
$$

where $c_{\lambda_1,\dots,\lambda_n}(\mu_1,\dots,\mu_n;p,\lambda)$ *is a constant independent of* $\beta_1,\dots,\beta_N$.

In order to determine $c_{\lambda_1,\dots,\lambda_n}(\mu_1,\dots,\mu_n;p,\lambda)$, we consider the asymptotics of D/E as

$$
\beta_1 \ll \cdots \ll \beta_N. \tag{3.9}
$$

This is in the next section.

§4. Asymptotics of the Solutions

First, we consider the asymptotics of $D_{\lambda_1,\dots,\lambda_n}$.

We denote the set of variables

$$(4.1)\qquad \gamma_{j,m}\qquad (0 \leqslant j \leqslant n-1; 1 \leqslant m \leqslant \nu_j)$$

by γ. Fix a set of permutations $\sigma = (\sigma_1, \dots, \sigma_{n-1})$; $\sigma_j \in S_{\nu_j}$ $(1 \leqslant j \leqslant n-1)$. We use $\sigma_0 = \mathrm{id}$. We denote

$$(4.2)\qquad \gamma_{j,\sigma_j(m)}\qquad (0 \leqslant j \leqslant n-1; 1 \leqslant m \leqslant \nu_j)$$

by γ_σ.

Consider

$$(4.3)\qquad F_{J,J',\sigma}(\gamma) = K(\gamma) g_J(\gamma) g_{J'}(\gamma_\sigma),$$

where $K(\gamma)$ is given by (2.13).

In the following, we use the abbreviation $\beta_{ij} = \beta_i - \beta_j$.

Proposition 4.1. *Suppose that $\beta_1 < \cdots < \beta_n$ and $\gamma_{j,m}$'s are all real. If λ is sufficiently large, then there exist positive constants ε, C, κ independent of the variables β and γ such that the following estimate holds.*

$$(4.4)\qquad |F_{J,J',\sigma}| < C \exp\left(-\kappa \sum_{\substack{1 \leqslant j \leqslant n-1 \\ 1 \leqslant m \leqslant \nu_j}} |\gamma_{j,m} - \gamma_{j-1,m^*(J,j,m)}| \right)$$

$$\times \exp\left(-\frac{2\pi^2}{\rho\lambda n} \sum_{1 \leqslant r < s \leqslant N} (1 - \delta_{J_r,J_s}) \beta_{sr} + \frac{2\pi}{\rho\lambda} \sum_{r=1}^{N} \mu_{J_r+1} \beta_r \right)$$

if

$$(4.5)\qquad \frac{2\pi}{\rho\lambda}(\mu_{j+1} - \mu_j) > \varepsilon, \qquad \frac{2\pi}{\rho\lambda}(\mu_n - \mu_1) < n\varepsilon.$$

Proof. Throughout the proof, we set $r_{j,m} = r^J_{j,m}$. We define new variables $\tilde{\gamma}$ by

$$(4.6)\qquad \gamma_{j,m} = \tilde{\gamma}_{j,m} + \beta_{r_{j,m}}.$$

Note that $\tilde{\gamma}_{0,m} = 0$. From (6.2), we have

$$\exp\left(\frac{2\pi}{\rho\lambda} \sum_{\substack{0 \leqslant j \leqslant n-1 \\ 1 \leqslant m \leqslant \nu_j}} (\mu_{j+1} - \mu_j) \gamma_{j,m} \right)$$

$$= \exp\left(\frac{2\pi}{\rho\lambda} \sum_{\substack{1 \leqslant j \leqslant n-1 \\ 1 \leqslant m \leqslant \nu_j}} (\mu_{j+1} - \mu_j) \tilde{\gamma}_{j,m} + \frac{2\pi}{\rho\lambda} \sum_{1 \leqslant r \leqslant N} \mu_{J_r+1} \beta_r \right),$$

