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Extended Affine Root System IV (Simply-Laced Elliptic Lie Algebras)

By

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Abstract

Let (R, G) be a pair consisting of an elliptic root system R and a marking G of R. Assume that the attached elliptic Dynkin diagram $\Gamma(R, G)$ is simply-laced. To the simply-laced elliptic root system, we associate three Lie algebras, explained in 1), 2) and 3) below. The main result of the present paper is to show that all three algebras are isomorphic.

1) The first one, studied in §3, is the subalgebra $\mathfrak{g}(R)$ generated by the highest vector e^α for all $\alpha \in R$ in the quotient Lie algebra $V_{Q(R)}/DV_{Q(R)}$ of the lattice vertex algebra attached to the elliptic root lattice $Q(R)$.

2) The second algebra $\mathfrak{e}(\Gamma_{\mathrm{ell}})$, studied in §4, is presented by the Chevalley generators and the generalized Serre relations attached to the elliptic Dynkin diagram $\Gamma_{\mathrm{ell}} = \Gamma(R, G)$.

3) The third algebra $\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'} * \mathfrak{g}_{\mathrm{af}}$, studied in §5, is defined as an amalgamation of an affine Heisenberg algebra and an affine Kac-Moody algebra together with the finite amalgamation relations.

Contents

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§1. Introduction

In the following (1.1) and (1.2), we give a brief overview of the history, the motivation and the results of the present article. Some readers may choose to skip to (1.3).

(1.1) The concepts of the generalized root system and, in particular, the extended affine and the elliptic (= two-extended affine) root system are introduced in the early eighties [Sa3-I]. Since then, there have been several attempts to construct Lie algebras realizing a given root system as the set of its "real roots." The answers were not unique, since there seemed to be no a priori constraint on the size of the center of the algebra. Let us recall some of these works.

The first attempt was due to P. Slodowy [Sl3], who looked at the tensor product of a simple algebra with the algebra of Laurent series of two variables. Wakimoto [W] has constructed some of the representations of these algebras with trivial center action (see also [ISW] for further development). The idea was extended by U. Pollmann [P]. For all types of elliptic root systems, she defined a twisted construction of, so called, biaffine algebras by using the affine-diagram automorphisms. The next attempt was due to H. Yamada [Y1], who constructed the tensor of the affine Kac-Moody algebra with the algebra of Laurent series of one variable by the use of vertex operators [F], [FK]. It is a certain infinite dimensional central extension of the two variables Laurent series extension of a simple algebra. He put a constraint on the size of the center of the algebra in order to get the action of the central extension of the elliptic Weyl group on the algebra. Still, the occurance of an infinite dimensional center in the algebra was a puzzle at the time.

The universal central extension of the tensor of a Lie algebra with a commutative algebra was systematically studied by Kassel [Kas] in terms of Kähler differentials of the algebra. Moody-Eswara-Yokonuma [MEY] studied the case where the commutative algebra is the Laurent series of several variables, and named it *toroidal algebra.* In [Sl2], [Sl4], Slodowy studied a certain generalization of the Kac-Moody algebra, called the *generalized intersection matrices algebra.* He defined the *intersection matrices algebra* as the quotient of the generalized intersection matrix algebra by the ideal generated by the root spaces whose roots have norms larger than two. Then, using the concept of the finite root system grading, Berman et al. [AABGP], [BGK] studied Lie algebras whose real roots are elliptic root systems. Modifying Cartan matrixes of simple Lie algebras, Berman and Moody [BM] constructed an intersection matrixes whose intersection matrix algebras are toroidal algebras.

The general idea to construct a Lie algebra $\mathfrak{g}(R)$ for an arbitrary homogeneous generalized root system R comes from vertex algebras, as we now explain. For an arbitrary even lattice Q Borcherds [Bo1], [Bo2] defined the vertex operator action for all the elements of V_Q, where V_Q is the total Fock space V_Q of level 1 representations of the Heisenberg algebra attached to Q. He has also axiomatized the structure on V_Q as the vertex algebra V and has shown that V/DV carries a Lie algebra structure for the derivation D of V. For a homogeneous generalized root system R, consider the Lie algebra $V_{Q(R)}/DV_{Q(R)}$ attached to the root lattice $Q(R)$. Then, we define $\mathfrak{g}(R)$ as its Lie subalgebra generated by all the highest weight vectors e^α attached to all the elements $\alpha \in R$. If the intersection form of the root lattice $Q(R)$ is degenerate (in fact it is the case for an elliptic root system), we embed the root lattice to a non-degenerate one. Accordingly, we embed $\mathfrak{g}(R)$ into $\tilde{\mathfrak{g}}(R)$, whose Cartan subalgebra $\tilde{\mathfrak{h}}$ is the non-degenerate extension of $\mathfrak{h}$. The algebra $\tilde{\mathfrak{g}}(R)$ admits a finite-dimensional root space decomposition with respect to $\tilde{\mathfrak{h}}$, where the set of the real roots of $\tilde{\mathfrak{g}}(R)$ contains R. The norms of the roots are less than or equal to two (if we normalize R to consist of elements of norms two). Thus, the algebra is a quotient of a certain intersection matrix algebra.

The correspondence $R \mapsto \tilde{\mathfrak{g}}(R)$ works as follows. If R is a finite or affine root system, then $\tilde{\mathfrak{g}}(R)$ is the corresponding finite or affine Kac-Moody algebra, respectively. If R is an extended affine root system, then $\mathfrak{g}(R)$ becomes a toroidal algebra.

In particular, let us call the algebras $\mathfrak{g}(R)$ and $\tilde{\mathfrak{g}}(R)$ attached to an elliptic root system R the *elliptic algebras*. The algebras considered by Yamada and Pollman are quotients of the elliptic algebra $\tilde{\mathfrak{g}}(R)$.

(1.2) We turn to the question of presentations of the algebra $\tilde{\mathfrak{g}}(R)$. If the Witt index of the root lattice $Q(R)$ is less than or equal to 1, then the root system is either finite, affine or hyperbolic. Then it admits Weyl chambers and the system of normal root vectors of the walls of the chamber define a simple root basis. Accordingly, the algebra $\tilde{\mathfrak{g}}(R)$ is presented by Serre relations. If Witt index of $Q(R)$ of a root system R is equal to or greater than two, we can not apply chamber theory. On the other hand, Moody-Eswara-Yokonuma [MEY] have found a presentation of the two-toroidal Lie algebra with infinitely many generators and relations.

One of the main goals of the present article is to give another presentation of the elliptic algebra with finite number of generators and relations (see (4.1) Definition 2), which are still locally nilpotent.

The elliptic root system R has the two-dimensional radical. An arbitrary one-dimensional subspace G of the radical is called a marking (2.3). The pair (R, G) admits a "root basis" $\Gamma(R, G)$, called the *elliptic root basis* (see Sect. 2, or [Sa3-I]), even though there is no longer a good analogue of the Weyl chambers.

The intersection matrix attached to the elliptic root basis is called an *elliptic Cartan matrix*. It contains some positive entries in its off-diagonal parts. Therefore it is not a generalized Cartan matrix [K]. The "classical" Serre relations which describe Kac-Moody algebras are not sufficient to describe the elliptic algebra. Thus it was necessary to find some new relations attached to the elliptic diagram.

A new impetus for the problem came from a description of the elliptic Weyl group. In [SaT3-III], the elliptic Weyl group was presented via a generalization of a Coxeter system as follows: the generators are involutions attached to all vertices of the elliptic diagram, while the relations are Coxeter relations involving two vertices of the elliptic diagram and new relations involving three and four vertices of the diagram. Inspired by this description we asked whether one could find presentations of the elliptic Lie algebras, the elliptic Artin groups, and the elliptic Hecke algebras where the defining relations involve only the same two, three and four vertices of the elliptic diagram (see Remark 2 below).

In the present article, we answer this problem affirmatively for the elliptic Lie algebras: the algebra $\mathfrak{e}(\Gamma_{\mathrm{ell}})$ is generated by a system of the Chevalley basis (i.e. $\mathfrak{sl}_2$-triplets attached at vertices of the elliptic diagram $\Gamma_{\mathrm{ell}} = \Gamma(R, G)$) and is defined by a generalization of the Serre relations involving three or four vertices of the diagram ((4.1) Definition and Theorem 1). We denote by $\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})$ the extension of $\mathfrak{e}(\Gamma_{\mathrm{ell}})$ by the non-degenerate Cartan subalgebra $\tilde{\mathfrak{h}}$. In the course of the proof of the isomorphisms: $\mathfrak{g}(R) \simeq \mathfrak{e}(\Gamma_{\mathrm{ell}})$ and $\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}}) \simeq \tilde{\mathfrak{g}}(R)$, we need to consider the amalgamations $\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'} * \mathfrak{g}_{\mathrm{af}}$ and $\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}}$, where $\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'}$ is the affine Heisenberg algebra, $\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}}$ is the extension of $\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'}$ by $\tilde{\mathfrak{h}}$ and $\mathfrak{g}_{\mathrm{af}}$ is the affine Kac-Moody Lie algebra. This gives the third description of the elliptic algebra. As a by-product of this third presentation, we obtain a generalized triangular decomposition of the elliptic algebra (5.2.2) ([BB] used a similar triangular decomposition to study their representations of the algebras).

Remark 1. The two-extended affine root systems [Sa3-I] describe the (transcendental) lattices generated by vanishing cycles for *simple elliptic singularities* [Sa1]. This is the reason why we call them the elliptic root systems [SaT3-III]. In fact, *the radical of the root system corresponds to the lattice of an elliptic curve, and a rank one subspace of the radical, called a marking, corresponds to a choice of a primitive form for the elliptic singularities* [Sa2]. The elliptic algebra should (conjecturally) serve to reconstruct the primitive form and the period mapping for the elliptic singularities (cf. the simple singularity case [Br], [Sl1], [Sl2], [Ya2]). From the marked elliptic root system, one has already reconstructed the *flat structure on the invariants of elliptic Weyl groups* [Sa3-II], [Sat1], [Sat2] and the *elliptic L-functions* [Sa3-V].

Remark 2. The problems raised in [SaT3-III] on the description of the elliptic Artin groups and Hecke algebras were affirmatively solved by H. Yamada

[Y3]. He rewrote the presentation of the fundamental group of the complement of the discriminant loci for simply elliptic singularities, which had been given by van der Lek, in terms of elliptic diagrams.

The relations for an elliptic Artin group naturally "cover" the relations for an elliptic Weyl group. Still, the relationship between the presentation of the elliptic Artin group and that of the elliptic algebra is not yet clear.

(1.3) Let us give an overview of the contents of the present article.

Section 2 reviews the material from [Sa3-I] on generalized root systems, elliptic root systems R and the elliptic diagrams $\Gamma_{\text{ell}} = \Gamma(R, G)$. Section 3 reviews Borcherds' description of lattice vertex algebras. Then we introduce the Lie algebras $\mathfrak{g}(R)$ and $\tilde{\mathfrak{g}}(R)$ for each homogeneous generalized root system R. In particular, if R is an extended affine root system, then $\mathfrak{g}(R)/\mathfrak{z}(\mathfrak{g}(R))$ turns out to be the finite dimensional simple Lie algebra tensored with the ring of Laurent series of two-variables, where $\mathfrak{z}(\mathfrak{g})$ is the center of the algebra $\mathfrak{g}$.

In (4.1), we introduce the algebras $\mathfrak{e}(\Gamma_{\text{ell}})$ and $\tilde{\mathfrak{e}}(\Gamma_{\text{ell}})$ attached to a simply-laced marked elliptic diagram Γ_{ell}. Both algebras $\tilde{\mathfrak{g}}(R)$ and $\tilde{\mathfrak{e}}(\Gamma_{\text{ell}})$ admit root space decompositions and there is a natural surjective homomorphism $\tilde{\mathfrak{e}}(\Gamma_{\text{ell}}) \to \tilde{\mathfrak{g}}(R)$ compatible with the root space decompositions. The root spaces $\tilde{\mathfrak{e}}(\Gamma_{\text{ell}})_\mu$ with roots μ belonging to the marking G span an extension $\tilde{\mathfrak{h}}_{\text{af}}^{\mathbb{Z}}$ by $\tilde{\mathfrak{h}}$ of the Heisenberg subalgebra $\mathfrak{h}_{\text{af}}^{\mathbb{Z}'}$ in $\tilde{\mathfrak{e}}(\Gamma_{\text{ell}})$. The subalgebras $\mathfrak{h}_{\text{af}}^{\mathbb{Z}'}$, $\tilde{\mathfrak{h}}_{\text{af}}^{\mathbb{Z}}$ and the affine Kac-Moody subalgebra $\mathfrak{g}_{\text{af}}$ attached to an affine subdiagram Γ_{af} of Γ_{ell} in $\tilde{\mathfrak{e}}(\Gamma_{\text{ell}})$ satisfy some simple relations (4.3.6). We consider, in Sect. 5, the abstract amalgamation $\mathfrak{h}_{\text{af}}^{\mathbb{Z}'} * \mathfrak{g}_{\text{af}}$ and $\tilde{\mathfrak{h}}_{\text{af}}^{\mathbb{Z}} * \mathfrak{g}_{\text{af}}$ of the those algebras satisfying those relations.

By definition there is a natural surjective homomorphism $\tilde{\mathfrak{h}}_{\text{af}}^{\mathbb{Z}} * \mathfrak{g}_{\text{af}} \to \tilde{\mathfrak{e}}(\Gamma_{\text{ell}})$ compatible with the root space decompositions. On the other hand, it turns out that the amalgamation algebra admits a generalized triangular decomposition:

$$\tilde{\mathfrak{h}}_{\text{af}}^{\mathbb{Z}} \oplus \mathfrak{n}_{\text{ell}}^{+} \oplus \mathfrak{n}_{\text{ell}}^{-}.$$

This fact leads to the proof that the set of roots of $\tilde{\mathfrak{h}}_{\text{af}}^{\mathbb{Z}} * \mathfrak{g}_{\text{af}}$ coincides with that of $\tilde{\mathfrak{g}}(R)$ and the multiplicities of the real roots are equal to one. It also implies that the derived algebra $\mathfrak{h}_{\text{af}}^{\mathbb{Z}'} * \mathfrak{g}_{\text{af}} = \left(\tilde{\mathfrak{h}}_{\text{af}}^{\mathbb{Z}} * \mathfrak{g}_{\text{af}}\right)'$ of $\tilde{\mathfrak{h}}_{\text{af}}^{\mathbb{Z}} * \mathfrak{g}_{\text{af}}$ is a central extension of $\mathfrak{g}(R)$. By the universality of the central extension $\mathfrak{g}(R) \to \mathfrak{g}(R)/\mathfrak{z}(\mathfrak{g}(R))$ and the perfectness of the algebra $\mathfrak{h}_{\text{af}}^{\mathbb{Z}'} * \mathfrak{g}_{\text{af}}$ we obtain the inverse homomorphism $\mathfrak{g}(R) \to \mathfrak{h}_{\text{af}}^{\mathbb{Z}'} * \mathfrak{g}_{\text{af}}$. These facts inmply the isomorphisms:

$$\mathfrak{g}(R) \simeq \mathfrak{e}(\Gamma_{\text{ell}}) \simeq \mathfrak{h}_{\text{af}}^{\mathbb{Z}'} * \mathfrak{g}_{\text{af}}$$

and, therefore, the isomorphism:

$$\tilde{\mathfrak{g}}(R) \simeq \tilde{\mathfrak{e}}(\Gamma_{\text{ell}}) \simeq \tilde{\mathfrak{h}}_{\text{af}}^{\mathbb{Z}} * \mathfrak{g}_{\text{af}}.$$

This proves the main result of the present paper.

Finally we note that the algebra $\tilde{\mathfrak{g}}(R)$ does not depend on the choice of the marking G, but the other algebras $\tilde{\mathfrak{e}}(\Gamma_{\rm ell})$ and $\tilde{\mathfrak{h}}^{\mathbb{Z}}_{\rm af} * \mathfrak{g}_{\rm af}$ do depend on this choice, i.e. that of an element of $PSL(2,\mathbb{Z})$. So, an ambiguity of the triangular decomposition of an elliptic algebra (more exactly, an ambiguity of the subalgebra $\tilde{\mathfrak{h}}^{\mathbb{Z}}_{\rm af} \oplus \mathfrak{n}^{+}_{\rm ell}$) depends on an element of $PSL(2,\mathbb{Z})$. Full study of this fact (i.e. $PSL(2,\mathbb{Z})$ action on the elliptic flag variety) is beyond the scope of the present article.

List of Relations. For the convenience of the reader, we list below the reference numbers for the relations of the algebras $\tilde{\mathfrak{g}}(R)$, $\tilde{\mathfrak{e}}(\Gamma(R,G))$, $\mathfrak{g}(A_{\triangle})$, $\mathfrak{g}_{\rm af} = \mathfrak{e}(\Gamma_{\rm af})$, $\mathfrak{h}^{\mathbb{Z}'}_{\rm af}$, $\tilde{\mathfrak{h}}^{\mathbb{Z}}_{\rm af}$ and $\tilde{\mathfrak{h}}^{\mathbb{Z}}_{\rm af} * \mathfrak{g}_{\rm af}$ studied in the present article.

