Publ. RIMS, Kyoto Univ.
36 (2000), 253–268

Schur Products and Module Maps on $B(\mathscr{H})$

By

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Abstract

We describe the isometry which induces a Schur product on $B(\mathscr{H})$ and characterize Schur product maps as completely bounded module maps from $K(\mathscr{H})$ to $B(\mathscr{H})$.

§ 1. Introduction

Let $\mathscr{H}$ be a complex Hilbert space, $B(\mathscr{H})$ the set of all bounded linear operators on $\mathscr{H}$, and $K(\mathscr{H})$ the set of all compact operators on $\mathscr{H}$. The notion of Schur products on $B(\mathscr{H})$ has been studied in several approaches. In this paper, we study an abstract characterization of the Schur products, and the module map property of associated Schur product maps.

For an orthonormal basis $\Xi = \{\xi_i\}$, we define an isometry $V = V_\Xi$ from $\mathscr{H}$ to $\mathscr{H} \otimes \mathscr{H}$ by the relation

$$V\xi_i = \xi_i \otimes \xi_i \qquad \text{for all } \xi_i \in \Xi.$$

Using this isometry, we define the Schur product $x \circ_\Xi y$ of $x, y \in B(\mathscr{H})$ relative to Ξ as follows:

$$x \circ_\Xi y = V^*(x \otimes y)V \in B(\mathscr{H}).$$

In this setting, V satisfies two conditions:

(1) $V^*(x \otimes 1)V = V^*(1 \otimes x)V$ for all $x \in B(\mathscr{H})$,

(2) $V^*(. \otimes 1)V$ is a projection of norm one from $B(\mathscr{H})$ to a $*$-subalgebra in $B(\mathscr{H})$.

Communicated by T. Kawai, August 18, 1999. Revised October 12, 1999 and December 2, 1999.

1991 Mathematics Subject Classification(s): 46L05, 46L10, 47D15, 47D25

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One of the aims of this paper is to find a necessary and sufficient condition for an isometry V from $\mathscr{H}$ to $\mathscr{H} \otimes \mathscr{H}$, to induce a Schur product on $B(\mathscr{H})$ relative to some Ξ. We show, in the next section, that if V satisfies the above two conditions, then V is of the form V_Ξ for some Ξ and thus induces the Schur product relative to Ξ [Theorem 5].

By using the Schur product, we can introduce, for each $a \in B(\mathscr{H})$, the Schur product map S_a from $B(\mathscr{H})$ to $B(\mathscr{H})$ by $S_a(x) = a \circ x$. One of the most important properties of Schur product maps is the module property. For example, let $M_n(\mathbf{C})$ be the $n \times n$ complex matrices and D_n the diagonal matrices in $M_n(\mathbf{C})$. If the Schur product on $M_n(\mathbf{C})$ is defined for the standard basis as usual, then the Schur product map S_a is a D_n-bimodule map. Moreover if φ is a D_n-bimodule map from $M_n(\mathbf{C})$ to $M_n(\mathbf{C})$, then there exists $a \in M_n(\mathbf{C})$ such that $\varphi = S_a$.

Given an integer n $(1 \le n < \infty)$ and a linear map φ from $B(\mathscr{H})$ to $B(\mathscr{H})$, we define a linear map φ_n from $M_n(B(\mathscr{H}))$ to $M_n(B(\mathscr{H}))$ by $\varphi_n([x_{ij}]) = [\varphi(x_{ij})]$. The map φ is defined to be completely bounded if

$$\|\varphi\|_{cb} = \sup_{n \in \mathbf{N}} \|\varphi_n\| < \infty.$$

We recall that φ is said to be n-positive if φ_n is positive, and completely positive if n-positive for all n. It was Haagerup who proved that the norm $\|S_a\|$ for S_a coincides with the completely bounded norm $\|S_a\|_{cb}$ for S_a on $M_n(\mathbf{C})$ [7]. Let D be a maximal abelian $*$-subalgebra of $B(\mathscr{H})$. Davidson and Power have shown that the (bounded) D-bimodule maps on $B(\mathscr{H})$ are completely bounded [3]. Moreover Blecher and Smith have proved that the set of all normal D-bimodule maps on $B(\mathscr{H})$ can be identified with the set of all D-bimodule maps from $K(\mathscr{H})$ to $B(\mathscr{H})$, denoted by $CB_D(K(\mathscr{H}), B(\mathscr{H}))$, and that $CB_D(K(\mathscr{H}), B(\mathscr{H}))$ is isometrically isomorphic to the extended (or weak*)-Haagerup tensor product $D \otimes_{eh} D$ [2]. When we take the Schur product on $B(\mathscr{H})$ relative to some orthonormal basis Ξ and denote by D the maximal abelian $*$-subalgebra of $B(\mathscr{H})$ diagonal to Ξ, it is clear that the Schur product maps are normal D-bimodule maps and are completely bounded.

The other aim of this paper is to determine the position of the Schur product maps $\{S_a | a \in B(\mathscr{H})\}$ in $CB_D(K(\mathscr{H}), B(\mathscr{H}))$. In other words, it is to solve the following problem: under what condition $\varphi \in CB_D(K(\mathscr{H}), B(\mathscr{H}))$ can be written as $\varphi = S_a$ for some $a \in B(\mathscr{H})$? Denoting by P the normal projection of norm one from $B(\mathscr{H})$ to D, we prove that if φ and $P - \varphi$ are 2-positive, then there exists a positive contraction $a \in B(\mathscr{H})$ such that $\varphi = S_a$ [Theorem 13]. Moreover we show that $\{S_a | a \in B(\mathscr{H}), a = a^*\}$ is isometrically (completely) order isomorphic to a self-ajoint subspace of $D \otimes_{eh} D$, which is characterized by using the element corresponding to P [Theorem 14].