$$
\begin{aligned}
(4.7)\qquad |\varphi(\gamma_{j,m}-\gamma_{j-1,m'})| &\leqslant \text{const. } e^{-\pi((1/\rho)+(1/\lambda)+(2\pi/\rho\lambda n))|\tilde{\gamma}_{j,m}-\tilde{\gamma}_{j-1\,m'}+\beta r_{j,m}r_{j-1,m'}|},\\
|\psi(\gamma_{j,m}-\gamma_{j,m'})| &\leqslant \text{const. } e^{-\pi((1/\rho)+(1/\lambda)-(4\pi/\rho\lambda n))|\tilde{\gamma}_{j,m}-\tilde{\gamma}_{j,m'}+\beta r_{j,m}r_{j,m'}|},\\
|g_J(\gamma)| &\leqslant \text{const. } \prod_{1\leqslant j\leqslant n-1}\Bigg\{\prod_{1\leqslant m\leqslant \nu_j} e^{-(\pi/\rho)(\tilde{\gamma}_{j,m}-\tilde{\gamma}_{j-1,m^*(J,j,m)})}\\
&\quad\times \prod_{\substack{1\leqslant m\leqslant \nu_j\\ 1\leqslant m'\leqslant \nu_{j-1}\\ m'\neq m^*(J,j,m)}} e^{(\pi/\rho)|\tilde{\gamma}_{j\,m}-\tilde{\gamma}_{j-1,m'}+\beta r_{j,m}r_{j-1,m'}|}\\
&\quad\times \prod_{1\leqslant m<m'\leqslant \nu_j} e^{(\pi/\rho)|\tilde{\gamma}_{j,m}-\tilde{\gamma}_{j,m'}+\beta r_{j\,m}r_{j,m'}|}\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
(4.8)\qquad |g_{J'}(\gamma_\sigma)| \leqslant \text{const. } &\prod_{1\leqslant j\leqslant n-1}\\
&\times\Bigg\{\prod_{1\leqslant m\leqslant \nu_j} e^{-(\pi/\lambda)(\tilde{\gamma}_{j,\sigma_j(m)}-\tilde{\gamma}_{j-1,\sigma_{j-1}(m^*(J',j,m))}+\beta r_{j,\sigma_j(m)}r_{j-1,\sigma_{j-1}(m^*(J',j,m))})}\\
&\times \prod_{\substack{1\leqslant m\leqslant \nu_j\\ m'\neq m^*(J',j,m)}} e^{(\pi/\lambda)|\tilde{\gamma}_{j,\sigma_j(m)}-\tilde{\gamma}_{j-1,\sigma_{j-1}(m')}+\beta r_{j,\sigma_j(m)}r_{j-1,\sigma_{j-1}(m')}|}\\
&\times \prod_{1\leqslant m<m'\leqslant \nu_j} e^{(\pi/\lambda)|\tilde{\gamma}_{j,m}-\tilde{\gamma}_{j,m'}+\beta r_{j,m}r_{j,m'}|}\Bigg\}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
(4.9)\qquad |F_{J,J',\sigma}(\gamma)| \leqslant \text{const. } &e^{(2\pi/\rho\lambda)\sum_{\substack{1\leqslant j\leqslant n-1\\ 1\leqslant m\leqslant \nu_j}}(\mu_{j+1}-\mu_j)\tilde{\gamma}_{j,m}+(2\pi/\rho\lambda)\sum_{1\leqslant \iota\leqslant N}\mu_{J_\iota+1}\beta_\iota}\\
&\times \prod_{\substack{1\leqslant j\leqslant n-1\\ 1\leqslant m\leqslant \nu_j\\ 1\leqslant m'\leqslant \nu_{j-1}}} e^{-(2\pi^2/\rho\lambda n)|\tilde{\gamma}_{j,m}-\tilde{\gamma}_{j-1,m'}+\beta r_{j,m}r_{j-1,m'}|}\\
&\times \prod_{\substack{1\leqslant j<n-1\\ 1\leqslant m<m'\leqslant \nu_j}} e^{(4\pi^2/\rho\lambda n)|\tilde{\gamma}_{j,m}-\tilde{\gamma}_{j,m'}+\beta r_{j,m}r_{j,m'}|}\\
&\times \prod_{\substack{1\leqslant j<n-1\\ 1\leqslant m\leqslant \nu_j}} e^{-(\pi/\rho)\xi(\tilde{\gamma}_{j,m}-\tilde{\gamma}_{j-1,m^*(J,j,m)})}\\
&\times \prod_{\substack{1\leqslant j\leqslant n-1\\ 1\leqslant m\leqslant \nu_j}} e^{-(\pi/\lambda)\xi(\tilde{\gamma}_{j,\sigma(m)}-\tilde{\gamma}_{j-1,\sigma(m^*(J',j,m))}+\beta r_{j,\sigma(m)}r_{j-1,\sigma(m^*(J',j,m))})}.
\end{aligned}
$$