- Relations in the lattice vertex Lie algebra $\tilde{\mathfrak{g}}(Q) = V_Q/DV_Q$:
  (3.1.8) **0**, **I**, **II.1**, **II.2**, **III** and **IV**.
- Relations for the elliptic Lie algebra $\tilde{\mathfrak{e}}(\Gamma(R,G))$:
  (4.1.1) **0**, **I**, **II.1**, **II.2**, **III**, **IV**, and **V**.
- Relations for the affine Kac-Moody algebra $\mathfrak{g}(A_{\triangle})$:
  (4.2.5) **A-0**, **A-I**, **A-II.1**, **A-II.2** and **A-II.3**.
- Relations for the affine Kac-Moody algebra $\mathfrak{g}_{\rm af} = \mathfrak{e}(\Gamma_{\rm af}) \simeq \mathfrak{g}(\Gamma_{\rm af})$:
  (4.1.1) **0**, **I**, **II.1** and **II.2**.
- Relations for the Heisenberg algebras $\mathfrak{h}^{\mathbb{Z}'}_{\rm af}$ and $\tilde{\mathfrak{h}}^{\mathbb{Z}}_{\rm af}$:
  (4.3.6) **H-I** and **H-II**.
- Amalgamation relations among $\mathfrak{h}^{\mathbb{Z}'}_{\rm af}$ and $\mathfrak{g}_{\rm af}$:
  (4.3.6), (5.1.1) **I***, **II*.1** and **II*.2**.

Notation. (1) For a sequence of elements $x_1, x_2, x_3, \dots, x_n$ of a Lie algebra, put:

$$[x_1, x_2, x_3, \dots, x_n] := [[\cdots[[x_1, x_2], x_3], \dots, x_{n-1}], x_n]$$

and call it a *multi-bracket of length n*. For any s with $1 < s \le n$ by successive applications of the Jacobi identity one gets an identity:

$$\begin{aligned}[x_1, x_2, x_3, \dots, x_n, y] \; = \; & [x_1, \dots, x_{s-1}, y, x_s, x_{s+1}, \dots, x_n] \\ & + [x_1, \dots, x_{s-1}, [x_s, y], x_{s+1}, \dots, x_n] \\ & + [x_1, \dots, x_{s-1}, x_s, [x_{s+1}, y], \dots, x_n] \\ & \qquad \cdots \\ & + [x_1, \dots, x_{s-1}, x_s, x_{s+1}, \dots, [x_n, y]].\end{aligned}$$

We shall refer to this transformation as *"delivering y to the left"*.

(2) For a subset S of a root system R in F, we put

$$\pm S := S \cup (-S) = S \cup \{-s \mid s \in S\}.$$

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§2. Generalized Root Systems and Elliptic Root Systems

The notions of a generalized root system and an extended affine root system were introduced in [Sa3-I]. We recall the classification of 2-extended affine root systems (= elliptic root systems) in terms of elliptic diagrams together with some related facts.

(2.1) Let (F, q) be a pair of $\mathbb{R}$-vector space F and a quadratic form q on it with bounded rank, that is, the set of dimensions of all the non-singular subspace of F is bounded from above. The bilinear form $I : F \times F \to \mathbb{R}$ is attached to q by $I(x, y) := q(x + y) - q(x) - q(y)$ and $q(x) = I(x, x)/2$. Let $\operatorname{rad} F := F^{\perp}$ be the *radical* of q. The signature $\operatorname{sig} q = (\mu_+, \mu_0, \mu_-)$ is the triplet with μ_+ (resp. μ_-) := the maximum rank of the positive (resp. negative) definite subspaces of F and $\mu_0 := \operatorname{rank}(\operatorname{rad} F)$. We call $I(\alpha, \alpha) = 2q(\alpha)$ the *norm* of $\alpha \in F$. A vector $\alpha \in F$ of nonzero norm is called *non-isotropic*. For a non-isotropic vector α, put $\alpha^\vee := \alpha/q(\alpha)$. We have $\alpha^{\vee\vee} = \alpha$. Define the reflection w_α on F by $w_\alpha(u) := u - I(\alpha^\vee, u)\alpha$, which satisfies $w_\alpha^2 = \mathrm{id}_F$ and $I \circ w_\alpha = I$.

Definition. A set R of non-isotropic vectors of F is called a *generalized root system* belonging to (F, q) if it satisfies the following conditions:

1) The additive subgroup $Q(R)$ generated in F by R is a *full lattice* of F (i.e. $Q(R) \otimes_{\mathbb{Z}} \mathbb{R} \simeq F$).

2) For all α and $\beta \in R$, one has $I(\alpha^\vee, \beta) \in \mathbb{Z}$.

3) For all $\alpha \in R$, the reflection w_α preserves the set R.

4) If $R = R_1 \cup R_2$ and $R_1 \perp R_2$ with respect to q, then either R_1 or R_2 is empty.

A root system R is called *reduced* if $\mathbb{Q}\alpha \cap R = \{\pm\alpha\}$ for any $\alpha \in R$. The subgroup $Q(R) \subset F$ is called the *root lattice* for the root system R. The group $W(R)$ generated by the reflections w_α for all $\alpha \in R$ is called the *Weyl group* of the root system R. Two root systems are *isomorphic* if there is an isomorphism between the ambient vector spaces which induces a bijection between the sets of roots. For any subspace H of $rad(q)$ which is defined over $\mathbb{Q}$, the image set of R in the quotient space F/H is a root system, called the *quotient root system* modulo H, and is denoted by R/H. In particular, $R/\operatorname{rad} q$ is called the *radical quotient* of R.

Definition. We say that a subset $\Pi \subset R$ *generates* the root system R if $R = W(\Pi)\Pi$, where $W(\Pi) := \langle w_\alpha \mid \alpha \in \Pi \rangle$.

If Π generates R then the following holds: i) $Q(R) = \mathbb{Z}\Pi := \sum_{\alpha\in\Pi} \mathbb{Z}\alpha$, ii) $W(R) = W(\Pi)$.

One can show that the set $\{q(\alpha) \mid \alpha \in R\}$ is finite and that the proportion $q(\alpha)/q(\beta)$ for any $\alpha, \beta \in R$ is a rational number. Hence the integer:

$$t(R) := \operatorname{lcm}\{q(\alpha) \mid \alpha \in R\}/\gcd\{q(\alpha) \mid \alpha \in R\}, \tag{2.1.1}$$

called the *total tier number* of the root system, is a well-defined positive integer. So, up to a constant factor, the bilinear form I may be assumed to take rational values on $Q(R)$. In particular, by choosing a constant factor c such that $\gcd\{c \cdot q(\alpha) \mid \alpha \in R\} = 1$, we define the normalized forms:

$$q_R := c \cdot q, \qquad I_R := c \cdot I \tag{2.1.2}$$

(the constant c will be referred to as $I_R : I$). Then $Q(R)$ becomes an even lattice with respect to the form q_R (and I_R). We shall always consider the vector space F to be equipped with the integral lattice structure $Q(R)$ and the rational structure $F_\mathbb{Q} := \mathbb{Q} \otimes_\mathbb{Z} Q(R)$. A root system R is called *homogeneous* if $t(R) = 1$, and, hence, $q_R(\alpha) = \pm 1$ for $\alpha \in R$.

Let R be a root system. The set $R^\vee := \{\alpha^\vee \mid \alpha \in R\}$, called the *dual* of R, is also a root system with respect to the same quadratic form q. It satisfies $q(\alpha^\vee) = 1/q(\alpha)$ for all $\alpha \in R$, and $t(R^\vee) = t(R)$. The sets R and $R^\vee$ span the same vector space F, but the lattice $Q(R^\vee) := \mathbb{Z}R^\vee$ (equipped with the normalized quadratic form $q_{R^\vee} = (I_{R^\vee} : I) \cdot q$ satisfying $q_R(\alpha) \cdot q_{R^\vee}(\alpha^\vee) = t(R)$) may define a different $\mathbb{Z}$-structure on F.

(2.2) We call R a *k-extended affine root system of rank l* if q is positive semi-definite with $\operatorname{rank}(\operatorname{rad} q) = k$ and $\operatorname{rank} Q(R) = l + k$.

One has the equivalence: $\#R < \infty \iff \#W(R) < \infty \iff q$ is definite $\iff k = 0$. This is the case studied in the classical literature. The root systems are classified into types A_l $(l \geq 1)$, B_l $(l \geq 2)$, C_l $(l \geq 3)$, D_l $(l \geq 4)$, E_l $(l = 6, 7, 8)$, F_4 and G_2.

If $k = 1$, then R turns out to be an affine root system in the sense of Macdonald [Mc] (cf. [K], [MP]). These are classified by the types $P^{(t_1)}$, where P is the type of the finite root system $R_{\mathrm{f}} := R/\operatorname{rad} q$ and t_1 is called the first tier number (an integer satisfying $t_1|t(R)$).

If $k = 2$, then R is called an elliptic root system [SaTa-III]. In most cases these are classified by the types $P^{(t_1,t_2)}$, where P is the type of the finite root system $R_{\mathrm{f}} = R/\operatorname{rad} q$ and t_1, t_2 are called the first and the second tier numbers (integers satisfying $t_1|t(R)$, $t_2|t(R)$) explained in (2.4).

In general, let R be a homogeneous k-extended affine root system. Then by a suitable choice of the basis $a_1, \dots, a_k$ of the radical $\operatorname{rad} q$, one has an expression of the set of roots

$$(2.2.1) \qquad \begin{aligned} R &= R_{\mathrm{f}} \oplus \mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_k \\ &= \{\alpha + n_1 a_1 + \cdots + n_k a_k \mid \alpha \in R_{\mathrm{f}},\ n_i \in \mathbb{Z} \text{ for } i = 1, \dots, k\} \end{aligned}$$

where R_{f} is a "splitting sub-root system of R" defined in a positive subspace F_{f} of F and is isomorphic to the radical quotient finite root system $R/\operatorname{rad} q$ of R.

If $k \leq 1$, then the Weyl group $W(R)$ acts properly discontinuously on a domain in F^*. The fundamental domain of the action bounded by the reflection hyperplanes is called a *Weyl chamber*. Then the set $\Gamma(R)$ of those roots which are normal to the walls of a fixed Weyl chamber gives a *simple root basis* of R and defines the classical or affine Dynkin diagram according to whether R is a finite or an affine root system. If $k \geq 2$, then the Weyl group acts nowhere properly discontinuously on F or F^*, so that there is no concept of a Weyl chamber. Nevertheless, for the elliptic root system R (i.e. $k = 2$), by a use of a marking G (see (2.3)), we introduce an *elliptic root basis* $\Gamma(R, G)$ of R which leads to a definition of an elliptic Dynkin diagram for (R, G). We recall some more details on this in the following (2.3)–(2.6).

(2.3) Let R be an elliptic root system of rank l. That is: R is a generalized root system whose quadratic form q is positive semi-definite with a two-dimensional radical, where $l := \operatorname{rank}(F/\operatorname{rad} q) = \operatorname{rank} F - 2$. If R is elliptic, so is $R^\vee$.

Definition. A *marking* of an elliptic root system R is a rank one subspace G of $\operatorname{rad} F$ defined over $\mathbb{Q}$. The pair (R, G) is called a *marked elliptic root system.*

The dual $R^\vee$ is also marked by the same space G. We choose and fix integral basis a and $a^\vee$ of the marking G:

$$(2.3.1) \qquad \mathbb{Z}a = G_{\mathbb{Z}} := G \cap Q(R) \quad \text{and} \quad \mathbb{Z}a^\vee = G^\vee_{\mathbb{Z}} := G \cap Q(R^\vee),$$

satisfying $a^\vee : a > 0$ (here $a^\vee$ is not $(a)^\vee$ for the operation $(\ )^\vee$ defined in (2.1)). The marking G induces a projection

$$(2.3.2) \qquad \pi_G : \ Q(R) \to Q_{\mathrm{af}} := Q(R)/G_{\mathbb{Z}}.$$

The image $R_{\mathrm{af}} := R/G := \pi_G(R)$ is an affine root system belonging to the affine root lattice Q_{af}. We assume R_{af} to be *reduced.* For the rest of the present paper, we choose and fix a simple root basis $\Gamma(R_{\mathrm{af}})$ of the affine root system R_{af} and a set $\Gamma_{\mathrm{af}} \subset R$ such that the projection π_G induces a bijection $\pi_G|\Gamma_{\mathrm{af}} \xrightarrow{\sim} \Gamma(R_{\mathrm{af}})$. In particular, the intersection matrix $\big(I(\alpha^\vee,\beta)\big)_{\alpha,\beta\in\Gamma_{\mathrm{af}}}$ equals the affine Cartan matrix attached to $\Gamma(R_{\mathrm{af}})$. The set Γ_{af} is unique up to an isomorphism of (R,G). We identify Γ_{af} with the affine Dynkin diagram $\Gamma(R_{\mathrm{af}})$, the lattice $\bigoplus_{\alpha\in\Gamma_{\mathrm{af}}}\mathbb{Z}\alpha$ with Q_{af}, and the set of roots $R\cap\bigoplus_{\alpha\in\Gamma_{\mathrm{af}}}\mathbb{Z}\alpha$ with R_{af}, respectively. The following three properties are well-known: i) Γ_{af} forms a basis of Q_{af} such that the affine roots R_{af} are contained in $Q^+_{\mathrm{af}} \cup Q^-_{\mathrm{af}}$ where $Q^\pm_{\mathrm{af}} := \big(\pm\sum_{\alpha\in\Gamma_{\mathrm{af}}}\mathbb{Z}_{\geq 0}\alpha\big)\backslash\{0\}$. ii) There exist positive integers $\{n_\alpha\in\mathbb{Z}_{>0}\}_{\alpha\in\Gamma_{\mathrm{af}}}$ such that $\sum_{\alpha\in\Gamma_{\mathrm{af}}} n_\alpha\alpha$ is a generator of the radical of Q_{af} (null roots of Q_{af}). iii) There exists $\alpha_0\in\Gamma_{\mathrm{af}}$ with $n_{\alpha_0}=1$ such that $\Gamma_{\mathrm{f}} := \Gamma_{\mathrm{af}}\setminus\{\alpha_0\}$ is a Dynkin diagram of the finite root system $R_{\mathrm{f}} = R/\operatorname{rad} q$ with the root lattice $Q_f := \bigoplus_{\alpha\in\Gamma_{\mathrm{f}}}\mathbb{Z}\alpha$. So, the root lattice $Q(R)$ and the radical $\operatorname{rad} Q(R)$ split over $\mathbb{Z}$ as:

$$Q(R) = Q_{\mathrm{af}}\oplus\mathbb{Z}a = Q_{\mathrm{f}}\oplus\mathbb{Z}b\oplus\mathbb{Z}a, \tag{2.3.3}$$

$$\operatorname{rad} Q(R) = \mathbb{Z}b\oplus\mathbb{Z}a, \tag{2.3.4}$$

where b is a lifting of the generator of the null roots in Q_{af}.

$$b := \sum_{\alpha\in\Gamma_{\mathrm{af}}} n_\alpha\alpha. \tag{2.3.5}$$

Similarly, $\Gamma^\vee_{\mathrm{af}} := \{\alpha^\vee \mid \alpha\in\Gamma_{\mathrm{af}}\} \subset R^\vee$ is bijective to a simple root basis of the dual affine root system $R^\vee_{\mathrm{af}}$. So, we get a generator $b^\vee := \sum_{\alpha^\vee\in\Gamma^\vee_{\mathrm{af}}} n_{\alpha^\vee}\alpha^\vee$ of the radical of $Q(R^\vee_{\mathrm{af}})$.

(2.4) We introduce the first and the second tier numbers t_1, t_2 of a marked elliptic root system (R,G) and its dual $(R^\vee,G)$ as follows. These will describe some subtle relations between the two integral structures $Q(R)$ and $Q(R^\vee)$, and will be used to define the type of a marked elliptic root system.

$$\begin{aligned} t_1(R,G) &:= (b^\vee : b)\cdot(I_{R^\vee} : I), \quad & t_1(R^\vee,G) &:= (b : b^\vee)\cdot(I_R : I),\\ t_2(R,G) &:= (a^\vee : a)\cdot(I_{R^\vee} : I), \quad & t_2(R^\vee,G) &:= (a : a^\vee)\cdot(I_R : I). \end{aligned} \tag{2.4.1}$$

These are positive integers satisfying the relation

$$t(R) = t_1(R,G)\cdot t_1(R^\vee,G) = t_2(R,G)\cdot t_2(R^\vee,G). \tag{2.4.2}$$

The isomorphism class of (R,G), in most cases, is determined by the triplet (P,t_1,t_2) where P is the type of the finite root system $R/\operatorname{rad} q$ and t_1 and t_2 are the first and the second tier numbers of (R,G). The symbol $P^{(t_1,t_2)}$ is called the *type* of (R,G).

Let us define some further numerical invariants which lead to the definition of the exponents and the definition of the elliptic root basis $\Gamma(R,G)$. For $\alpha \in R$, put

(2.4.3) $$k(\alpha) := \inf\{k \in \mathbb{Z}_{>0} \mid \alpha + k \cdot a \in R\} \quad \text{and} \quad \alpha^* := \alpha + k(\alpha) \cdot a$$

(and define $k^\vee(\alpha^\vee)$ similarly for $\alpha^\vee \in R^\vee$). Then, one has the following proportionality relations: for $\alpha \in \Gamma_{\mathrm{af}}$,

(2.4.4) $$\frac{n_{\alpha^\vee}}{n_\alpha} = \frac{t_1(R,G)}{q_{R^\vee}(\alpha^\vee)}, \quad \frac{k^\vee(\alpha^\vee)}{k(\alpha)} = \frac{q_{R^\vee}(\alpha^\vee)}{t_2(R,G)}.$$

Definition. (1) The set of *exponents* (resp. *dual exponents*) of (R,G) is the union of $\{0\}$ and $\{m_\alpha \mid \alpha \in \Gamma_{\mathrm{af}}$ (resp. $\{m_{\alpha^\vee} \mid \alpha \in \Gamma_{\mathrm{af}}$), where

(2.4.5) $$m_\alpha := \frac{q(\alpha)}{k(\alpha)} \cdot n_\alpha \quad (\text{resp.} \quad m_{\alpha^\vee} := \frac{q(\alpha^\vee)}{k^\vee(\alpha^\vee)} \cdot n_{\alpha^\vee})$$

for $\alpha \in \Gamma_{\mathrm{af}}$.
(2) Put $m_{\max} := \max\{m_\alpha \mid \alpha \in \Gamma_{\mathrm{af}}\}$ and

(2.4.6) $$\Gamma_{\max} := \{\alpha \in \Gamma_{\mathrm{af}} \mid m_\alpha = m_{\max}\}, \quad \Gamma^*_{\max} := \{\alpha^* \mid \alpha \in \Gamma_{\max}\}.$$

An *elliptic root basis* is the union of $\Gamma_{\max}$ and $\Gamma^*_{\max}$:

(2.4.7) $$\Gamma(R,G) := \Gamma_{\mathrm{af}} \cup \Gamma^*_{\max}.$$

Fact 1. *The set $\Gamma(R,G)$ generates ((2.1) Definition) the elliptic root system R.*

Note that the proportionalities in (2.4.4) imply the proportionalities: $m_{\alpha^\vee}/m_\alpha = t_1(R,G) \cdot t_2(R,G)/t(R)$ and therefore $\Gamma(R^\vee,G) = \Gamma(R,G)^\vee$. The $\left(a_{\alpha,\beta} := I(\alpha^\vee,\beta)\right)_{\alpha,\beta\in\Gamma(R,G)}$, called the *elliptic Cartan matrix,* is not a generalized Cartan matrix in the sense of Kac-Moody theory [K] because of the positive off-diagonal entry $a_{\alpha,\alpha^*} = 2$ for $\alpha \in \Gamma_{\max}$.

