Publ. RIMS, Kyoto Univ.
36 (2000), 321–336

Codimension Two Compact Hausdorff Foliations by Hyperbolic Surfaces Are Not Stable

By

Kazuhiko FUKUI* and Atsuhide MORI**

§1. Introduction

The set of codimension q C^r-foliations of a C^r-n-manifold M carries the fine C^r-topology described in [6]. We denote this space by $Fol_q^r(M)$. We say a foliation $\mathcal{F}$ is C^r-stable if there is a neighborhood V of $\mathcal{F}$ in $Fol_q^r(M)$ such that any foliation in V has a compact leaf. $\mathcal{F}$ is C^r-unstable if not. A foliation in a small neighborhood of $\mathcal{F}$ in $Fol_q^r(M)$ is said to be a C^r-perturbation of $\mathcal{F}$.

There are many works about the stability of foliations. Let L be a compact leaf of $\mathcal{F}$. Langevin-Rosenberg[12] showed, generalizing the Reeb stability theorem [15], that $\mathcal{F}$ is C^1-stable if $H^1(L;\mathbf{R}) = 0$. Let $\pi_1(L) \to GL(q,\mathbf{R})$ be the action determined from the linear holonomy of L. Stowe[18] showed, generalizing the results of Hirsch[11] and Thurston[19], that $\mathcal{F}$ is C^1-stable if the cohomology group $H^1(\pi_1(L);\mathbf{R}^q)$ is trivial.

On the other hand, as for the foliation $\mathcal{F}$ of an orientable S^1-bundle over a closed surface B by fibres, Seifert[17] showed that $\mathcal{F}$ is C^0-stable if $\chi(B) \neq 0$, where $\chi(B)$ is the euler characteristic of B. This result was generalized by Fuller[10] and Fukui[8] : Fuller generalized this to orientable S^1-bundles over arbitrary closed manifolds with $\chi(B) \neq 0$, and Fukui classified all codimension two compact C^r-foliations of closed C^r-3-manifolds ($r \geq 1$) into C^1-stable or C^r-unstable ones. Langevin-Rosenberg[13] considered a fibration $\pi : M \to B$ with fibre L and showed that the foliation by fibres is C^0-stable if 1) $\pi_1(L) \cong \mathbf{Z}$, 2) B is a closed surface with $\chi(B) \neq 0$, and 3) $\pi_1(B)$ acts trivially on $\pi_1(L)$. Fukui[6] generalized this result to compact codimension

Communicated by K. Saito, May 25, 1998. Revised July 12, 1999.
1991 Mathematics Subject Classification(s): 57R30.
*This research was partially supported by Grant-in Aid for Scientific Research (No. 10640096), Ministry of Education, Science and Culture, Japan.
*Department of Mathematics, Kyoto Sangyo University, Kyoto 603-8555, Japan.
e-mail: fukui@cc.kyoto-su.ac.jp
**Department of Mathematics, Kyoto Sangyo University, Kyoto 603-8555, Japan.
e-mail: rimo@cc.kyoto-su.ac.jp

two foliations. Plante[14] classified all transversely orientable C^r-foliations of closed C^r-3-manifolds by closed orientable surfaces ($r \geq 1$) into C^1-stable or C^r-unstable ones. Especially, foliations by hyperbolic surfaces are C^r-unstable, where a closed orientable surface is said to be *elliptic, parabolic,* or *hyperbolic* if it is of genus 0, 1 , or ≥ 2 respectively. Morover, Tsuboi[20] showed, improving the result of Bonatti-Firmo[2], that a transversely orientable foliation $\mathcal{F}$ ($\in Fol_1^2(M)$) of a 3-manifold M is C^1-unstable if all compact leaves of $\mathcal{F}$ are hyperbolic. As for the stability of foliations of closed 4-manifolds by closed orientable surfaces, we can summarize as follows (Bonatti[1],Fukui[9] et al):

A) The case where $\mathcal{F}(\in Fol_2^1(M))$ is a foliation by elliptic surfaces S^2:
In this case the Reeb stability theorem implies that $\mathcal{F}$ is C^1-stable.

B) The case where $\mathcal{F}(\in Fol_2^1(M))$ is a foliation by parabolic surfaces T^2:
1) The product foliation $\mathcal{F}$ of $M = T^2 \times B$ is C^1-stable if and only if $\chi(B) \neq 0$.
2) $\mathcal{F}$ is C^1-stable if $\mathcal{F}$ has a rotation or a dihedral leaf.
3) $\mathcal{F}(\in Fol_2^2(M))$ is C^2-stable if $\mathcal{F}$ has reflection leaves with

$$h_* = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $h_* : H_1(T^2; \mathbf{Z}) \to H_1(T^2; \mathbf{Z})$ is the automorphism induced from the attaching diffeomorphism $h : T^2 \to T^2$ and we regard the connected components of the reflection leaves as $T^2 \times [0,1]/h$.

C) The case where $\mathcal{F}(\in Fol_2^1(M))$ is a foliation by hyperbolic surfaces:
In this case $\mathcal{F}$ is C^r-unstable if the genus of the generic leaf is sufficiently large.

We study here the case of C). Our main result is the following.

Theorem (Theorem 11). *All compact Hausdorff C^r-foliations of C^r-4-manifolds by hyperbolic surfaces are C^r-unstable* ($1 \leq r \leq \infty$).

Note that M need not be compact, but if it is compact, then by the results of Epstein[4], Edwards, Millett and Sullivan[3], codimension two compact foliations of compact manifolds are always Hausdorff ones. Thus our result means that $\mathcal{F}$ is always C^r-unstable in the case of C).

The authers would like to thank the referee for pointing out a gap in the first version and for helpful suggestions. Especially, the referee suggested the argument in §3 taking local sections to prove the non-existence of compact leaves, which made the explanations more precise.

§2. Compact Hausdorff Foliations and Singular Leaves

Let M be a C^r-n-manifold without boundary and $\mathcal{F}$ a codimension q compact Hausdorff C^r-foliation of M ($1 \leq r \leq \infty$), where $\mathcal{F}$ is said to be *Hausdorff* if the leaf space $M/\mathcal{F}$ is Hausdoff. Then we have a nice picture of the local behavior of $\mathcal{F}$ as follows.

Proposition 1 (Epstein[5]). *There is a generic leaf L_0 with property that there is an open dense subset of M, where the leaves have all trivial holonomy and are all diffeomorphic to L_0. Given a leaf L, we can describe a neighborhood $U(L)$ of L, together with the foliation on the neighborhood as follows. There is a finite subgroup $G(L)$ of $O(q)$ such that $G(L)$ acts freely on L_0 on the right and $L_0/G(L) \cong L$. Let D^q be the unit disk. We foliate $L_0 \times D^q$ with leaves of the form $L_0 \times \{pt\}$. This foliation is preserved by the diagonal action of $G(L)$, defined by $g(x,y) = (x \cdot g^{-1}, g \cdot y)$ for $g \in G(L)$, $x \in L_0$ and $y \in D^q$ where $G(L)$ acts linearly on D^q. So we have a foliation induced on $U = L_0 \times_{G(L)} D^q$. The leaf corresponding to $y = 0$ is $L_0/G(L)$. Then there is a C^r-imbedding $\varphi : U \to M$ with $\varphi(U) = U(L)$, which preserves leaves and $\varphi(L_0/G(L)) = L$.*

Hereafter we consider the case of codimension 2 in this paper. Since $G(L)$ is a finite subgroup of $O(2)$, the action of $G(L)$ is isomorphic to that of a rotation group $\mathbf{Z}_k$ ($k > 1$), a reflection group $\mathbf{D}$ or a dihedral group $\mathbf{D}_l$ ($l > 1$).