Here

$$\xi(x) = x + |x|. \tag{4.10}$$

We apply $-|A+B| \leqslant |A| - |B|$ to the second line of (4.9), and $|A+B| \leqslant |A| + |B|$ to the third line. Then, we use

$$\sum_{\substack{1\leqslant j\leqslant n-1\\ 1\leqslant m\leqslant \nu_j\\ 1\leqslant m'\leqslant \nu_{j-1}}} |\beta_{r_{j,m}r_{j-1,m'}}| = \sum_{1\leqslant r<s\leqslant N} (2\min(J_r, J_s) + 1 - \delta_{J_r,J_s})\beta_{sr}, \tag{4.11}$$

$$\sum_{\substack{1\leqslant j\leqslant n-1\\ 1\leqslant m<m'\leqslant \nu_j}} |\beta_{r_{j,m}r_{j,m'}}| = \sum_{1\leqslant r<s\leqslant N} \min(J_r, J_s)\beta_{sr}. \tag{4.12}$$

We ignore the last line of (4.9). After all these steps, it is enough to show

$$\begin{aligned}\frac{2\pi}{\rho\lambda} \sum_{\substack{1\leqslant j\leqslant n-1\\ 1\leqslant m\nu_j}} (\mu_{j+1} - \mu_j)\tilde{\gamma}_{j,m} &+ \frac{2\pi^2}{\rho\lambda n} \sum_{\substack{1\leqslant j\leqslant n-1\\ 1\leqslant m\leqslant \nu_j\\ 1\leqslant m'\leqslant \nu_{j-1}}} |\tilde{\gamma}_{j,m} - \tilde{\gamma}_{j-1,m'}| \\ &+ \frac{4\pi^2}{\rho\lambda n} \sum_{\substack{1\leqslant j\leqslant n-1\\ 1\leqslant m<m'\leqslant \nu_j}} |\tilde{\gamma}_{j,m} - \tilde{\gamma}_{j,m'}| - \frac{\pi}{\rho} \sum_{\substack{1\leqslant j\leqslant n-1\\ 1\leqslant m\leqslant \nu_j}} \xi(\tilde{\gamma}_{j,m} - \tilde{\gamma}_{j-1,m^*(J,j,m)}) \\ &< -\kappa \sum_{\substack{1\leqslant j\leqslant n-1\\ 1\leqslant m\leqslant \nu_j}} |\gamma_{j,m} - \gamma_{j-1,m^*(J,j,m)}|.\end{aligned} \tag{4.13}$$

The left hand side is not larger than

$$\begin{aligned}\sum_{\substack{1\leqslant j\leqslant n-1\\ 1\leqslant m\leqslant \nu_j}} \bigg\{ &\frac{2\pi}{\rho\lambda} (\mu_{j_{\max}(J,j,m)+1} - \mu_j)(\gamma_{j,m} - \gamma_{j-1,m^*(J,j,m)}) \\ &+ \frac{K}{\rho\lambda} |\gamma_{j,m} - \gamma_{j-1,m^*(J,j,m)}| - \frac{\pi}{\rho} \xi(\gamma_{j,m} - \gamma_{j-1,m^*(J,j,m)}) \bigg\},\end{aligned} \tag{4.14}$$

where

$$K = \frac{2\pi^2}{n} \sum_{\substack{1\leqslant j\leqslant n-1\\ 1\leqslant m\leqslant \nu_j\\ 1\leqslant m'\leqslant \nu_{j-1}}} 1 + \frac{4\pi^2}{n} \sum_{\substack{1\leqslant j\leqslant n-1\\ 1\leqslant m<m'\leqslant \nu_j}} 1, \tag{4.15}$$

and

$$j_{\max}(J, j, m) = \max\{j'; r_{j,m} \in \mathcal{N}_{j'}^{J}\}. \tag{4.16}$$

Choose ε, κ so that

$$n\varepsilon + \frac{K}{\rho\lambda} - \frac{2\pi}{\rho} < -\kappa, \tag{4.17}$$

$$\varepsilon - \frac{K}{\rho\lambda} > \kappa. \tag{4.18}$$

This is possible if

$$\frac{2K}{\rho\lambda} < \frac{2\pi}{\rho}. \tag{4.19}$$

Then, the estimate (4.13) follows from (4.10) and (4.5). □

For $J \in \mathscr{Z}_{\nu_1,\dots,\nu_{n-1}}$, we set

$$P_J = \exp\left(\frac{2\pi^2}{\rho\lambda n} \sum_{1 \leqslant r < s \leqslant N} (1 - \delta_{J_r, J_s})\beta_{sr} - \frac{2\pi}{\rho\lambda} \sum_{r=1}^{N} \mu_{J_r+1} \beta_r\right). \tag{4.20}$$

The following is an obvious consequence of Proposition 4.1.