(2.5) The *elliptic diagram* (which we shall identify with the elliptic root basis $\Gamma(R,G)$) is defined by the following rule:

i) vertices are in one-to-one correspondence with $\Gamma(R,G)$,

ii) the type of the bond between the vertices α, $\beta \in \Gamma(R,G)$ is defined according to the value $a_{\alpha,\beta}/a_{\beta,\alpha}$ by the usual convention (e.g. [B Chap.VI, §4 n°4.2]), except for the new additional convention: a double dotted bond o===o if $a_{\alpha,\beta} = a_{\beta,\alpha} = 2$ (i.e. between vertices α and α^* for $\alpha \in \Gamma_{\max}$).

Fact 2 ([Sa3-I] Theorem (9.6)). *The elliptic diagram is uniquely determined by the isomorphism class of (R, G). Conversely, the elliptic diagram $\Gamma(R, G)$ determines uniquely the isomorphism class of the marked elliptic root system (R, G) together with an elliptic root basis which is identified with the vertices of $\Gamma(R, G)$.*

Let us review briefly the reconstruction of the root system (R, G) from the diagram $\Gamma(R, G)$.

a) Put $\hat{F} := \bigoplus_{\alpha\in\Gamma(R,G)} \mathbb{R}\alpha$.

b) Difine a symmetric bilinear form $\hat{I} : \hat{F} \times \hat{F} \to \mathbb{R}$ satisfying the relations $\frac{2\hat{I}(\alpha,\beta)}{\hat{I}(\alpha,\alpha)} = a_{\alpha,\beta}$ and $\hat{I}(\alpha, \alpha) > 0$ for $\alpha, \beta \in \Gamma(R, G)$.

c) Put $\widehat{W} := \langle \widehat{w}_\alpha \mid \alpha \in \Gamma(R, G)\rangle$ where $\widehat{w}_\alpha$ is the reflection on $\widehat{F}$ with respect to α.

d) Define the pre-Coxeter element $\widehat{c} := \prod_{\alpha\in\Gamma(R,G)} \widehat{w}_\alpha$ where $\widehat{w}_{\alpha^*}$ comes next to $\widehat{w}_\alpha$ for $\alpha \in \Gamma_{\max}$. Then one has:

(i) *The eigenvalues of $\hat{c}$ are given by* 1 *and* $\exp(2\pi\sqrt{-1}m_\alpha/m_{\max})$ *for* $\alpha \in \Gamma_{\mathrm{af}}$.

(ii) *The image of* $1 - \hat{c}^{m'_{\max}}$ *is contained in* $\mathrm{rad}\,\hat{I}$ *and is spanned by* $\frac{\alpha^*-\alpha}{k(\alpha)} - \frac{\beta^*-\beta}{k(\beta)}$ *for* $\alpha,\ \beta \in \Gamma_{\max}$, *where* $m'_{\max} :=$ *the least common denominator of the* $m_\alpha/m_{\max}$ *for* $\alpha \in \Gamma_{\mathrm{af}}$.

Fact 3. *Put* $F := \hat{F}/(1 - \hat{c}^{m'_{\max}})\hat{F}$, $I :=$*the form on* F *induced from* $\hat{I}$, $G :=$ *the subspace in* F *spanned by* $\alpha^* - \alpha$ *for* $\alpha \in \Gamma_{\max}$ *and* $R :=$*the image set of* $\hat{W} \cdot \Gamma(R, G)$. *Then* R *is an elliptic root system belonging to* (F, I) *with the marking* G. *The image set in* F *of the vertices of* $\Gamma(R, G)$ *forms an elliptic root basis of the elliptic root system. The root lattice in* F *generated by* $\Gamma(R, G)$ *is given by*

$$Q(R) = \mathbb{Z}\Gamma(R, G)/\langle\frac{\alpha^* - \alpha}{k(\alpha)} - \frac{\beta^* - \beta}{k(\beta)} \mid \alpha,\ \beta \in \Gamma_{\max}\rangle, \tag{2.5.1}$$

$$Q(R^\vee) = \mathbb{Z}\Gamma(R, G)^\vee/\langle\frac{\alpha^{*\vee} - \alpha^\vee}{k^\vee(\alpha^\vee)} - \frac{\beta^{*\vee} - \beta^\vee}{k^\vee(\beta^\vee)} \mid \alpha,\ \beta \in \Gamma_{\max}\rangle. \tag{2.5.2}$$

(2.6) A marked elliptic root system (R, G) is called *simply-laced* if its diagram $\Gamma(R, G)$ consists only of simply-laced bonds o——o and doubly dotted bonds o===o.

Fact 4. *A simply-laced elliptic root system* (R, G) *is homogeneous. Hence* $t(R) = t_1(R) = t_2(R) = 1$, $m'_{max} = m_{max}$, *and* $k(\alpha) = 1$ *for all* $\alpha \in R$. *The set of roots decomposes* (*cf.* (2.3.3)) :

$$R = R_{\mathrm{af}} + \mathbb{Z}a = R_{\mathrm{f}} + \mathbb{Z}b + \mathbb{Z}a. \tag{2.6.1}$$

The simply-laced elliptic root systems are of types $A_l^{(1,1)}$ for $l \geq 2$, $D_l^{(1,1)}$ for $l \geq 4$ and $E_l^{(1,1)}$ for $l = 6, 7, 8$, whose diagrams and exponents are exhibited in Appendix A.

§3. The Lie Algebra $\tilde{\mathfrak{g}}(R)$ Associated to a Generalized Root System

Borcherds [Bo1] has introduced a Lie algebra V_Q/DV_Q as a quotient of the vertex algebra V_Q attached to an even lattice Q. The aim of this section is to introduce the Lie algebras $\mathfrak{g}(R)$ and $\tilde{\mathfrak{g}}(R)$ attached to a generalized root system R as subalgebras of $V_{Q(R)}/DV_{Q(R)}$ ((3.2) Definition 1). We recall the construction of V_Q/DV_Q in the first half of this section (cf. [G], [GN], [K2] and [MN]).

(3.1) We review the construction of the lattice vertex algebra in our context. Let Q be an even lattice with an integral symmetric bilinear form I attached to a quadratic form q such that $q(x) = I(x,x)/2$. There is a canonical central extension: $0 \to \mathbb{Z}/2\mathbb{Z} \to \hat{Q} \to Q \to 0$ defined by the skew symmetric form $I \bmod 2$. Fixing a section $e : Q \mapsto \hat{Q}$, $\alpha \mapsto e^\alpha$, we have the product rule: $e^\alpha e^\beta = \kappa^{I(\alpha,\beta)} e^\beta e^\alpha$, where κ is the multiplicative generator of the center $\mathbb{Z}/2\mathbb{Z}$. The following two are equivalent: 1) giving an additive cocycle $\varepsilon : Q \times Q \to \mathbb{Z}/2\mathbb{Z}$ such that $\varepsilon(\alpha,\beta) + \varepsilon(\beta,\alpha) \equiv I(\alpha,\beta) \bmod 2$, and 2) giving a product rule: $e^\alpha e^\beta = \kappa^{\varepsilon(\alpha,\beta)} e^{\alpha+\beta}$. In this article, we shall assume $\varepsilon(\alpha,\alpha) = I(\alpha,\alpha)/2 \bmod 2$ for any $\alpha \in Q$. Let $\mathbb{Q}\{Q\}$ be the quotient of the group ring $\mathbb{Q}[\hat{Q}]$ divided by the ideal generated by $1 + \kappa$. The image of the section $\{e^\alpha \mid \alpha \in Q\}$ (denoted by the same symbol) gives a basis of $\mathbb{Q}\{Q\}$ with the product rule $e^\alpha e^\beta = (-1)^{\varepsilon(\alpha,\beta)} e^{\alpha+\beta}$.

Put $F_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} Q$ and let $\tilde{F}_{\mathbb{Q}}$ be a $\mathbb{Q}$ vector space equipped with a non-degenerate symmetric bilinear form $\tilde{I}$ such that i) $\tilde{F}_{\mathbb{Q}}$ contains $F_{\mathbb{Q}}$ as a subspace, ii) the restriction of $\tilde{I}$ on $F_{\mathbb{Q}}$ coincides with I. Such $\tilde{F}_{\mathbb{Q}}$, having the lowest rank (=rank Q + rank(rad Q)) is unique, and shall be called the *non-degenerate hull* of $F_{\mathbb{Q}}$. We identify $\tilde{F}_{\mathbb{Q}}$ with its dual space $\tilde{F}_{\mathbb{Q}}^*$ by $\tilde{I} : \tilde{F}_{\mathbb{Q}} \to \tilde{F}_{\mathbb{Q}}^*$; $x \mapsto \tilde{I}(x)(y) = \langle \tilde{I}(x), y \rangle := \tilde{I}(x,y)$. Put $\tilde{\mathfrak{h}} := \tilde{F}_{\mathbb{Q}}^* = \tilde{I}(\tilde{F}_{\mathbb{Q}})$, $\mathfrak{h} := \tilde{I}(F_{\mathbb{Q}})$ (note that $\mathfrak{h}$ is not the dual space $F_{\mathbb{Q}}^*$ of $F_{\mathbb{Q}}$ if I is degenerate) and $h_x := \tilde{I}(x) \in \tilde{\mathfrak{h}}$ for any $x \in \tilde{F}_{\mathbb{Q}}$. We introduce the form $\tilde{I}^*$ on $\tilde{\mathfrak{h}}$ by $\tilde{I}^*(\tilde{I}(x), \tilde{I}(y)) := \tilde{I}(x,y)$ for any $x, y \in \tilde{F}_{\mathbb{Q}}$.

Put

$$(3.1.1)\qquad V_Q := S\left(\bigoplus_{n\in\mathbb{Z}_{>0}} \tilde{\mathfrak{h}}(-n)\right)\otimes \mathbb{Q}\{Q\},$$

where $\mathcal{S} := S\big(\bigoplus_{n\in\mathbb{Z}_{>0}} \tilde{\mathfrak{h}}(-n)\big)$ is the symmetric tensor algebra of the direct sum of an infinite sequence $\{\tilde{\mathfrak{h}}(-n)\}_{n\in\mathbb{Z}_{>0}}$ of copies of $\tilde{\mathfrak{h}}$ (copies of an element $h\in\tilde{\mathfrak{h}}$ are denoted by $h(-1), h(-2), \dots$). Then V_Q has the following structures i)–iv).

i) As the tensor product of algebras, V_Q is an algebra.

ii) For any $h\in\tilde{\mathfrak{h}}$ and $n\in\mathbb{Z}$, we define the left-operator $h(n)$ on V_Q as follows: if $n<0$ then $h(n)$ is multiplication by $h(n)$. If $n=0$ then $h(0)e^\alpha = \langle h,\alpha\rangle e^\alpha$ for any $\alpha\in Q$. If $n>0$ then $h(n)e^\alpha = 0$ for any $\alpha\in Q$. One has the rule: $[\,h(m), g(n)] = m\delta_{m+n,0}\tilde{I}^*(h,g)$ for any $h,g\in\tilde{\mathfrak{h}}$ and $m,n\in\mathbb{Z}$.

iii) The algebra V_Q has the Cartan involution ω : $\omega(e^\alpha) = e^{-\alpha}$, $\omega(h) = -h$ for any $\alpha\in Q$ and $h\in\tilde{\mathfrak{h}}$.

iv) There is a linear map deg : $V_Q\to V_Q$ such that $\deg e^\alpha = q(\alpha)e^\alpha$ and $\deg h(-n)v = h(-n)(nv+\deg v)$ for any $\alpha\in Q$, $h\in\tilde{\mathfrak{h}}$ and $v\in V_Q$. We say $u\in V_Q$ is *homogeneous of degree* n if $\deg u = nu$ $(n\in\mathbb{Z})$. V_Q is $\mathbb{Z}$-graded by this degree: $V_Q = \bigoplus_{n\in\mathbb{Z}} V_{Qn}$.

v) The algebra V_Q is Q-graded. That is: $V_Q = \bigoplus_{\alpha\in Q}(V_Q)_\alpha$, where $(V_Q)_\alpha := \mathcal{S}\otimes e^\alpha$. An element $u\in(V_Q)_\alpha$ is said to have the *grade* α.

For any $n\in\mathbb{Z}$ we define the n-th product, denoted by $u_{(n)}v$, of $u = h_1(-n_1)\cdots h_k(-n_k)e^\alpha\in V_Q$ ($h_1,\dots,h_k\in\tilde{\mathfrak{h}}$, $\alpha\in Q$ and $k\geq 0$) and $v\in V_Q$ by

$$(3.1.2)\qquad \begin{aligned} u_{(n)}v \;:=\;& \text{the coefficient of } z^{-n-1} \text{ in} \\ & \left({}^\circ_\circ Q(h_1(-n_1),z)\cdots Q(h_k(-n_k),z)\exp\left(Q(h_\alpha(0),z)\right)e^\alpha {}^\circ_\circ\right)v, \end{aligned}$$

where for $h\in\tilde{\mathfrak{h}}$ and $n\geq 0$, we put

$$(3.1.3)\qquad Q(h(-n),z) := \frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^n\left(\sum_{i\neq 0}\frac{h(-i)}{i}z^i + h(0)\log(z)\right).$$

Here "${}^\circ_\circ X{}^\circ_\circ$" is the "normal ordering of X ", where one rearranges the ordering of products in the formal expression of X in such a way that the creation operators $h(-i)$ $(i\geq 1)$ occur to the left of all annihilation operators $h(i)$ $(i\geq 1)$ and e^α occur to the left of operators $h(0)$. Note that $\exp(h_\alpha(0)\log(z))e^\beta = e^\beta z^{I(\alpha,\beta)}$. Extending (3.1.2) linearly in u, we make the vector space V_Q equipped with countably many bilinear operations (n). Then, the system of these operations defines the vertex algebra structure on V_Q [Bo1], [FLM].

For any $h \in \tilde{\mathfrak{h}}$, $u \in V_Q$ and $n \in \mathbb{Z}$, we have $h(-n)_{(-1)}u = h(-n)u$ by definition. So, V_Q is generated as (-1)-th product algebra by $h(-n)$ and e^α for $h \in \tilde{\mathfrak{h}}$, $n \in \mathbb{Z}_{>0}$ and $\alpha \in Q$. Define an operator

$$D: \ V_Q \to V_Q\ , \quad a \mapsto \ a_{(-2)}1. \tag{3.1.4}$$

Then, for any $n \in \mathbb{Z}$ and $u, v \in V_Q$ we have

$$D(u_{(n)}v) = (Du)_{(n)}v + u_{(n)}Dv. \tag{3.1.5}$$

In addition, for any $h \in \tilde{\mathfrak{h}}$, $\alpha \in Q$ and $n \in \mathbb{Z}_{>0}$,

$$De^\alpha = h_\alpha(-1)e^\alpha, \quad Dh(-n) = nh(-n-1), \tag{3.1.6}$$

so D is a homogeneous operator of degree 1.

Fact 5 (Borcherds[Bo1]). *The product $u_{(0)}v$ for $u, v \in V_Q$ induces a Lie algebra structure on the quotient space V_Q/DV_Q and a left V_Q/DV_Q-module structure on V_Q.*

In this section, let us tentatively denote the algebra V_Q/DV_Q by $\tilde{\mathfrak{g}}(Q)$.

We shall use the same symbols to express an element in $\tilde{\mathfrak{g}}(Q)$ as an element in V_Q. The Lie bracket of $\tilde{\mathfrak{g}}(Q)$ is given by $[u, v] = u_{(0)}v$. The algebra $\tilde{\mathfrak{g}}(Q)$ inherits $\mathbb{Z}$- and Q-grading structures since D is homogeneous and preserves the Q-grading.

If $u, v \in V_Q$ have degrees $l, m \in \mathbb{Z}$ respectively then $u_{(n)}v$ has degree $l + m - n - 1$. So, the subspace V_{Q1} is closed under the 0-th product.