§ 2. Characterization of Schur Product

For $\xi, \eta \in \mathscr{H}$, we define a rank 1 operator $\omega_{\xi,\eta} \in B(\mathscr{H})$ by

$$\omega_{\xi,\eta}(\zeta) = (\zeta|\eta)\xi, \qquad \text{for all } \zeta \in \mathscr{H}.$$

Let $\Xi = \{\xi_i | i \in I\}$ be a complete orthonormal system of $\mathscr{H}$. Given $x \in B(\mathscr{H})$, the sum

$$\sum_{i,j \in I} (x\xi_j|\xi_i)\omega_{\xi_i,\xi_j}$$

is weakly convergent to x. We call $\{x_{ij}\}_{i,j \in I}$, $x_{ij} = (x\xi_j|\xi_i)$ a matrix representation of x by Ξ. By $D_\Xi(\mathscr{H})$, we denote the weakly closed $*$-subalgebra of $B(\mathscr{H})$ generated by

$$\{\omega_{\xi_i,\xi_i} | \xi_i \in \Xi\}.$$

Then the following easily follows:

Lemma 1. *$D_\Xi(\mathscr{H})$ is a maximal abelian $*$-subalgebra of $B(\mathscr{H})$, that is, if M is an abelian $*$-subalgebra of $B(\mathscr{H})$ containing $D_\Xi(\mathscr{H})$, then $M = D_\Xi(\mathscr{H})$.*

Lemma 2. *For $x, y \in B(\mathscr{H})$, the operator $x \circ_\Xi y$ has the following matrix representation*

$$\{(x\xi_j|\xi_i)(y\xi_j|\xi_i)\}_{i,j \in I}.$$

Moreover, we have

$$\|x \circ_\Xi y\| \le \|x\| \, \|y\| \qquad \textit{and} \qquad x \circ_\Xi y = y \circ_\Xi x.$$

Proof. The following estimation is easy:

$$\|x \circ_\Xi y\| \le \|V^*\| \, \|x \otimes y\| \, \|V\| \le \|x\| \, \|y\|.$$

It is sufficient to show that

$$((x \circ_\Xi y)\xi_j|\xi_i) = ((y \circ_\Xi x)\xi_j|\xi_i) = (x\xi_j|\xi_i)(y\xi_j|\xi_i).$$

Indeed, we have

$$\begin{aligned} ((x \circ_\Xi y)\xi_j|\xi_i) &= (V^*(x \otimes y)V\xi_j|\xi_i) \\ &= ((x \otimes y)V\xi_j|V\xi_i) \\ &= ((x \otimes y)(\xi_j \otimes \xi_j)|\xi_i \otimes \xi_i) \\ &= (x\xi_j|\xi_i)(y\xi_j|\xi_i). \end{aligned}$$ □

Let B be a closed $*$-subalgebra of $B(\mathscr{H})$ and P a linear map from $B(\mathscr{H})$ to B. We call P a projection of norm one onto B if

$$P|_B = id_B \qquad \text{and} \qquad \|P\| = 1.$$

It is known that a projection P of norm one onto B is automatically completely positive and satisfies the bimodule map property:

$$P(axb) = aP(x)b \qquad \text{for all } a, b \in B \qquad \text{and} \qquad x \in B(\mathscr{H}).$$

We define a map P_Ξ from $B(\mathscr{H})$ to $B(\mathscr{H})$ by the relation

$$P_\Xi(x) = x \circ_\Xi I = I \circ_\Xi x. \qquad x \in B(\mathscr{H}).$$

Then P_Ξ is the unique normal projection of norm one onto $D_\Xi(\mathscr{H})$ satisfying the relation

$$P_\Xi(\omega_{\xi_i, \xi_j}) = \delta_{ij}\omega_{\xi_i, \xi_i},$$

where δ_{ij} is Kronecker's symbol.

Proposition 3. *Let D be a maximal abelian $*$-subalgebra of $B(\mathscr{H})$. If there is a normal projection P of norm one onto D, then there exists a complete orthonormal system Ξ such that $D = D_\Xi(\mathscr{H})$ and $P = P_\Xi$.*

Proof. Since $B(\mathscr{H})$ is atomic and P is normal, there exists a minimal projection e in D ([12, Proposition 1.1]). By the maximality of D, e is also minimal in $B(\mathscr{H})$, that is, $\dim e\mathscr{H} = 1$. Using Zorn's lemma, we can choose a maximal family $\{e_i | i \in I\}$ of orthogonal projections with $\dim e_i\mathscr{H} = 1$ for each $i \in I$. If we set $\Xi = \{\xi_i | i \in I\}$ with

$$e_i\xi_i = \xi_i, \qquad \|\xi_i\| = 1,$$

then we obtain $D = D_\Xi$ and $P = P_\Xi$. □

For another complete orthonormal system $\Phi = \{\phi_i | i \in I\}$ in $\mathscr{H}$, we have a Schur product $x \circ_\Phi y$ of $x, y \in B(\mathscr{H})$ with the following matrix representation relative to Φ:

$$((x \circ_\Phi y)\phi_j | \phi_i) = (x\phi_j | \phi_i)(y\phi_j | \phi_i) \qquad \text{for all } i, j \in I.$$

Let U be the unitary operator on $\mathscr{H}$ satisfying

$$U\xi_i = \phi_i \qquad i \in I,$$

and π the $*$-isomorphism of $B(\mathscr{H})$ defined by

$$\pi(x) = U^*xU \qquad \text{for all } x \in B(\mathscr{H}).$$

Lemma 4. *For $x, y \in B(\mathscr{H})$, we have the following relation*:

$$\pi(x \circ_\Phi y) = \pi(x) \circ_\Xi \pi(y).$$

In particular, π is a $*$-isomorphism from $D_\Phi(\mathscr{H})$ onto $D_\Xi(\mathscr{H})$.

Proof. For $\xi_i, \xi_j \in \Xi$, we have

$$(\pi(x \circ_\Phi y)\xi_j|\xi_i) = ((x \circ_\Phi y)\phi_j|\phi_i)$$
$$= (x\phi_j|\phi_i)(y\phi_j|\phi_i),$$

and

$$(\pi(x) \circ_\Xi \pi(y)\xi_j|\xi_i) = (\pi(x)\xi_j|\xi_i)(\pi(y)\xi_j|\xi_i)$$
$$= (x\phi_j|\phi_i)(y\phi_j|\phi_i).$$

By this calculation, we have

$$\pi(x \circ_\Phi y) = \pi(x) \circ_\Xi \pi(y).$$

By the identity

$$\omega_{\xi_t,\xi_t} \circ_\Xi \omega_{\xi_t,\xi_t} = \omega_{\xi_t,\xi_t}, \qquad \omega_{\phi_t,\phi_t} \circ_\Phi \omega_{\phi_t,\phi_t} = \omega_{\phi_t,\phi_t}$$

and

$$\pi(\omega_{\phi_t,\phi_t}) = \omega_{\xi_t,\xi_t},$$

we can get

$$\pi(D_\Phi(\mathscr{H})) = D_\Xi(\mathscr{H}).$$ □

If we have no confusion, we denote $x \circ y$ instead of $x \circ_\Xi y$.