Definition 2. A leaf L is *singular* if $G(L)$ is not trivial. The order of $G(L)$ is called the order of holonomy of L. We say such an L is a *rotation leaf*, a *reflection leaf* or a *dihedral leaf* if the action of $G(L)$ is isomorphic to that of $\mathbf{Z}_k$ ($k > 1$), $\mathbf{D}$ or $\mathbf{D}_l$ ($l > 1$) respectively.

§3. Instability of a Foliation by Fibres

In this section, we consider the stability of bundle foliations. Let L be a hyperbolic surface and $\pi : M \to B$ a C^r-fibre bundle with fibre L over a closed surface B. Let $\mathcal{F}$ be a foliation of M by fibres and $g(L)$ the genus of the hyperbolic surface L. Then we have the following theorem by the similar way as Plante[14] and Fukui[9].

Theorem 3. *$\mathcal{F}$ is C^r-unstable if $g(L) \geq 3$ ($0 \leq r \leq \infty$).*

We can prove this theorem by using the following Remark 4 and Lemma 5.

Remark 4 (Construction of C^r-perturbed cassette). Let U be an open neighborhood in a C^r-2-manifold M which is C^r-diffeomorphic to the open unit disk $intD^2 = \{(x,y) \in \mathbf{R}^2 \mid x^2 + y^2 < 1\}$ via a diffeomorphism φ. Take a family of diffeomorphisms $\{f_t : U \to U \mid t \in [\frac{1}{3}, \frac{2}{3}]\}$ such that

1) f_t is C^r-close to the identity in the fine C^r-topology for $t \in [\frac{1}{3}, \frac{2}{3}]$,
2) $f_t = id$ for $t \in [\frac{1}{3}, \frac{4}{9}]$,
3) $f_t = f_{\frac{2}{3}}$ for $t \in [\frac{5}{9}, \frac{2}{3}]$ and
4) for fixed $p(\in U)$, $(t, f_t(p)) : [\frac{1}{3}, \frac{2}{3}] \to [\frac{1}{3}, \frac{2}{3}] \times U$ is a C^r-imbedding.
5) $f = f_{\frac{2}{3}}$ has no periodic points.

We say that the C^r-foliation of $[\frac{1}{3}, \frac{2}{3}] \times U$ by curves of the above 4) is a C^r-perturbed cassette for the product foliation of $[\frac{1}{3}, \frac{2}{3}] \times U$ and denote it by $P^r(U, f)$. For example, for a function $h_1(t) = \exp((1/t(t-1))$ $(0 < t < 1)$, $h_1(t) = 0$ $(t \leq 0$ or $t \geq 1)$ and small $\epsilon > 0$, putting

$$h_2(s) = \int_0^s h_1(t)dt / \int_0^1 h_1(t)dt, \quad h(x, y) = (x + \epsilon h_2(2 - 2\sqrt{x^2 + y^2}), y),$$

we can take a family of diffeomorphisms $\{f_t : U \to U; t \in [\frac{1}{3}, \frac{2}{3}]\}$ defined by

$$\varphi \circ f_t \circ \varphi^{-1}(x, y) = (x, y) + h_2(9t - 4)(h(x, y) - (x, y)).$$

Lemma 5. *Let L be a closed orientable surface and g the genus of L. For given $(2g-2)$ simple closed curves $\delta_1, \delta_2, \cdots, \delta_{2g-2}$ in L, there is a simple closed curve δ such that*

$$[\delta] \neq 0, [\delta] \neq [\delta_1], [\delta] \neq [\delta_1], \cdots, [\delta] \neq [\delta_{2g-2}]$$

and

$$\langle [\delta], [\delta_1] \rangle = \langle [\delta], [\delta_2] \rangle = \cdots = \langle [\delta], [\delta_{2g-2}] \rangle = 0,$$

where [ ] denotes the homology class of a simple closed curve in L and $\langle , \rangle$ denotes the algebraic intersection number of 1-cycles in L.

Proof. Let α_i, β_i $(1 \leq i \leq g)$ be non-null-homologous simple closed curves in L such that

$$\langle [\alpha_i], [\alpha_j] \rangle = \langle [\beta_i], [\beta_j] \rangle = 0, \ \langle [\alpha_i], [\beta_j] \rangle = \delta_{ij} \ (\forall i, j).$$

Then $[\alpha_i]$ and $[\beta_i]$ $(1 \leq i \leq g)$ form a canonical symplectic basis for $H_1(L; \mathbf{Z})$. The homology class of δ_j is expressed as $[\delta_j] = \sum_{i=1}^{g}(k_{ij}[\alpha_i] + l_{ij}[\beta_i])$ $(j = 1, 2, \cdots, 2g-2)$. Put $[\delta] = \sum_{i=1}^{g}(k_i[\alpha_i] + l_i[\beta_i])$. Then equations $\langle [\delta], [\delta_j] \rangle = 0$ $(j = 1, 2, \cdots, 2g-2)$ can be solved with $[\delta] \neq 0$, $[\delta] \neq [\delta_1], [\delta] \neq [\delta_2], \cdots, [\delta] \neq [\delta_{2g-2}]$. Note that δ can be realized as a simple closed curve in L if $k_1, l_1, \cdots, k_g, l_g$ are relatively prime. □

In the above proof, $[\delta_1], [\delta_2], \cdots, [\delta_{2g-2}]$ are linearly independent when the dimension of the solution space of equations $\langle [\delta], [\delta_j] \rangle = 0$ $(j = 1, 2, \cdots, 2g-2)$

is two and then $[\delta_1], [\delta_2], \cdots, [\delta_{2g-2}]$ are different from each other. Thus, the following corollary holds.

Corollary 6. *Let L be a closed orientable surface and g the genus of L. For given $(2g-1)$ simple closed curves $\delta_1, \delta_2, \cdots, \delta_{2g-1}$ in L, there is a simple closed curve δ such that* 1) $[\delta] \neq 0$,

2) $[\delta]$ *is equal to at most one of* $[\delta_1], [\delta_2], \cdots, [\delta_{2g-1}]$ *and*

3) $\langle[\delta], [\delta_1]\rangle = \langle[\delta], [\delta_2]\rangle = \cdots = \langle[\delta], [\delta_{2g-1}]\rangle = 0$.

Now we are in a position to prove Theorem 3.

Proof of Theorem 3. We consider the dual cell decomposition of a small triangulation of the base space B and denote each of these polygons by P_i $(i \in I)$. Let V_i $(\supset P_i)$ be a neighborhood diffeomorphic to an open disk $(i \in I)$. We can assume that π is trivial over each V_i and let $\psi_i : \pi^{-1}(V_i) \to L \times V_i$ be a trivialization of π over $V_i (i \in I)$. Put $\psi_{ji} = \psi_j \circ \psi_i^{-1}$. Let δ_i be a simple closed curve in L of $L \times V_i$, then $\psi_{ji}(\delta_i \times \{p\})$ $(p \in V_i \cap V_j)$ is homologous, in each fibre, to some simple closed curve δ_i' in L of $L \times V_j$, that is, $\psi_{ji}([\delta_i]) = [\delta_i']$.

If $P_i \cap P_j \cap P_k$ is a vertex, take a neighborhood U_{ijk} of the vertex such that $U_{ijk} \subset V_i \cap V_j \cap V_k$, U_{ijk} is diffeomorphic to an open disk and $U_{ijk} \cap U_{i'j'k'} = \emptyset$ for a neighborhood $U_{i'j'k'}$ of another vertex.