Fact 6. *$\tilde{\mathfrak{g}}(Q)_1 = (V_Q/DV_Q)_1 \simeq V_{Q1}/DV_{Q0}$ is a Lie subalgebra of $\tilde{\mathfrak{g}}(Q)$.*

Here, we recall some terminologies from [K], [MP]. Given a Lie algebra $\mathfrak{g}$ and its abelian subalgebra $\tilde{\mathfrak{h}}$, we say an element $x \in \mathfrak{g}$ is a *weight vector* of *weight* $\alpha \in \tilde{\mathfrak{h}}^*$ if $[h, x] = \langle h, \alpha\rangle x$ for all $h \in \tilde{\mathfrak{h}}$. Let $\mathfrak{g}_\alpha$ be the set of all elements which have weight α. If $\mathfrak{g}_\alpha \neq \{0\}$, α is called a *root* and $\mathfrak{g}_\alpha$ is called its *root space.* If $\mathfrak{g}$ is spanned (as a vector space) by root spaces, then we say that $\mathfrak{g}$ *has a root space decomposition* with respect to $\tilde{\mathfrak{h}}$.

Let $u \in V_Q$ be an element of the grade $\alpha \in Q$. Then $h(-1)_{(0)}u = \langle h, \alpha\rangle u$ for any $h \in \tilde{\mathfrak{h}}$ (use (3.1.2)) and u is a weight vector of weight α with respect to $\tilde{\mathfrak{h}}(-1)$. This means that the concepts weight and grade coincide. Therefore $\tilde{\mathfrak{g}}(Q)$ has the root space decomposition with respect to $\tilde{\mathfrak{h}}(-1)$ and $\tilde{\mathfrak{g}}(Q)_\alpha = \big(\mathcal{S}/(D+h_\alpha(-1))\mathcal{S}\big) \otimes e^\alpha$ for $\alpha \in Q(R)$. The subalgebra $\tilde{\mathfrak{g}}(Q)_1$ has a root space decomposition with respect to $\tilde{\mathfrak{h}}(-1)$. The norms $I(\alpha, \alpha) = 2q(\alpha)$ of every root α of $\tilde{\mathfrak{g}}(R)$ is less than or equal to two (since if $v = h_1(-n_1) \cdots h_k(-n_k)e^\alpha$ is a nontrivial weight vector, then $q(\alpha) \leq q(\alpha) + \sum_i n_i =: \deg v = 1)$). Some of the root spaces are described easily as follows.

$$\begin{array}{lrcll} (3.1.7) & \tilde{\mathfrak{g}}(Q)_{1,\alpha} & = & \mathbb{Q}e^{\alpha} & \text{for} \quad q(\alpha) = 1, \\ & \tilde{\mathfrak{g}}(Q)_{1,\mu} & = & \big(\tilde{\mathfrak{h}}(-1)/\mathbb{Q}\mu(-1)\big)\, e^{\mu} & \text{for} \quad q(\mu) = 0. \end{array}$$

A root $\alpha \in Q$ such that $q(\alpha) = 1$ (resp. $q(\alpha) \leq 0$) is called a *real root* (resp. an *imaginary root*).

Let us list some bracket formulae in $\tilde{\mathfrak{g}}(Q)$, which can be calculated from the definition (3.1.2) (or refer [GN] (2.119)–(2.123)). We shall use them in the present article without mentioning it explicitly. Put $Q_1 := \{\alpha \in Q \mid q(\alpha) = 1\}$. In the following formulae, $\alpha, \beta \in Q$, $\bar{h}, \bar{g} \in \mathfrak{h}$, $h, g \in \tilde{\mathfrak{h}}$ and $\mu, \lambda \in \operatorname{rad} Q$:

$$\begin{array}{lll} & \mathbf{0.} & [h(-1), g(-1)] = 0, \\ & \mathbf{I.} & [h(-1), e^{\alpha}] = \langle h, \alpha \rangle e^{\alpha}, \\ & & [h(-1), \bar{g}(-1)e^{\mu}] = \langle h, \mu \rangle \bar{g}(-1)e^{\mu}, \\ (3.1.8) & \mathbf{II.1.} & [e^{\alpha}, e^{\beta}] = \begin{cases} 0 & \text{if} \quad I(\alpha,\beta) \geq 0 \\ (-1)^{\varepsilon(\alpha,\beta)} e^{\alpha+\beta} & \text{if} \quad I(\alpha,\beta) = -1 \\ (-1)^{\varepsilon(\alpha,\beta)} h_{\alpha^\vee}(-1) e^{\alpha+\beta} & \text{if} \quad I(\alpha,\beta) = -2 \end{cases} \\ & & \text{for} \quad \alpha, \beta \in Q_1, \\ & \mathbf{II.2.} & \operatorname{ad}(e^{\alpha})^{1-\langle h_{\alpha^\vee}, \beta \rangle} e^{\beta} = 0 \quad \text{if} \quad I(\alpha,\beta) \leq 0 \text{ and } \alpha \in Q_1, \\ & \mathbf{III.} & [\bar{h}(-1)e^{\mu}, e^{\alpha}] = (-1)^{\varepsilon(\mu,\alpha)} \langle \bar{h}, \alpha \rangle e^{\alpha+\mu}, \\ & \mathbf{IV.} & [\bar{h}(-1)e^{\mu}, \bar{g}(-1)e^{\lambda}] = (-1)^{\varepsilon(\mu,\lambda)} \tilde{I}^*(\bar{h}, \bar{g}) h_{\mu}(-1) e^{\mu+\lambda}. \end{array}$$

Remark 3. The above construction of the vertex algebra V_Q and the Lie algebra $\tilde{\mathfrak{g}}(Q)$ attached to an even lattice Q works completely parallel, even when we replace the space $F_{\mathbb{Q}}$ at the begining by its quotient space $F_{\mathbb{Q}}/H$ for an arbitrary linear subspace $H \subset \operatorname{rad} F_{\mathbb{Q}}$ and replace I by its induced form on $F_{\mathbb{Q}}/H$ but keep $\mathbb{Q}\{Q\}$ the same. Tentatively, let the resulting algberas be denoted by $V_{Q,H}$ and $\tilde{\mathfrak{g}}_H(Q)$, respectively. They are naturally the quotient algebras of V_Q and $\tilde{\mathfrak{g}}(Q)$.

(3.2) Let R be a homogeneous generalized root system and I_R the even lattice structure (2.1.2) on the root lattice $Q(R)$. In the rest of this section we use the normalized bilinear form I_R but will denote it by I for short.

Definition 1. Define subalgebras $\tilde{\mathfrak{g}}(R)$ and $\mathfrak{g}(R)$ of $\tilde{\mathfrak{g}}(Q(R))$ by

$$\begin{array}{lrcl} (3.2.1) & \tilde{\mathfrak{g}}(R) & := & \langle h(-1), e^{\alpha} \mid h \in \tilde{\mathfrak{h}},\ \alpha \in R \rangle, \\ & \mathfrak{g}(R) & := & \langle e^{\alpha} \mid \alpha \in R \rangle, \end{array}$$

where $\langle * \rangle$ denotes the Lie algebra over $\mathbb{Q}$ generated by the set $*$.

The algebra $\tilde{\mathfrak{g}}(R)$ is a subalgebra of $\tilde{\mathfrak{g}}(Q(R))_1$ (since all the generators are in $\tilde{\mathfrak{g}}(Q(R))_1$ and in view of (3.1) Fact 6). It inherits the $Q(R)$-grading structure from $\tilde{\mathfrak{g}}(Q(R))_1$. Since all the generators are weight vectors, the algebra $\tilde{\mathfrak{g}}(R)$ has a root space decomposition with respect to $\tilde{\mathfrak{h}}(-1)$. If α is a root of $\tilde{\mathfrak{g}}(R)$, then $q(\alpha) \leq 1$. For every $\alpha \in R$, $\{e := e^\alpha, h := h_{\alpha^\vee}, f := -e^{-\alpha}\}$ is a standard $\mathfrak{sl}_2$-triplet, i.e. $[h, e] = 2e, [h, f] = -2f$ and $[e, f] = h$. Note that $\mathfrak{g}(R)$ is the derived algebra of $\tilde{\mathfrak{g}}(R)$ (i.e. $\mathfrak{g}(R) = [\tilde{\mathfrak{g}}(R), \tilde{\mathfrak{g}}(R)]$) and it is perfect (i.e. $\mathfrak{g}(R) = [\mathfrak{g}(R), \mathfrak{g}(R)]$).

For $S \subset R$, we consider the subalgebra $\mathfrak{g}(S)$ of $\mathfrak{g}(R)$:

$$\mathfrak{g}(S) := \langle e^\alpha \mid \alpha \in \pm S \rangle. \tag{3.2.2}$$

Assertion 1. *For any subset $S \subset R$, one has $\mathfrak{g}(S) = \mathfrak{g}(W(S)S)$. In particular, if Π generates R ((2.1) Definition), then $\{e^\alpha \mid \alpha \in \pm\Pi\}$ generates the algebra $\mathfrak{g}(R)$.*

Proof. First, we show that $e^{w_\alpha \beta} \in \langle e^\alpha, e^\beta \rangle$. Since the degree one space is closed under the 0-th product, we have $e^{w_\alpha(\beta)} = \text{const} \cdot (\operatorname{ad} e^\alpha)^{-I(\alpha^\vee, \beta)} e^\beta$ for roots $\alpha, \beta \in R$ s.t. $I(\alpha, \beta) \leq 0$. It is enough to show $(\operatorname{ad} e^\alpha)^{-I(\alpha^\vee, \beta)} e^\beta \neq 0$ since the real root space is one dimensional (3.1.7). If it were zero then applying $(\operatorname{ad} e^{-\alpha})^{-I(\alpha^\vee, \beta)}$ we would get $e^\beta = 0$ and a contradiction. If $I(\alpha, \beta) > 0$, consider the pair $-\alpha$ and β.

Let us return to the proof. Any element of $W(S)S$ is an image of S by successive applications of reflections with respect to elements of S. Applying the claim above successively, we obtain $\mathfrak{g}(W(S)S) = \mathfrak{g}(S)$. □

Let R be a homogeneous k-extended affine root system of rank l (2.2). In this case, we have an explicit description of $\tilde{\mathfrak{g}}(R)$:

$$\tilde{\mathfrak{g}}(R) = \tilde{\mathfrak{h}}(-1) \oplus \bigoplus_{\alpha \in R} \mathbb{Q} e^\alpha \oplus \bigoplus_{\mu \in \operatorname{rad} Q(R) \setminus \{0\}} (\mathfrak{h}(-1)/\mathbb{Q}\mu(-1)) e^\mu. \tag{3.2.3}$$

If we replace $\tilde{\mathfrak{h}}(-1)$ by $\mathfrak{h}(-1)$ in (3.2.3), then we get an explicit description of $\mathfrak{g}(R)$. Put $N := \tilde{I}(\operatorname{rad} F_{\mathbb{Q}}) \subset \tilde{\mathfrak{h}}$. Then it is not hard to see the following:
i) $\mathfrak{z}(\mathfrak{g}(R)) = \oplus_{\mu \in \operatorname{rad} Q(R)} N(-1) e^\mu$,
ii) $\mathfrak{g}(R)/\mathfrak{z}(\mathfrak{g}(R))$ is isomorphic to $\mathfrak{g}_f \otimes \mathbb{Q}[e^{\pm a_1}, \dots, e^{\pm a_k}]$, where $\mathfrak{g}_f$ is the simple Lie algebra associated to the finite root system $R/\operatorname{rad} F$.

The following lemma is due to [MEY].

Lemma 1. *$\mathfrak{g}(R)$ is the universal central extension of $\mathfrak{g}(R)/\mathfrak{z}(\mathfrak{g}(R))$ and perfect. That is: by definition, $\mathfrak{g}(R)$ is the k-toroidal Lie algebra.*

The next assertion is easy to show.

Assertion 2. *If R is a simply-laced finite or affine root system, then $\mathfrak{g}(R)$ is isomorphic to a finite or affine Kac-Moody algebra, respectively.*

Proof. Take a simple root basis Γ of R. Take a proper cocycle ε as in (3.1). Then the Serre relations are satisfied by the Chevalley generator system $\{\alpha^\vee := h_{\alpha^\vee}(-1), e_\alpha := e^\alpha, f_\alpha := -e^{-\alpha}\}_{\alpha\in\Gamma}$ and $h_{\alpha^\vee}(-1)$ for $\alpha \in \Gamma$ are linearly independent. Apply Gabber-Kac's theorem. □

For a root lattice $Q(R) = Q_{\mathrm{af}} \oplus \mathbb{Z}a$ of a simply-laced elliptic root system R, we always use $\mathbb{Z}$-bilinear cocycle ε satisfying:

$$\text{(3.2.4)}\qquad \left.\begin{matrix} \varepsilon(\alpha, a) = \varepsilon(a, \alpha) = 0 \\ \varepsilon(\alpha,\alpha) = \frac{I(\alpha,\alpha)}{2} \end{matrix}\right\} \text{ for any } \alpha \in Q(R).$$

This choice simplifies the formulae in $\tilde{\mathfrak{g}}(R)$ as

$$\text{(3.2.5)}\qquad [e^\alpha, e^{-\alpha}] = -h_{\alpha^\vee} \text{ for any } \alpha \in R.$$

The following is a construction of such a cocycle ε. Recall from the splitting (2.3.3) that the set $\Gamma_{\mathrm{af}} \cup \{a\}$ forms a $\mathbb{Z}$-basis of $Q(R)$. Index $\Gamma_{\mathrm{af}} = \{\alpha_0, \dots, \alpha_l\}$ and $\alpha_{-1} := a$ tentatively, and define $\varepsilon(\alpha_i, \alpha_j)$ to be $I(\alpha_i, \alpha_j)$ mod 2 if $i > j$, $I(\alpha_i, \alpha_i)/2$ mod 2 if $i = j$, otherwise 0. Extend them $\mathbb{Z}$-linearly to $Q(R)$.

Remark 4. Recall the notation $\tilde{\mathfrak{g}}_H(Q)$ of Remark 3 at the end of (3.1). Let us denote by $\tilde{\mathfrak{g}}_H(R)$ the subalgebra of $\tilde{\mathfrak{g}}_H(Q(R))$ generated by e^α for all $\alpha \in R$. Then $\tilde{\mathfrak{g}}_{\mathrm{rad}\, F_{\mathbb{Q}}}(R) = \tilde{\mathfrak{g}}(R)/\mathfrak{z}(\mathfrak{g}(R))$. For a marked elliptic root system (R, G), $\tilde{\mathfrak{g}}_G(R)$ is the algebra studied by Yamada [Y1].

§4. The Elliptic Lie Algebra Presented by Generators and Relations

In section 3, we have introduced the Lie algebras $\mathfrak{g}(R)$ and $\tilde{\mathfrak{g}}(R)$ attached to a simply-laced elliptic root system R. In this section, we introduce the second Lie algebras $\mathfrak{e}(\Gamma_{\mathrm{ell}})$ and $\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})$ attached to the ellpitic diagram $\Gamma_{\mathrm{ell}} = \Gamma(R, G)$ of a simply-laced *marked* elliptic root system (R, G). These algebras are presented by generators and relations determined by Γ_{ell}. In (4.1) Theorem 1, we state the main result of the present article: the isomorphism of the two algebras $\tilde{\mathfrak{g}}(R)$ and $\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})$.

The rest of this article is devoted to the proof of the theorem. This section gives a preparation by studying subalgebras of $\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})$. In (4.2), we study the subalgebras $\mathfrak{e}(A)$ of $\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})$ attached to the A-parts A of the diagram Γ_{ell}, which turn out to be the affine Kac-Moody algebras $\mathfrak{g}(A_\triangle)$ with the generator system $A_\triangle$ associated to A. In (4.3) we consider the subalgebra $\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'}$ of $\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})$. The subalgebra $\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'}$ turns out to be a Heisenberg algebra and all the weights of the elements of $\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'}$ belong to the marking $\mathbb{Z}a$.

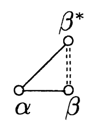
(4.1) Let $\Gamma_{\mathrm{ell}} = \Gamma(R, G)$ be the elliptic diagram of a simply-laced marked elliptic root system (R, G) (2.6). Let us fix or recall some notation. As in (2.5) a)–d), we reconstruct the root lattice $Q(R)$ (2.5.1) from Γ_{ell}, where Γ_{ell} is identified with an elliptic root basis of R as a subset of $Q(R)$. As in (3.1) put $F_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} Q(R)$ and let $(\tilde{F}_{\mathbb{Q}}, \tilde{I})$ be its non-degenerate hull. The space $\tilde{F}_{\mathbb{Q}}$ is identified with its dual space $\tilde{\mathfrak{h}} := \mathrm{Hom}_{\mathbb{Q}}(\tilde{F}_{\mathbb{Q}}, \mathbb{Q})$ by $\tilde{I} : x \mapsto h_x$, where $h_x \in \tilde{\mathfrak{h}}$ is defined by $h_x(y) := \tilde{I}(x, y)$ for any $y \in \tilde{F}_{\mathbb{Q}}$. Recall that $a \in Q(R)$ denotes the base of the marking space G (2.3.1) and $\alpha^* - \alpha = a$ for any $\alpha \in \Gamma_{\max}$ (see (2.4.3) and (2.6) Fact 4). Hence we have $h_{\alpha^{*\vee}} - h_{\alpha^\vee} = h_{a^\vee}$.