Let V be an isometry from $\mathscr{H}$ to $\mathscr{H} \otimes \mathscr{H}$. For a complete orthonormal system $\{\xi_i | i \in I\}$, we set $\alpha_{j,k}^i = (V\xi_i|\xi_j \otimes \xi_k)$. Then V and V^* have the following form:

$$V\xi_i = \sum_{j,k \in I} \alpha_{j,k}^i \xi_j \otimes \xi_k, \qquad V^*(\xi_j \otimes \xi_k) = \sum_{i \in I} \overline{\alpha_{j,k}^i} \xi_i.$$

We need the following identities later:

$$(V^*(I \otimes \omega_{\xi_s,\xi_t})V\xi_a|\xi_b) = \sum_j \alpha_{j,t}^a \overline{\alpha_{j,s}^b},$$

$$(V^*(\omega_{\xi_s,\xi_t} \otimes I)V\xi_a|\xi_b) = \sum_k \alpha_{t,k}^a \overline{\alpha_{s,k}^b}.$$

We call V an S-isometry on $\mathscr{H}$ if V is an isometry from $\mathscr{H}$ to $\mathscr{H} \otimes \mathscr{H}$ and the map P_V defined by the relation

$$P_V(x) = V^*(x \otimes 1)V = V^*(1 \otimes x)V \qquad (x \in B(\mathscr{H}))$$

becomes a projection of norm one onto some $*$-subalgebra A_V of $B(\mathscr{H})$. A typical example of an S-isometry V on $\mathscr{H}$ is an isometry which defines a Schur product on $B(\mathscr{H})$, and then

$$P_V = P_\Xi, \qquad A_V = D_\Xi$$

for some complete orthonormal system Ξ.

Our main result in this section is the following:

Theorem 5. *If V is an S-isometry on $\mathscr{H}$, then there exists a complete orthonormal system $\Xi = \{\xi_i | i \in I\}$ such that*

$$V\xi_i = \xi_i \otimes \xi_i \ \ (i \in I), \qquad P_V = P_\Xi, \qquad A_V = D_\Xi.$$

To prove this theorem, we need some preparation.

Lemma 6. *Let V be an isometry from $\mathscr{H}$ to $\mathscr{H} \otimes \mathscr{H}$ and e a projection on $\mathscr{H}$ satisfying*

$$e = V^*(e \otimes 1)V = V^*(1 \otimes e)V.$$

Then Ve becomes an isometry from $e\mathscr{H}$ to $e\mathscr{H} \otimes e\mathscr{H}$.

Proof. It suffices to show that $Ve = (e \otimes e)Ve$. Since V is an isometry, we have

$$\begin{aligned}(Ve - (e \otimes 1)Ve)^*(Ve - (e \otimes 1)Ve) &= eV^*((1-e) \otimes 1)Ve \\ &= eV^*Ve - eV^*(e \otimes 1)Ve = 0.\end{aligned}$$

Hence $Ve = (e \otimes 1)Ve$. Similarly we have $Ve = (1 \otimes e)Ve$. Hence

$$Ve = (e \otimes 1)Ve = (e \otimes 1)(1 \otimes e)Ve = (e \otimes e)Ve. \qquad \square$$

This statement says that, for an S-isometry V on $\mathscr{H}$, a projection $e \in A_V$ is automatically central as follows: for any $x \in A_V$,

$$\begin{aligned} xe &= P_V(x)e = (e + (1-e))V^*(x \otimes 1)Ve \\ &= eV^*(x \otimes 1)Ve + (1-e)V^*((1-e) \otimes (1-e))(x \otimes 1)(e \otimes e)Ve \\ &= exe. \end{aligned}$$

Corollary 7. *Let $\Xi = \{\xi_i | i \in I\}$ be a complete orthonormal system. If V is an S-isometry on $\mathscr{H}$ with $A_V = D_\Xi$, then there exist complex numbers α_i $(i \in I)$ such that*

$$|\alpha_i| = 1, \qquad V\xi_i = \alpha_i(\xi_i \otimes \xi_i)$$

for all $i \in I$, that is,

$$P_V(x) = I \circ_\Phi x \qquad \text{for all } x \in B(\mathscr{H}),$$

where $\Phi = \{\alpha_i \xi_i | i \in I\}$.

Lemma 8. *If* $\dim \mathscr{H} > 1$, *then there is no S-isometry* V *on* $\mathscr{H}$ *such that* $A_V = \mathbf{C}1_{\mathscr{H}}$.

Proof. Suppose that $A_V = \mathbf{C}1_{\mathscr{H}}$. In this case P_V has the form $\varphi(\cdot)1_{\mathscr{H}}$ for some normal state φ on $B(\mathscr{H})$. Let $\varphi(\cdot) = \sum_i \lambda_i(\cdot\xi_i|\xi_i)$ where $\{\xi_i\}$ is a complete orthonormal system of $\mathscr{H}$, $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$. Let $J = \{j|\lambda_j > 0\}$ and $n = |J|$. Set

$$V\xi_i = \sum_{j,k} \alpha^i_{j,k}(\xi_j \otimes \xi_k).$$

By $\varphi(x) = V^*(1 \otimes x)V = V^*(1 \otimes x)V$, we have

$$(*) \qquad \sum_j \alpha^a_{j,t}\overline{\alpha^b_{j,s}} = \sum_k \alpha^a_{t,k}\overline{\alpha^b_{s,k}} = \delta_{ab}\delta_{st}\lambda_s.$$

If $s \notin J$, $\lambda_s = 0$ hence $\alpha^b_{j,s} = 0$ and $\alpha^b_{s,k} = 0$. Therefore, we can restrict the sum over j and over k in $(*)$ to $j \in J$ and $k \in J$.

