Conjugacy of Z^2-subshifts and Textile Systems

By

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Abstract

It will be shown that any topological conjugacy of Z^2-subshifts is factorized into a finite number of bipartite codes, and that in particular when textile shifts which are Z^2-subshifts arising from textile systems introduced by Nasu are taken each bipartite code appearing in this factorization is given by a bipartite graph code of textile shifts which is defined in terms of textile systems. The latter result extends the Williams result on strong shift equivalence of Z^1-topological Markov shifts to a Z^2-shift case.

§ 1. Introduction

It is known that any conjugacy of topological Markov shifts is described by Williams strong shift equivalence [8], which is an equivalence relation of their adjacency matrices. Topological Markov shifts are defined on the set of one dimensional lattice points. On the other hand, on the set of two dimensional lattice points, notions of shift space and shift of finite type (SFT) are analogously defined and called a Z^2-shift space and a Z^2-SFT respectively. However, a notion of a two dimensional topological Markov shift is not available. So, it is natural to ask what a two dimensional analogue of a topological Markov shift is. One idea is to consider the class of textile shifts U_T arising from textile systems T introduced by Nasu [5].

A textile system T consists of graph homomorphisms p and q of a graph Γ into a graph G. The textile shift U_T is the set of all $(x_{i,j})_{i,j \in Z}$ satisfying $p(x_{i,j}) = q(x_{i+1,j})$ and $t_\Gamma(x_{i,j}) = s_\Gamma(x_{i,j+1})$ for all $i, j \in Z$ where s_Γ and t_Γ mean the source and target maps of the graph Γ. Every textile

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shift is a Z^2-SFT and any Z^2-SFT is conjugate with a textile shift (Theorem 4.1). These are known in one dimensional case if a topological Markov shift and a Z^1-SFT are taken instead of a textile shift and a Z^2-SFT. Speaking about strong shift equivalence of topological Markov shifts, a natural question about what is available as a conjugacy invariant of textile shifts, comes up. In this paper we will discuss about conjugacy of textile shifts. We indeed obtain a textile shift analogue of the Williams strong shift equivalence (Theorem 3.1). For this, the notions of a bipartite textile system, a bipartite graph code (Definition 3.3) and a bipartite relation (Definition 3.4) are introduced. We remark that a bipartite graph code of textile shifts does correspond to one-step strong shift equivalence by Williams. As a matter of fact, we will obtain

Main theorem (Theorem 3.1):
Let $T = (p, q)$, $T' = (p', q')$ be textile systems. Suppose U_T and $U_{T'}$ are conjugate under a conjugacy ϕ. Then, T and T' are bipartitely related and ϕ is a composition of the corresponding bipartite graph codes and the symbolic conjugacies arising from essentially identical isomorphisms.

The content of this paper is as follows: In Section 2, bipartite code of two dimensional shift spaces is defined and it is shown that any conjugacy of two dimentional shift spaces is factorized into a finite number of bipartite codes (Theorem 2.1). In Section 3, the notions of an essential textile system, a bipartite textile system, a bipartite graph code and a bipartite relation are defined, and the main theorem stated above is obtained in Theorem 3.1. In Section 4, every Z^2-SFT is shown to be conjugate with some textile shift (Theorem 4.1). In order to understand Theorem 4.1, we will show in Example 4.1 that the three dots model is conjugate with some textile shift.

After writing up the first version of the manuscript, the author was informed by A. Johnson and K. Mardden [1] that they also had a similar result of our main theorem in terms of a decomposition of a conjugacy by a finite number of state splitting codes.

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§2. Conjugacy of Z^2-subshifts

Let A be a finite alphabet. For integers $m, n \geq 1$, each element in the product space $A^{[1,m] \times [1,n]}$ is called a block over A of size $m \times n$. Consider the space $A^{Z^2} = \{(x_{i,j})_{i,j \in Z} \in Z^2 \mid x_{i,j} \in A\}$. Let F be a collection of blocks. We define the subset $X_F = \{x \in A^{Z^2} \mid$ any block in F does not appear in $x\}$. On X_F, a Z^2-action $\sigma = \sigma(i,j)_{i,j \in Z^2}$ is defined by $[\sigma(i,j)x]_{m,n} = x_{m+i,n+j}$ which is called a shift mapping. We call the pair (X_F, σ) a Z^2-**shift space** (sometimes simply denoted by X_F). When F is a finite collection, X_F is said to be a Z^2-**shift of finite type** (Z^2-SFT).

In this section, we introduce a notion of a bipartite code of Z^2-subshifts and prove that any conjugacy of Z^2-subshifts is factorized into a finite number of bipartite codes (Definition 2.2, Theorem 2.1). This is a generalization of the bipartite factorization theorem of Z^1-shifts by Nasu [4].

Let X be a Z^2-shift space. Throughout this paper, we denote the alphabet of X by $A(X)$ and the set of all $m \times n$ blocks appearing in an element in X by $B_{m,n}(X)$. Let B be a finite alphabet. We call any map $f : B_{m,n}(X) \to B$ a **block map**. Let p, q, r, s be non-negative integers with $p+q = m-1$, $r+s = n-1$. We set $Y = \{y = (y_{i,j})_{i,j \in Z} \in B^{Z^2} \mid$ there exists an $x \in X$ such that $y_{i,j} = f(x_{[i-p, i+q] \times [j-r, j+s]})\}$, where $x_{[k,l] \times [m,n]} = (x_{i,j})_{(i,j) \in [k,l] \times [m,n]}$. Then the set Y is a Z^2-shift space. By $f_\infty^{(p,q,r,s)}$, we denote a mapping $X \to Y$ defined by $f_\infty^{(p,q,r,s)}(x) = y$ where $y_{i,j} = f(x_{[i-p, i+q] \times [j-r, j+s]})$, $i,j \in Z$. We call $f_\infty^{(p,q,r,s)}$ a **sliding block code** of vertical memory p, vertical anticipation q, horizontal memory r and horizontal anticipation s (briefly called a (p,q,r,s)-type sliding block code). If $p = q = r = s = 0$, we call $f_\infty^{(0,0,0,0)}$ a 1-block sliding block code and simply denote it by f_∞. A bijective sliding block code is called a **conjugacy**. When a 1-block sliding block code has an inverse which is also a 1-block sliding block code, it is called a **symbolic conjugacy**.

Definition 2.1. *Let X be a Z^2-shift space.*
(1) *The mapping ρ_f (resp. ρ_b): $X \to (B_{1,2}(X))^{Z^2}$ defined by*

$$\rho_f(x)_{i,j \in Z} = (x_{i,j} x_{i,j+1})_{i,j \in Z} \; (resp. \; \rho_b(x) = (x_{i,j-1} x_{i,j})_{i,j \in Z})$$

for $x = (x_{i,j})_{i,j \in Z} \in X$, is called the forward (resp. backward) 2-higher block code of X. The image $\rho_f(X) = \rho_b(X)$ is called the horizontal 2-higher block sytem of X.
(2) *The mapping ρ_u (resp. ρ_d): $X \to (B_{2,1}(X))^{Z^2}$ defined by*

$$\rho_u(x) = \begin{bmatrix} x_{i+1,j} \\ x_{i,j} \end{bmatrix}_{i,j \in Z} (resp.\ \rho_d(x) = \begin{bmatrix} x_{i,j} \\ x_{i-1,j} \end{bmatrix}_{i,j \in Z})$$

for $x = (x_{i,j})_{i,j \in Z} \in X$, is called the upward (resp. downward) 2-higher block code of X. The image $\rho_u(X) = \rho_d(X)$ is called the vertical 2-higher block system of X.