If $P_i \cap P_j$ is an edge, take a neighborhood U_{ij} of $(P_i \cap P_j) - (U_{ijk} \cup U_{ijl})$ such that $U_{ij} \subset V_i \cap V_j$, U_{ij} is diffeomorphic to an open disk and $U_{ij} \cap U_{i'j'} = \emptyset$ for a neighborhood $U_{i'j'}$ of another (shortened) edge, where U_{ijk} and U_{ijl} are the neighborhoods of the ends of the edge.

We can take an open set U_i $(\subset P_i)$ diffeomorphic to an open disk for each $i \in I$ such that $\{U_i; i \in I\} \cup \{U_{ijk}; P_i \cap P_j \cap P_k$ is a vertex$\} \cup \{U_{ij}; P_i \cap P_j$ is an edge$\}$ is an open covering of the base space B as shown in Fig.1(a).

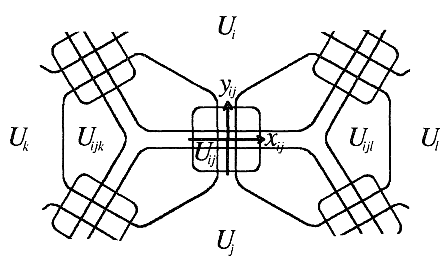


Fig. 1(a).

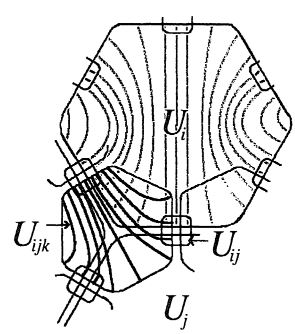


Fig. 1(b).

Take a local coodinate (x_{ij}, y_{ij}) on each U_{ij} such that ∂U_i and ∂U_j (resp. ∂U_{ijk} and ∂U_{ijl}) are two lines parallel to the x_{ij} axis (resp. y_{ij} axis) in U_{ij}.

Step 1. We perturb $\mathcal{F}$ on $L \times U_i$ as follows. Let γ_i be a non-null-homologous simple closed curve in L of $L \times U_i$ and Q an open tubular neighborhood of γ_i in L such that its closure $\bar{Q}$ is homeomorphic to $\gamma_i \times [0,1]$ with coordinate (s,t), $s \in \gamma_i$, $t \in [0,1]$. Since the holonomy of L in $\mathcal{F}$ along γ_i is trivial, the foliation induced on $\bar{Q} \times U_i$ has leaves of form $\bar{Q} \times \{y\}$. In the restriction of $\mathcal{F}$ to $\gamma_i \times [0,1] \times U_i$, we change the part $\gamma_i \times [\frac{1}{3}, \frac{2}{3}] \times U_i$ to $\gamma_i \times P^r(U_i, f_i)$, where $P^r(U_i, f_i)$ is a C^r-perturbed cassette in Remark 4. By such a C^r-perturbation for each i, we obtain a new foliation $\mathcal{F}'$ which has no compact leaves in each $\pi^{-1}(U_i)$. We can choose each (γ_i, f_i) such that $\langle [\gamma_i], \psi_{ij}([\gamma_j]) \rangle \neq 0$ and f_i coincides with the time-1 map determined by a vector field of the form $\lambda_{ij}(y_{ij}) \frac{\partial}{\partial y_{ij}}$ on $U_i \cap U_{ij}$ for any pair (i,j) defining U_{ij} (see Fig.1(b)).

Step 2. We perturb $\mathcal{F}'$ on $\pi^{-1}(U_{ijk})$ as follows. Before the perturbation of the above step 1, take a simple closed curve γ_{ijk} in L of $L \times U_{ijk}$ ($\subset L \times V_i$) which satisfies

$$[\gamma_{ijk}] \neq 0, [\gamma_{ijk}] \neq [\gamma_i], [\gamma_{ijk}] \neq \psi_{ij}([\gamma_j]), [\gamma_{ijk}] \neq \psi_{ik}([\gamma_k])$$

and

$$(*) \qquad \langle [\gamma_{ijk}], [\gamma_i] \rangle = \langle [\gamma_{ijk}], \psi_{ij}([\gamma_j]) \rangle = \langle [\gamma_{ijk}], \psi_{ik}([\gamma_k]) \rangle = 0.$$

This is possible by Lemma 5. Since the holonomy of $\mathcal{F}'$ (after the first perturbation) along γ_{ijk} is still trivial, we obtain, by a similar perturbation along this γ_{ijk} using $P^r(U_{ijk}, f_{ijk})$ as in Step 1, a new foliation $\mathcal{F}''$.

In order to show that $\mathcal{F}''$ has no compact leaves in each $\pi^{-1}(U_{ijk})$ nor in each $\pi^{-1}(U_i)$, we take a point p_i on γ_i (resp. p_{ijk} on γ_{ijk}) and a local section $s_i : U_i \to M$ (resp. $s_{ijk} : U_{ijk} \to M$) which satisfies $s_i(U_i) = \{p_i\} \times \{0\} \times U_i$ (resp. $s_{ijk}(U_{ijk}) = \{p_{ijk}\} \times \{0\} \times U_{ijk}$). We may assume that each local section does not intersect with the interior of $(\gamma_i \times [0,1] \times U_i) \cup (\gamma_{ijk} \times [0,1] \times U_{ijk})$. Then any leaf of $\mathcal{F}'$ intersecting with $s_i(U_i)$ is not compact; To see this, take a closed curve $c : \mathbf{R} \to \mathbf{R}/\mathbf{Z} \to M$ which has the geometric intersection number $+1$ (or -1) with γ_i on a leaf of $\mathcal{F}$ intersecting with $s_i(U_i)$ and consider the lift of c to the corresponding leaf curve on $\mathcal{F}'$, then we see that the leaf curve accumulates to a point of $s_i(\partial U_i)$. Next we take the lift of c to the corresponding leaf curve on $\mathcal{F}''$. If the leaf containing the lift does not intersect with $s_{ijk}(U_{ijk})$, then the lift accumulates to a point of $s_i(\partial U_i)$. Thus any leaf of $\mathcal{F}''$ which does not intersect with $s_{ijk}(U_{ijk})$ but intersects with $s_i(U_i)$ is not compact. We can also show that any leaf of $\mathcal{F}''$ intersecting with $s_{ijk}(U_{ijk})$ is not compact as follows; Take a closed curve $c' : \mathbf{R} \to \mathbf{R}/\mathbf{Z} \to M$ on a leaf of $\mathcal{F}'$ which has the geometric intersection number $+1$ (or -1) with one of the components of the lift of γ_{ijk} to the corresponding leaf curve on $\mathcal{F}'$ and does not intersect with the other (possibly empty) components, then the lift of c' to the corresponding leaf curve on $\mathcal{F}''$ accumulates to a point of $s_{ijk}(\partial U_{ijk})$. We can always take the above c' since the lift of γ_{ijk} does not separate the leaf of $\mathcal{F}'$. Thus $\mathcal{F}''$ has no compact leaves in each $\pi^{-1}(U_{ijk})$ nor in each $\pi^{-1}(U_i)$.

Since the rank of the solution space of the above equations $(*)$ is greater than one, we can choose each (γ_{ijk}, f_{ijk}) such that $\langle[\gamma_{ijk}], [\gamma_{ijl}]\rangle \neq 0$ for $l(\neq k)$ defining U_{ijl} and f_{ijk} coincides with the time-1 map determined by a vector field of the form $\mu_{ijk}(x_{ij})\frac{\partial}{\partial x_{ij}}$ on $U_{ijk} \cap U_{ij}$ (see Fig.1(b)).