Definition 2. The Lie algebra $\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})$ is the algebra presented by the following generators and relations:

Generators: $\tilde{\mathfrak{h}}$ and $\{E^\alpha \mid \alpha \in \pm\Gamma_{\mathrm{ell}}\}$

Relations:

(4.1.1)
$$\begin{array}{ll} \mathbf{0.} & \tilde{\mathfrak{h}} \text{ is abelian,} \\ \mathbf{I.} & [h, E^\alpha] = \langle h, \alpha \rangle E^\alpha, \\ \mathbf{II.1.} & [E^\alpha, E^{-\alpha}] = -h_{\alpha^\vee}, \\ & [E^\alpha, E^\beta] = 0 \quad \text{for} \quad I(\alpha, \beta) \geq 0, \\ \mathbf{II.2.} & (ad\, E^\alpha)^{1 - \langle h_{\alpha^\vee}, \beta \rangle} E^\beta = 0 \quad \text{for} \quad I(\alpha, \beta) \leq 0, \end{array}$$

III. $$[E^\alpha, E^\beta, E^{\beta^*}] = 0 \qquad [E^{-\alpha}, E^{-\beta}, E^{-\beta^*}] = 0$$ for 

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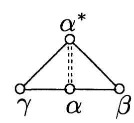
IV. $$[E^\beta, E^\alpha, E^\gamma, E^{\alpha^*}] = 0 \qquad [E^{-\beta}, E^{-\alpha}, E^{-\gamma}, E^{-\alpha^*}] = 0$$ for 

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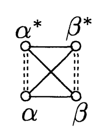
V. $$[E^{\alpha^*}, E^{-\alpha}, E^\beta] = E^{\beta^*} \qquad [E^{-\alpha^*}, E^\alpha, E^{-\beta}] = E^{-\beta^*}$$ for 

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where h runs over $\tilde{\mathfrak{h}}$ in **I**, α, β run over $\pm\Gamma_{\mathrm{ell}}$ in **I**, **II**, and α, β, γ run over $\pm\Gamma_{\mathrm{af}}$ in **III**, **IV** and **V**.

Remark 5. The definition of a root system is invariant under the scalar multiplication cI of the form I for $c \in \mathbb{Q} \setminus \{0\}$. We use $h_{\alpha^\vee}$ instead of h_α for

$\alpha \in R$ so that the result does not depend on the choice of c, where $\langle h_{\alpha^\vee}, \alpha\rangle = 2$. Similarly, $h_{\alpha^\vee}$ does not move after rescaling of I (see (2.4.1) where $t_2(R,G)$ is fixed to 1 in the present article). See also the proof of (4.3) Lemma 4 **H-II**.

Remark 6. The relations **0–II** are the well-known Kac-Moody type relations. The relations **III–V** are new relations caused by the new type bonds $\circ === \circ$ in $\Gamma(R,G)$. Note that the relations **III–V** reduce to the classical relations if we replace $E^{\pm\alpha^*}$ by $E^{\pm\alpha}$ for $\alpha \in \Gamma_{\max}$.

The following is the main theorem of the present article.

Theorem 1. *The following correspondence extends to an isomorphism from $\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})$ to $\tilde{\mathfrak{g}}(R)$:*

$$\text{(4.1.2)} \qquad \begin{array}{cccll} h & \mapsto & h(-1) & \text{for} & h \in \tilde{\mathfrak{h}}, \\ E^\alpha & \mapsto & e^\alpha & \text{for} & \alpha \in \pm\Gamma_{\mathrm{ell}}. \end{array}$$

Here, the cocycle ε is so chosen to satisfy $\varepsilon(\alpha,a) = \varepsilon(a,\alpha) = 0$ and $\varepsilon(\alpha,\alpha) = I(\alpha,\alpha)/2$ for any $\alpha \in Q(R)$.

The rest of this article is devoted to the proof of the theorem.

Assertion 3. *The map defined in* (4.1.2) *extends to a surjective Lie homomorphism φ*

$$\text{(4.1.3)} \qquad \varphi : \tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}}) \to \tilde{\mathfrak{g}}(R).$$

Proof. We can check the vanishing of the φ-images of the defining relations (4.1.1) of $\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})$ using (3.1.8). The surjectivity follows from (2.4) Fact 1 and (3.2) Assertion 1. □

The following facts follow immediately from the definitions.

Facts. i) The algebra $\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})$ has the root space decomposition with respect to $\tilde{\mathfrak{h}}$ and the set of roots is contained in $Q(R)$. The root space of a root α is denoted by $\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})_\alpha$.
ii) $\phi(\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})_\alpha) = \tilde{\mathfrak{g}}(R)_\alpha$ for any $\alpha \in Q(R)$.
iii) There exists and involution ω, called the Cartan involution, on $\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})$ defined by $h \mapsto -h$ for $h \in \tilde{\mathfrak{h}}$ and $E^\alpha \mapsto E^{-\alpha}$ for $\alpha \in \pm\Gamma_{\mathrm{ell}}$.
iv) Since $\mathrm{ad}\, E^\alpha : x \mapsto [E^\alpha, x]$ is a locally nilpotent derivation for every $\alpha \in \pm\Gamma_{\mathrm{ell}}$, we have a Lie algebra automorphism of $\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})$:

$$\mathfrak{n}_\alpha := (\exp \mathrm{ad}\, E^\alpha)(\exp \mathrm{ad}\, E^{-\alpha})(\exp \mathrm{ad}\, E^\alpha).$$

(Note that the sign of the middle factor is not equal to the usual one because of the minus sign in the relation **II.1.**) One verifies directly that the restriction of $\mathfrak{n}_\alpha$ on $\tilde{\mathfrak{h}}$ coincides with the reflection $\tilde{w}_\alpha$ on $\tilde{\mathfrak{h}}$: $\tilde{w}_\alpha(h) := h - \langle h, \alpha \rangle h_{\alpha^\vee}$. This implies $\mathfrak{n}_\alpha$ induces an isomorphism $\tilde{\mathfrak{e}}_\beta \simeq \tilde{\mathfrak{e}}_{w_\alpha \beta}$. Therefore the elliptic Weyl group $W(R)$ acts on the set of all roots of $\tilde{\mathfrak{e}}(\Gamma_{\text{ell}})$.

For a subset $S \subset \Gamma_{\text{ell}}$, we consider a subalgebra $\mathfrak{e}(S)$ of $\tilde{\mathfrak{e}}(\Gamma_{\text{ell}})$:

$$\mathfrak{e}(S) := \langle E^\alpha | \, \alpha \in \pm S \rangle. \tag{4.1.4}$$

The subalgebra $\mathfrak{e}(\Gamma_{\text{ell}})$ is an example. Note that the derived algebra of $\tilde{\mathfrak{e}}(\Gamma_{\text{ell}})$ is $\mathfrak{e}(\Gamma_{\text{ell}})$.

Assertion 4. *If a subset $S \subset \Gamma_{\text{ell}}$ is linearly independent in $F_{\mathbb{Q}}$ and its intersection matrix $(I(\alpha^\vee, \beta))_{\alpha,\beta \in S}$ forms a generalized Cartan matrix (i.e. off-diagonal parts are non positive), then $\mathfrak{e}(S)$ is canonically isomorphic to the (derived) Kac-Moody algebra constructed from the Cartan matrix, and also to $\mathfrak{g}(S)$.*

Proof. The elements $\{E^\alpha, h_{\alpha^\vee}, -E^{-\alpha}\}_{\alpha \in S}$ satisfy the Kac-Moody relations and linear independency of S implies the isomorphism. For the last claim, see (3.2) Assertion 2. □

Applying this assertion to Γ_{af}, we have $\mathfrak{e}(\Gamma_{\text{af}}) \simeq \mathfrak{g}(\Gamma_{\text{af}})$.

(4.2) This subsection is aimed to prove Lemma 2, which plays a crucial role in the proof of Theorem 1 in the sequel.

Definition 3. *An A-part A* of $\Gamma(R, G)$ is a union of a maximal linear subdiagram A_{af} of Γ_{af} and $\{\alpha^* \mid \alpha \in A_{\text{af}} \cap \Gamma_{\max}\}$. That is: $A = A_{\text{af}} \cup (A_{\text{af}} \cap \Gamma_{\max})^*$.

Explicit list of A-parts is given in Appendix A. The following facts follow directly from the definition.

Facts. i) For any $\alpha, \beta \in \Gamma(R, G)$, there exists an A-part A which contains α and β (note that $A_1^{(1,1)}$ is not simply-laced).
ii) $A_{\text{af}} \cap \Gamma_{\max} \neq \varnothing$.

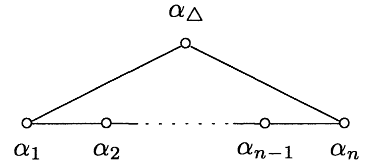
Lemma 2. *For any given A-part A, the restriction of φ* (4.1.5) *on $\mathfrak{e}(A)$ gives an isomorphism onto $\mathfrak{g}(A)$.*

Proof. Since $\varphi|_{\mathfrak{e}(A)}$ is a surjective homomorphism, it is sufficient to show an existence of a surjective homomorphism $\Upsilon : \mathfrak{g}(A) \to \mathfrak{e}(A)$ such that $\Upsilon \circ \varphi = \text{id}_{\mathfrak{e}(A)}$.

Put $A_{\mathrm{af}} = A \cap \Gamma_{\mathrm{af}} =: \{\alpha_1, \dots, \alpha_n\}$ such that $I(\alpha_i^\vee, \alpha_{i+1}) = -1$ for $i = 1, \dots, n-1$. We fix an element $\alpha_s \in A \cap \Gamma_{\max}$ for $1 \le s \le n$. Put $A_{s^*} := A_{\mathrm{af}} \cup \{\alpha_s^*\}$,

$$\text{(4.2.1)} \qquad \begin{aligned} \alpha_\triangle &:= -\alpha_1 - \cdots - \alpha_{s-1} - \alpha_s^* - \alpha_{s+1} - \cdots - \alpha_n, \\ &= -a - \sum_{\alpha \in A_{\mathrm{af}}} \alpha \end{aligned}$$

and $A_\triangle := A_{\mathrm{af}} \cup \{\alpha_\triangle\}$. The diagram of $A_\triangle$ is given below.



Then, $W(A_{s^*})A_{s^*} = W(A)A = W(A_\triangle)A_\triangle$ and hence $\mathfrak{g}(A_{s^*}) = \mathfrak{g}(A_\triangle)$ ((3.2) Assertion 1). Note $\mathfrak{e}(A_{s^*}) = \mathfrak{e}(A)$ due to the relation $\mathbf{V}$ in (4.1.1).

Let us define a map Υ_s: $\{e^\alpha, e^{-\alpha} \mid \alpha \in A_\triangle\} \to \tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})$. First, we put

$$\text{(4.2.2)} \qquad \Upsilon_s : e^\alpha \mapsto E^\alpha \quad \text{for} \quad \alpha \in \pm A_{\mathrm{af}}.$$

In order to define the images of $e^{\pm\alpha_\triangle}$, we consider an automorphism $M_{\alpha_\triangle}$ of $\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})$:

$$\text{(4.2.3)} \qquad M_{\alpha_\triangle} := \mathfrak{n}_{\alpha_1} \cdots \mathfrak{n}_{\alpha_{s-1}} \mathfrak{n}_{\alpha_s^*} \mathfrak{n}_{\alpha_{s+1}} \cdots \mathfrak{n}_{\alpha_{n-1}},$$

and put

$$E_s^{\alpha_\triangle} := v M_{\alpha_\triangle} E^{-\alpha_n}, \qquad E_s^{-\alpha_\triangle} := v M_{\alpha_\triangle} E^{\alpha_n},$$

where $v \in \{\pm 1\}$ is defined by the relation

$$\text{(4.2.4)} \qquad \begin{aligned} e^{\alpha_\triangle} &= v[e^{-\alpha_1}, \dots, e^{-\alpha_{s-1}}, e^{-\alpha_s^*}, e^{-\alpha_{s+1}}, \dots, e^{-\alpha_n}], \\ e^{-\alpha_\triangle} &= v[e^{\alpha_1}, \dots, e^{\alpha_{s-1}}, e^{\alpha_s^*}, e^{\alpha_{s+1}}, \dots, e^{\alpha_n}]. \end{aligned}$$

Put $\Upsilon_s : e^{\pm\alpha_\triangle} \mapsto E_s^{\pm\alpha_\triangle}$.

Let us see that the map Υ_s extends to a homomorphism from $\mathfrak{g}(A)$ to $\mathfrak{e}(A)$. Since $\mathfrak{g}(A) = \mathfrak{g}(A_\triangle)$ is a Kac-Moody algebra with respect to the generators $\{e^\alpha, h_{\alpha^\vee}, e^{-\alpha} \mid \alpha \in A_\triangle\}$, it is enough to show the vanishing of the Υ_s-images in $\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})$ of the defining relations of the affine Kac-Moody algebra $\mathfrak{g}(A_\triangle)$:

$$\text{(4.2.5)} \qquad \begin{array}{lll} \textbf{A-0.} & [h_{\alpha^\vee}(-1), h_{\beta^\vee}(-1)] = 0, & \\ \textbf{A-I.} & [h_{\alpha^\vee}(-1), e^\beta] = \langle h_{\alpha^\vee}, \beta \rangle e^\beta, & \\ \textbf{A-II.1.} & [e^\alpha, e^{-\alpha}] = -h_{\alpha^\vee}(-1), & \\ \textbf{A-II.2.} & [e^\alpha, e^\beta] = 0 & \text{for} \quad I(\alpha, \beta) \ge 0, \\ \textbf{A-II.3.} & (ad\, e^\alpha)^{1 - \langle h_{\alpha^\vee}, \beta \rangle} e^\beta = 0 & \text{for} \quad I(\alpha, \beta) \le 0, \end{array}$$

where $\alpha, \beta \in \pm A_\triangle$. Furthermore, it is enough to check only the relations involving $\alpha_\triangle$. **A-0** and **A-I** vanish by **0** and **I** in (4.1.1). **A-II.1**: $[E_s^{\alpha_\triangle}, E_s^{-\alpha_\triangle}] = [vM_{\alpha_\triangle}E^{-\alpha_n}, vM_{\alpha_\triangle}E^{\alpha_n}] = M_{\alpha_\triangle}[E^{-\alpha_n}, E^{\alpha_n}] = \tilde{w}_{\alpha_1}\cdots\tilde{w}_{\alpha_{n-1}}h_{\alpha_n} = -h_{\alpha_\triangle}$. **A-II.2**: Let us show $[E_s^{-\alpha_\triangle}, E^{\alpha_i}] = 0$ for $i = 1, \ldots, n$ ($[E_s^{\alpha_\triangle}, E^{-\alpha_i}] = 0$ can be shown similarly). First we consider the case $i \neq s$. The relation is inside a finite Kac-Moody subalgebra $\mathfrak{e}(A_{s^*} \setminus \{\alpha_s\})$ of type A_n (see (4.1) Assertion 3). We know $\alpha_\triangle - \alpha_i$ is not a root of $\mathfrak{e}(A_{s^*} \setminus \{\alpha_s\})$, and we have done. Next, the case $i = s$. We use the following formulae, which are easily shown.

Formula. Let $\alpha, \beta \in \pm\Gamma(R, G)$.
If $I(\alpha, \beta^\vee) = 0$, $\mathfrak{n}_\alpha E^\beta = E^\beta$. If $I(\alpha, \beta^\vee) = -1$,

$$\begin{aligned} \mathfrak{n}_\alpha E^\beta &= [E^\alpha, E^\beta], & \mathfrak{n}_\alpha E^{-\beta} &= [E^{-\alpha}, E^{-\beta}], \\ \mathfrak{n}_\alpha^{-1} E^\beta &= -[E^\alpha, E^\beta], & \mathfrak{n}_\alpha^{-1} E^{-\beta} &= -[E^{-\alpha}, E^{-\beta}]. \end{aligned} \tag{4.2.6}$$

$$\mathfrak{n}_\alpha E^{-\alpha} = E^\alpha, \quad \mathfrak{n}_\alpha E^\alpha = E^{-\alpha}. \tag{4.2.7}$$

Using these formulae, the definition (4.2.4) is rewritten as

$$\begin{aligned} E_s^{\alpha_\triangle} &= v[E^{-\alpha_1}, \ldots, E^{-\alpha_{s-1}}, E^{-\alpha_s^*}, E^{-\alpha_{s+1}}, \ldots, E^{-\alpha_n}], \\ E_s^{-\alpha_\triangle} &= v[E^{\alpha_1}, \ldots, E^{\alpha_{s-1}}, E^{\alpha_s^*}, E^{\alpha_{s+1}}, \ldots, E^{\alpha_n}]. \end{aligned} \tag{4.2.8}$$

Now we show the case $i = s$. In $[E^{-\alpha_\triangle}, E^{\alpha_s}]$, expanding $E_s^{-\alpha_\triangle}$ as in (4.2.8) and delivering E^{α_s} to the left, we have two surviving terms

$$[\cdots, E^{\alpha_{s-2}}, [E^{\alpha_{s-1}}, E^{\alpha_s}], E^{\alpha_s^*}, \cdots] + [\cdots, E^{\alpha_{s-1}}, E^{\alpha_s^*}, [E^{\alpha_{s+1}}, E^{\alpha_s}], E^{\alpha_{s+2}}, \cdots].$$

For the first term, delivering $E^{\alpha_s^*}$ to the left, we have

$$[\cdots, E^{\alpha_{s-2}}, [E^{\alpha_{s-1}}, E^{\alpha_s}, E^{\alpha_s^*}], \cdots] \overset{\mathbf{(III)}}{=} 0.$$

For the second term, delivering $[E^{\alpha_{s+1}}, E^{\alpha_s}]$ to the left, using the fact that $[E^{\alpha_i}, [E^{\alpha_{s+1}}, E^{\alpha_s}]] = 0$ for $i < s - 1$ and **III**, we find that only one term

$$[\ldots, E^{\alpha_{s-2}}, [E^{\alpha_{s-1}}, [E^{\alpha_{s+1}}, E^{\alpha_s}]], E^{\alpha_s^*}, \cdots]$$

survives. Finally, delivering $E^{\alpha_s^*}$ to the left, we see that it is equal to **IV** and hence 0.

To show **A-II.3**, we study $\mathfrak{e}(A)$ in more detail.