Next we introduce a bipartite code. Let P_i ($i = 1, 2$) be finite sets. We write a point in $P_1 \times P_2$ by ab or sometimes $\begin{smallmatrix} a \\ b \end{smallmatrix}$ where $a \in P_1$, $b \in P_2$.

Definition 2.2. Let $l: A(X) \to P_1 \times P_2$ be a 1-1 map.
(1) The forward (resp. backward) bipartite code induced by l is a map l_f (resp. l_b): $X \to (P_2 \times P_1)^{Z^2}$ defined by

$$l_f(x) = (b_{i,j} a_{i,j+1})_{i,j \in Z} (resp.\ l_b(x) = (b_{i,j-1} a_{i,j})_{i,j \in Z})$$

for $x = (x_{i,j})_{i,j \in Z} \in X$, where $l(x_{i,j}) = a_{i,j} b_{i,j}$.

(2) The upward (resp. downward) bipartite code induced by l is a map l_u (resp. l_d): $X \to (P_2 \times P_1)^{Z^2}$ defined by

$$l_u(x) = \begin{bmatrix} b_{i+1,j} \\ a_{i,j} \end{bmatrix}_{i,j \in Z} (resp.\ l_d(x) = \begin{bmatrix} b_{i,j} \\ a_{i-1,j} \end{bmatrix}_{i,j \in Z})$$

for $x = (x_{i,j})_{i,j \in Z} \in X$, where $l(x_{i,j}) = \begin{smallmatrix} a_{i,j} \\ b_{i,j} \end{smallmatrix}$.

We remark that a bipartite code is a conjugacy and that a higher block code is a bipartite code and that the inverse of a higher block code is also a bipartite code.

Lemma 2.1. Let $\phi = h_\infty^{(0,0,0,0)} = h_\infty: X \to Y$ be a conjugacy and let $\phi^{-1} = k_\infty^{(p,q,r,s)}$. Let $l: A(X) \to A(Y) \times A(X)$ be a 1-1 map defined by $l(a) = h(a)a$ (resp. $l(a) = ah(a)$), $a \in A(X)$. Suppose $s \neq 0$ (resp. $r \neq 0$). Then there exist block maps $H: A(l_f(X)) \to A(\rho_f(Y))$ and $K: B_{p+q+1,\ r+s+1}(\rho_f(Y)) \to A(l_f(X))$ such that $(H_\infty)^{-1} = K_\infty^{(p,q,r,s-1)}$ (resp. $(H_\infty)^{-1} = K_\infty^{(p,q,r-1,s)}$) and such that the following commutative diagram holds:

$$\begin{array}{ccc} X & \xrightarrow{\ l_f\ } & l_f(X) \\ \phi \downarrow & & \downarrow H_\infty \\ Y & \xrightarrow[\ \rho_f\ ]{} & \rho_f(Y) \end{array}$$

Proof. We suppose $f(a) = h(a)a$, $a \in A(X)$. From our assumption, $h_\infty((x_{i,j})_{i,j \in Z}) = (h(x_{i,j}))_{i,j \in Z}$ for $(x_{i,j})_{i,j \in Z} \in X$ and

$$k\begin{pmatrix} h(x_{p+q+1,1}) & \cdots & h(x_{p+q+1,\,r+s+1}) \\ \vdots & & \vdots \\ h(x_{1,1}) & \cdots & h(x_{1,r+s+1}) \end{pmatrix} = x_{q+1,r+1}$$

for $\quad\begin{matrix} x_{p+q+1,1} & \cdots & x_{p+q+1,\,r+s+1} \\ \vdots & & \vdots \\ x_{1,1} & \cdots & x_{1,r+s+1} \end{matrix} \in B_{p+q+1,\,r+s+1}(X)$.

Note that $l_f(X) = \{[x_{i,j}h(x_{i,j+1})]_{i,j \in Z} \mid (x_{ij})_{i,j \in Z}\}$. We define a block map $H: A(l_f(X) \to A(\rho_f(Y))$ by $H(x_{i,j}h(x_{i,j+1})) = h(x_{i,j})h(x_{i,j+1})$, for $x_{i,j}h(x_{i,j+1}) \in A(l_f(X))$. Then, H_∞ is a conjugacy. Define a block map

$$K: B_{p+q+1,\,r+s+1}(\rho_f(Y)) \to A(l_f(X)) \text{ by}$$

$$K\begin{pmatrix} h(x_{p+q+1,1})h(x_{p+q+1,2}) & \cdots & h(x_{p+q+1,\,r+s})h(x_{p+q+1,\,r+s+1}) \\ \vdots & & \vdots \\ h(x_{1,1})h(x_{1,2}) & \cdots & h(x_{1,r+s})h(x_{1,r+s+1}) \end{pmatrix}$$

$= x_{q+1,\,r+1}h(x_{q+1,\,r+2})$. Then, $H_\infty \circ K_\infty^{(p,q,r,s-1)} = id$, $K_\infty^{(p,q,r,s-1)} \circ H_\infty = id$. So, $H_\infty^{-1} = K_\infty^{(p,q,r,s-1)}$. The other case is similarly done and the proof is omitted. Q.E.D.

In Lemma 2.1, if we define a map l by $l(a) = \dfrac{a}{h(a)}$, (*resp.* $l(a) = \dfrac{h(a)}{a}$) and use a upward (resp. downward) higher block code and a bipartite code, we get the conjugacy $K_\infty^{(p-1,\,q,r,s)}$ (*resp.* $K_\infty^{(p,q-1,r,s)}$).

Theorem 2.1. *Any conjugacy of Z^2-shift spaces is factorized into a finite number of bipartite codes.*

Proof. Let $\phi: X \to Y$ be a conjugacy. If necessary, by taking a higher block system, we may and do assume $\phi = h_\infty$ where $\phi^{-1} = k_\infty^{(p,q,r,s)}$. If $s \neq 0$, then by applying Lemma 2.1, we have a commutative diagram in Lemma 2.1. When $s = 0$, do the same things for the other parameters. Repeat this argument, then after a finite number of steps, we have a commutative diagram:

$$X \xrightarrow{\lambda_1} X_1 \xrightarrow{\lambda_2} X_2 \cdots \xrightarrow{\lambda_n} X_n$$
$$\phi \downarrow \quad\quad \downarrow (H_1)_\infty \quad\quad \downarrow (H_2)_\infty \quad\quad \downarrow (H_n)_\infty$$
$$Y \xrightarrow{\rho_1} Y_1 \xrightarrow{\rho_2} Y_2 \cdots \xrightarrow{\rho_n} Y_n$$

where λ_i's are bipartite codes and ρ_i's are higher block codes and $(H_n)_\infty$ is a symbolic conjugacy. Finally note that $h_\infty = \rho_1^{-1} \circ \cdots \circ \rho_n^{-1} \circ (H_n)_\infty \circ \lambda_n \circ \cdots \circ \lambda_1$ and that ρ_i^{-1} are bipartite codes. Thus h_∞ is factorized into a finite number of bipartite codes as above.   Q.E.D.

§3. Conjugacy of Textile Shifts

In this section, we see a conjugacy of textile shifts (see [5]). For this, we introduce a notion of essentially identical isomorphism, bipartite textile systems and a bipartite graph code of textile shifts. As a matter of fact, a bipartite graph code of textile shifts corresponds to a one-step strong shift equivalence for Z^1-topological Markov shifts considered by R. F. Williams [8] and is a central ingredient for the proof of the main Theorem.