Corollary 4.2. *The integral* (2.13) *is absolutely convergent. The convergence is uniform in the variables* β *if we multiply* P_J *to the integrand.*

Define a partial order in $\mathscr{Z}_{\nu_1,\dots,\nu_n}$:

$$J \leqslant J' \qquad \text{if and only if } J_r + \cdots + J_N \leqslant J'_r + \cdots + J'_N \qquad \text{for all } r. \tag{4.21}$$

Proposition 4.3. *If* $J \nleqslant J'$, *then we have*

$$\lim\nolimits_{\beta_1 \ll \cdots \ll \beta_N} P_J \left(\prod_{j=1}^{n-1} \prod_{m=1}^{\nu_j} \int_{C_j} d\gamma_{j,m}\right) F_{J,J',\sigma}(\gamma) = 0. \tag{4.22}$$

Proof. We follow the estimate in the proof of Proposition 4.1. When we go from (4.9) to (4.13), we dropped the last line in (4.9). This time we use that term. Namely, we can claim that (4.22) holds unless for some $\sigma = (\sigma_1, \dots, \sigma_{n-1})$

$$r^J_{j,\sigma_j(m)} \leqslant r^J_{j-1,\sigma_{j-1}(m^*(J',j,m))} \tag{4.23}$$

holds for all j and m. This is clear because

$$\xi(x + y) = 2(x + y) \qquad \text{if } y > -x. \tag{4.24}$$

We show that (4.23) implies $J \leqslant J'$. This will complete the proof.

First we prove

$$r^J_{j,\sigma_j(m)} \leqslant r^{J'}_{j,m} \qquad \text{for all } m \tag{4.25}$$

by induction on j. The case $j = 0$ is obvious. Suppose that (4.25) is true for $j - 1$. Then we have

$$r^J_{j,\sigma_j(m)} \leqslant r^J_{j-1,\sigma_{j-1}(m^*(J',j,m))} \leqslant r^{J'}_{j-1,m^*(J',j,m)} = r^{J'}_{j,m}. \tag{4.26}$$

Therefore, (4.25) is true for all j. It follows from (4.25) that

$$r^J_{j,m} \leqslant r^{J'}_{j,m} \qquad \text{for all } j, m. \tag{4.27}$$

Finally, we prove $J \leqslant J'$. This is clear because

$$J_r + \cdots + J_N = \#\{(j,m); r^J_{j,m} \geqslant r.\} \tag{4.28}$$

The proof of Proposition 4.3 is over. □

This proposition shows that in the asymptotic limit the matrix $(I(w^{(\rho)}_J, w^{(\lambda)}_{J'}))_{J,J'}$ is triangular.

We have also

Proposition 4.4.

$$\lim_{\beta_1 \ll \cdots \ll \beta_N} P_J \left(\prod_{j=1}^{n-1} \prod_{m=1}^{\nu_j} \int_{C_j} d\gamma_{j,m} \right) F_{J,J,\sigma}(\gamma) = 0 \tag{4.29}$$

unless $\sigma_j = \mathrm{id}$ *for all* j.

Proof. Suppose that

$$r^J_{j,\sigma_j(m)} \leqslant r^J_{j-1,\sigma_{j-1}(m^*(J,j,m))} \tag{4.30}$$

for all j, m. From the proof of Proposition 4.3 we have

$$r^J_{j,\sigma_j(m)} \leqslant r^J_{j,m} \qquad \text{for all } j, m. \tag{4.31}$$

This implies that $\sigma_j = \mathrm{id}$ for all j. □

Define

$$\nu^{J,+}_{j,r} = \#\{s \in \mathcal{N}^J_j; r < s\}, \qquad \nu^{J,-}_{j,r} = \#\{s \in \mathcal{N}^J_j; r > s\}. \tag{4.32}$$

From (6.2), we have

Proposition 4.5.