Step 3. Similarly as above, we can perturb $\mathcal{F}''$ by using Lemma 5. Before the perturbations in the above two steps, take a simple closed curve γ_{ij} in L of $L \times U_{ij}$ ($\subset L \times V_i$) which satisfies

$$[\gamma_{ij}] \neq 0, [\gamma_{ij}] \neq [\gamma_i], [\gamma_{ij}] \neq \psi_{ij}([\gamma_j]), [\gamma_{ij}] \neq [\gamma_{ijk}], [\gamma_{ij}] \neq [\gamma_{ijl}]$$

and

$$\langle[\gamma_{ij}], [\gamma_i]\rangle = \langle[\gamma_{ij}], \psi_{ij}([\gamma_j])\rangle = \langle[\gamma_{ij}], [\gamma_{ijk}]\rangle = \langle[\gamma_{ij}], [\gamma_{ijl}]\rangle = 0,$$

for any pair (k, l) defining U_{ijk} and U_{ijl} $(k \neq l)$. Then, automatically $[\gamma_i], [\gamma_{ijk}]$ and $[\gamma_{ij}]$ are linearly independent. By the choice of f_i on $U_i \cap U_{ij}$ in Step 1 and f_{ijk} on $U_{ij} \cap U_{ijk}$ in Step 2, we may assume that f_i and f_{ijk} are commutative on the intersection of their domains in U_{ij} if the length of a sequence consisting of f_i, f_{ijk} and their inverses is less than or equal to the geometric intersection number of γ_{ij} with γ_i and γ_{ijk}. This ensures that the holonomy of $\mathcal{F}''$ along γ_{ij} is trivial. Thus we obtain, by the perturbation along this γ_{ij}, a new foliation $\mathcal{F}'''$ which has no compact leaves in each $\pi^{-1}(U_{ij})$.

In order to show that $\mathcal{F}'''$ has no compact leaves in M, we take a point p_{ij} on γ_{ij} and a local section $s_{ij} : U_{ij} \to M$ which satisfies $s_{ij}(U_{ij}) = \{p_{ij}\} \times \{0\} \times U_{ij}$. We may assume that each local section does not intersect with the interior of $(\gamma_i \times [0,1] \times U_i) \cup (\gamma_{ijk} \times [0,1] \times U_{ijk}) \cup (\gamma_{ij} \times [0,1] \times U_{ij})$. Then any leaf of $\mathcal{F}'''$ intersecting with $s_{ij}(U_{ij})$ is not compact; To see this, take a closed curve $c'' : \mathbf{R} \to \mathbf{R}/\mathbf{Z} \to M$ on a leaf of $\mathcal{F}''$ which has the geometric intersection number $+1$ (or -1) with one of the components of the lift of γ_{ij} to the corresponding leaf curve on $\mathcal{F}''$ and does not intersect with the other (possibly empty) components, then the lift of c'' to the corresponding leaf curve on $\mathcal{F}'''$ accumulates to a point of $s_{ij}(\partial U_{ij})$. We can always take the above c'' since the lift of γ_{ij} does not separate the leaf of $\mathcal{F}''$ as a consequence of the linear independence of $[\gamma_i], [\gamma_{ijk}]$ and $[\gamma_{ij}]$. Any leaf of $\mathcal{F}'''$ which does not intersect with $s_{ij}(U_{ij})$ but intersects with $s_i(U_i)$ or $s_{ijk}(U_{ijk})$ can be identified with the corresponding leaf of $\mathcal{F}''$. This leaf has been shown not to be compact in Step 2. Thus $\mathcal{F}'''$ has no compact leaves in M. This completes the proof. □

Theorem 3 can be extended to the case of $g(L) = 2$ as follows.

Theorem 7. *$\mathcal{F}$ is C^r-unstable if $g(L) = 2$ $(0 \leq r \leq \infty)$.*

Proof. In the same setting as in the proof of Theorem 3, we choose simple closed curves γ_i ($[\gamma_i] \neq 0$) in L of $L \times U_i$ ($i \in I$) such that

1) $\langle [\gamma_i], \psi_{ij}([\gamma_j]) \rangle \neq 0$ for any pair (i,j) defining U_{ij}.

2) $[\gamma_i], \psi_{ij}([\gamma_j]), \psi_{ik}([\gamma_k])$ and $\psi_{il}([\gamma_l])$ are linearly independent for any quartet (i,j,k,l) defining U_{ij}, U_{ijk} and U_{ijl} $(k \neq l)$.

Putting $\langle [\gamma_i], \psi_{ij}([\gamma_j]) \rangle = p$, $\langle \psi_{ij}([\gamma_j]), \psi_{ik}([\gamma_k]) \rangle = q$, $\langle \psi_{ik}([\gamma_k]), [\gamma_i] \rangle = r$, we determine the solution of equations

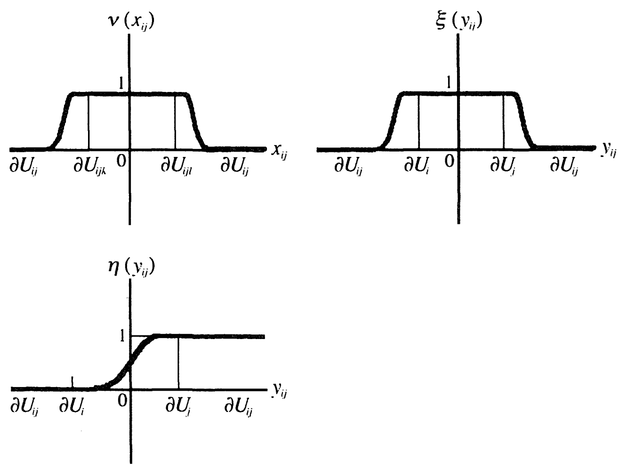
$$\langle [\gamma_{ijk}], [\gamma_i] \rangle = \langle [\gamma_{ijk}], \psi_{ij}([\gamma_j]) \rangle = \langle [\gamma_{ijk}], \psi_{ik}([\gamma_k]) \rangle = 0$$

by $[\gamma_{ijk}]$ (or its multiple) $= q[\gamma_i] + r\psi_{ij}([\gamma_j]) + p\psi_{ik}([\gamma_k])$. We can perturb $\mathcal{F}$ by Steps 1,2 in the above proof of Theorem 3 and obtain a new foliation $\mathcal{F}''$.

But we cannot take γ_{ij} as in the proof of Theorem 3 in this case. So we take a null-homologous simple closed curve γ_{ij} representing the homotopy class $\{\alpha_1\}\{\beta_1\}\{\alpha_1\}^{-1}$ $\{\beta_1\}^{-1}$. Then γ_{ij} separates each leaf of $\mathcal{F}$ into two once punctured tori T_1 and T_2. We may assume that f_i, f_j, f_{ijk} and f_{ijl} are commutative on the intersection of their domains in U_{ij} if the length of a sequence consisting of $f_i, f_j, f_{ijk}, f_{ijl}$ and their inverses are less than or equal to the geometric intersection number of γ_{ij} with $\gamma_i, \gamma_j, \gamma_{ijk}$ and γ_{ijl}, thus the holonomy of $\mathcal{F}''$ along γ_{ij} is still trivial. Let F_1 (resp. F_2) be a subset of U_{ij} such that $s_{ij}(F_1)$ (resp. $s_{ij}(F_2)$) is the common fixed point set of the holonomies on $s_{ij}(U_{ij})$ along loops homotopic to α_1 and β_1 (resp. α_2 and β_2).