(Case 1) $n = 1$: Then,

$$V\xi_i = \alpha^i_{kk}(\xi_k \otimes \xi_k)$$

where J consists of a single index k and α^i_{kk} is a complex number of modulus 1 for each i. Since $\dim \mathscr{H} > 1$, this contradicts with

$$(V\xi_1|V\xi_2) = (\xi_1|\xi_2) = 0.$$

(Case 2) $1 < n < \infty$: The following vectors are mutually orthogonal with norm $\sqrt{\lambda_k}$ for any fixed k and span $\mathbf{C}^n$:

$$(\alpha^j_{ik})_{i\in J} \qquad (j \in J).$$

So they are a basis of $\mathbf{C}^n$. Let $l \in J$, $l \neq k$. The vectors

$$(\alpha^j_{il})_{i\in J} \qquad (j \in J)$$

are orthogonal to the above basis due to $(*)$. Hence $\alpha^j_{il} = 0$ for $i \in J$, $j \in J$, $l \neq k$, $l \in J$. Therefore

$$(\alpha^j_{k,i})_{i\in J} = (0, \ldots, \alpha^j_{k,k}, \ldots, 0) \qquad (j \in J).$$

Thus these vectors are all proportional to a vector

$$(\delta_{ik})_{i\in J} = (0, \ldots, 1, \ldots, 0)$$

and are linearly dependent for $n > 1$. On the other hand, using the same argument as for $(\alpha^j_{ik})_{i\in J}$ $(j \in J)$ and (*), the vectors $(\alpha^j_{k,i})_{i\in J}$ $(j \in J)$ are shown to be mutually orthogonal with norm $\sqrt{\lambda_k}$ and hence are linearly independent. This is a contradiction.

(Case 3) $n = \infty$: Consider the Hilbert space $\ell^2(\mathbb{N})$ with the orthonormal system $\{\xi_j | j \in J\}$ (which can be identified as the subspace of $\mathscr{H}$ spanned by $\{\xi_j | j \in J\}$). Define $v_i \in B(\ell^2(\mathbb{N}))$ by

$$v_i \xi_k = \sum_{j \in J} \alpha^i_{j,k} \xi_j.$$

Then we have

$$(v_i \xi_j | v_i \xi_k) = \delta_{jk} \lambda_j = (v_i^* \xi_j | v_i^* \xi_k)$$

and hence $v_i^* v_i = v_i v_i^*$ and $|v_i| \xi_j = |v_i^*| \xi_j = \sqrt{\lambda_j} \xi_j$.

By the polar decomposition $v_i = u_i |v_i|$, we obtain a unitary u_i commuting with $|v_i|$. This means that, for any $k, l \in J$ with $\lambda_k \neq \lambda_l$,

$$(v_i \xi_k | \xi_l) = 0,$$

because

$$\sqrt{\lambda_k}(u_i \xi_k | \xi_l) = (u_i |v_i| \xi_k | \xi_l) = (v_i \xi_k | \xi_l) = (u_i \xi_k | \, |v_i| \xi_l) = \sqrt{\lambda_l}(u_i \xi_k | \xi_l).$$

Hence $\alpha^i_{kl} = 0$ for all $i \in J$ if $\lambda_k \neq \lambda_l$ by definition of v_i. Due to $\sum \lambda_i = 1$, the number of l with $\lambda_l = \lambda_k$ for a given k is finite. Therefore, the vectors

$$(\alpha^i_{kj})_{j \in J} \qquad (i \in J)$$

has non-zero elements only for a finite number of j (satisfying $\lambda_j = \lambda_k$) and has a finite basis. This contradicts with the fact that they are mutually orthogonal vectors of norm $\sqrt{\lambda_k} \neq 0$ and are of an infinite number due to $|J| = n = \infty$. □

Now we can prove Theorem 5.

Proof. Since P_V is a normal projection of norm one onto A_V (c.f. [12]), A_V has a minimal projection p. By Lemma 6, Vp is also an S-isometry on $p\mathscr{H}$. Applying Lemma 8 to Vp, we get that p is 1-dimensional. If we continue this argument to $(1-p)A_V(1-p)$, we can show that A_V is D_Ξ for some complete orthonormal system Ξ. The remaining part follows from Corollary 7.

§ 3. Characterization of Schur Product Maps

Throughout this section we restrict to separable Hilbert spaces for notational simplicity, although the results remain true with any index sets replacing the integers. Let ℓ^∞ be the set of all bounded sequences with the norm $\|a\|_\infty = \sup |a_n|$, ℓ^1 the set of all absolutely summable sequences with $\|f\|_1 = \sum_{n=1}^\infty |f(n)|$ and c_0 the set of all sequences tend to 0 with the same norm in ℓ^∞.

We fix a complete orthonormal system $\Xi = \{\xi_n | n = 1, 2, \dots\}$ of $\mathscr{H}$ in this section, and use the notation ℓ^∞ instead of $D_\Xi(\mathscr{H})$.

As in the usual operator space setting, ℓ^∞ (resp. ℓ^1) is the (standard) dual operator space of ℓ^1 (resp. c_0), that is, $M_n(\ell^1) = CB(c_0, M_n)$ and $M_n(\ell^\infty) = CB(\ell^1, M_n)$ (c.f. [1], [5]). We consider the Haagerup tensor product to the algebraic tensor product $\ell^1 \otimes \ell^1$ and denote its completion by $\ell^1 \otimes_h \ell^1$, where

$$\|v\|_h = \inf\left\{ \|[f_1, \dots, f_n]\| \left\| \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} \right\| \;\middle|\; v = \sum_{i=1}^{n} f_i \otimes g_i \in \ell^1 \otimes_h \ell^1 \right\}.$$

We recall the extended Haagerup tensor product which is the dual operator space of $\ell^1 \otimes_h \ell^1$. Given $u \in \ell^\infty \otimes_{eh} \ell^\infty$, if we identify u with the bilinear form, then there exist $\{a_i\}_{i=1}^\infty$, $\{b_i\}_{i=1}^\infty \subset \ell^\infty$ such that $\|\sum_{i=1}^\infty a_i a_i^*\| < \infty$, $\|\sum_{i=1}^\infty b_i^* b_i\| < \infty$, and $u(f, g) = \sum_{i=1}^\infty a_i(f) b_i(g)$ for $f, g \in \ell^1$ (c.f. [6]). We denote u by $\sum_i a_i \otimes b_i$. The extended Haagerup norm is also given by

$$\|u\|_{eh} = \inf\left\{ \left\|\sum_{i=1}^\infty a_i a_i^*\right\|^{1/2} \left\|\sum_{i=1}^\infty b_i^* b_i\right\|^{1/2} \;\middle|\; u = \sum_{i=1}^\infty a_i \otimes b_i \right\}.$$