$$\begin{aligned} &\lim_{\beta_1 \ll \cdots \ll \beta_N} P_J \left(\prod_{j=1}^{n-1} \prod_{m=1}^{\nu_j} \int_{C_j} d\gamma_{j,m} \right) F_{J,J,\mathrm{id}}(\gamma) \\ &\quad = 2^{-d_{\nu_1, \ \nu_{n-1}}} \exp\left(\left(\frac{1}{\rho} + \frac{1}{\lambda} \right) \frac{\pi^2 i}{n} \sum_{j=1}^{n-1} \{\nu_j(\nu_{j-1} - 1) - \nu_j(\nu_j - 1)\} \right) \\ &\quad\quad \times \prod_{r=1}^{N} G_{J_r}(\tilde{\mu}^J_{1,r}, \ldots, \tilde{\mu}^J_{n,r}) \end{aligned} \tag{4.33}$$

where

$$G_k(\mu_1,\dots,\mu_n)=\prod_{j=1}^{k}\int_{C_j}\frac{d\gamma_j}{2\pi i}\prod_{j=1}^{k}\varphi(\gamma_j-\gamma_{j-1})e^{(2\pi/\rho\lambda)\sum_{j=1}^{k}(\mu_{j+1}-\mu_j)\gamma_j}, \tag{4.34}$$

$$\tilde{\mu}^{J}_{j,r}=\mu_j+\frac{\pi}{n}\sum_{\varepsilon=\pm}\varepsilon(v^{J,\varepsilon}_{j,r}-v^{J,\varepsilon}_{j-1,r})-\delta_{j,J_r+1}\frac{\rho+\lambda}{2}. \tag{4.35}$$

In the above formula of $G_k, \gamma_0=0$ *and the contour* C_j *for* γ_j *is a deformation of the real line* $(-\infty,\infty)$ *such that the poles at*

$$\gamma_{j-1}-\frac{\pi i}{n}+\rho i\mathbf{Z}_{\geqslant 0}+\lambda i\mathbf{Z}_{\geqslant 0} \tag{4.36}$$

are above C_j *and the poles at*

$$\gamma_{j-1}+\frac{\pi i}{n}-\rho i\mathbf{Z}_{\geqslant 0}-\lambda i\mathbf{Z}_{\geqslant 0} \tag{4.37}$$

are below C_j.

This proposition shows that in the asymptotic limit the diagonal element $I(w^{(\rho)}_J,w^{(\lambda)}_{J'})$ reduces to the one point functions G_{J_r} $(1\leqslant r\leqslant N)$.

Now we consider the asymptotics of $D_{\lambda_1,\dots,\lambda_n}/E_{\lambda_1,\dots,\lambda_n}$. Hereafter we use the notation $\sim$ as follows:

$$f(\beta_1,\dots,\beta_N)\sim g(\beta_1,\dots,\beta_N)\overset{\text{def}}{\Longleftrightarrow}\lim\nolimits_{\beta_1\ll\cdots\ll\beta_N}\left\{\frac{f(\beta_1,\dots,\beta_N)}{g(\beta_1,\dots,\beta_N)}\right\}=1. \tag{4.38}$$

From (6.1), we have

$$\begin{aligned}E_{\lambda_1,\dots,\lambda_n}(\beta_1,\dots,\beta_N)&\sim\exp\left(\frac{2\pi}{\rho\lambda}\left(\sum_{j=1}^{n}\binom{N-1}{\lambda_1,\dots,\lambda_j-1,\dots,\lambda_n}\mu_j\right)\sum_{m=1}^{N}\beta_m\right)\\&\quad\times\exp\left(\left\{\left(\frac{1}{\rho}+\frac{1}{\lambda}\right)\frac{2\pi^2 i}{n}-\frac{4\pi^2}{n\rho\lambda}\sum_{1\leqslant r<s\leqslant N}\beta_{sr}\right\}\Lambda^{(2)}_{\lambda_1,\dots,\lambda_n}\right).\end{aligned} \tag{4.39}$$

We note that

$$\begin{aligned}\prod_{J\in\mathscr{Z}_{\nu_1,\dots,\nu_{n-1}}}P_J=\exp\Bigg(&\frac{4\pi^2}{n\rho\lambda}\sum_{1\leqslant r<s\leqslant N}\beta_{sr}\Lambda^{(2)}_{\lambda_1,\dots,\lambda_n}\\&-\frac{2\pi}{\rho\lambda}\left(\sum_{j=1}^{n}\binom{N-1}{\lambda_1,\dots,\lambda_j-1,\dots,\lambda_n}\mu_j\right)\sum_{m=1}^{N}\beta_m\Bigg).\end{aligned} \tag{4.40}$$