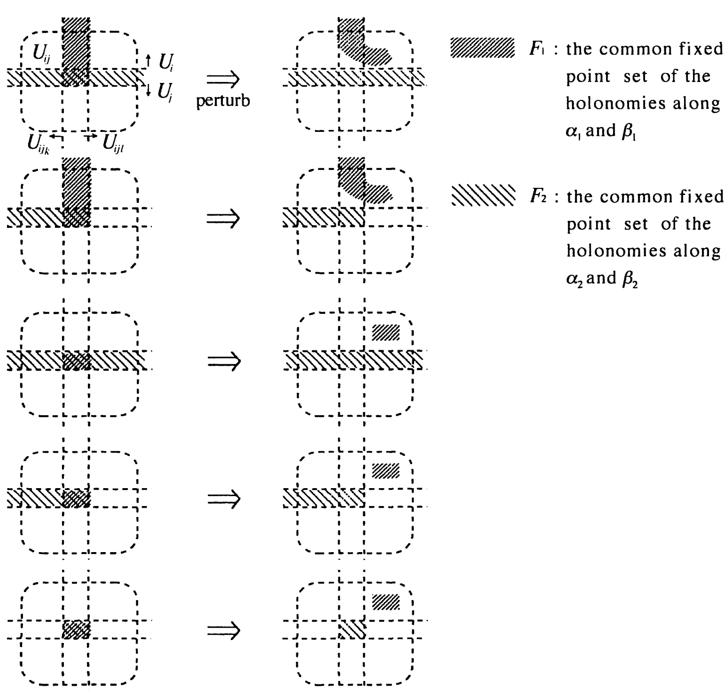
i) First, we consider the case where $F_1 = U_{ij} - (U_i \cup U_j)$ and $F_2 = U_{ij} - (U_{ijk} \cup U_{ijl})$. Let γ_i' and γ_j' (resp. γ_{ijk}' and γ_{ijl}') be simple closed curves in T_1 (resp. T_2) each of which is homologous to the corresponding one of γ_i and γ_j (resp. γ_{ijk} and γ_{ijl}). From the commutativity of the holonomies, we see that the foliation $\mathcal{F}''$, which has been perturbed along $\gamma_i, \gamma_j, \gamma_{ijk}$ and γ_{ijl}, is C^r-diffeomorphic in a neighborhood N_{ij} of $L \times (U_{ij} - (U_i \cup U_j \cup U_{ijk} \cup U_{ijl}))$ to the foliation which is obtained from $\mathcal{F}$ by the perturbations along $\gamma_i', \gamma_j', \gamma_{ijk}'$ and γ_{ijl}' in the same way. We may assume $L \times U_{ij} \subset N_{ij}$ by taking U_{ij} sufficiently small if necessary.

Let $\nu(x_{ij})$, $\xi(y_{ij})$ and $\eta(y_{ij}) : U_{ij} \to [0,1]$ be C^r-functions as shown in Fig.2(a) and $g_{ij}^i : U_{ij} \to U_{ij}$ (resp. $g_{ij}^j : U_{ij} \to U_{ij}$) the time-1 map determined by $\eta(y_{ij})\nu(x_{ij})\xi(y_{ij})\frac{\partial}{\partial x_{ij}}$ (resp. $(1-\eta(y_{ij}))\nu(x_{ij})\xi(y_{ij})\frac{\partial}{\partial x_{ij}}$)). Let S_{ij} be the support of the function $\nu(x_{ij})\xi(y_{ij})$. We define a perturbation of $\mathcal{F}$ as follows: Cut T_1 of $T_1 \times U_{ij}$ along γ_i' and $\psi_{ij}(\gamma_j')$ and paste them along γ_i' (resp. $\psi_{ij}(\gamma_j')$) with sliding by g_{ij}^i (resp. g_{ij}^j). This is possible since g_{ij}^i and g_{ij}^j commute. We can regard this perturbation as a modification of Step 1 as follows. Change f_i (resp. f_j) into the time-1 map determined by $\lambda_{ij}(y_{ij})\frac{\partial}{\partial y_{ij}} + \eta(y_{ij})\nu(x_{ij})\xi(y_{ij})\frac{\partial}{\partial x_{ij}}$ (resp. $-\lambda_{ji}(-y_{ij})\frac{\partial}{\partial y_{ij}} + (1-\eta(y_{ij}))\nu(x_{ij})\xi(y_{ij})\frac{\partial}{\partial x_{ij}}$). Then we can extend the domain of f_i (resp. f_j) to U_{ij}. Since this modification is different from the original $\mathcal{F}''$ only in N_{ij}, we can define a new perturbation



$\nu(x_{ij})$
1
∂U_{ij}
∂U_{ijk}
0
∂U_{ijl}
∂U_{ij}
x_{ij}
$\xi(y_{ij})$
1
∂U_{ij}
∂U_i
0
∂U_j
∂U_{ij}
y_{ij}
$\eta(y_{ij})$
1
∂U_{ij}
∂U_i
0
∂U_j
∂U_{ij}
y_{ij}

Fig. 2(a).



U_{ij}
U_i
U_j
U_{ijk}
U_{ijl}
perturb
F_1 : the common fixed point set of the holonomies along α_1 and β_1
F_2 : the common fixed point set of the holonomies along α_2 and β_2

Fig. 2(b).

$\mathcal{F}_0'''$.

Then a leaf of $\mathcal{F}_0'''$ intersecting with $s_{ij}(int(S_{ij}) - (U_i \cup U_j))$ accumulates to a point of $s_{ij}(\partial S_{ij})$. Another leaf intersecting with $s_{ij}(U_{ij})$ intersects with $s_i(U_i)$, $s_j(U_j)$, $s_{ijk}(U_{ijk})$ or $s_{ijl}(U_{ijl})$. By the same argument as in Step 2 in the proof of Theorem 3, a leaf intersecting with $s_i(U_i)$ or $s_j(U_j)$ accumulates to a point of $s_i(\partial U_i)$ or $s_j(\partial U_j)$ since U_i, U_j are invariant under the new f_i, f_j. In order to show that any leaf which does not intersect with $s_{ij}(int(S_{ij})-(U_i \cup U_j))$ but intersects with $s_{ij}(U_{ij})$ and $s_{ijk}(U_{ijk})$ (resp. $s_{ijl}(U_{ijl})$) accumulates to a point of $s_{ijk}(\partial U_{ijk})$ (resp. $s_{ijl}(\partial U_{ijl})$), take a closed curve $c : \mathbf{R} \to \mathbf{R}/\mathbf{Z} \to M$ on T_2 in a leaf of $\mathcal{F}''$ which has the geometric intersection number $+1$ (or -1) with γ'_{ijk} (resp. γ'_{ijl}), then the lift of c to the corresponding leaf curve on $\mathcal{F}''$ accumulates to a point of $s_{ijk}(\partial U_{ijk})$.

ii) If we consider the case where $F_1 = U_{ij} - (U_{ijk} \cup U_{ijl})$ and $F_2 = U_{ij} - (U_i \cup U_j)$, we conclude the same result as in i).