Here, let us recall a directed graph and a textile system (see [3], [5]). Let $G = (V(G), E(G))$ be a directed graph, where $V(G)$, $E(G)$ are the vertex set and the edge set. Let s_G and $t_G : E(G) \to V(G)$ are the source map and the target map. For a graph G, we denote the shift space $X(G) = \{(r_i)_{i \in Z} \mid r_i \in E(G), t_G(r_i) = s_G(r_{i+1})\}$. We call it a graph shift [3]. If s_G and t_G are onto, we say that G is nondegenerate. We define a 2-higher block graph and a bipartite graph. Let $B_2(G) = \{ab \mid t_G(a) = s_G(b)\ a, b \in E_G\}$. Let $G^{[2]}$ denote the graph with $E(G^{[2]}) = B_2(G)$, $V(G^{[2]}) = E(G)$ such that the source and target maps are defined by $s_{G^{[2]}}(ab) = a$ and $t_{G^{[2]}}(ab) = b$ for $ab \in E(G^{[2]})$. We call it the 2-higher block graph of G.

Next, a directed graph G is called a **bipartite graph**, if $V(G)$ is decomposed into disjoint sets V_1 and V_2 and if for any edge $a \in E(G)$, $s_G(a) \in V_i(G)$ implies $t_G(a) \in V_j(G)\ i, j = 1, 2,\ i \neq j$. We set $E_i = \{a \in E(G) \mid s_G(a) \in V_i(G)\}\ i = 1, 2$, $E_{12} = \{ab \in B_2(G) \mid a \in E_1,\ b \in E_2\}$, $E_{21} = \{ba \in B_2(G) \mid b \in E_2,\ a \in E_1\}$. Now we denote the bipartite graph G by $(V_1(G), E_1(G), V_2(G), E_2(G))$ and we define the graphs G_{ij} ($i, j = 1, 2, i \neq j$) by $G_{ij} = (V_i(G), E_{ij})$ where the source and target maps are defined by $s_{G_{ij}}(ab) = s_G(a)$ and $t_{G_{ij}}(ab) = t_G(b)$ ($ab \in E_{ij}$).

For directed graphs Γ and G, a **graph homomorphism** $h : \Gamma \to G$ is a pair of maps $h_E : E(\Gamma) \to E(G)$ and $h_V : V(\Gamma) \to V(G)$ such that $s_G(h_E(a)) = h_V(s_\Gamma(a))$ and $t_G(h_E(a)) = h_V(t_\Gamma(a))$ for all edges $a \in E(\Gamma)$. We call

h_E the edge map of h and h_V the vertex map of h. If both h_E and h_V are bijective, h is called a **graph isomorphism** and we say that G and G' are graph isomorphic.

Let Γ and G be directed graphs and p and $q : \Gamma \to G$ be graph homomorphisms. If the mapping $a \in E(\Gamma) \to (p(a), q(a), s_\Gamma(a), t_\Gamma(a))$ is injective, the pair of graph homomorphisms $T = (p, q)$ is called a **textile system** and this condition is called the condition of a textile system. A textile system T induces a Z^2-shift space U_T defined by $U_T = \{(x_{i,j})_{i,j \in Z} | x_{i,j} \in E(\Gamma), t_\Gamma(x_{i,j}) = s_\Gamma(x_{i,j+1})$ and $p(x_{i,j}) = q(x_{i+1,j})\}$. Then it is a Z^2-SFT. We call it a textile shift and denote it by (U_T, σ) or simply by U_T. By $(p(a), q(a), s_\Gamma(a), t_\Gamma(a))$, we mean the tile for $a \in \Gamma$ and call a tile a. Each $x \in U_T$ is considered to be a configuration consisting of tiles $x_{i,j}$ such that $x_{i,j+1}$ is next to $x_{i,j}$ to right $(t_\Gamma(x_{i,j}) = s_\Gamma(x_{i,j+1}))$ and such that $x_{i+1,j}$ is next to $x_{i,j}$ to the top $(p(x_{i,j}) = q(x_{i,j+1}))$.

A textile system $T = (p, q)$ induces the 2-higher block sytem $T^{[2]} = (p^{[2]}, q^{[2]})$ which is a textile system defined by $p^{[2]}$ and $q^{[2]} : \Gamma^{[2]} \to G^{[2]}$, $p^{[2]}(ab) = p(a)p(b)$ and $q^{[2]}(ab) = q(a)q(b)$, $ab \in E(\Gamma^{[2]})$. Also, it induces the dual textile system $T^* = (p^t, q^t)$, which is a textile system defined as follows. Let Γ^t and G^t be the graph defined by $E(\Gamma^t) = E(\Gamma)$, $V(\Gamma^t) = E(G), s_{\Gamma^t} = p_E, t_{\Gamma^t} = q_E, E(G^t) = V(\Gamma), V(G^t) = V(G), s_{G^t} = p_V$, $t_{G^t} = q_V$, and $p^t, q^t : \Gamma^t \to G^t$ be the graph homomorphisms defined by $p_E^t = s_\Gamma, p_V^t = s_G, q_E^t = t_\Gamma, q_V^t = t_G$.

In the definition of a textile system, the graphs Γ, G, Γ^t and G^t are not assumed to be nondegenerate. We say that a textile system T is **standard** if all the graphs of T and T^* are nondegenerate (that is, Γ and G are nondegenerate and the vertex and edge maps of p, q are onto). For a textile system T, we can always construct a standard textile system which provides the same textile shift as U_T (see [5]). So, all textile systems considered in the paper are always assumed to be standard. We say that a tile a is **essential** if $a \in B_{1,1}(U_T)$ and denote by $E(\Gamma)_0$ the set of all essential tiles. If $E(\Gamma)_0$ is not empty, by restricting the graph Γ to the subset $E(\Gamma)_0$ of $E(\Gamma)$, we have the subgraph Γ_0, that is, $E(\Gamma_0) = E(\Gamma)_0$, $V(\Gamma_0) = s_\Gamma(E(\Gamma_0)), s_{\Gamma_0} = s_\Gamma, t_{\Gamma_0} = t_\Gamma$ on $E(\Gamma_0)$. Then Γ_0 is nondegenerate. Also, by restricting the graph homomorphisms p and q to the graph Γ_0, denoting the restrictions by p_0 and q_0 respectively, and letting $G_0 = p_0(\Gamma_0)$, we have a standard textile system $T_0 = (p_0, q_0)$.

Definition 3.1. *We call T_0 the essential part of T and if $\Gamma = \Gamma_0$, then T is called an essential textile system.*

It is known in [7] that Γ_0 is undecidable. In the next example, we show an example of a standard textile system T with no essential tile,

that is, $U_T = \phi$. The existence of such a textile system is pointed out by Nasu [5].