Let $\{e_i\}_{i=1}^\infty$ be the canonical basis in ℓ^∞. For $x = [\alpha_{ij}] \in B(\mathscr{H})$ we define the bilinear form $\sum_{i,j=1}^\infty \alpha_{ij} e_i \otimes e_j$, which converges pointwisely in the following sense:

$$\ell^1 \times \ell^1 \ni (f, g) \mapsto \sum_{i,j=1}^\infty \alpha_{ij} f(e_i) g(e_j) \in \mathbb{C}.$$

Moreover we have:

Lemma 9. *$\sum_{i,j=1}^\infty \alpha_{ij} e_i \otimes e_j$ belongs to $\ell^\infty \otimes_{eh} \ell^\infty$.*

Proof. It is enough to show that $\sum_{i,j=1}^\infty \alpha_{ij} e_i \otimes e_j \in (\ell^1 \otimes_h \ell^1)^*$. For $[e_1, \dots, e_n] \in M_{1,n}(\ell^\infty)$, it is clear that $\|[e_1, \dots, e_n]\| = 1$. By the definition of the norm for $[f_1, \dots, f_m] \in M_{1,m}(\ell^1)$, we have

$$\begin{aligned} \|[f_1, \dots, f_m]\| &= \sup\{\|[f_1(x_{kl}), \dots, f_m(x_{kl})]\| \mid \\ &\qquad [x_{kl}] \in M_n(c_0), \|[x_{kl}]\| \le 1, n \in \mathbb{N}\} \\ &\ge \|[f_1(e_1), \dots, f_1(e_n), \dots, f_m(e_1), \dots, f_m(e_n)]\|. \end{aligned}$$

Then given $\sum_{k=1}^m f_k \otimes g_k \in \ell^1 \otimes_h \ell^1$, we obtain that

$$\left|\sum_{k=1}^{m}\sum_{i,j=1}^{n}\alpha_{ij}f_k(e_i)g_k(e_j)\right|$$
$$\leq \|[f_1(e_1),\dots,f_1(e_n),\dots,f_m(e_1),\dots,f_m(e_n)]\|$$
$$\times\left\|\begin{bmatrix}[\alpha_{ij}]_{i,j=1}^{n} & 0 & \dots & 0\\ 0 & [\alpha_{ij}]_{i,j=1}^{n} & 0 & \vdots\\ \vdots & 0 & \ddots & 0\\ 0 & \dots & 0 & [\alpha_{ij}]_{i,j=1}^{n}\end{bmatrix}\right\|\left\|\begin{bmatrix}g_1(e_1)\\ \vdots\\ g_1(e_n)\\ \vdots\\ g_m(e_1)\\ \vdots\\ g_m(e_n)\end{bmatrix}\right\|$$
$$\leq \|[f_1,\dots,f_m]\|\,\|[\alpha_{ij}]_{i,j=1}^{\infty}\|\left\|\begin{bmatrix}g_1\\ \vdots\\ g_m\end{bmatrix}\right\|.$$

This implies that $\sum_{i,j=1}^{\infty}\alpha_{ij}e_i\otimes e_j$ is in $(\ell^1\otimes_h\ell^1)^*$. □

We define the $*$-operation to $x=\sum_{i=1}^{\infty}a_i\otimes b_i\in\ell^\infty\otimes_{eh}\ell^\infty$ by $x^*=\sum_{i=1}^{\infty}b_i^*\otimes a_i^*$ and also introduce a positive cone $(\ell^\infty\otimes_{eh}\ell^\infty)^+$ as

$$\left\{\sum_{i=1}^{\infty}a_i\otimes a_i^*\in\ell^\infty\otimes_{eh}\ell^\infty\,\middle|\,a_i\in\ell^\infty\right\}.$$

The multiplication for $\sum_{i=1}^{\infty}a_i\otimes b_i$, $\sum_{j=1}^{\infty}c_j\otimes d_j\in\ell^\infty\otimes_{eh}\ell^\infty$ is defined by

$$\sum_{i,j=1}^{\infty}a_ic_j\otimes b_id_j$$

(c.f. [2]).

Proposition 10. *If we introduce the multiplication on $B(\mathscr{H})$ defined by Schur product, then the map*

$$B(\mathscr{H})\ni[\alpha_{ij}]\mapsto\sum_{i,j=1}^{\infty}\alpha_{ij}e_i\otimes e_j\in\ell^\infty\otimes_{eh}\ell^\infty$$

is faithful, $*$-preserving, homomorphic and contractive, i.e.,

$$\left\|\sum_{i,j=1}^{\infty}\alpha_{ij}e_i\otimes e_j\right\|_{eh}\leq\|[\alpha_{ij}]\|.$$

Proof. It is easy to check that the map is faithful, $*$-preserving, and homomorphic. The inequality has been already done in Lemma 9. □

Lemma 11. *If a sequence $x_n = [\alpha_{ij}^{(n)}] \in B(\mathscr{H})$ converges to $x = [\alpha_{ij}]$ weakly, then $\sum_{i,j=1}^{\infty} \alpha_{ij}^{(n)} e_i \otimes e_j$ converges to $\sum_{i,j=1}^{\infty} \alpha_{ij} e_i \otimes e_j$ in the $\sigma(\ell^\infty \otimes_{eh} \ell^\infty, \ell^1 \otimes_h \ell^1)$—topology.*