iii) In the other cases, we may assume that the foliation $\mathcal{F}''$ is C^r-diffeomorphic in a neighborhood of $L \times U_{ij}$ to the foliation which is obtained from $\mathcal{F}$ by the perturbation along $\alpha_1, \beta_1, \alpha_2$ and β_2 by taking U_{ij} sufficiently small if necessary, and let $H(\alpha_m) = f_i^{a_i^m} \circ f_j^{a_j^m} \circ f_{ijk}^{a_{ijk}^m} \circ f_{ijl}^{a_{ijl}^m}$ (resp. $H(\beta_m) = f_i^{b_i^m} \circ f_j^{b_j^m} \circ f_{ijk}^{b_{ijk}^m} \circ f_{ijl}^{b_{ijk}^m}$) be the holonomy on $s_{ij}(U_{ij})$ along a loop homotopic to α_m (resp. β_m), where a_*^m (resp. b_*^m) are some integers and we regard each f_* as the identity out of the domain, for $m = 1, 2$. Since $[\gamma_i], \psi_{ij}([\gamma_j]), [\gamma_{ijk}]$ and $[\gamma_{ijl}]$ are linearly independent, $g(L) = 2$ and $\langle [\gamma_i], [\gamma_{ijk}] \rangle = \langle [\gamma_i], [\gamma_{ijl}] \rangle = \langle \psi_{i=j}([\gamma_j]), [\gamma_{ijk}] \rangle = \langle \psi_{ij}([\gamma_j]), [\gamma_{ijl}] \rangle = 0$, we see $\langle [\gamma_i], \psi_{ij}[\gamma_j] \rangle \neq 0$, $\langle [\gamma_{ijk}], [\gamma_{ijl}] \rangle \neq 0$. Thus either $(a_i^1 b_j^1 - a_j^1 b_i^1)(a_{ijk}^2 b_{ijl}^2 - a_{ijl}^2 b_{ijk}^2) \neq 0$ or $(a_i^2 b_j^2 - a_j^2 b_i^2)(a_{ijk}^1 b_{ijl}^1 - a_{ijl}^1 b_{ijk}^1) \neq 0$ holds. Then it is sufficient to consider the five cases (up to rotations and reflections) shown in Fig.2(b) by assuming, for example, $(a_i^2 b_j^2 - a_j^2 b_i^2)(a_{ijk}^1 b_{ijl}^1 - a_{ijl}^1 b_{ijk}^1) \neq 0$. We can always eliminate the intersection of F_1 and F_2 by a perturbation using $P^r(U_{ij}, f_{ij})$ along γ_{ij} for example as shown in Fig.2(b) if we take $U_{ij} - (U_i \cup U_j \cup U_{ijk} \cup U_{ijl})$ sufficiently small. We denote this new foliation by $\mathcal{F}_1'''$ and let F_1' be the image of F_1 by the perturbation.

A leaf of $\mathcal{F}_1'''$ intersecting with $s_{ij}(U_{ij})$ accumulates to a point of $s_{ij}(\partial F_1')$ or $s_{ij}(\partial F_2)$); To see this, for example in the fourth case shown in Fig.2(b), we can choose loops $\delta_1, \delta_2, \ldots, \delta_8$ (resp. $\delta_1', \delta_2', \ldots, \delta_5'$) on each T_1 (resp. T_2) of $\mathcal{F}$ over 1) $U_{ijk} \cap U_i$, 2) $U_{ijk} - (U_i \cup U_j)$, 3) $U_{ijk} \cap U_j$, 4) $U_j - (U_{ijk} \cup U_{ijl})$, 5) $U_j \cap U_{ijl}$, 6) $U_{ijl} - (U_i \cup U_j)$, 7) $U_i \cap U_{ijl}$ and 8) $U_i - (U_{ijk} \cup U_{ijl})$ (resp. 1) $U_i - U_{ijl}$, 2) $U_i \cap U_{ijl}$, 3) $U_{ijl} - (U_i \cup U_j)$, 4) $U_{ijl} \cap U_j$ and 5) $U_j - U_{ijl}$) in U_{ij} satisfying that each $[\delta_p]$ (resp. $[\delta_q']$) is expressed as $\kappa_p[\alpha_1] + \lambda_p[\beta_1]$ (resp. $\kappa_q'[\alpha_2] + \lambda_q'[\beta_2]$) where κ_p and λ_p (resp. κ_q' and λ_q') are integers such that $(H(\alpha_1))^{\kappa_p} \circ (H(\beta_1))^{\lambda_p}$ (resp. $(H(\alpha_2))^{\kappa_q'} \circ (H(\beta_2))^{\lambda_q'}$) is a contraction to $s_{ij}(\partial F_1)$ (resp. $s_{ij}(\partial F_2)$) for $p = 1, 2, \ldots, 8$ (resp. $q = 1, 2, \ldots, 5$). Then the lift of each loop to the corresponding leaf curve on $\mathcal{F}''$ accumulates to a point of $s_{ij}(\partial F_1)$ (resp. $s_{ij}(\partial F_2)$). The other cases can be observed more easily. In any case after the perturbation along γ_{ij}, a leaf of $\mathcal{F}_1'''$ intersecting $s_{ij}(U_{ij})$ accumulates to a point of $s_{ij}(\partial F_1')$ or $s_{ij}(\partial F_2)$, thus this leaf is not compact.

Combining i),ii) and iii), we define a new folliation $\mathcal{F}'''$ which has no compact leaves in the saturation of each $s_{ij}(U_{ij})$. Another leaf can be identified with the corresponding non-compact leaf of $\mathcal{F}''$. This completes the proof. $\square$

§4. Instability of a Foliation with Singular Leaves

In this section, we consider the stability of a Hausdorff C^r-foliation $\mathcal{F}$ of a C^r-4-manifold by hyperbolic surfaces with singular leaves. We can generalize the above theorems as follows.

A) The case of foliations with only rotation leaves as singular leaves.

Theorem 8. *If all the singular leaves of $\mathcal{F}$ are rotation leaves, then $\mathcal{F}$ is C^r-unstable* $(1 \leq r \leq \infty)$.

Proof. In this case, note that the genus of the generic leaves satisfies $g \geq 3$. Let $B = M/\mathcal{F}$ be the leaf space of $\mathcal{F}$. B is a two dimensional V-manifold without boundaries and is also a topological manifold. The quotient map $\pi : M \to B$ is a V-bundle (see Satake[16] for definitions). We denote the rotation leaves of $\mathcal{F}$ by L_j $(j \in J)$. Choose saturated neighborhoods $U(L_j)$ as in Proposition 1 to be disjoint. We denote its interior by $int(U(L_j))$. The map π restricted to $M - \cup_{j \in J} L_j$ is a fibre bundle with fibre being the generic leaf L. So we take a sufficiently fine open covering of B as in the proof of Theorem 3,

$$\{U_i; i \in I\} \cup \{U_{ijk}; P_i \cap P_j \cap P_k \text{ is a vertex}\} \cup \{U_{ij}; P_i \cap P_j \text{ is an edge}\}$$

and modify it partially near the points corresponding to rotation leaves as follows (see Fig. 3) :

1) $int(U(L_j)) = \pi^{-1}(U_j)$ $(j \in J \subset I)$ and
2) $U_j \cap U_k = \emptyset$ $(j, k \in J)$.

First we perturb $\mathcal{F}$ on $int(U(L_j))$ as follows. Let $p_j : L \to L_j$ be the covering map and τ_j a simple closed curve in L_j representing a generator of the holonomy groups $\mathbf{Z}_{k(j)}$ of L_j $(j \in J)$. By using Lemma 5, we can take a simple closed curve δ_j in each L_j such that $[\delta_j] \neq 0$, $[\delta_j] \neq [\tau_j]$ and $\langle[\tau_j], [\delta_j]\rangle = 0$. Since the holonomy of L_j along δ_j is trivial, we can perturb $\mathcal{F}$ on $int(U(L_j))$ as in the proof of Theorem 3 and obtain a new foliation $\mathcal{F}'$ without compact leaves in each $int(U(L_j))$.