Example 3.1. Consider the following matrices $A(\Gamma)$ and $A(G)$ which present graphs Γ and G.

$$A(\Gamma) = \begin{bmatrix} 0 & \alpha & 0 & \gamma \\ 0 & 0 & \beta & 0 \\ \epsilon & 0 & 0 & 0 \\ 0 & 0 & \delta & 0 \end{bmatrix}, \ A(G) = \begin{bmatrix} 0 & a+c & 0 \\ 0 & 0 & b+d \\ e & 0 & 0 \end{bmatrix}$$

We define the graph homomorphisms p and q by $p(\alpha) = a$, $p(\beta) = b$, $p(\gamma) = c$, $p(\delta) = d$, $p(\epsilon) = e$, $q(\alpha) = a$, $q(\beta) = d$, $q(\gamma) = c$, $q(\delta) = b$, $q(\epsilon) = e$. Now Γ and G are nondegenerate and $p(E(\Gamma)) = q(E(\Gamma)) = E(G)$. We show that U_T is empty. This is because, otherwise let $x \in U_T$ then either $\alpha\beta$ or $\gamma\delta$ appears on the block $x_{0,0} \, x_{0,1} \, x_{0,2} \, x_{0,3}$. Meanwhile $\alpha\beta$ (resp. $\gamma\delta$) can not admit a block $uv \in B_2(G)$ such that $p(\alpha) = u$ and $p(\beta) = v$ (resp. $p(\gamma) = u$ and $p(\delta) = v$), which contradicts $p(x_{0,i}) = q(x_{1,i})$ with $x_{1,i} x_{1,i+1} \in B_2(\Gamma)$.

Definition 3.2. *Let $T = (p,q)$ and $T' = (p',q')$ be textile systems and $f = (f_E, f_V): \Gamma \to \Gamma'$ and $g = (g_E, g_V): G \to G'$ be graph homomorphisms. We say that the pair (f, g) is a textile homomorphism of T onto T' if the following conditions are fulfilled;*
  (1) $p' \circ f_E = g_E \circ p$ and $q' \circ f_E = g_E \circ q$.
  (2) All these maps f_E, g_E, f_V and g_V are surjective.
  If those maps are all injective, then we call (f, g) a textile isomorphism. A 1-block sliding block code $x \in U_T \to x' \in U_{T'}$ is induced from a textile homomorphism (f, g), by setting, $x'_{i,j} = f_E(x_{i,j})$. We denote it by f_∞ instead of $(f_E)_\infty$ and call it a 1-block code induced by a textile homomorphism (f, g). For textile systems S and S', if there exists a textile isomorphism (f, g) between S_0 and S'_0, we say that S and S' are **essentially identical** *and denote $S \approx S'$, and that (f, g) is an* **essentially identical isomorphism** *of S and S'.*

We are ready to define a bipartite textile system and a bipartite graph code.

Definition 3.3. *Let Γ and G be bipartite graphs and p and q be graph homomorphisms from Γ into G. We suppose the following three conditions (bipartite textile condition);*
  (1) The restrictions p_i and q_i of p and q to $E_i(\Gamma) i = 1, 2$ satisfy $p_i(E_i(\Gamma)) \subset E_i(G)$ and $q_i(E_i(\Gamma)) \subset E_i(G)$.

(2) *Both the mappings*

$$\alpha \in E_i(\Gamma) \to (p_i(\alpha),\ q_i(\alpha),\ s_\Gamma(\alpha),\ t_\Gamma(\alpha)),\ (i = 1, 2)$$

are injective.

(3) *Both the mappings*

$$\alpha\beta \in E_{12}(\Gamma) \to (p_1(\alpha),\ p_2(\beta),\ q_1(\alpha),\ q_2(\beta),\ s_\Gamma(\alpha),\ t_\Gamma(\beta))\ and$$
$$\beta\alpha \in E_{21}(\Gamma) \to (p_2(\beta),\ p_1(\alpha),\ q_2(\beta),\ q_1(\alpha),\ s_\Gamma(\beta),\ t_\Gamma(\alpha))$$

are injective.

In this case, we say that the textile system $T = (p,q)$ is a **bipartite textile system** and denote the textile systems $(p_1 \otimes p_2,\ q_1 \otimes q_2)$ and $(p_2 \otimes p_1,\ q_2 \otimes q_1)$ by T_{12} and T_{21}. Here $p_1 \otimes p_2(\alpha\beta) = p_1(\alpha)p_2(\beta)$, $\alpha\beta \in E_{12}(\Gamma)$, $p_2 \otimes p_1(\beta\alpha) = p_2(\beta)p_1(\alpha)$, $\beta\alpha \in E_{21}(\Gamma)$. The others $q_1 \otimes q_2$ and $q_2 \otimes q_1$ are similarly understood. We note that $U_{T_k} = \{(r_{i,j}r_{i,j+1})_{i,j \in Z}\ |\ r = (r_{i,j})_{i,j \in Z} \in U_T$ and $s_\Gamma(r_{i,j}) \in V(\Gamma_k)\}$ $(k = 12, 21)$.

The conjugacy ϕ_f (resp. ϕ_b): $U_{T_{12}} \to U_{T_{21}}$ defined by

$$\phi_f((r_{i,j}r_{i,j+1})_{i,j \in Z}) = (r_{i,j+1}r_{i,j+2})_{i,j \in Z}$$

$$(resp.\ \phi_b((r_{i,j}r_{i,j+1})_{i,j \in Z}) = (r_{i,j-1}r_{i,j})_{i,j \in Z}$$

is called the forward (resp. backward) **bipartite graph code** *induced by the bipartite textile system T. We note that the inverse of a forward (resp. backward) bipartite graph code is a backward (resp. forward) bipartite graph code.*

If the dual textile system $T^* = (p^t, q^t)$ of a textile system T is a bipartite textile system, the conjugacy ϕ_u (resp. ϕ_d): $U_{((T^*)_{12})^*} \to U_{((T^*)_{21})^*}$ is analogously defined. We call it the upward (resp. downward) bipartite graph code. We note that the inverse of a upward (resp. downward) bipartite graph code is a downward (resp. upward) bipartite graph code.

Definition 3.4. *Textile systems T and T' are* **1-step bipartitely related** $T \sim T'$ *if there exists a bipartite textile system S such that $T \approx S_{12}$ and $T' \approx S_{21}$ or $T^* \approx S_{12}$ and $T'^* \approx S_{21}$. Moreover, T and T' are* **n-step bipartitely related** *if there are $n-1$ textile systems $S_i (i = 1, 2, \cdots, n-1)$ satisfying $T \sim S_1 \sim \cdots \sim S_{n-1} \sim T'$. T and T' are n-step bipartitely related for some $n \geq 1$, we say they are* **bipartitely related.**

As a warm-up for proving Theorem 3.1, we show

Lemma 3.1. *For any textile system T, T and $T^{[2]}$ are 1-step bipartitely related.*

Proof. We set $E_1(\Gamma) = \{s_\Gamma(\alpha)\alpha \mid \alpha \in E(\Gamma)\}$, $E_2(\Gamma) = E(\Gamma)$, $V_1(\Gamma) = V(\Gamma)$ and $V_2(\Gamma) = \{s_\Gamma(\alpha)\alpha \mid \alpha \in E(\Gamma)\}$. We define a source map s and a target map t by letting $s(s_\Gamma(\alpha)\alpha) = s_\Gamma(\alpha)$, $t(s_\Gamma(\alpha)\alpha) = s_\Gamma(\alpha)\alpha$ for $s_\Gamma(\alpha)\alpha \in E_1(\Gamma)$ and $s(\alpha) = s_\Gamma(\alpha)\alpha$, $t(\alpha) = t_\Gamma(\alpha)$, for $\alpha \in E_2(\Gamma)$. Thus we have a bipartite graph $(V_1(\Gamma), E_1(\Gamma), V_2(\Gamma), E_2(\Gamma))$. Similarly G induces a bipartite graph $(V_1(G), E_1(G), V_2(G), E_2(G))$. The graph homomorphisms $p : \Gamma \to G$ is extended to a graph homomorphisms between the bipartite graphs;

$$p_1 : s_\Gamma(\alpha)\alpha \in E_1(\Gamma) \to s_G(p(\alpha))p(\alpha) \in E_1(G),$$
$$p_2 : \alpha \in E_2(\Gamma) \to p(\alpha) \in E_2(G).$$

Similarly so does q. Then it is easily seen that these p_1, p_2, q_1 and q_2 satisfy the bipartite textile condition. By letting S be the bipartite textile system defined by p_1, p_2, q_1 and q_2, we have $S_{12} \approx T$ and $S_{21} \approx T^{[2]}$.