Proof. Let $f_k = (f_k(1), f_k(2), \dots)$ and $g_k = (g_k(1), g_k(2), \dots) \in \ell^1$. It is clear that f_k and g_k are in ℓ^2. Then given $\sum_{k=1}^{m} f_k \otimes g_k \in \ell^1 \otimes \ell^1$, we have

$$\left(\sum_{i,j=1}^{\infty} \alpha_{ij}^{(n)} e_i \otimes e_j - \sum_{i,j=1}^{\infty} \alpha_{ij} e_i \otimes e_j\right)\left(\sum_{k=1}^{m} f_k \otimes g_k\right)$$

$$= \sum_{k=1}^{m} \sum_{i,j=1}^{\infty} f_k(i)(\alpha_{ij}^{(n)} - \alpha_{ij}) g_k(j)$$

$$= \sum_{k=1}^{m} ((x_n - x) g_k | \overline{f_k}).$$

Since $\{\sum_{i,j=1}^{\infty} \alpha_{ij}^{(n)} e_i \otimes e_j\}$ are uniformly bounded, it follows that the convergence is valid. □

Lemma 12. *If $x = [\alpha_{ij}] \in B(\mathscr{H})$ is positive, then $\sum_{i,j=1}^{\infty} \alpha_{ij} e_i \otimes e_j$ belongs to $(\ell^\infty \otimes_{eh} \ell^\infty)^+$.*

Proof. Since $x = [\alpha_{ij}] \geq 0$ in $B(\mathscr{H})$, there exists $[\beta_{ij}] \in B(\mathscr{H})$ such that $[\alpha_{ij}] = [\beta_{ij}]^*[\beta_{ij}]$. Let $b_k = \sum_{i=1}^{\infty} \beta_{ki} e_i$. It follows that $b_k \in \ell^2$ and $\sum_{k=1}^{n} [\overline{\beta_{ki}} \beta_{kj}]$ converges to x strongly when n tends to ∞. Hence we have by the previous Lemma,

$$\sum_{i,j=1}^{\infty} \alpha_{ij} e_i \otimes e_j = \lim_{n \to \infty} \sum_{i,j=1}^{\infty} \sum_{k=1}^{n} \overline{\beta_{ki}} \beta_{kj} e_i \otimes e_j$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} b_k^* \otimes b_k \in (\ell^\infty \otimes_{eh} \ell^\infty)^+. \qquad \square$$

Theorem 13. *Let P be the normal projection of norm one from $B(\mathscr{H})$ onto ℓ^∞ and S a 2-positive map from $B(\mathscr{H})$ to $B(\mathscr{H})$. If $P - S$ is 2-positive, then there exists a positive contraction $x \in B(\mathscr{H})$ such that*

$$S(a) = a \circ x \qquad \text{for all } a \in B(\mathscr{H}).$$

In particular, S is a completely positive ℓ^∞-bimodule map.

Proof. For elements ξ_i, ξ_j in the fixed basis of $\mathscr{H}$, we use the notation $e_{ij} \in B(\mathscr{H})$ instead of the operator ω_{ξ_i, ξ_j}, that is,

$$e_{ij}\xi = (\xi|\xi_j)\xi_i \qquad \text{for any } \xi \in \mathscr{H}.$$

Then we show that there exists a complex number x_{ij} such that

$$S(e_{ij}) = x_{ij}e_{ij}.$$

By the relation $0 \leq S(e_{ii}) \leq P(e_{ii}) = e_{ii}$, there exists a positive real x_{ii} such that

$$S(e_{ii}) = x_{ii}e_{ii} \qquad \text{for all } i.$$

Since the matrix

$$\begin{pmatrix} e_{ii} & e_{ij} \\ e_{ji} & e_{jj} \end{pmatrix} \in M_2(B(\mathscr{H}))$$

is positive, by the 2-positivity of S, we have

$$0 \leq \begin{pmatrix} S(e_{ii}) & S(e_{ij}) \\ S(e_{ji}) & S(e_{jj}) \end{pmatrix} \leq \begin{pmatrix} P(e_{ii}) & P(e_{ij}) \\ P(e_{ji}) & P(e_{jj}) \end{pmatrix},$$

that is,

$$0 \leq \begin{pmatrix} x_{ii}e_{ii} & S(e_{ij}) \\ S(e_{ji}) & x_{jj}e_{jj} \end{pmatrix} \leq \begin{pmatrix} e_{ii} & 0 \\ 0 & e_{jj} \end{pmatrix}.$$

So we have that there exists a complex number x_{ij} such that

$$S(e_{ij}) = x_{ij}e_{ij}.$$

If we define an operator x by

$$x\xi_j = \sum_i x_{ij}\xi_i \qquad \text{for } \xi_j \in \mathscr{H},$$

then we show that x becomes a positive contraction.

For any positive integer n, $\sum_{j,k=1}^n e_{jk}$ is positive. So we have

$$0 \leq S\left(\sum_{j,k=1}^n e_{jk}\right) \leq P\left(\sum_{j,k=1}^n e_{jk}\right) = \sum_{j=1}^n e_{jj},$$

and

$$0 \leq S\left(\sum_{j,k=1}^n e_{jk}\right)^2 \leq \sum_{j=1}^n e_{jj}.$$

For a vector $\eta = \sum_{k=1}^n \alpha_k\xi_k$ with $\|\eta\| = 1$, we have

$$\left\|S\left(\sum_{j,k=1}^n e_{jk}\right)\eta\right\|^2 = \left(S\left(\sum_{j,k=1}^n e_{jk}\right)^2 \eta|\eta\right) \leq \left(\sum_{j=1}^n e_{jj}\eta|\eta\right) = 1.$$

Since

$$\left\| S\left(\sum_{j,k=1}^{n} e_{jk}\right)\eta\right\|^2 = \left\|\sum_{j,k,l=1}^{n} x_{jk}\alpha_l e_{jk}\xi_l\right\|^2$$
$$= \left\|\sum_{j,k=1}^{n} x_{jk}\alpha_k\xi_j\right\|^2 \leq 1,$$

we have

$$\|x\eta\|^2 = \left\|\sum_{k=1}^{n}\sum_{j=1}^{n} x_{jk}\alpha_k\xi_j\right\|^2 \leq 1,$$

that is, $x \in B(\mathscr{H})$ and $\|x\| \leq 1$. Moreover, the relation

$$\sum_{j,k=1}^{n} x_{jk}\alpha_j\overline{\alpha_k} = \left(\sum_{j,k=1}^{n} x_{jk}e_{jk}\eta|\eta\right) = \left(S\left(\sum_{j,k=1}^{n} e_{jk}\right)\eta|\eta\right) \geq 0$$

implies that $x \geq 0$.