Next we perturb $\mathcal{F}'$ on $L \times U_{ji}$ and $L \times U_{jik}$. Each δ_j induces $k(j)$ simple closed curves $p_j^{-1}(\delta_j) = \delta_{j1} \cup \delta_{j2} \cup \cdots \cup \delta_{jk(j)}$ in L of $L \times U_{ji}$ and $L \times U_{jik}$. Since

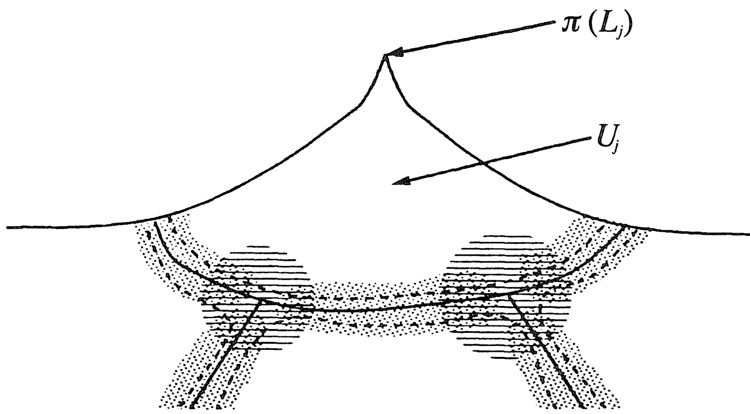


Fig. 3.

the genus of $L \geq k(j)+1$, we can take a simple closed curve γ_{ji} on each U_{ji} for the perturbation, by using Corollary 6 and the relation $2k(j)+1 \geq k(j)+3$ ($k(j) \geq 2$), where "+3" means that each U_{ji} meets three other neighborhoods except U_j. Similarly, we can take a simple closed curve γ_{jik} for the perturbation on each U_{jik} by Lemma 5. Since $g \geq 3$, we can perturb $\mathcal{F}'$ on the other neighborhoods in B as in the proof of Theorem 3. This completes the proof. □

B) The case of foliations with only reflection leaves as singular leaves.

Theorem 9. *If all the singular leaves of $\mathcal{F}$ are reflection leaves, then $\mathcal{F}$ is C^r-unstable* ($1 \leq r \leq \infty$).

Proof. In this case, note that the genus of the generic leaves satisfies $g \geq 3$. The leaf space $B = M/\mathcal{F}$ is a 2-manifold with boundaries and the quotient map $\pi : M \to B$ is a V-bundle. Reflection leaves correspond to the boundary points of B. Each boundary component of B is homeomorphic to the line $\mathbf{R}$ or the circle S^1. First we consider the case where B has only one boundary component and denote this component by ∂B. Let L_x ($x \in \partial B$) be reflection leaves of $\mathcal{F}$. Choose saturated neighborhoods $U(L_x)$ as in Proposition 1 to be sufficiently small and choose an open covering $\{int(U(L_j)); j \in J = \mathbf{Z}/2m\}$ (m is a natural number, but if $\partial B \cong \mathbf{R}$, then $J = \mathbf{Z}$, $m = +\infty$) of $\pi^{-1}(\partial B)$ such that $k \equiv j$ or $j \pm 1 \pmod{2m}$ if $int(U(L_k)) \cap int(U(L_j)) \neq \emptyset$. Since π restricted to $M - \pi^{-1}(\partial B)$ is a fibre bundle with fibre L, take a sufficiently fine open covering of B as in the proof of Theorem 3,

$$\{U_i; i \in I\} \cup \{U_{ijk}; P_i \cap P_j \cap P_k \text{ is a vertex}\} \cup \{U_{ij}; P_i \cap P_j \text{ is an edge}\}$$

and modify it partially near the boundary as follows (Fig. 4) :

1) $int(U(L_j)) = \pi^{-1}(U_j)$ $(j \in J \subset I)$,

2) If U_i $(i \in I - J)$ meets $U_j \cap U_{k'}$ $(j \in J, k' \equiv j+1 \pmod{2m})$, we denote such i by $i(j)$,

3) $U_{ji(j)i(k)}$ is a sufficiently small neighborhood which meets U_j, $U_{i(j)}$ and $U_{i(k)}$, and $U_{i(j)i(k)}$ meets $U_{i(k)}$, $U_{i(j)}$, $U_{ji(j)i(k)}$ and $U_{i(j)i(k)l}$ $(j \in J, k \equiv j-1 \pmod{2m})$ and

4) π restricted to the intersection of U_j and the neighborhoods which appear in the above 2) and 3) $(j \in J)$ is also a fibre bundle and we can define the transformation map ψ_{**} over there as in the proof of Theorem 3.

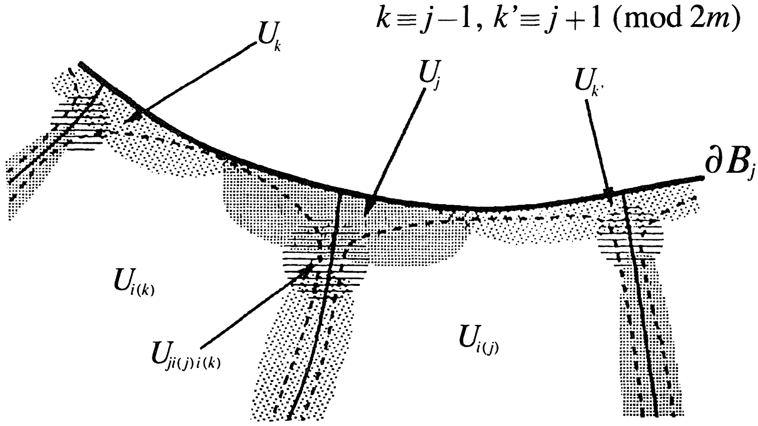


Fig. 4.

Let τ_j be a simple closed curve in L_j representing a generator of the holonomy group $\mathbf{D}$ of L_j $(j \in J)$.

Step 1. In the case where j is odd, take an arbitrary simple closed curve δ_j on L_j such that $[\delta_j] \neq 0$, $[\delta_j] \neq [\tau_j]$ and $\langle [\tau_j], [\delta_j] \rangle = 0$ hold and take a tubular neighborhood S_j homeomorphic to $\delta_j \times (0,1)$. Since the holonomy along δ_j is trivial, $U(L_j)\,|_{S_j}$ can be regarded as $S_j \times D$, where D is the unit 2-disk. So, we can perturb $\mathcal{F}$ to a foliation which has no compact leaves on $int(U(L_j))$ as in the proof of Theorem 3.

Step 2. In the case where j is even, we choose a coordinate (x, y) of the unit disk D^2 in Proposition 1 such that the linear action is given by $g \cdot (x, y) = (x, -y)$. We apply the diffeomorphism h in Remark 4 in the interior of the disk D^2.

Then, $g \cdot h(x, y) = h(g \cdot (x, y))$ holds. So we obtain a C^r-perturbation of the foliation of the form $L \times \{pt\}$ by using h which is still preserved by the diagonal action in Proposition 1. This induces a C^r-perturbation of $\mathcal{F}$ which has no compact leaves on $int(U(L_j))$.

We use a simple closed curve with trivial holonomy in Step 1. On the other hand, we use a simple closed curve with non-trivial holonomy in Step 2. Therefore we obtain a new foliation which has no compact leaves on $\cup_{j\in J} int(U(L_j))$ after Steps 1 and 2. We have to note that on neighborhoods near ∂B , at most 4 simple closed curves are induced from each δ_j in each reflection leaves L_j since $g \geq 3$, but we can perturb on the other neighborhoods as in the proof of Theorem 3.