<div align="right">Q.E.D.</div>

In Lemma 3.1, if we consider the dual textile system T^*, we get $T \sim ((T^*)^{[2]})^*$.

As seen in Theorem 2.1, bipartite code is a fundamental tool for conjugacy of Z^2-shifts. If a conjugacy of textile shifts is concerned, then the corresponding bipartite codes come up as bipartite graph codes as shown in the following. We are going to establish a textile shift analogue of Lemma 2.1.

Proposition 3.1. *Let $T = (p, q)$, $T' = (p', q')$ be textile systems and suppose that there exists a conjugacy $h_\infty = h_\infty^{(0,0,0,0)} : U_T \to U_{T'}$ whose inverse $(h_\infty)^{-1}$ is of (p, q, r, s)-type, where $p \geq 0, q \geq 0, r \geq 0, s > 0$. Then there exist a textile system T_1 and a conjugacy $H_\infty = H_\infty^{(0,0,0,0)} : U_{T_1} \to U_{(((T'^*)^{[2]})^*)^{[2]}}$ satisfying the following properties:*
  (1) $((T^*)^{[2]})^* \sim T_1$ *(1-step bipartite relation)*
  (2) *The following commutative diagram holds up to essentially identical isomorphisms*

$$
\begin{array}{ccccc}
U_T & \xrightarrow{\rho_u} & U_{((T^*)^{[2]})^*} & \xrightarrow{\phi_f} & U_{T_1} \\
{\scriptstyle h_\infty}\downarrow & & & & \downarrow{\scriptstyle H_\infty} \\
U_{T'} & \xrightarrow{\rho_u} & U_{((T'^*)^{[2]})^*} & \xrightarrow{\rho_f} & U_{(((T'^*)^{[2]})^*)^{[2]}}
\end{array}
$$

where the inverse $(H_\infty)^{-1}$ is of $(p, q, r, s-1)$-type.

Proof. We begin with the proof of Lemma 2.1 for $U_{((T^*)^{[2]})_*}$ and $U_{((T'^*)^{[2]})_*}$ instead of X and Y. The reason why we do not start from U_T (and $U_{T'}$) is that we must construct bipartite textile system S so that

$$((T^*)^{[2]})^* \sim S_{21}$$

and that the inverse of the conjugacy from $U_{((T^*)^{[2]})_*}$ to $U_{((T'^*)^{[2]})_*}$ which is naturally induced from h_∞ is of the same type as $(h_\infty)^{-1}$.

Firstly we construct graphs $\overline{\Gamma}$ and $\overline{G}$ and graph homomorphisms $\overline{p}$, $\overline{q} : \overline{\Gamma} \to \overline{G}$. Set

$$V(\overline{\Gamma}) = \left\{ \begin{matrix} s_\Gamma(\alpha) \\ s_\Gamma(\beta) \end{matrix} \,\Big|\, \begin{matrix} \alpha \\ \beta \end{matrix} \in B_{2,1}(U_T) \right\}, \quad E(\overline{\Gamma}) = B_{2,1}(U_T)$$

and define

$$s_{\overline{\Gamma}}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{matrix} s_\Gamma(\alpha) \\ s_\Gamma(\beta) \end{matrix}, \quad t_{\overline{\Gamma}}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{matrix} t_\Gamma(\alpha) \\ t_\Gamma(\beta) \end{matrix}, \quad \begin{matrix} \alpha \\ \beta \end{matrix} \in E(\overline{\Gamma}).$$

We also set

$$V(\overline{G}) = \{ s_\Gamma(\alpha) \mid \alpha \in B_{1,1}(U_T) \}, \quad E(\overline{G}) = B_{1,1}(U_T)$$

and define

$$s_{\overline{G}}(\alpha) = s_\Gamma(\alpha), \quad t_{\overline{G}}(\alpha) = t_\Gamma(\alpha), \quad \alpha \in E(\overline{G}).$$

Likewise $\overline{\Gamma}'$ and $\overline{G}'$ are defined. Since the 2-higher block system $((T^*)^{[2]})^*$ was taken at the beginning instead of T, the following graph homomorphisms $\overline{p}$, $\overline{q} : \overline{\Gamma} \to \overline{G}$ can be defined:

$$\overline{p}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha, \quad \overline{q}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \beta, \quad \begin{matrix} \alpha \\ \beta \end{matrix} \in E(\overline{\Gamma}).$$

Corresponding to the 1-1 mapping l in Lemma 2.1, we consider the injection $k : E(\overline{\Gamma}) \to E(\overline{\Gamma}') \times E(\overline{\Gamma})$ defined by

$$k\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{matrix} h(\alpha)\alpha \\ h(\beta)\beta \end{matrix}, \quad \begin{matrix} \alpha \\ \beta \end{matrix} \in E(\overline{\Gamma})$$

where the right hand side means the pair of $\begin{matrix} h(\alpha) \\ h(\beta) \end{matrix} \in E(\overline{\Gamma}')$ and $\begin{matrix} \alpha \\ \beta \end{matrix} \in E(\overline{\Gamma})$.

In order to define an equivalence relation of edges of $\overline{\Gamma}$ and $\overline{G}$, we rename in 1-1 fashion each edge $\begin{matrix} \alpha \\ \beta \end{matrix}$ of $\overline{\Gamma}$ and each edge α of $\overline{G}$, by $\begin{matrix} h(\alpha)\alpha \\ h(\beta)\beta \end{matrix}$

and $h(\alpha)\alpha$ respectively.

Secondary we will construct a bipartite textile system $S = (\tilde{p}, \tilde{q})$ satisfying $((T^*)^{[2]})^* \sim S_{21}$. Edges $\dfrac{h(\alpha)\alpha}{h(\beta)\beta}$ and $\dfrac{h(\gamma)\gamma}{h(\delta)\delta}$ are said to be equivalent if $\dfrac{s_\Gamma(\alpha)}{s_\Gamma(\beta)} = \dfrac{s_\Gamma(\gamma)}{s_\Gamma(\delta)}$ and $\dfrac{h(\alpha)}{h(\beta)} = \dfrac{h(\gamma)}{h(\delta)}$. We note that such an equivalence relation is considered in [6]. The equivalence class of $\dfrac{h(\alpha)\alpha}{h(\beta)\beta}$ is written by $\dfrac{s_\Gamma(\alpha)h(\alpha)}{s_\Gamma(\beta)h(\beta)}$. We set

$$E_1 = \left\{ \dfrac{s_\Gamma(\alpha)h(\alpha)}{s_\Gamma(\beta)h(\beta)} \mid \dfrac{\alpha}{\beta} \in B_{2,1}(U_T) \right\}, \quad E_2 = E(\overline{\Gamma}), \quad V_1 = V(\overline{\Gamma}), \text{ and } V_2 = E_1.$$