First, let us see the set $\{E^{\pm\alpha} \mid \alpha \in A_{\mathrm{af}}\} \cup E_s^{\pm\alpha_\triangle}$ generates $\mathfrak{e}(A)$. It is enough to show $E^{\pm\alpha_s^*}$ is generated by those elements. The next formula proves this (the case $E^{\alpha_s^*}$ can be shown similarly).

$$[E^{-\alpha_{s-1}}, \ldots, E^{-\alpha_1}, E_s^{-\alpha_\triangle}, E^{-\alpha_n}, \ldots, E^{-\alpha_{s+1}}] = (-1)^{n-1} E^{\alpha_s^*}. \tag{4.2.9}$$

This is shown by expanding $E_s^{-\alpha_\triangle}$ as in (4.2.8) and using the relations $\mathbf{0}$, $\mathbf{I}$ and $\mathbf{II}$ in (4.1.1).

In the previous proof, we know that $\mathfrak{e}(A)$ has the triangular decomposition with respect to the generators $\{E^\alpha \,|\, \alpha \in \pm A_{\mathrm{af}}\} \cup \{E_s^{\pm\alpha_\triangle}\}$. Define an automorphism $\mathfrak{n}_{\alpha_\triangle} := M_{\alpha_\triangle}\mathfrak{n}_{\alpha_n}M_{\alpha_\triangle}^{-1}$ of $\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})$, which is equal to the reflection $\tilde{w}_{\alpha_\triangle}$ on $\tilde{\mathfrak{h}}$. Together with $\{\mathfrak{n}_\alpha \mid \alpha \in A_{\mathrm{af}}\}$, we know the action of the group $W(A_\triangle) := \langle \tilde{w}_\alpha \mid \alpha \in A_\triangle \rangle$ on $\tilde{\mathfrak{h}}$ extends to the action on $\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})$. Restricting these actions to on $\mathfrak{e}(A)$, we know the set $\Delta(\mathfrak{e}(A))$ of all roots of $\mathfrak{e}(A)$ is invariant under the action of $W(A_\triangle)$.

A-II.3: It follows from the fact that for any $\alpha, \beta \in A_\triangle$ such that $I(\alpha,\beta) < 0$, $\beta + (1 - I(\alpha^\vee, \beta))\alpha$ is not a root of $\mathfrak{e}(A)$. Let us prove this. Any root is a sum of elements of $A_\triangle$ (called positive) or sum of elements of $-A_\triangle$ (called negative), especially $\beta - \alpha$ is not a root because $\alpha \neq \beta$. So, $w_\alpha(\beta - \alpha) = \beta + (1 - I(\alpha^\vee, \beta))\alpha$ is not a root.

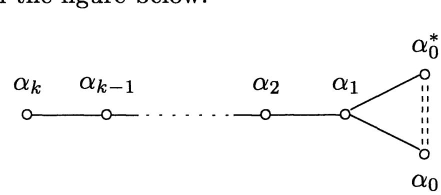
Finally, let us see that the homomorphism Υ_s satisfies $\Upsilon_s \circ \varphi = \mathrm{id}_{\mathfrak{e}(A)}$. It is enough to show $\Upsilon_s(e^{\pm\alpha_s^*}) = E^{\pm\alpha_s^*}$ because $\mathfrak{e}(A_{s^*}) = \mathfrak{e}(A)$.

$$
\begin{aligned}
\Upsilon_s(e^{\alpha_s^*}) &= \Upsilon_s(\tau[\, e^{-\alpha_{s-1}}, \dots, e^{-\alpha_1}, e^{-\alpha_\triangle} e^{-\alpha_n}, \dots, e^{-\alpha_{s+1}}]) \\
&= \tau v[\, E^{-\alpha_{s-1}}, \dots, E^{-\alpha_1}, E_s^{-\alpha_\triangle}, E^{-\alpha_n}, \dots, E^{-\alpha_{s+1}}] \\
&= \tau v(-1)^{n-1} E^{\alpha_s^*}.
\end{aligned}
\tag{4.2.10}
$$

Similar calculation also shows $e^{\alpha_s^*} = \tau v(-1)^{n-1} e^{\alpha^*}$. So, $\Upsilon_s(e^{\alpha_s^*}) = E^{\alpha_s^*}$. Similarly we can show $\Upsilon_s(e^{-\alpha_s^*}) = E^{-\alpha_s^*}$. This finishes the proof of Lemma 2. □

Remark 7. In [Sl2], Slodowy has shown a weaker statement than that of Lemma 2: the diagram $A_{\mathrm{af}} \cup \{\alpha_s^*\}$ is braid equivalent to the diagram $A_\triangle$, and hence the intersection matrix algebra for $A_{\mathrm{af}} \cup \{\alpha_s^*\}$ is isomorphic to the Kac-Moody algebra for $A_\triangle$.

(4.3) We construct elements of $\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})$ whose weight belong to the marking $G_{\mathbb{Z}} = \mathbb{Z}a$ (2.3.1). For the purpose, let us define $E^{\pm\alpha^*}$ not only for $\alpha \in \Gamma_{\max}$ but also for $\alpha \in \Gamma_{\mathrm{af}} \setminus \Gamma_{\max}$. First recall $\alpha^* := \alpha + a$ ((2.4.3), (2.6) Fact 4) and $\Gamma_{\mathrm{ell}} = \Gamma_{\mathrm{af}} \cup \Gamma_{\max}^*$ (2.4.7), where $\Gamma_{\max}$ is a connected subdiagram of the affine diagram Γ_{af} such that the complement $\Gamma_{\mathrm{af}} \setminus \Gamma_{\max}$ is a union $\bigcup_{j=1}^r \Gamma_j$ of A_{l_j}-type diagrams (Appendix A). Let $\alpha_0 \in \Gamma_{\max}$ be an element connected to a component Γ_j. Let the elements of Γ_j be ordered from the side α_0 as $\alpha_1, \alpha_2, \dots, \alpha_k$, as in the figure below.



Inductively, we define $E^{\alpha_i^*} := [E^{\alpha_{i-1}^*}, E^{-\alpha_{i-1}}, E^{\alpha_i}]$ and $E^{-\alpha_i^*} := [E^{-\alpha_{i-1}^*}, E^{\alpha_{i-1}}, E^{-\alpha_i}]$ for $i = 1, \dots, k$. Then we have $[h, E^{\pm\alpha^*}] = \langle h, \pm\alpha^* \rangle E^{\pm\alpha^*}$ for any $h \in \tilde{\mathfrak{h}}$ and $\alpha \in \Gamma_{\mathrm{af}}$. If an A-part contains α, $\tilde{\mathfrak{e}}(A)$ contains $E^{\pm\alpha^*}$. Using the relation **V** in (4.1.1) for $\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})$ and (3.1.8) for $\tilde{\mathfrak{g}}(R)$, one easily checks that (4.1.3) induces the correspondence:

$$\varphi : E^{\pm\alpha^*} \mapsto e^{\pm\alpha^*} \qquad \text{for any} \quad \alpha \in \Gamma_{\mathrm{af}}. \tag{4.3.1}$$

Assertion 5. *Let m be a positive integer and $\alpha \in \Gamma_{\mathrm{af}}$, then,*

$$\begin{aligned} [\underbrace{e^{-\alpha}, e^{\alpha^*}, \dots, e^{-\alpha}, e^{\alpha^*}}_{m-\text{pairs}}] &= (-2)^{m-1} h_{\alpha^\vee}(-1) e^{ma}, \\ [\underbrace{e^{-\alpha^*}, e^{\alpha}, \dots, e^{-\alpha^*}, e^{\alpha}}_{m-\text{pairs}}] &= (-2)^{m-1} h_{\alpha^\vee}(-1) e^{-ma}. \end{aligned} \tag{4.3.2}$$

Proof. One can show this by induction on m using (3.1.8). □

In view of Assertion 5, we make the next definition.

Definition 4. Define $H_{\alpha^\vee}^{(n)}$ for any $\alpha \in \Gamma_{\mathrm{af}}$ and $n \in \mathbb{Z}$ by

$$\begin{cases} H_{\alpha^\vee}^{(m)} &:= (-2)^{-(m-1)} [\underbrace{E^{-\alpha}, E^{\alpha^*}, \dots, E^{-\alpha}, E^{\alpha^*}}_{m-\text{pairs}}], \\ H_{\alpha^\vee}^{(-m)} &:= (-2)^{-(m-1)} [\underbrace{E^{-\alpha^*}, E^{\alpha}, \dots, E^{-\alpha^*}, E^{\alpha}}_{m-\text{pairs}}], \\ H_{\alpha^\vee}^{(0)} &:= h_{\alpha^\vee}, \end{cases} \tag{4.3.3}$$

where $m \in \mathbb{Z}_{>0}$. Put $H_{-\alpha^\vee}^{(n)} := -H_{\alpha^\vee}^{(n)}$ for any $\alpha \in \Gamma_{\mathrm{af}}$ and any $n \in \mathbb{Z}$.

By (4.3.1), $\varphi(H_{\alpha^\vee}^{(n)}) = h_{\alpha^\vee}(-1)e^{na}$ for any $\alpha \in \pm\Gamma_{\mathrm{af}}$ and $n \in \mathbb{Z}$. The following is straightforward.

Lemma 3. *For any given A-part $A \subset \Gamma_{\mathrm{ell}}$, Υ in* (4.2) *Lemma* 2 *satisfies:*

$$\Upsilon : \quad \begin{array}{ccc} h_{\alpha^\vee}(-1) & \mapsto & h_{\alpha^\vee}, \\ e^{\alpha} & \mapsto & E^{\alpha}, \\ h_{\alpha^\vee}(-1)e^{na} & \mapsto & H_{\alpha^\vee}^{(n)}, \end{array} \tag{4.3.4}$$

where α runs over $\{\pm\alpha, \pm\alpha^* \mid \alpha \in A\} = \pm A \cup \pm A^*$.

According to (4.2) Lemma 2, the algebra $\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})$ inherits all the relations in $\mathfrak{g}(A)$. So, let us list up some formulae in $\tilde{\mathfrak{g}}(R)$ obtained by a use of (3.1.8).

Assertion 6. *Let $n, m \in \mathbb{Z}$, $\alpha, \beta \in R$ and $h, g \in \mathfrak{h}$, then one has:*

$$\begin{aligned}
&\textbf{H-II.} && [h(-1)e^{ma}, g(-1)e^{na}] = \tilde{I}^*(h,g) m\delta_{m+n,0} h_a(-1)e^{(m+n)a},\\
&\textbf{I*.} && [e^\alpha, h(-1)e^{ma}, g(-1)e^{na}]\\
(4.3.5)\quad & && \qquad = -\tfrac{1}{2}\langle h,\alpha\rangle\langle g,\alpha\rangle [e^\alpha, h_{\alpha^\vee}(-1)e^{(n+m)a}],\\
&\textbf{II*.1.} && [h(-1)e^{na}, e^\alpha, e^{-\alpha}]\\
& && \qquad = [h(-1)e^{na}, e^{-\alpha}, e^\alpha] = \langle h,\alpha\rangle h_{\alpha^\vee}(-1)e^{na},\\
&\textbf{II*.2.} && [h(-1)e^{na}, e^\alpha, e^\beta] = 0 \quad \textit{for} \quad I(\alpha,\beta) \geq 0.
\end{aligned}$$

Relations in $\tilde{\mathfrak{g}}(R)$ imply the relations in $\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})$ below:

Lemma 4. *The following relations hold in $\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})$:*

$$\begin{aligned}
&\textbf{H-I.} && [h, H^{(n)}_{\alpha^\vee}] = \langle h, na\rangle H^{(n)}_{\alpha^\vee},\\
&\textbf{H-II.} && [H^{(m)}_{\alpha^\vee}, H^{(n)}_{\beta^\vee}] = I(\alpha^\vee,\beta) m\delta_{m+n,0} h_{a^\vee},\\
(4.3.6)\quad &\textbf{I*.} && [E^\alpha, H^{(m)}_{\beta^\vee}, H^{(n)}_{\gamma^\vee}] = -\tfrac{1}{2} I(\beta^\vee,\alpha) I(\gamma^\vee,\alpha) [E^\alpha, H^{(n+m)}_{\alpha^\vee}],\\
&\textbf{II*.1.} && [H^{(n)}_{\alpha^\vee}, E^\beta, E^{-\beta}] = [H^{(n)}_{\alpha^\vee}, E^{-\beta}, E^\beta] = I(\alpha^\vee,\beta) H^{(n)}_{\beta^\vee},\\
&\textbf{II*.2.} && [H^{(n)}_{\alpha^\vee}, E^\beta, E^\gamma] = 0 \quad \textit{for} \quad I(\beta,\gamma) \geq 0,
\end{aligned}$$

where $h \in \tilde{\mathfrak{h}}$, $n, m \in \mathbb{Z}$ and $\alpha, \beta, \gamma \in \pm\Gamma_{\mathrm{af}}$.

Proof. Since the formula **H-II** in (4.3.5) contains a basis $h_{a^\vee}$, we first treat this case separately. The relation **H-II** in (4.3.5) is a specialization of the relation **IV** in (3.1.8):

$$[h(-1)e^{ma}, g(-1)e^{na}] = \tilde{I}_R^*(h,g) m\delta_{m+n,0} \tilde{I}_R(a,\cdot)(-1)e^{(m+n)a}.$$

Replacing $\tilde{I}_R$ by $\tilde{I}$ (where $I_R = c \cdot I$ for any constant c (2.1.2)), we see that the right-hand side becomes $\frac{1}{c}\tilde{I}^*(h,g) m\delta_{m+n,0} h_{a^\vee}(-1)e^{(m+n)a}$, since we have $h_a(-1) = h_{a^\vee}(-1)$ (we use the normalized form I_R in $\tilde{\mathfrak{g}}(R)$) and $c \cdot \tilde{I}_R^* = \tilde{I}^*$ on $\tilde{\mathfrak{h}}$ (use $\alpha^\vee = c \cdot \alpha$ for $\alpha \in R$). One has $(1/c)I(\alpha^\vee, \beta^\vee) = I(\alpha^\vee, \beta) = I(\beta^\vee, \alpha) \in \mathbb{Z}$ for any $\alpha, \beta \in R$. See also (4.1) Remark 5.

H-I: Clear from the definition of $H^{(n)}_{\alpha^\vee}$. **II*.1**: Take A-part A containing $\{\alpha, \beta\}$ (4.2), and apply (4.2) Lemma 2 for A. These are Υ-images of the relations **II*.1** in (4.3.5). Similar argument shows the following relation holds:

$$(4.3.7) \qquad [H^{(n)}_{\alpha^\vee}, E^\beta] = \frac{1}{2} I(\alpha^\vee, \beta) [H^{(n)}_{\beta^\vee}, E^\beta].$$

(Note that the relations can also be induced from the relation **I*** by substituting $m = 0$). **I*** and **II*.2**: Use (4.3.7) for the first bracket, then take an A-part A

containing $\{\beta, \gamma\}$. They are Υ-images of the relations **I*** and **II*.2** in (4.3.6), respectively. □

(4.4) We introduce three Lie subalgebras of $\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})$: $\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'}$, $\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}}$ and $\mathfrak{g}_{\mathrm{af}}$. They are the subalgebras generated by $\mathfrak{B}^0 \cup \{h_{a^\vee}\}$, $\mathfrak{B}^0 \cup \tilde{\mathfrak{h}}$ and $\mathfrak{B}^+ \cup \mathfrak{B}^-$, respectively, where

$$\text{(4.4.1)} \qquad \begin{array}{lll} \mathbb{Z}' & := & \mathbb{Z} \setminus \{0\}, \\ \mathfrak{B}^0 & := & \{H_{\alpha^\vee}^{(n)} \mid \alpha \in \Gamma_{\mathrm{af}}, n \in \mathbb{Z}'\}, \\ \mathfrak{B}^+ & := & \{E^\alpha \mid \alpha \in \Gamma_{\mathrm{af}}\}, \\ \mathfrak{B}^- & := & \{E^{-\alpha} \mid \alpha \in \Gamma_{\mathrm{af}}\}. \end{array}$$

Lemma 5. (1) $\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'}$ *is a Heisenberg algebra:*

$$\text{(4.4.2)} \qquad \mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'} = \mathbb{Q}h_{a^\vee} \oplus \bigoplus_{n \in \mathbb{Z}'} \mathfrak{h}_{\mathrm{af}}^{(n)} \quad \textit{where} \quad \mathfrak{h}_{\mathrm{af}}^{(n)} := \bigoplus_{\alpha \in \Gamma_{\mathrm{af}}} \mathbb{Q}H_{\alpha^\vee}^{(n)}.$$

(2) $\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}}$ *is the extension of* $\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'}$:

$$\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} = \tilde{\mathfrak{h}} \oplus \bigoplus_{n \in \mathbb{Z}'} \mathfrak{h}_{\mathrm{af}}^{(n)}.$$

(3) $\mathfrak{g}_{\mathrm{af}} = \mathfrak{e}(\Gamma_{\mathrm{af}})$ *is isomorphic to the affine Kac-Moody algebra* $\mathfrak{g}(\Gamma_{\mathrm{af}})$.
(4) $\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})$ *is generated by* $\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}}$ *and* $\mathfrak{g}_{\mathrm{af}}$.