We can prove similarly in the case where B has several boundary components. This completes the proof. □

C) The case of foliations with dihedral leaves.

Theorem 10. *If $\mathcal{F}$ has no rotation leaves, but has dihedral leaves, then $\mathcal{F}$ is C^r-unstable* $(1 \leq r \leq \infty)$.

Proof. The leaf space $B = M/\mathcal{F}$ is a 2-V-manifold with boundaries and the quotient map $\pi : M \to B$ is a V-bundle. Reflection leaves and dihedral leaves correspond to the boundary points of B. Now, we may assume that B has only one boundary component ∂B and it is homeomorphic to the line $\mathbf{R}$ or the circle S^1. Let $L_x\,(x \in \partial B)$ be singular leaves of $\mathcal{F}$. Choose saturated neighborhoods $U(L_x)$ as in Proposition 1 and choose an open covering $\{int(U(L_j)); j \in J = \mathbf{Z}/2m\}$ (m is a natural number, but if $\partial B \cong \mathbf{R}$, then $J = \mathbf{Z}$, $m = \infty$) of $\pi^{-1}(\partial B)$ such that 1) $k \equiv j$ or $j \pm 1 \pmod{2m}$ if $int(U(L_k)) \cap int(U(L_j)) \neq \emptyset$ and 2) L_j is a dihedral leaf if and only if j is odd. The open covering of B in the proof of Theorem 9 is modified as in the following Fig. 5.

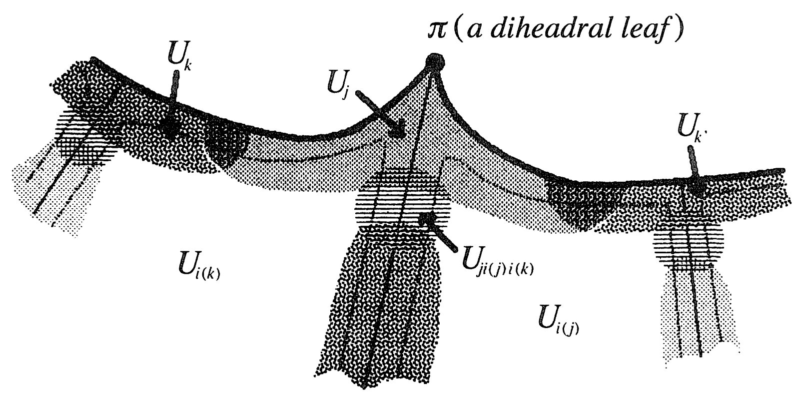


Fig. 5.

First, we perturb $\mathcal{F}$ on $int(U(L_j))$. The holonomy group of a dihedral leaf L_j is generated by generators τ_j, $\tau_{j'}$ which are generators of adjoining reflection leaves on both sides. Let $p_j : L \to L_j$ be the covering maps $(j \in J)$. Then

Step 1. In the case where j is odd, take an arbitrary simple closed curve δ_j on L_j such that

$$[\delta_j] \neq 0,\ [\delta_j] \neq [\tau_j],\ [\delta_j] \neq [\tau_{j'}],\ \langle[\delta_j],[\tau_j]\rangle = \langle[\delta_j],[\tau_{j'}]\rangle = 0.$$

This is possible from Lemma 5. Then we can perturb $\mathcal{F}$ to a foliation which has no compact leaves on $int(U(L_j))$ as in Step 1 of the proof of Theorem 9.

Step 2. In the case where j is even, we can perturb $\mathcal{F}$ to a foliation which has no compact leaves on $int(U(L_j))$ as in Step 2 of the proof of Theorem 9.

Hence we obtain a new foliation which has no compact leaves on $\cup_{j\in J} int(U(L_j))$. We have to note that on neighborhoods meeting $int(U(L_j))$, at most $n(j)+2$ simple closed curves are induced from τ_j and $\tau_{j'}$ in each dihedral leaf , but we can perturb it in the same way as in the proof of Theorem 3 since the genus of the generic leaf $\geq n(j)+1$ and $2\{n(j)+1\}-2 \geq n(j)+2$, where $n(j)$ is the order of the holonomy group of L_j. This completes the proof. □

D) The general case.

Combining Theorems 3, 7, 8, 9 and 10, we have the following.

Theorem 11. *All compact Hausdorff C^r-foliations of C^r-4-manifolds by hyperbolic surfaces are C^r-unstable* $(1 \leq r \leq \infty)$.

M need not be compact, but if M is compact, then codimension two compact foliations of M are always Hausdorff by the results of Epstein[4], Edwards, Millett and Sullivan[3]. Thus we have the following.

Corollary 12. *All compact C^r-foliations of compact C^r-4-manifolds by hyperbolic surfaces are C^r-unstable* $(1 \leq r \leq \infty)$.

References

[1] Bonatti, C., Difféomorphismes commutants des surfaces et stabilité des fibrations en tores, *Topology*, **29** (1990), 101–126.

[2] Bonatti, C. and Firmo, S., Feuilles compactes d'un feuilletage générique en codimension un, *Ann. Sc. Norm. Sup.*, **27** (1994), 407–462.

[3] Edwards, R., Millett, K. and Sullivan, D., Foliation with all leaves compact, *Topology*, **16** (1977), 13–32.

[4] Epstein, D. B. A., Periodic flows on three-manifolds, *Ann. of Math.*, **95** (1976), 68–82.

[5] ———, Foliations with all leaves compact, *Ann. Inst. Fourier, Grenoble*, **26** (1976), 265–282.

[6] ———, A topology for the space of foliations, Geometry and Topology, *Lecture Notes in Math.*, **597** (1976), 132–150.

[7] Fukui, K., Perturbations of compact foliations, *Proc. Japan Acad.*, **58** (1982), 341–344.

[8] ———, Stability of foliations of 3-manifolds by circles, *J. Math. Soc. Japan*, **39** (1987), 117–126.
[9] ———, Stability and instability of certain foliations of 4-manifolds by closed orientable surfaces, *Publ. RIMS, Kyoto Univ.*, **22** (1986), 1155–1171.
[10] Fuller, F., An index of fixed point type for periodic orbits, *Amer. J. Math.*, **89** (1967), 133–148.
[11] Hirsch, M., Stability of compact leaves of foliations, *Dynamical Systems, Acad. Press*, (1971), 135–155.
[12] Langevin, R. and Rosenberg, H., On stability of compact leaves and fibrations, *Topology*, **16** (1977), 107–112.
[13] ———, Integral perturbations of fibrations and a theorem of Seifert, Differential topology, foliations and Gelfand-Fuks cohomology, *Lecture Notes in Math.*, **652** (1978), 122–127.
[14] Plante, J., Stability of codimension one foliations by compact leaves, *Topology*, **22** (1983), 173–177.
[15] Reeb, G., Sur certaines propriétés topologiques des variétés feuilletées, *Act. Sc. et Ind.*, **1183**, Herman, Paris, 1952.
[16] Satake, I., The Gauss-Bonnet theorem for V-manifolds, *J. Math. Soc. Japan*, **9–4** (1957), 464–492.
[17] Seifert, H., Closed integral curves in 3-spaces and isotopic two dimensional deformations, *Proc. A. M. S.*, **1** (1950), 287–301.
[18] Stowe, D., The stationary set of a group action, *Proc. A. M. S.*, **79** (1980), 139–146.
[19] Thurston, W., A generalization of the Reeb stability theorem, *Topology*, **13** (1974), 347–352.
[20] Tsuboi, T., Hyperbolic compact leaves are not C^1-stable, *Geometric study of foliations*, World Sci., Singapore, (1994), 437–455.