We define a bipartite graph $\widetilde{\Gamma}$ with $V(\widetilde{\Gamma}) = V_1 \cup V_2$ and $E(\widetilde{\Gamma}) = E_1 \cup E_2$. When $\dfrac{s_\Gamma(\alpha)h(\alpha)}{s_\Gamma(\beta)h(\beta)} \in E_1$, the source and target vertices are $s_{\widetilde{\Gamma}}\left(\dfrac{s_\Gamma(\alpha)h(\alpha)}{s_\Gamma(\beta)h(\beta)}\right)$

$= \dfrac{s_\Gamma(\alpha)}{s_\Gamma(\beta)}$ and $t_{\widetilde{\Gamma}}\left(\dfrac{s_\Gamma(\alpha)h(\alpha)}{s_\Gamma(\beta)h(\beta)}\right) = \dfrac{s_\Gamma(\alpha)h(\alpha)}{s_\Gamma(\beta)h(\beta)}$. For $\dfrac{\alpha}{\beta} \in E_2$, the source and

target vertices are $s_{\widetilde{\Gamma}}\left(\dfrac{\alpha}{\beta}\right) = \dfrac{s_\Gamma(\alpha)h(\alpha)}{s_\Gamma(\beta)h(\beta)}$ and $t_{\widetilde{\Gamma}}\left(\dfrac{\alpha}{\beta}\right) = \dfrac{t_\Gamma(\alpha)}{t_\Gamma(\beta)}$. Edges $h(\alpha)\alpha$

and $h(\gamma)\gamma$ of $\overline{G}$ are said to be equivalent if $s_\Gamma(\alpha) = s_\Gamma(\gamma)$ and $h(\alpha) = h(\gamma)$. The equivalence class of $h(\alpha)\alpha$ is written by $s_\Gamma(\alpha)h(\alpha)$. We set $E_1' = \{s_\Gamma(\alpha)h(\alpha) \mid \alpha \in E(\overline{G})\}$, $E_2' = E(\overline{G})$, $V_1' = V(\overline{\Gamma})$ and $V_2' = E_1'$. We define a bipartite graph $\widetilde{G}$ with $V(\widetilde{G}) = V_1' \cup V_2'$ and $E(\widetilde{G}) = E_1' \cup E_2'$. When $s_\Gamma(\alpha)h(\alpha) \in E_1'$, the source and target vertices are $s_{\widetilde{G}}(s_\Gamma(\alpha)h(\alpha))$ $= s_\Gamma(\alpha)$ and $t_{\widetilde{G}}(s_\Gamma(\alpha)h(\alpha)) = s_\Gamma(\alpha)h(\alpha)$. For an edge $\alpha \in E_2'$, the source and target vertices are $s_{\widetilde{G}}(\alpha) = s_\Gamma(\alpha)h(\alpha)$ and $t_{\widetilde{G}}(\alpha) = t_\Gamma(\alpha)$. Figure 1 describes the above construction of the graphs $\widetilde{\Gamma}$ and $\widetilde{G}$.

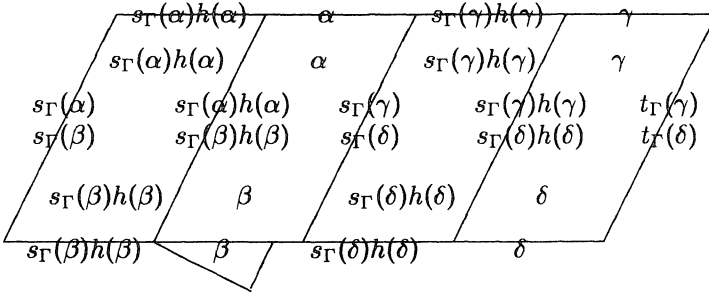**Figure 1**

We define graph homomorphisms $\tilde{p}$ and $\tilde{q}: \tilde{\Gamma} \to \tilde{G}$ by $\tilde{p}\begin{pmatrix} s_\Gamma(\alpha)h(\alpha) \\ s_\Gamma(\beta)h(\beta) \end{pmatrix}$

$= s_\Gamma(\alpha)h(\alpha)$ and $\tilde{q}\begin{pmatrix} s_\Gamma(\alpha)h(\alpha) \\ s_\Gamma(\beta)h(\beta) \end{pmatrix} = s_\Gamma(\beta)h(\beta)$, $\begin{matrix} s_\Gamma(\alpha)h(\alpha) \\ s_\Gamma(\beta)h(\beta) \end{matrix} \in E_1$ and $\tilde{p}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$= \alpha$ and $\tilde{q}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \beta$, $\begin{matrix} \alpha \\ \beta \end{matrix} \in E_2$. As usual, we let $(\tilde{p})_i$, $(\tilde{q})_i$ be the restrictions

of $\tilde{p}$ and $\tilde{q}$ to E_i, $i = 1, 2$. Then, we easily see that these $(\tilde{p})_i$, $(\tilde{q})_i$, $i = 1, 2$ satisfy the bipartite textile condition. Thus we have the bipartite textile system $S = (\tilde{p}, \tilde{q})$.

Thirdly we show an interesting fact that $((T^*)^{[2]})^* \approx S_{12}$. For this we consider the graph homomorphisms $f = (f_V, f_E): (\tilde{\Gamma}_{12})_0 \to (\tilde{\Gamma})_0$ and $g = (g_V, g_E): (\tilde{G}_{12})_0 \to (\overline{G})_0$ defined by

$$f_E\begin{pmatrix} s_\Gamma(\alpha)h(\alpha)\alpha \\ s_\Gamma(\beta)h(\beta)\beta \end{pmatrix} = \frac{\alpha}{\beta}, \; \frac{s_\Gamma(\alpha)h(\alpha)\alpha}{s_\Gamma(\beta)h(\beta)\beta} \in E((\tilde{\Gamma}_{12})_0),$$

$$f_V\begin{pmatrix} s_\Gamma(\alpha) \\ s_\Gamma(\beta) \end{pmatrix} = \frac{s_\Gamma(\alpha)}{s_\Gamma(\beta)}, \; \frac{s_\Gamma(\alpha)}{s_\Gamma(\beta)} \in V((\tilde{\Gamma}_{12})_0), \text{ and}$$

$$g_E(s_\Gamma(\alpha)h(\alpha)\alpha) = \alpha, \; s_\Gamma(\alpha)h(\alpha)\alpha \in E((\overline{G})_0) \text{ and}$$

$$g_V(s_\Gamma(\alpha)) = s_\Gamma(\alpha), \; s_\Gamma(\alpha) \in V((\overline{G})_0).$$

Then f and g give graph homomorphisms. This is because,

$$s_{\overline{\Gamma}} \circ f_E\begin{pmatrix} s_\Gamma(\alpha)h(\alpha)\alpha \\ s_\Gamma(\beta)h(\beta)\beta \end{pmatrix} = s_{\overline{\Gamma}}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{s_\Gamma(\alpha)}{s_\Gamma(\beta)} \text{ and}$$

$$f_V \circ s_{\tilde{\Gamma}_{12}}\begin{pmatrix} s_\Gamma(\alpha)h(\alpha)\alpha \\ s_\Gamma(\beta)h(\beta)\beta \end{pmatrix} = f_V\begin{pmatrix} s_\Gamma(\alpha) \\ s_\Gamma(\beta) \end{pmatrix} = \frac{s_\Gamma(\alpha)}{s_\Gamma(\beta)}.$$

So, $s_{\overline{\Gamma}} \circ f_E = f_V \circ s_{\tilde{\Gamma}_{12}}$. Likewise, we have $t_{\overline{\Gamma}} \circ f_E = f_V \circ t_{\tilde{\Gamma}_{12}}$. Similarly, g is a graph homomorphism. Moreover, f and g are bijective and satisfy

$$g_E\left((\tilde{p})_1 \otimes (\tilde{p})_2 \begin{pmatrix} s_\Gamma(\alpha)h(\alpha)\alpha \\ s_\Gamma(\beta)h(\beta)\beta \end{pmatrix} \right) = g_E(s_\Gamma(\alpha)h(\alpha)\alpha) = \alpha = \overline{p}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= \overline{p}\left(f_E\begin{pmatrix} s_\Gamma(\alpha)h(\alpha)\alpha \\ s_\Gamma(\beta)h(\beta)\beta \end{pmatrix} \right).$$

Likewise, we have $g_E \circ (\tilde{q})_1 \otimes (\tilde{q})_2 = \overline{q} \circ f_E$. Hence, S_{12} and $((T^*)^{[2]})^*$ are essentially identical and f_∞ is the 1-block sliding block code induced by the textile isomorphism (f, g).