Hence we find

$$\begin{aligned}(4.41)\quad &\frac{D_{\lambda_1,\dots,\lambda_n}(\beta_1,\dots,\beta_N)}{E_{\lambda_1,\dots,\lambda_n}(\beta_1,\dots,\beta_N)}\\ &\qquad\sim \exp\left(-\left(\frac{1}{\rho}+\frac{1}{\lambda}\right)\frac{2\pi^2 i}{n}\Lambda^{(2)}_{\lambda_1,\dots,\lambda_n}\right)\det(P_J I(w_J^{(\rho)},w_{J'}^{(\lambda)}))_{J,J'\in\mathscr{Z}_{\nu_1,\dots,\nu_{n-1}}}.\end{aligned}$$

From Propositions 4.3, 4.4, and 4.5, we see that

(4.42)

$$\det(P_J I(w_J^{(\rho)},w_{J'}^{(\lambda)}))_{J,J'\in\mathscr{Z}_{\nu_1,\dots,\nu_{n-1}}} \sim \prod_{J\in\mathscr{Z}_{\nu_1,\dots,\nu_{n-1}}} (\text{the right hand side of (4.33)}).$$

Therefore, we get

Proposition 4.6.

$$\begin{aligned}(4.43)\quad c_{\lambda_1,\dots,\lambda_n}(\mu_1,\dots,\mu_n;\rho,\lambda) &= 2^{-d_{\nu_1,\dots,\nu_{n-1}}\Lambda^{(0)}_{\lambda_1,\dots,\lambda_n}}\\ &\times\exp\left((N^2-N-2)\left(\frac{1}{\rho}+\frac{1}{\lambda}\right)\frac{\pi^2 i}{n}\Lambda^{(2)}_{\lambda_1,\dots,\lambda_n}\right)\\ &\times\prod_{J\in\mathscr{Z}_{\nu_1,\dots,\nu_{n-1}}}\prod_{r=1}^{N} G_{J,}(\tilde{\mu}^J_{1,r},\dots,\tilde{\mu}^J_{n,r}).\end{aligned}$$

Proof. Note that

$$(4.44)\qquad \#\mathscr{Z}_{\nu_1,\dots,\nu_{n-1}} = \Lambda^{(0)}_{\lambda_1,\dots,\lambda_n}.$$

We get the term

$$(4.45)\qquad \exp\left((N^2-N-2)\left(\frac{1}{\rho}+\frac{1}{\lambda}\right)\frac{\pi^2 i}{n}\Lambda^{(2)}_{\lambda_1,\dots,\lambda_n}\right)$$

by using the following formulae:

$$\begin{aligned}(4.46)\qquad &\sum_{j=1}^{n-1}\{\nu_j(\nu_{j-1}-1)-\nu_j(\nu_j-1)\} = \frac{1}{2}\left(N^2-\sum_{j=1}^{n}\lambda_j^2\right),\\ &\frac{1}{2}\left(N^2-\sum_{j=1}^{n}\lambda_j^2\right)\Lambda^{(0)}_{\lambda_1,\dots,\lambda_n} = N(N-1)\Lambda^{(2)}_{\lambda_1,\dots,\lambda_n}. \quad \square\end{aligned}$$

§5. Proof of Theorem 3.1

First, we find an explicit formula for $G_k(\mu_1,\dots,\mu_n)$. We set

$$(5.1)\qquad H_k(x_1,\dots,x_k) = \left(\prod_{j=1}^{k}\int_{C_j} d\gamma_j\right)\prod_{j=1}^{k}\varphi(\gamma_j-\gamma_{j-1})e^{(2\pi/\rho\lambda)\sum_{j=1}^{k}x_j(\gamma_j-\gamma_{j-1})},$$

where $\gamma_0 = 0$. Then we have

$$G_k(\mu_1, \dots, \mu_n) = H_k(\mu_{k+1} - \mu_1, \dots, \mu_{k+1} - \mu_k). \tag{5.2}$$

The integral (5.1) is absolutely convergent if

$$|\mathrm{Re}\, x_j| < \frac{\rho+\lambda}{2} + \frac{\pi}{n}, \qquad (j = 1, \dots, n). \tag{5.3}$$