Proof. (1), (2): The relations **H-I** and **H-II** in (4.3.6) are the defining relations for the Heisenberg algebra and its extension. Linear independence of components of direct sum follows from linear independence of their φ-images in $\tilde{\mathfrak{g}}(R)$. (3): See (4.1) Assertion 4. (4): Notice that the following relations prove (4): for $\alpha \in \Gamma_{\mathrm{af}}$, one has $[H_{\alpha^\vee}^{(1)}, E^\alpha] = 2E^{\alpha^*}$ and $[H_{\alpha^\vee}^{(-1)}, E^{-\alpha}] = -2E^{-\alpha^*}$. Let us prove the first relation. The second relation can be proved similarly. By the definition of $H_{\alpha^\vee}^{(1)}$, we have $[H_{\alpha^\vee}^{(1)}, E^\alpha] = [E^{-\alpha}, E^{\alpha^*}, E^\alpha]$. Delivering E^α to the left, we get $[h_{\alpha^\vee}, E^{\alpha^*}]$ ($[E^{\alpha^*}, E^\alpha] = 0$ since $\alpha + \alpha^*$ is not a root of $\mathfrak{e}(A)$ for some A-part A and apply (4.2) Lemma 2) and it is $2E^{\alpha^*}$. □

§5. The Amalgamation Algebra $\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}}$

We introduce the third Lie algebras $\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'} * \mathfrak{g}_{\mathrm{af}}$ and $\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}}$ attached to a simply-laced marked elliptic root system, where $\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'}$ is the affine Heisenberg algebra, $\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}}$ is the extension of $\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'}$ by the non-degenerate Cartan subalgebra $\tilde{\mathfrak{h}}$ and $\mathfrak{g}_{\mathrm{af}}$ is the affine Kac-Moody Lie algebra (see Notation below).

In this section, we prove that three Lie algebras $\tilde{\mathfrak{g}}(R)$, $\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})$ and $\tilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}}*\mathfrak{g}_{\mathrm{af}}$ are isomorphic. We first show that the amalgamation algebra admits a generalized triangular decomposition. Then it implies that the natural surjective homomorphism $\varrho : \tilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}} \to \tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})$ induces a central extension of $\mathfrak{g}(R)/\mathfrak{z}(\mathfrak{g}(R))$. We already know that $\mathfrak{g}(R)$ is the universal central extension of $\mathfrak{g}(R)/\mathfrak{z}(\mathfrak{g}(R))$ (3.1). Using this universality, we can show that the composition $\varrho \circ \varphi$ is an isomorphism.

Notation. By the *amalgamation* $\mathfrak{g}_1 * \mathfrak{g}_2$ of two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, we shall mean the Lie subalgebra generated by $\mathfrak{g}_i\,(i = 1, 2)$ in $T(\mathfrak{g}_1 \oplus \mathfrak{g}_2)/\mathcal{I}$, where $T(V)$ is the tensor algebra of a vector space V, and $\mathcal{I}$ is the both-side ideal generated by the elements $g_i \otimes h_i - h_i \otimes g_i - [g_i, h_i]$ for all $g_i, h_i \in \mathfrak{g}_i$. Further, if there are Lie algebra homomorphisms $\varphi_i : \mathfrak{g} \to \mathfrak{g}_i\,(i = 1, 2)$, we denote by $\mathfrak{g}_1 *_{\mathfrak{g}} \mathfrak{g}_2$ the Lie algebra defined similarly, but adding more relations $\varphi_1(g) - \varphi_2(g)$ for $g \in \mathfrak{g}$ to the generators of the ideal I. Abusing the notation, we sometimes call a quotient algebra of $\mathfrak{g}_1 * \mathfrak{g}_2$ also an amalgamation of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ and denote it by $\mathfrak{g}_1 * \mathfrak{g}_2$.

(5.1) Let (R, G) and $\Gamma_{\mathrm{ell}} := \Gamma(R, G)$ be as before. Recall $\Gamma_{\mathrm{ell}} := \Gamma_{\mathrm{af}} \cup \Gamma^*_{\mathrm{max}}$ (2.4.7), the extension $\tilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}}$ of the Heisenberg algebra $\mathfrak{h}^{\mathbb{Z}'}_{\mathrm{af}}$, $\mathfrak{g}_{\mathrm{af}} = \mathfrak{e}(\Gamma_{\mathrm{af}}) \simeq \mathfrak{g}(\Gamma_{\mathrm{af}})$ in (4.4), and $\tilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} \cap \mathfrak{g}_{\mathrm{af}} = \mathfrak{h}_{\mathrm{af}} := \bigoplus_{\alpha\in\Gamma_{\mathrm{af}}} \mathbb{Q}h_{\alpha^\vee}$.

Definition 5. We define the Lie algebra $\tilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}}$ as the quotient algebra of the amalgamation $\tilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} *_{\mathfrak{h}_{\mathrm{af}}} \mathfrak{g}_{\mathrm{af}}$ of $\tilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}}$ and $\mathfrak{g}_{\mathrm{af}}$ divided by the ideal defined by the following relations **0***, **I*** and **II***:

$$(5.1.1)\qquad \begin{array}{ll} \mathbf{0^*}. & [h, E^\alpha] = \langle h, \alpha\rangle E^\alpha, \\ \mathbf{I^*}. & [E^\alpha, H^{(m)}_{\beta^\vee}, H^{(n)}_{\gamma^\vee}] = -\frac{1}{2}I(\beta^\vee,\alpha)I(\gamma^\vee,\alpha)[E^\alpha, H^{(n+m)}_{\alpha^\vee}], \\ \mathbf{II^*.1.} & [H^{(n)}_{\alpha^\vee}, E^\beta, E^{-\beta}] = [H^{(n)}_{\alpha^\vee}, E^{-\beta}, E^\beta] = -I(\alpha^\vee,\beta)H^{(n)}_{\beta^\vee}, \\ \mathbf{II^*.2.} & [H^{(n)}_{\alpha^\vee}, E^\beta, E^\gamma] = 0 \quad \text{for} \quad I(\beta,\gamma) \geq 0. \end{array}$$

where $h \in \tilde{\mathfrak{h}}$, $\alpha, \beta, \gamma \in \pm\Gamma_{\mathrm{af}}$, $m, n \in \mathbb{Z}$ and $H^{(0)}_{\alpha^\vee} := h_{\alpha^\vee}$ for any $\alpha \in \pm\Gamma_{\mathrm{af}}$.

Thanks to (4.3) Lemma 4, all the relations in (5.1.1) are satisfied in $\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})$. So, due to (4.4) Lemma 5 (4) the next fact follows.

Assertion 7. *The natural inclusion homomorphisms from $\tilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}}$ and $\mathfrak{g}_{\mathrm{af}}$ to $\tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}})$ induce a surjective homomorphism*

$$(5.1.2)\qquad \varrho \;:\; \tilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}} \;\to\; \tilde{\mathfrak{e}}(\Gamma).$$

As a consequence of the above assertion the algebras $\tilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}}$ and $\mathfrak{g}_{\mathrm{af}}$ can be considered as subalgebras of $\tilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}}$. We can also consider the root space decomposition of $\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}}$ with respect to $\tilde{\mathfrak{h}}$ since all the generators of $\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}}$ are weight vectors (**I** in (4.1.1) and **H-I** in (4.3.6)) and their weights are in $\mathbb{Z}a \oplus Q_{\mathrm{af}} = Q(R)$ (2.3.3). The set of all roots of the algebra $\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}}$ is denoted by Δ. By definition, ϱ is compatible with the root space decompositions.

Due to the symmetry of the defining relations (5.1.1) there exists an involution ω, called the Cartan involution, on $\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}}$ defined by $h \mapsto -h$ for $h \in \tilde{\mathfrak{h}}$, $H_{\alpha^\vee}^{(n)} \mapsto -H_{\alpha^\vee}^{(-n)}$ for $\alpha \in \pm\Gamma_{\mathrm{af}}$, $n \in \mathbb{Z}$ and $E^\alpha \mapsto E^{-\alpha}$ for $\alpha \in \pm\Gamma_{\mathrm{af}}$. The Cartan involution brings the root space for α to the root space for $-\alpha$.

(5.2) Recall the cones Q_{af}^+ and Q_{af}^- in the affine lattice Q_{af} (2.3). Put

(5.2.1) $$Q^+ := \mathbb{Z}a \oplus Q_{\mathrm{af}}^+, \quad Q^0 := \mathbb{Z}a = G_{\mathbb{Z}}, \quad Q^- := \mathbb{Z}a \oplus Q_{\mathrm{af}}^-.$$

Lemma 6. $\Delta \subset Q^+ \cup Q^0 \cup Q^-$. *So we have the decomposition*

(5.2.2) $$\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}} = \tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} \oplus \mathfrak{n}_{\mathrm{ell}}^+ \oplus \mathfrak{n}_{\mathrm{ell}}^-,$$

where $\mathfrak{n}_{\mathrm{ell}}^\sigma := \bigoplus_{\alpha \in \Delta \cap Q_{\mathrm{af}}^\sigma} (\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}})_\alpha$ *for* $\sigma \in \{\pm\}$. *The* $\mathfrak{n}_{\mathrm{ell}}^\sigma$ *is the ideal of the algebra* $\mathfrak{u}_{\mathrm{ell}}^\sigma := \langle \tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}}, \mathfrak{n}_{\mathrm{af}}^\sigma \rangle = \tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} \oplus \mathfrak{n}_{\mathrm{ell}}^\sigma$ *generated by* $\mathfrak{n}_{\mathrm{af}}^\sigma := \langle E^\alpha \mid \alpha \in \sigma\Gamma_{\mathrm{af}} \rangle$ *and is nilpotent in the sense*:

(5.2.3) $$\bigcap_{m=1}^{\infty} [\underbrace{\mathfrak{n}_{\mathrm{ell}}^\sigma, \cdots, \mathfrak{n}_{\mathrm{ell}}^\sigma}_{m\text{-times}}] = \{0\} \quad \textit{for} \quad \sigma \in \{\pm\}.$$

Proof. Let us introduce additional notaion:

(5.2.4) $$\begin{array}{rcl} \mathfrak{B} & := & \mathfrak{B}^+ \cup \mathfrak{B}^0 \cup \mathfrak{B}^-, \\ \mathfrak{C} & := & \tilde{\mathfrak{h}} \cup \{[B_1, \ldots, B_k] \mid B_i \in \mathfrak{B}^+ \cup \mathfrak{B}^0, k \in \mathbb{Z}_{>0}\} \\ & & \quad \cup \{[B_1, \ldots, B_k] \mid B_i \in \mathfrak{B}^- \cup \mathfrak{B}^0, k \in \mathbb{Z}_{>0}\}, \\ \mathfrak{D} & := & \text{the subspace of } \tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}} \text{ spanned linearly by } \mathfrak{C}. \end{array}$$

We want to show that $\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}} = \mathfrak{D}$. It is enough to prove $[B_1, \ldots, B_k] \in \mathfrak{D}$ for any sequence $B_i \in \mathfrak{B}$ with $k \in \mathbb{Z}_{>0}$. We show this by induction on k. Cases $k = 1, 2$ are clear. Assume $k \geq 3$. Then $[B_1, \ldots, B_k, B_{k+1}] = [[B_1, \ldots, B_k], B_{k+1}]$. Expressing $[B_1, \ldots, B_k] \in \mathfrak{D}$ as a linear combination of elements in G, it is enough to show the following three cases: (i) $[C_1^+, \ldots, C_m^+, B_{k+1}] \in \mathfrak{D}$ for any $C_i^+ \in \mathfrak{B}^+ \cup \mathfrak{B}^0$ and any $m \in \mathbb{Z}_{>0}$, (ii) $[C_1^-, \ldots, C_m^-, B_{k+1}] \in \mathfrak{D}$ for any $C_i^- \in \mathfrak{B}^- \cup \mathfrak{B}^0$ and any $m \in \mathbb{Z}_{>0}$, and (iii) $[h, B_{k+1}] \in \mathfrak{D}$ for any $h \in \tilde{\mathfrak{h}}$. The case (iii) is clear by **I** and **H-I**. Let us consider the case (i). The case (ii) can be shown similarly. If $B_{k+1} \in \mathfrak{B}^+ \cup \mathfrak{B}^0$, it is clearly in $\mathfrak{D}$. So we assume $B_{k+1}^- := B_{k+1} \in \mathfrak{B}^-$. Delivering B_{k+1}^- to the left, we have $\sum_{i=1}^{k}[\dots, C_{i-1}^{+}, [C_i^+, B_{k+1}^-], C_{i+1}^+, \cdots]$. If $C_i^+ \in \mathfrak{B}^+$, $[C_i^+, B_{k+1}^-]$ is in $\tilde{\mathfrak{h}}$ by **II** and the i-th term is in $\mathfrak{D}$. If $H_i := C_i^+ \in \mathfrak{B}^0$, the i-th term is $[\dots, C_{i-1}^+, [H_i, B_{k+1}^-], C_{i+1}^+, \cdots]$. If $i = 1$, then using either relations of the **I***, **II*.1** or **II*.2** in (5.1.1), we can reduce the 1st-term to the case $\leq k$. If $i > 1$, deliver $[H_i, B_{k+1}^-]$ further to the left. Again, using relations in (5.1.1), we can reduce the ith-term to the case $\leq k$. □

As a consequence of Lemma 6, we can determine some of root spaces of $\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}}$.

Lemma 7. (1) $(\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}})_0 = \tilde{\mathfrak{h}}$,

(2) $(\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}})_{\alpha+na} = \mathbb{Q}[E^\alpha, H_{\alpha^\vee}^{(n)}]$ *for all* $\alpha \in \Gamma_{\mathrm{af}}$ *and* $n \in \mathbb{Z}$,

(3) $(\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}})_{m\alpha+na} = 0$ *for all* $\alpha \in \pm\Gamma_{\mathrm{af}}$, $m \in \mathbb{Z}$ *such that* $|m| \geq 2$, *and* $n \in \mathbb{Z}$.

(4) $(\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}})_{na} = \oplus_{\alpha\in\Gamma_{\mathrm{af}}} \mathbb{Q}H_\alpha^{(n)}$ *for any* $n \in \mathbb{Z} \setminus \{0\}$.

Proof. Recall the relations in (4.3.6).

(1) According to Lemma 6, $(\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}})_0$ is spanned by $\tilde{\mathfrak{h}}$ and elements $[H_{\beta_1^\vee}^{(n_1)}, \cdots, H_{\beta_k^\vee}^{(n_k)}]$ for $\beta_i \in \Gamma_{\mathrm{af}}$ satisfying $n_1 + \cdots + n_k = 0$. If $k \geq 2$, the first bracket is in $\mathbb{Q}h_{a^\vee}$ by **H-II**. So if $k > 2$, it is 0 by **H-I**.

(2) If $\alpha \in \Gamma_{\mathrm{af}}$, $(\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}})_{\alpha+na}$ is spanned by elements $[\dots, H_{\beta_{s-1}^\vee}^{(n_{s-1})}, E^\alpha, H_{\beta_s^\vee}^{(n_s)}, \cdots]$. That is: one entry is $E^\alpha \in \mathfrak{B}^+$ and the others are in $\mathfrak{B}^0$ satisfying $n_1 + \cdots + n_k = n$. Using **H-II** and **I*** repeatedly, we finally find it is either in $\mathbb{Q}E^\alpha$ (if $n = 0$) or in $\mathbb{Q}[E^\alpha, H_{\alpha^\vee}^{(n)}]$ (if $n \neq 0$).

(3) $(\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}})_{m\alpha+na}$ for $m \geq 2$ is spanned by elements $[A, E^\alpha, B, E^\alpha, \cdots]$, where A and B are sequences of $\mathfrak{B}^0$. Similarly to (2) above, this becomes $[E^\alpha, E^\alpha, \cdots]$ or $[E^\alpha, H_{\alpha^\vee}^{(n)}, E^\alpha, \cdots]$ and it is 0 by **II*.2**.

(4) $(\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}})_{na}$ is spanned by elements $[H_{\beta_1^\vee}^{(n_1)}, \dots, H_{\beta_k^\vee}^{(n_k)}]$, where $\beta_1, \dots, \beta_k \in \Gamma_{\mathrm{af}}$ and $n_1 + \cdots + n_k = n$. Use **H-II**. □

(5.3) As in (4.1), we lift the action of the subgroup $W_{\mathrm{af}} := \langle \tilde{w}_\alpha \mid \alpha \in \Gamma_{\mathrm{af}} \rangle$ of $W(R)$ to automorphisms of $\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}}$.

Fisrt note that the action of $\mathrm{ad}\, E^\alpha$ on $\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}}$ is a locally nilpotent derivation for all $\alpha \in \pm\Gamma_{\mathrm{af}}$, since it is a nilpotent for any generators by $\mathbb{I}$, $\mathbb{II}$ (4.1.1) and **II*.2** (5.1.1). Thanks to this fact, we can define the exponential of $\mathrm{ad}\, E^\alpha$: $\exp(\mathrm{ad}\, E^\alpha) := \sum_{i=0}^{\infty}(1/i!)(\mathrm{ad}\, E^\alpha)^i$ for any $\alpha \in \pm\Gamma_{\mathrm{af}}$. This is an automorphism of the algebra $\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}}$. Composing these automorphisms, we define

$$\mathfrak{n}_\alpha := \exp(\mathrm{ad}\, E^\alpha) \exp(\mathrm{ad}\, E^{-\alpha}) \exp(\mathrm{ad}\, E^\alpha) \tag{5.3.1}$$

for any $\alpha \in \Gamma_{\mathrm{af}}$. Then, it coincides with $\tilde{w}_\alpha$ on $\tilde{\mathfrak{h}}$ and maps $(\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}})_\beta$ to $(\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}})_{w_\alpha\beta}$ isomorphically for any $\alpha \in \Gamma_{\mathrm{af}}$ and $\beta \in Q(R)$. So,

Fact 7. *The set of roots Δ of $\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}}$ is W_{af}-invariant.*

Define a *height* of an element $x = \sum_{\alpha \in \Gamma_{\mathrm{af}}} m_\alpha \alpha + na \in Q_{\mathrm{af}} \oplus \mathbb{Z}a = Q(R)$ by $h(x) := \sum_{\alpha \in \Gamma_{\mathrm{af}}} m_\alpha \in \mathbb{Z}$. Let us call the element x *positive* (resp. *negative*) if $h(x) > 0$ (resp. $h(x) < 0$). Due to Lemma 6, each root in $\Delta \setminus \mathbb{Z}a$ is either positive or negative.