Finally we lift $h_\infty : U_T \to U_{T'}$ to a conjugacy $(h_1)_\infty : U_{((T^*)^{[2]})^*} \to$ $U_{((T'^*)^{[2]})^*}$ by defining a block map $h_1 : B_{2,1}(U_T) \to B_{2,1}(U_{T'})$ by $h_1\begin{pmatrix}\alpha\\\beta\end{pmatrix}$ $=\dfrac{h(\alpha)}{h(\beta)}$, for $\dfrac{\alpha}{\beta} \in B_{2,1}(U_T)$. Then we have a commutative diagram:

$$
\begin{array}{ccc}
U_T & \xrightarrow{\ \rho_u\ } & U_{((T^*)^{[2]})^*} \\
h_\infty \downarrow & & \downarrow (h_1)_\infty \\
U_{T'} & \xrightarrow{\ \rho_u\ } & U_{((T'^*)^{[2]})^*}
\end{array}
$$

We note that $((h_1)_\infty)^{-1}$ is of $(p,\ q,\ r,\ s)$-type. Now we apply Lemma 2.1 for $(h_1)_\infty = (h_1)_\infty^{(0,0,0,0)} : U_{((T^*)^{[2]})^*} \to U_{((T'^*)^{[2]})^*}$ instead of $h_\infty = h_\infty^{(0,0,0,0)} :$ $X \to Y$ in the lemma. As to the mapping H in the lemma, we let

$$
H\begin{pmatrix} \alpha s_\Gamma(\gamma)h(\gamma) \\ \beta s_\Gamma(\delta)h(\delta) \end{pmatrix} = \frac{h(\alpha)h(\gamma)}{h(\beta)h(\delta)}, \quad \frac{\alpha s_\Gamma(\gamma)h(\gamma)}{\beta s_\Gamma(\delta)h(\delta)} \in E_{21}(\widetilde{\Gamma}).
$$

Then, we have a commutative diagram:

$$
\begin{array}{ccccc}
U_{((T^*)^{[2]})^*} & \xrightarrow{\ f_\infty\ } & U_{S_{12}} & \xrightarrow{\ \phi_f\ } & U_{S_{21}} \\
(h_1)_\infty \downarrow & & & & \downarrow H_\infty \\
U_{((T'^*)^{[2]})^*} & & \xrightarrow{\qquad\qquad \rho_f \qquad\qquad} & & U_{(((T'^*)^{[2]})^*)^{[2]}}
\end{array}
$$

We also see from the proof of Lemma 2.1 that H_∞ is a conjugacy of $(0,\ 0,\ 0,\ 0)$-type whose inverse is of $(p,\ q,\ r,\ s-1)$-type. In order to emphasize the essentially identical relation of $((T^*)^{[2]})^*$ and S_{12}, in the above diagram we write the corresponding symbolic conjugacy by f_∞, though. But for simplicity we do not write each symbolic conjugacy appearing in the other part (say, along the line under the map ρ_f). We note that ϕ_f is corresponds to l_f in Lemma 2.1. By Lemma 3.1, each 2-higher block system of textile system is 1-step bipartitely related to the textile system. Thus, the proof of the proposition is complete.

<div align="right">Q.E.D.</div>

With minor modifications in the statement, Proposition 3.1 is true when any one of $p,\ q,\ r,\ s$ is positive, because the proof is symmetric for these parameters.

The following lemma is a special case of the main theorem but is useful for completing the proof of the main theorem.

Lemma 3.2. *Let* T *and* T' *be textile systems. If* U_T *and* $U_{T'}$ *are*

conjugate under a symbolic conjugacy, then $(((T^*)^{[2]})^*)^{[2]} \approx (((T'^*)^{[2]})^*)^{[2]}$.

Proof. The set $B_{2,2}(U_T)$ is considered to be an edge set of a graph by putting for $\dfrac{\alpha\gamma}{\beta\delta} \in B_{2,2}(U_T)$, $s_v\!\begin{pmatrix}\alpha\gamma\\\beta\delta\end{pmatrix} = \dfrac{\alpha}{\beta}$, $t_v\!\begin{pmatrix}\alpha\gamma\\\beta\delta\end{pmatrix} = \dfrac{\gamma}{\delta}$. Thus we have a graph $\Gamma_v = (B_{2,1}(U_T), B_{2,2}(U_T),\ s_v,\ t_v)$. Similarly a graph $G_h = (B_{1,1}(U_T), B_{1,2}(U_T),\ s_h,\ t_h)$ is obtained, where $s_h(\alpha\gamma) = \alpha$, $t_h(\alpha\gamma) = \gamma$. Then the graph homomorphisms $\bar{p},\ \bar{q} : \Gamma_v \to G_h$ are defined by $\bar{p}\!\begin{pmatrix}\alpha\gamma\\\beta\delta\end{pmatrix} = \alpha\gamma$ and $\bar{q}\!\begin{pmatrix}\alpha\gamma\\\beta\delta\end{pmatrix} = \beta\delta$. Likewise for T', we have $\gamma'_v,\ G'_h,\ \bar{p}',\ \bar{q}'$. Then we immediately see that $(((T^*)^{[2]})^*)^{[2]} \approx (\bar{p},\ \bar{q})$ and $(((T'^*)^{[2]})^*)^{[2]} \approx (\bar{p}',\ \bar{q}')$. Now let $\theta_\infty : U_T \to U_{T'}$ be a symbolic conjugacy. Then θ_∞ naturally induces a textile isomorphism $(f,\ g)$ of the textile systems $(\bar{p},\ \bar{q})$ and $(\bar{p}',\ \bar{q}')$ by letting $f_E\!\begin{pmatrix}\alpha\gamma\\\beta\delta\end{pmatrix} = \dfrac{\theta(\alpha)\theta(\gamma)}{\theta(\beta)\theta(\delta)}$, $f_V\!\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \dfrac{\theta(\alpha)}{\theta(\beta)}$ and

$g_E(\alpha\gamma) = \theta(\alpha)\theta(\gamma),\ g_V(\alpha) = \theta(\alpha)$. $\qquad\qquad$ Q.E.D.

Now we are ready to prove the main theorem.