By changing the integration variables γ_j to

$$u_j = \gamma_j - \gamma_{j-1}, \qquad (j = 1, \dots, n), \tag{5.4}$$

we can see that

$$H_k(x_1, \dots, x_k) = \prod_{j=1}^{k} H(x_j), \tag{5.5}$$

where

$$H(x) = \int_C du \varphi(u) e^{(2\pi/\rho\lambda)xu}. \tag{5.6}$$

In the above formula, the contour C is a deformation of the real line $(-\infty, \infty)$ such that the poles at

$$-\frac{\pi i}{n} + \rho i \mathbf{Z}_{\geqslant 0} + \lambda i \mathbf{Z}_{\geqslant 0} \tag{5.7}$$

are above C and the poles at

$$\frac{\pi i}{n} - \rho i \mathbf{Z}_{\geqslant 0} - \lambda i \mathbf{Z}_{\geqslant 0} \tag{5.8}$$

are below C.

The explicit formula for the function H is obtained in [MT1].

Proposition 5.1.

$$H(x) = \frac{\sqrt{\rho\lambda}}{S_2\left(-\frac{2\pi}{n}\right)} \frac{S_2\left(x + \frac{\rho+\lambda}{2} - \frac{\pi}{n}\right)}{S_2\left(x + \frac{\rho+\lambda}{2} + \frac{\pi}{n}\right)} \tag{5.9}$$

From (5.2), (5.5) and (5.9), we get

Proposition 5.2.

$$G_k(\mu_1, \dots, \mu_n) = \left(\frac{\sqrt{\rho\lambda}}{S_2\left(-\frac{2\pi}{n}\right)}\right)^k \prod_{j=1}^{k} \frac{S_2\left(\mu_{k+1} - \mu_j + \frac{\rho+\lambda}{2} - \frac{\pi}{n}\right)}{S_2\left(\mu_{k+1} - \mu_j + \frac{\rho+\lambda}{2} + \frac{\pi}{n}\right)}. \tag{5.10}$$

Now it remains to calculate

$$\prod_{J\in\mathscr{Z}_{\nu_1,\ldots,\nu_{n-1}}}\prod_{r=1}^{N}G_{J,}(\tilde{\mu}^J_{1,r},\ldots,\tilde{\mu}^J_{n,r}). \tag{5.11}$$

We set

$$M^{J,+}_{j,k}=\#\{r;J_r=j,k<r\},\qquad M^{J,-}_{j,k}=\#\{r;J_r=j,k>r\}. \tag{5.12}$$

Note that

$$\sum_{r=1}^{n}J_r=\sum_{j=1}^{n}(j-1)\lambda_j,\qquad \text{for all } J\in\mathscr{Z}_{\nu_1,\ldots,\nu_{n-1}}. \tag{5.13}$$

From (4.33) and (5.10), we have

$$\begin{aligned}\prod_{J\in\mathscr{Z}_{\nu_1,\ldots,\nu_{n-1}}}\prod_{r=1}^{N}&G_{J,}(\tilde{\mu}^J_{1,r},\ldots,\tilde{\mu}^J_{n,r})\\ &=\left(\frac{\sqrt{\rho\lambda}}{S_2\left(-\frac{2\pi}{n}\right)}\right)^{\Lambda^{(0)}_{\lambda_1,\ldots,\lambda_n}\sum_{j=1}^{n}(j-1)\lambda_j}\prod_{1\leqslant r'<r\leqslant n}\prod_{J\in\mathscr{Z}_{\nu_1,\ldots,\nu_{n-1}}}\prod_{\substack{k\\ J_k+1=r}}\\ &\quad\times\frac{S_2\left(\mu_r-\mu_{r'}+\frac{\pi}{n}(D^J_{r',r,k}-1)\right)}{S_2\left(\mu_r-\mu_{r'}+\frac{\pi}{n}(D^J_{r',r,k}+1)\right)},\end{aligned} \tag{5.14}$$

where $D^J_{r',r,k}$ is given by

$$D^J_{r',r,k}=\sum_{\varepsilon=\pm}\varepsilon(M^{J,\varepsilon}_{r'-1,k}-M^{J,\varepsilon}_{r-1,k})=\lambda_{r'}-\lambda_r+1-2(M^{J,-}_{r'-1,k}-M^{J,-}_{r-1,k}) \tag{5.15}$$

for k satisfying $J_k+1=r$.