By the normality of P, S is also normal. So we can show that $S(a) = a \circ x$ for all $a \in B(\mathscr{H})$. □

Let $B(\mathscr{H})_1^+$ be the set of all positive contractions. For a pair of selfadjoint elements $x, y \in \ell^\infty \otimes_{eh} \ell^\infty$, we write $x \geq y$ when $x - y \in (\ell^\infty \otimes_{eh} \ell^\infty)^+$.

We define $(\ell^\infty \otimes_{eh}^P \ell^\infty)^+ = \{x \in \ell^\infty \otimes_{eh} \ell^\infty \mid 0 \leq x \leq \sum_{i=1}^\infty e_i \otimes e_i\}$. Let $CB_{\ell^\infty}(K(\mathscr{H}), B(\mathscr{H}))$ be the set of ℓ^∞-bimodule completely bounded maps from $K(\mathscr{H})$ to $B(\mathscr{H})$.

For $\varphi \in CB(K(\mathscr{H}), B(\mathscr{H}))$, we define $*$-operation by $\varphi^*(x) = \varphi(x^*)^*$ for $x \in K(\mathscr{H})$. For a pair of selfadjoint maps $\varphi, \psi \in CB_{\ell^\infty}(K(\mathscr{H}), B(\mathscr{H}))$, we write $\varphi \geq \psi$ when $\varphi - \psi$ is completely positive. We define

$$CP_{\ell^\infty}^P(K(\mathscr{H}), B(\mathscr{H})) = \{\varphi \in CB_{\ell^\infty}(K(\mathscr{H}), B(\mathscr{H})) | 0 \leq \varphi \leq P\}.$$

Now we can show the main theorem in this section.

Theorem 14. *There exist affine isomorphisms among the following three closed convex sets*:

(1) $B(\mathscr{H})_1^+$

(2) $(\ell^\infty \otimes_{eh}^P \ell^\infty)^+$

(3) $CP_{\ell^\infty}^P(K(\mathscr{H}), B(\mathscr{H}))$.

Proof. First, by Proposition 10, we have the map

$$B(\mathscr{H})_1^+ \ni [\alpha_{ij}] \mapsto \sum_{i,j=1}^{\infty} \alpha_{ij} e_i \otimes e_j \in (\ell^\infty \otimes_{eh}^P \ell^\infty)^+$$

is affine and injective.

Next, for $\sum_{i=1}^{\infty} a_i \otimes b_i \in \ell^\infty \otimes_{eh} \ell^\infty$, we define the map $\langle \sum_{i=1}^{\infty} a_i \otimes b_i \rangle$ by

$$\left\langle \sum_{i=1}^{\infty} a_i \otimes b_i \right\rangle (x) = \sum_{i=1}^{\infty} a_i x b_i$$

for $x \in K(\mathscr{H})$. Then $\ell^\infty \otimes_{eh} \ell^\infty$ is isometric isomorphic to $CB_{\ell^\infty}(K(\mathscr{H}), B(\mathscr{H}))$ by [2]. Moreover we have $\langle \sum_{i=1}^{\infty} e_i \otimes e_i \rangle = P$, so the map

$$(\ell^\infty \otimes_{eh}^P \ell^\infty)^+ \ni \sum_{i=1}^{\infty} a_i \otimes a_i^* \mapsto \left\langle \sum_{i=1}^{\infty} a_i \otimes a_i^* \right\rangle \in CP_{\ell^\infty}^P(K(\mathscr{H}), B(\mathscr{H}))$$

is affine isomorphic.

Finally, for $\varphi \in CP_{\ell^\infty}^P(K(\mathscr{H}), B(\mathscr{H}))$, there exists $x \in B(\mathscr{H})_1^+$, such that $\varphi = S_x$. Therefore the first correspondence is surjective. We are done. □

Corollary 15. *If $x = [\alpha_{ij}] \in B(\mathscr{H})$ is selfadjoint, then*

$$\begin{aligned} \|x\| &= \inf\{\lambda \geq 0 \mid -\lambda P \leq S_x \leq \lambda P\} \\ &= \inf\left\{\lambda \geq 0 \,\middle|\, -\lambda \sum_{i=1}^{\infty} e_i \otimes e_i \leq \sum_{i=1}^{\infty} \alpha_{ij} e_i \otimes e_j \leq \lambda \sum_{i=1}^{\infty} e_i \otimes e_i \right\}. \end{aligned}$$

Remark 1. If $\varphi \in CB_{\ell^\infty}(K(\mathscr{H}), B(\mathscr{H}))$ is selfadjoint, by [8], we have

$$\begin{aligned} \|\varphi\| &= \|\varphi\|_{cb} \\ &= \inf\{\|\varphi\|_{cb} \mid -\psi \leq \varphi \leq \psi, \psi = \psi^*, \psi \in CB_{\ell^\infty}(K(\mathscr{H}), B(\mathscr{H}))\}. \end{aligned}$$

Moreover since $x \in B(\mathscr{H})$ is positive if and only if S_x is positive if and only if S_x is completely positive, we have for $x = x^*$

$$\|S_x\| = \|S_x\|_{cb} = \inf\{\|S_y\|_{cb} \mid -S_y \leq S_x \leq S_y, y = y* \in B(\mathscr{H})\}.$$

In the rest of this section, we discuss the relation between the 2-positive map and the module map, which leads the crucial part to get the main theorem. For the convenience of the reader, we show the next proposition, however it might be well known.