Theorem 3.1. *Let $T = (p, q)$, $T' = (p', q')$ be textile systems. Suppose U_T and $U_{T'}$ are conjugate under a conjugacy ϕ. Then, T and T' are bipartitely related and ϕ is a composition of the corresponding bipartite graph codes and the symbolic conjugacies arising from essentially identical isomorphisms.*

Proof. If necessary, by taking higher block systems, we may and do assume that ϕ is a conjugacy h_∞, whose inverse is of (p, q, r, s)-type for some $p, q, r, s \geq 0$. Here we may assume that at least one of p, q, r, s is positive. Otherwise, we can apply Lemma 3.2 to get the theorem. Then by applying repeatedly Proposition 3.1, we have a finite number of textile sytems $S_1, \cdots, S_d, S'_1, \cdots, S'_d$, satisfying
  (1) $T \sim S_1 \sim \cdots \sim S_d$, $T' \sim S'_1 \sim \cdots \sim S'_d$ and
  (2) the following commutative diagram holds:

$$U_T \xrightarrow{\phi_1} U_{S_1} \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_d} U_{S_d}$$

$$\phi \downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow H_\infty$$

$$U_{T'} \xrightarrow{\phi'_1} U_{S'_1} \xrightarrow{\phi'_2} \cdots \xrightarrow{\phi'_d} U_{S'_d}$$

where H_∞ is a symbolic conjugacy and ϕ_i and ϕ'_i are corresponding bipartite graph codes. Here, as usual we omit to write the symbolic conjugacies arising from each essentially identical isomorphisms appearing the line under ϕ_i and ϕ'_i, $i = 1, \cdots d$. Finally if we apply Lemma 3.2 for the symbolic conjugacy H_∞, the proof of the theorem is complete.

<div align="right">Q.E.D.</div>

§ 4. Z^2-shifts of Finite Type

Let (X_F, σ) be a Z^2-SFT. It is known that any Z^1-SFT is conjugate with some Z^1-topological Markov shift [3]. In this section, we prove that any Z^2-SFT is conjugate with some textile shift.

Theorem 4.1. *Any Z^2-SFT (X_F, σ) is conjugate with some textile shift.*

Proof. By taking a higher block system, we may assume that F is a set of 2×2 blocks. Firstly we define a graph Γ. The vertex and edge sets $V(\Gamma)$ and $E(\Gamma)$ are defined by

$$V(\Gamma) = \left\{ \begin{matrix} a_{2,1} \\ a_{1,1} \end{matrix} \in B_{2,1}(X_F) \mid \exists b_1, b_2 \in A(X_F) \text{s.t.} \begin{matrix} a_{2,1} b_2 \\ a_{1,1} b_1 \end{matrix} \notin F \right\} \text{ and}$$

$$E(\Gamma) = B_{2,2}(X_F).$$

The source and target maps s_Γ and t_Γ are defined by $s_\Gamma \begin{pmatrix} a_{2,1} a_{2,2} \\ a_{1,1} a_{1,2} \end{pmatrix} = \begin{matrix} a_{2,1} \\ a_{1,1} \end{matrix}$ and $t_\Gamma \begin{pmatrix} a_{2,1} a_{2,2} \\ a_{1,1} a_{1,2} \end{pmatrix} = \begin{matrix} a_{2,2} \\ a_{1,2} \end{matrix}$. Next we define a graph G. The vertex and edge sets $V(G)$ and $E(G)$ are defined by

$$V(G) = \left\{ a_{11} \in B_{1,1}(X_F) \mid \exists b_1, b_2, c_1 \in A(X_F) \text{s.t.} \begin{matrix} c_1 b_2 \\ a_{1,1} b_1 \end{matrix} \notin F \right\} \text{ and}$$

$$E(G) = \left\{ a_{1,1} a_{1,2} \in B_{1,2}(X_F) \mid \exists c_1, c_2 \in A(X_F) \text{s.t.} \begin{matrix} c_1 c_2 \\ a_{1,1} a_{1,2} \end{matrix} \notin F \right\}.$$

The source and target maps of s_G and t_G are defined by $s_G(a_{1,1} a_{1,2}) = a_{1,1}$ and $t_G(a_{1,1} a_{1,2}) = a_{1,2}$. We define graph homomorphisms p and $q : \Gamma \to G$

by $p_V\begin{pmatrix} a_{2,1} \\ a_{1,1} \end{pmatrix} = a_{2,1}$, $p_E\begin{pmatrix} a_{2,1}a_{2,2} \\ a_{1,1}a_{1,2} \end{pmatrix} = a_{2,1}a_{2,2}$, $q_V\begin{pmatrix} a_{2,1} \\ a_{1,1} \end{pmatrix} = a_{1,1}$ and $q_E\begin{pmatrix} a_{2,1}a_{2,2} \\ a_{1,1}a_{1,2} \end{pmatrix}$

$= a_{1,1}a_{1,2}$ for $\begin{matrix} a_{2,1} \\ a_{1,1} \end{matrix} \in V(\Gamma)$, $\begin{matrix} a_{2,1}a_{2,2} \\ a_{1,1}a_{1,2} \end{matrix} \in E(\Gamma)$. Thus we immediately see that the textile system $T = (p, q)$ satisfies $U_T = \rho_u \circ \rho_f(X_F)$. Q.E.D.

In the next example, we will see that the 2-higher block system of the three dots model is a textile shift.

Example 4.1. Let $A = \{0, 1\}$, and X be the SFT consisting of all $(x_{i,j})_{i,j \in z} \in A^{Z^2}$ is satisfying $x_{i,j} + x_{i+1,j} + x_{i,j+1} \equiv 0 \pmod 2$ for all $i, j \in Z$. This model is introduced by Ledrappier [2] and called the three dots model.

The textile system which we construct has $\rho_u \circ \rho_f(X)$. It is as follows. Firstly we define a graph Γ. The vertex and edge set of Γ are given by

$$V(\Gamma) = \left\{ \begin{matrix} 0 \\ 0 \end{matrix}, \begin{matrix} 0 \\ 1 \end{matrix}, \begin{matrix} 1 \\ 0 \end{matrix}, \begin{matrix} 1 \\ 1 \end{matrix} \right\} \text{ and}$$

$$E(\Gamma) = \left\{ \begin{matrix} 00 \\ 00 \end{matrix}, \begin{matrix} 01 \\ 00 \end{matrix}, \begin{matrix} 10 \\ 01 \end{matrix}, \begin{matrix} 11 \\ 01 \end{matrix}, \begin{matrix} 10 \\ 10 \end{matrix}, \begin{matrix} 11 \\ 10 \end{matrix}, \begin{matrix} 00 \\ 11 \end{matrix}, \begin{matrix} 01 \\ 11 \end{matrix} \right\}.$$

The source and target maps s_Γ and t_Γ are defined by $s_\Gamma\begin{pmatrix} ab \\ cd \end{pmatrix} = \begin{matrix} a \\ c \end{matrix}$ and $t_\Gamma\begin{pmatrix} ab \\ cd \end{pmatrix} = \begin{matrix} b \\ d \end{matrix}$ for $\begin{matrix} ab \\ cd \end{matrix} \in E(\Gamma)$. Next we define the graph G. The vertex and edge set of G are given by $V(G) = \{0, 1\}$, and $E(G) = \{00, 10, 01, 11\}$. The source and target maps s_G and t_G are defined by $s_G(ab) = a$ and $t_G(ab) = b$ for $ab \in E(G)$. The graph homomorphisms p and $q : \Gamma \to G$ are $p_E\begin{pmatrix} ab \\ cd \end{pmatrix} = ab$, $p_V\begin{pmatrix} a \\ c \end{pmatrix} = a$, $q_E\begin{pmatrix} ab \\ cd \end{pmatrix} = cd$ and $q_V\begin{pmatrix} a \\ c \end{pmatrix} = c$.

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