Now we rewrite

$$\prod_{J\in\mathscr{Z}_{\nu_1,\ldots,\nu_{n-1}}}\prod_{\substack{k\\ J_k+1=r}}\frac{S_2\left(\mu_r-\mu_{r'}+\frac{\pi}{n}(D^J_{r',r,k}-1)\right)}{S_2\left(\mu_r-\mu_{r'}+\frac{\pi}{n}(D^J_{r',r,k}+1)\right)}. \tag{5.16}$$

Let us consider the following set:

$$\mathscr{F}^{(r',r)}=\bigsqcup_{\substack{(J,k)\\ J\in\mathscr{Z}_{\nu_1,\ldots,\nu_{n-1}}\\ J_k+1=r}}\{M^{J,-}_{r'-1,k}-M^{J,-}_{r-1,k}\},\qquad(1\leqslant r'<r\leqslant n), \tag{5.17}$$

where $\bigsqcup$ means a disjoint union. For $a \in \mathbf{Z}$, we set

$$\operatorname{mult}^{(r',r)}(a) = \#\{t \in \mathscr{F}^{(r',r)}; t = a\}. \tag{5.18}$$

Then we have

$$(5.16) = \prod_{a \in \mathbf{Z}} S_2\Big(\mu_r - \mu_{r'} + \frac{\pi}{n}(\lambda_{r'} - \lambda_r - 2a)\Big)^{\operatorname{mult}^{(r',r)}(a) - \operatorname{mult}^{(r',r)}(a+1)}. \tag{5.19}$$

We can show

$$\operatorname{mult}^{(r',r)}(a) - \operatorname{mult}^{(r',r)}(a+1)$$
$$= \begin{cases} -\Lambda^{(0)}_{\lambda_1,\dots,\lambda_n} \binom{\lambda_r + \lambda_{r'}}{\lambda_r + a} \Big/ \binom{\lambda_r + \lambda_{r'}}{\lambda_r}, & -\lambda_r \leqslant a \leqslant \min\{-1, \lambda_r - \lambda_{r'}\}, \\ \Lambda^{(0)}_{\lambda_1,\dots,\lambda_n} \binom{\lambda_r + \lambda_{r'}}{\lambda_{r'} - a} \Big/ \binom{\lambda_r + \lambda_{r'}}{\lambda_r}, & \max\{0, \lambda_{r'} - \lambda_r + 1\} \leqslant a \leqslant \lambda_{r'}, \\ 0, & \text{otherwise}. \end{cases}$$

This completes the proof. □

Appendix

Here we summarize the property of the double sine function $S_2(x) = S_2(x|\omega_1, \omega_2)$ following [JM].

We assume that $\operatorname{Re}\omega_1 > 0$, $\operatorname{Re}\omega_2 > 0$. $S_2(x|\omega_1, \omega_2)$ is a meromorpic function of x and symmetric with respect to ω_1, ω_2. Its zeros and poles are given by

$$\text{zeros at } x = \omega_1 \mathbf{Z}_{\leqslant 0} + \omega_2 \mathbf{Z}_{\leqslant 0}, \qquad \text{poles at } x = \omega_1 \mathbf{Z}_{\geqslant 1} + \omega_2 \mathbf{Z}_{\geqslant 1}.$$

Its asymptotic behavior is as follows (note that we corrected a sign in the formula (6.1) cited from [JM]):

$$\log S_2(x) = \pm \pi i \left(\frac{x^2}{2\omega_1\omega_2} - \frac{\omega_1 + \omega_2}{2\omega_1\omega_2} x + \frac{1}{12}\left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} + 3 \right) \right) + o(1), \tag{6.1}$$
$$(x \to \infty, \pm \operatorname{Im} x > 0).$$

This implies that

(6.2)
$$\log S_2(a + x) S_2(a - x) = \pm \pi i \frac{2a - \omega_1 - \omega_2}{\omega_1\omega_2} x + o(1), \qquad (x \to \infty, \pm \operatorname{Im} x > 0).$$

The following formulae hold:

$$\frac{S_2(x+\omega_1)}{S_2(x)}=\frac{1}{2\sin\frac{\pi x}{\omega_2}}, \tag{6.3}$$

$$S_2(x)=\frac{2\pi}{\sqrt{\omega_1\omega_2}}x+O(x^2) \qquad (x\to 0). \tag{6.4}$$

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