Proposition 16. *Let M be a von Neumann algebra with a cyclic vector and φ an M-bimodule map from $B(\mathscr{H})$ to $B(\mathscr{H})$. If φ is positive, then φ is completely positive.*

Proof. Given any $n \times n$ positive operator $[a_{ij}] \in M_n(B(\mathscr{H}))$, and any $\xi_1, \ldots, \xi_n \in \mathscr{H}$. Then there exist $\xi \in \mathscr{H}$ and $\{b_i^{(k)}\} \subset M$ $(k = 1, \ldots, \infty)$ such that

$$\lim_{k \to \infty} \|b_i^{(k)} \xi - \xi_i\| = 0 \qquad \text{for } i = 1, \ldots, n.$$

Since $\sum_{i,j=1}^{n} b_i^{(k)^*} a_{ij} b_j^{(k)}$ is positive, we have

$$\begin{aligned}
&\left([\varphi(a_{ij})] \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \middle| \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \right) \\
&= \lim_{k\to\infty} \left([\varphi(a_{ij})] \begin{bmatrix} b_1^{(k)}\xi_1 \\ \vdots \\ b_n^{(k)}\xi_n \end{bmatrix} \middle| \begin{bmatrix} b_1^{(k)}\xi_1 \\ \vdots \\ b_n^{(k)}\xi_n \end{bmatrix} \right) \\
&= \lim_{k\to\infty} \left(\varphi\left(\sum_{i,j=1}^{n} b_i^{(k)^*} a_{ij} b_j^{(k)} \right) \xi | \xi \right) \\
&\geq 0.
\end{aligned}$$

□

Proposition 17. *Let M be a von Neumann algebra of type I, P a normal projection of norm one from $B(\mathscr{H})$ to M, and φ a 2-positive map of $B(\mathscr{H})$ to $B(\mathscr{H})$. If $P - \varphi$ is 2-positive, then φ is a normal $M \cap M'$—bimodule map.*

Proof. Let q be a central projection in M. Since the positivity of $P - \varphi$ and the normality of P lead that φ is normal, it is enough to show that $\varphi(xq) = \varphi(x)q = q\varphi(x)$ for all $x \in B(\mathscr{H})$. By the module property of P, it is clear that $\varphi(qxq) \in qB(\mathscr{H})q$ for all $x \in B(\mathscr{H})$.

Let $\{e_i\}$ and $\{f_j\}$ be minimal projections such that $q = \sum e_i$, $1 - q = \sum f_j$. Choose the partial isometry v_{ij} which has e_i as the initial projection and f_j as the final projection, then the 2×2-matrix

$$\begin{bmatrix} e_i & v_{ji}^* \\ v_{ji} & f_j \end{bmatrix}$$

is positive.

Since $P(v_{ji}) = P(v_{ji}q) = P(v_{ji})q = qP(v_{ji}) = 0$, we have

$$0 \leq \begin{bmatrix} \varphi(e_i) & \varphi(v_{ji}^*) \\ \varphi(v_{ji}) & \varphi(f_j) \end{bmatrix} \leq \begin{bmatrix} P(e_i) & 0 \\ 0 & P(f_j) \end{bmatrix} = \begin{bmatrix} e_i & 0 \\ 0 & f_j \end{bmatrix}.$$

Thus, for $\xi, \eta \in \mathscr{H}$,

$$\begin{aligned}
0 &\leq \left(\begin{bmatrix} \varphi(e_i) & \varphi(v_{ji}^*) \\ \varphi(v_{ji}) & \varphi(f_j) \end{bmatrix} \begin{bmatrix} (1-q)\xi \\ q\eta \end{bmatrix} \middle| \begin{bmatrix} (1-q)\xi \\ q\eta \end{bmatrix} \right) \\
&\leq \left(\begin{bmatrix} e_i & 0 \\ 0 & f_j \end{bmatrix} \begin{bmatrix} (1-q)\xi \\ q\eta \end{bmatrix} \middle| \begin{bmatrix} (1-q)\xi \\ q\eta \end{bmatrix} \right) = 0
\end{aligned}$$

and

$$\varphi(e_i)(1-q)\xi = \varphi(f_j)q\eta = 0.$$

Therefore we have that

$$(\varphi(v_{ji}^*)q\eta|(1-q)\xi) + (\varphi(v_{ji})(1-q)\xi|q\eta) = 0.$$

This implies that $\mathrm{Re}(\varphi(v_{ji})(1-q)\xi|q\eta) = 0$. By the same argument, we also have $\mathrm{Im}(\varphi(v_{ji})(1-q)\xi|q\eta) = 0$. Hence we have $\varphi(v_{ji}) \in (1-q)B(\mathscr{H})q$. Thus it turns out $\varphi(f_jxe_i) \in (1-q)B(\mathscr{H})q$, moreover $\varphi((1-q)xq) \in (1-q)B(\mathscr{H})q$. □

Remark 2. In Proposition 17, we cannot replace the assumption of the 2-positivity by the 1-positivity: indeed, let θ be the map

$$M_2(\mathbb{C}) \ni \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \mapsto \frac{1}{2}\begin{bmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & \alpha_{22} \end{bmatrix} \in M_2(\mathbb{C}).$$

Then θ and $P - \theta$ are positive, but θ is not an ℓ^∞-bimodule map.

References

[1] Blecher, D. P. and Paulsen, V. I., Tensor products of operator spaces, *J. Funct. Anal.*, **99** (1991), 262–292.

[2] Blecher, D. P. and Smith, R. R., The dual of the Haagerup tensor product, *J. London Math. Soc.*, **45** (1992), 126–144.

[3] Davidson, K. R. and Power, S. C., Isometric automorphisms and homology for non-self-adjoint operator algebras, *Quart. J. Math.*, **42** (1991), 271–292.

[4] Effros, E. G. and Kishimoto, A., Module maps and Hochschild-Johnson cohomology, *Indiana Math. J.*, **36** (1987), 257–276.

[5] Effros, E. G. and Ruan, Z.-J., A new approach to operator spaces, *Canad. Math. Bull.*, **34** (1991), 329–337.

[6] ———, Operator convolution algebras: An approach to Quantum groups, *Preprint*.

[7] Haagerup, U., *Decomposition of completely bounded maps on operator algebras*, unpublished manuscript.

[8] ———, Injectivity and decomposition of completely bounded maps, *Lecture Notes in Math.*, Springer-Verlag, **1132** (1983), 170–222.

[9] Paulsen, V. I., Completely bounded maps and dilations, *Pitman Res. Notes in Math. Ser.*, **146** (1986).

[10] Smith, R. R., Completely bounded module maps and the Haagerup tensor product, *J. Funct. Anal.*, **102** (1991), 156–175.

[11] Takesaki, M., *Theory of Operator Algebras I*, Springer-Verlag.

[12] Tomiyama, J., On some types of maximal abelian subalgebras, *J. Funct. Anal.*, **10** (1972), 373–386.