Lemma 8. $\Delta = R \cup \operatorname{rad} Q(R)$. *The multiplicity of a root in R is equal to one.*

Proof. Consider the W_{af}-orbit of a root $x \in \Delta$. If the orbit contains an element y with $h(y) = 0$. Then $y = na$ for some $n \in \mathbb{Z}$ by (5.2) Lemma 6 and hence $x = na \in \operatorname{rad} Q(R)$.

If the orbit contains both positive and negative elements, then there exist an element y in the orbit and $\alpha \in \Gamma_{\mathrm{af}}$ such that $h(y) > 0 > h(w_\alpha(y))$. Express $y = \sum_{\beta \in \Gamma_{\mathrm{af}}} m_\beta \beta \ + \ na$, where all $m_\beta \in \mathbb{Z}$ are non-negative. $w_\alpha(y) := y - I(\alpha^\vee, y)\alpha = \sum_{\beta \in \Gamma_{\mathrm{af}} \setminus \{\alpha\}} m_\beta \beta + (m_\alpha - I(\alpha^\vee, y)\ )\alpha + na$, and $w_\alpha(y)$ is negative, hence $m_\beta = 0$ for $\beta \neq \alpha$. So, $y = m_\alpha \alpha + na$ and hence $m_\alpha = 1$ by (5.2) Lemma 7 (3). Thus, $y \in R$ (recall (2.6.1)) and hence $x \in W_{\mathrm{af}} \cdot R = R$. The multiplicity of $(\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}})_y$ is equal to one because of (5.2) Lemma 7 (2).

Assume that all elements of the orbit $W_{\mathrm{af}} \cdot x$ have positive heights and $y \in W_{\mathrm{af}} \cdot x$ attains the minimal height. Then, for any $\beta \in \Gamma_{\mathrm{af}}$, one has $h(y) \leq h(w_\beta(y)) = h(y - I(y,\beta)\beta) = h(y) - I(y,\beta)$. So, $I(y,\beta) \leq 0$. Using the expression: $y = \sum_{\beta \in \Gamma_{\mathrm{af}}} m_\beta \beta + na$, one has $I(y,y) = \sum_{\beta \in \Gamma_{\mathrm{af}}} m_\beta I(y,\beta) \leq 0$, and so, $y \in \operatorname{rad} Q(R)$. The case where all elements of $W_{\mathrm{af}} \cdot x$ have negative weights is reduced to the positive case by the Cartan involution (5.1). □

A root in R is called a *real root* and a root in $\operatorname{rad} Q(R)$ is called an *imaginary root* (see (3.1.7)).

(5.4) We are ready to prove our main result (4.1) Theorem 1.

Fact. (i) The derived algebra $(\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}})'$ of $\tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}}$ is equal to the amalgamation $\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'} * \mathfrak{g}_{\mathrm{af}}$ of $\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'}$ and $\mathfrak{g}_{\mathrm{af}}$.

(ii) $\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'} * \mathfrak{g}_{\mathrm{af}}$ is generated by $\{[E^\alpha, H_{\alpha^\vee}^{(n)}] \mid \alpha \in \pm\Gamma_{\mathrm{af}}, n \in \mathbb{Z}\}$, because of **0*** and **II*.1** in (5.1.1). Especially, $\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'} * \mathfrak{g}_{\mathrm{af}}$ is perfect (i.e. the derived algebra is itself).

Let us denote the composition map $\varphi \circ \varrho$ by ξ:

(5.4.1)
$$\xi := \varphi \circ \varrho : \ \tilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}} \to \tilde{\mathfrak{g}}(R).$$

By restricting ξ on $\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'} * \mathfrak{g}_{\mathrm{af}}$, we have

(5.4.2)
$$\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'} * \mathfrak{g}_{\mathrm{af}} \xrightarrow{\xi} \mathfrak{g}(R) \xrightarrow{p} \mathfrak{g}(R)/\mathfrak{z}(\mathfrak{g}(R)),$$

where p is the natural projection (recall $\tilde{\mathfrak{g}}(R)' = \mathfrak{g}(R)$ (3.2)). Put $\nu := p \circ \xi$. Let us show that ν is a central extention. Since ν preserves the $Q(R)$-grading, $\ker \nu$ is $Q(R)$-graded ideal. For each real root α, the restriction of ν on the root space of α is an isomorphism, because of (3.1.7) and (5.3) Lemma 8. So, we have

$$\ker \nu = \bigoplus_{\mu \in \operatorname{rad} Q(R)} (\ker \nu)_\mu. \tag{5.4.3}$$

One calculates further for $\mu \in \operatorname{rad} Q(R)$, $n \in \mathbb{Z}$ and $\alpha \in \pm\Gamma_{\mathrm{af}}$:

$$[\,[\,E^\alpha, H^{(n)}_{\alpha^\vee}], (\ker \nu)_\mu] \subset (\ker \nu)_{\mu+na+\alpha} = 0, \tag{5.4.4}$$

since $\mu + na + \alpha$ is a real root. Due to the above Fact ii), one has $\ker \nu \subset \mathfrak{z}\big(\mathfrak{h}^{\mathbb{Z}'}_{\mathrm{af}}\big)$.

Since p is the universal central extension ((3.2) Lemma 1), there is a unique homomorphism ψ such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g}(R) & \xrightarrow{p} & \mathfrak{g}(R)/\mathfrak{z}(\mathfrak{g}(R)) \\ \psi \downarrow & \circlearrowleft & \| \\ \mathfrak{h}^{\mathbb{Z}'}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}} & \xrightarrow{\nu} & \mathfrak{g}(R)/\mathfrak{z}(\mathfrak{g}(R)) \end{array} \tag{5.4.5}$$

Let us prove $f := \psi \circ \xi$ is the identity map of $\mathfrak{h}^{\mathbb{Z}'}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}}$. The homomorphism f satisfies $\nu = \nu \circ f$. Since ν is a central extention, $[x, y] = [x', y']$ if $\nu(x) = \nu(x')$ and $\nu(y) = \nu(y')$ for $x, y, x', y' \in \mathfrak{h}^{\mathbb{Z}'}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}}$. The equality $\nu(f(x)) = \nu(x)$ for all $x \in \mathfrak{h}^{\mathbb{Z}'}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}}$ implies $f([x, y]) = [f(x), f(y)] = [x, y]$. Since $\mathfrak{h}^{\mathbb{Z}'}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}}$ is perfect, f is the identity and we are done the proof.

It is easy to see that ψ extends to an isomorphism from $\tilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}}$ to $\tilde{\mathfrak{g}}(R)$. This completes the proof of the main result (4.1) Theorem 1 and

Theorem 2. *We have the following isomorphisms of algebras*:

$$\mathfrak{g}(R) \simeq \mathfrak{e}(\Gamma_{\mathrm{ell}}) \simeq \mathfrak{h}^{\mathbb{Z}'}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}},$$

$$\tilde{\mathfrak{g}}(R) \simeq \tilde{\mathfrak{e}}(\Gamma_{\mathrm{ell}}) \simeq \tilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}}.$$

Let us call the isomorphism classes the *elliptic algebras.*

Appendix A. Table of the Simply-Laced Elliptic Diagrams

For each type of the root system, we exhibit the following data:

1) $m_i := m_{\alpha_i}$: the exponent of α_i.

2) Explicit description of Γ_{af} and Γ_{max}.

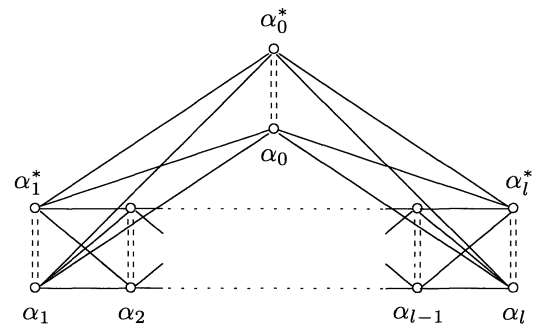
3) The list of the A-parts.

Type $A_l^{(1,1)}$ $(l \geq 2)$.

1) $m_\imath = 1\,(0 \geq i \geq l)$; $m'_{\max} = m_{\max} = 1$.

2) $\Gamma_{\text{af}} = \{\alpha_0, \dots, \alpha_l\}$, $\Gamma_{\max} = \{\alpha_0, \dots, \alpha_l\}$.

3) $\Gamma(R,G) \setminus \{\alpha_\jmath, \alpha_j^*\}$ for $j = 0, \dots, l$

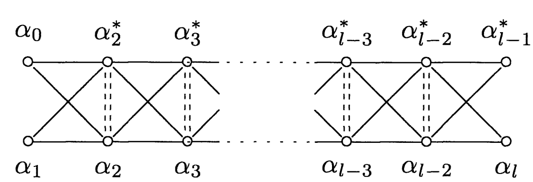


Type $D_l^{(1,1)} (l \geq 4)$.

1) $m_0 = 1$, $m_1 = 1$, $m_\imath = 2\,(2 \leq i \leq l-1)$, $m_{l-1} = 1$, $m_l = 1$; $m'_{\max} = m_{\max} = 2$.

2) $\Gamma_{\text{af}} = \{\alpha_0, \dots, \alpha_l\}$, $\Gamma_{\max} = \{\alpha_2, \dots, \alpha_{l-2}\}$.

3) $\{\alpha_0, \alpha_1, \alpha_2, \alpha_2^*\}$, $\{\alpha_{l-2}, \alpha_{l-2}^*, \alpha_{l-1}, \alpha_l\}$, $\{\alpha_p, \alpha_q\} \cup \{\alpha_i, \alpha_i^* \mid i = 2, \dots, l-2\}$ for $p \in \{0,1\}, q \in \{l-1, l\}$.

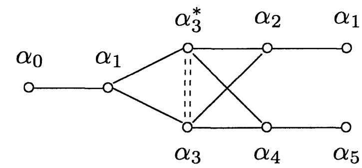


Type $E_6^{(1,1)}$.

1) $m_0 = 1$, $m_1 = 1$, $m_2 = 2$, $m_3 = 3$, $m_4 = 2$, $m_5 = 1$, $m_6 = 1$; $m'_{\max} = m_{\max} = 3$.

2) $\Gamma_{\text{af}} = \{\alpha_0, \dots, \alpha_6\}$, $\Gamma_{\max} = \{\alpha_3\}$.

3) $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_3^*, \alpha_6\}$, $\{\alpha_0, \alpha_3, \alpha_3^*, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$, $\{\alpha_1, \alpha_2, \alpha_3, \alpha_3^* \alpha_4, \alpha_5, \alpha_6\}$.

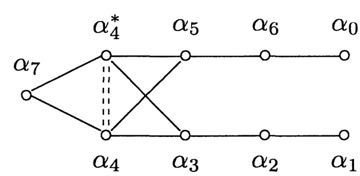


Type $E_7^{(1,1)}$.

1) $m_0 = 1$, $m_1 = 1$, $m_2 = 2$, $m_3 = 3$, $m_4 = 4$, $m_5 = 3$, $m_6 = 2$, $m_7 = 1$; $m'_{\max} = m_{\max} = 4$.

2) $\Gamma_{\text{af}} = \{\alpha_0, \dots, \alpha_7\}$, $\Gamma_{\max} = \{\alpha_4\}$.

3) $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_4^*, \alpha_7\}$, $\{\alpha_0, \alpha_6, \alpha_5, \alpha_4, \alpha_4^*, \alpha_7\}$, $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_4^*, \alpha_5, \alpha_6, \alpha_0\}$.

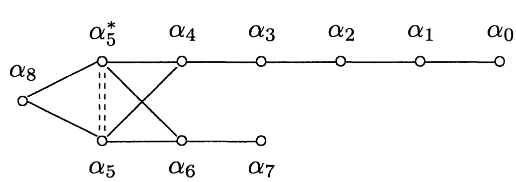


Type $E_8^{(1,1)}$.

1) $m_0 = 1$, $m_1 = 2$, $m_2 = 3$, $m_3 = 4$, $m_4 = 5$, $m_5 = 6$, $m_6 = 4$, $m_7 = 2$, $m_8 = 3$; $m'_{\max} = m_{\max} = 6$.

2) $\Gamma_{\text{af}} = \{\alpha_0, \dots, \alpha_8\}$, $\Gamma_{\max} = \{\alpha_5\}$.

3) $\{\alpha_7, \alpha_6, \alpha_5, \alpha_5^*, \alpha_8\}$, $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_5^*, \alpha_8\}$, $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_5^*, \alpha_6, \alpha_7\}$.



Appendix B. An Explicit Description of $\tilde{\mathfrak{g}}(R)$

Recall from (2.2) that there exists a sub-diagram Γ_{f} of Γ_{af} for a finite root system R_{f} such that the root lattice has the splitting $Q(R) = Q_{\mathrm{f}} \oplus \mathbb{Z}b \oplus \mathbb{Z}a$ (2.3.3) and the set of roots decomposes as $R = R_{\mathrm{f}} \oplus \mathbb{Z}b \oplus \mathbb{Z}a$ (2.6.1). Then, one can define a basis Λ_a and Λ_b of the non-degenerate hull $\tilde{F}_{\mathbb{Q}}$ of $F_{\mathbb{Q}}$ (3.1) such that

$$\tilde{F}_{\mathbb{Q}} = F_{\mathbb{Q}} \oplus \mathbb{Q}\Lambda_b \oplus \mathbb{Q}\Lambda_a,$$

where $\tilde{I}(\Lambda_a, a) = \tilde{I}(\Lambda_b, b) = 1$, $\tilde{I}(\Lambda_a, b) = \tilde{I}(\Lambda_b, a) = 0$ and $\tilde{I}(\Lambda_a, \Gamma_{\mathrm{f}}) = \tilde{I}(\Lambda_b, \Gamma_{\mathrm{f}}) = 0$. Recall the identification: h: $\tilde{F}_{\mathbb{Q}} \overset{\sim}{\to} \tilde{\mathfrak{h}}$; $x \mapsto h_x$ such that $\langle h_x, y\rangle = \tilde{I}(x,y)$ for $x, y \in \tilde{F}_{\mathbb{Q}}$. Then we put:

$$da^{\vee} := h_{a^{\vee}}, \quad db^{\vee} := h_{b^{\vee}}, \quad , \frac{\partial}{\partial a} := h_{\Lambda_a}, \quad \frac{\partial}{\partial b} := h_{\Lambda_b} \quad \text{and}$$

$$\mathfrak{h}_{\mathrm{f}} := \underset{\alpha\in\Gamma_{\mathrm{f}}}{\oplus} \mathbb{Q}h_{\alpha^{\vee}}, \quad \mathfrak{g}_{\alpha} := \mathbb{Q}e^{\alpha} \quad \text{for} \quad \alpha \in R_{\mathrm{f}}.$$

Then the elliptic algebra is given by

$$\tilde{\mathfrak{g}}(R) = \mathbb{Q}\tfrac{\partial}{\partial a} \oplus \mathbb{Q}\tfrac{\partial}{\partial b} \oplus \textstyle\bigoplus_{\alpha\in R_{\mathrm{f}}} \left(\mathfrak{g}_{\alpha} \otimes \mathbb{Q}[e^{a}, e^{-a}, e^{b}, e^{-b}]\right) \oplus$$
$$\textstyle\bigoplus_{m,n\in\mathbb{Z}} \left(\mathfrak{h}_{\mathrm{f}} \oplus \frac{\mathbb{Q}da^{\vee} + \mathbb{Q}db^{\vee}}{\mathbb{Q}(mda^{\vee}+ndb^{\vee})}\right) \otimes \mathbb{Q}e^{ma+nb}.$$

The non-degenerate hull $\tilde{\mathfrak{h}}$ of $\mathfrak{h}$, the Heisenberg subalgebra $\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'}$, the affine Kac-Moody subalgebra $\mathfrak{g}_{\mathrm{af}}$ and "nilpotent" subalgebras $\mathfrak{n}_{\mathrm{ell}}^{\pm}$ are given by the following.

$$\begin{aligned}
\tilde{\mathfrak{h}} &= \mathbb{Q}\frac{\partial}{\partial a} \oplus \mathbb{Q}\frac{\partial}{\partial b} \oplus \mathfrak{h}_{\mathrm{f}} \oplus \mathbb{Q}da^{\vee} \oplus \mathbb{Q}db^{\vee},\\
\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'} &= \mathbb{Q}da^{\vee} \oplus \bigoplus_{n\in\mathbb{Z}\setminus\{0\}} \left(\mathfrak{h}_{\mathrm{f}} \oplus \mathbb{Q}db^{\vee}\right) \otimes e^{na},\\
\mathfrak{g}_{\mathrm{af}} &= \bigoplus_{\alpha\in R_{\mathrm{f}}} \left(\mathfrak{g}_{\alpha} \otimes \mathbb{Q}[e^{b}, e^{-b}]\right) \oplus \bigoplus_{m\in\mathbb{Z}\setminus\{0\}} \left(\mathfrak{h}_{\mathrm{f}} \otimes e^{mb}\right) \oplus \mathfrak{h}_{\mathrm{f}} \oplus \mathbb{Q}db^{\vee},\\
\mathfrak{n}_{\mathrm{ell}}^{\pm} &= \bigoplus_{\alpha\in R_{\mathrm{f}}} \bigoplus_{m\in\pm\mathbb{Z}_{>0}} \left(\mathfrak{g}_{\alpha} \otimes e^{mb} \otimes \mathbb{Q}[e^{a}, e^{-a}]\right) \oplus \bigoplus_{\alpha\in R_{\mathrm{f}}^{\pm}} \left(\mathfrak{g}_{\alpha} \otimes \mathbb{Q}[e^{a}, e^{-a}]\right) \oplus\\
&\qquad \bigoplus_{m\in\pm\mathbb{Z}_{>0}} \left((\mathfrak{h}_{\mathrm{f}} \oplus \mathbb{Q}da^{\vee}) \otimes \mathbb{Q}e^{mb} \otimes \mathbb{Q}[e^{a}, e^{-a}]\right).
\end{aligned}$$

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