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Asymptotic Expansion of Singular Solutions and the Characteristic Polygon of Linear Partial Differential Equations in the Complex Domain

Dedicated to Professor Kunihiko Kajitani on his sixtieth birthday

By

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Abstract

Let $P(z,\partial)$ be a linear partial differential operator with holomorphic coefficients in a neighborhood Ω of $z=0$ in $\mathbb{C}^{d+1}$. Consider the equation $P(z,\partial)u(z)=f(z)$, where $u(z)$ admits singularities on the surface $K=\{z_0=0\}$ and $f(z)$ has an asymptotic expansion of Gevrey type with respect to z_0 as $z_0 \to 0$. We study the possibility of asymptotic expansion of $u(z)$. We define the characteristic polygon of $P(z,\partial)$ with respect to K and characteristic indices. We discuss the behavior of $u(z)$ in a neighborhood of K, by using these notions. The main result is a generalization of that in [6].

KEY WORDS: complex partial differential equations, solutions with asymptotic expansion

§0. Introduction

Let $P(z,\partial)$ be a linear partial differential operator with order m and holomorphic coefficients in a domain containing the origin $z=0$ in $\mathbb{C}^{d+1}$ and $K=\{z_0=0\}$. In the present paper we consider

$$P(z,\partial)u(z)=f(z), \qquad \text{(Eq)}$$

where $u(z)$ and $f(z)$ are holomorphic except on K. The purpose of the present paper is to study the behavior of singular solutions near K. First we note that for given $P(z,\partial)$ we can define the characteristic polygon Σ with respect to K

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and exponents γ_i $(0 \le i \le p)$, $0=\gamma_p<\gamma_{p-1}<\cdots<\gamma_1<\gamma_0=+\infty$, which are the slopes of the characteristic polygon Σ and called characteristic indices with respect to K. We study the relations between behaviors of singular solutions near K and the characteristic indices.

Roughly speaking, the main result is the following:

Consider (Eq) *and suppose that the following conditions* (1), (2) *and* (3) *are satisfied.*

(1) $u(z)$ *grows at most in some exponential order near* $z_0=0$, *that is, there is a* $\gamma>0$ *such that for any* $\varepsilon>0$ $|u(z)| \le C_\varepsilon \exp(\varepsilon |z_0|^{-\gamma})$.

(2) $f(z)$ *has a Gevrey type asymptotic expansion,* $f(z) \sim \sum_{n=0}^{+\infty} f_n(z') z_0^n$ *with* $|f_n(z')| \le AB^n\Gamma(n/\gamma+1)$, *as* $z_0 \to 0$ *in a sectorial region* $\Omega(\theta)$ *with respect to* z_0.

(3) *The traces of* $u(z)$ *on some hypersurface* S *have also the asymptotic expansion of the same type as* $f(z)$.

Then we conclude under some conditions on $P(z,\partial)$, γ *and* S *that* $u(z)$ *has also an asymptotic expansion of the same Gevrey type as* $f(z)$, *as* z_0 *tends to* 0.

In the above assertion γ is one of γ_i $(i=0,1,\cdots,p-1)$. It is a generalization of the main results of [5] and [6]. We treated the case $\gamma=\gamma_{p-1}$ in those papers. We studied in [5] the behavior of $u(z)$ by analysis of an integral representation of solutions with singularities on K. In [6] we did not use the representation but showed the possibility of asymptotic expansion of $u(z)$, by estimating the derivatives $(\partial/\partial z_0)^n u(z)$. So the arguments in [6] were less complicated and completely different from [5]. In the present paper we show the main result by the estimate of derivatives of $u(z)$, which follows [6].

In §1 we first define the characteristic polygon Σ. From Σ we determine the indices γ_i and polynomials $\chi_{P,i}(z',\xi')$ $(0 \le i \le p-1)$. Next we introduce function spaces $\mathcal{O}(\Omega(\theta))$ and $Asy_{\{\kappa\}}(\Omega(\theta))$ which are subspaces of holomorphic functions except on K. We give the main results (Theorem 1.4) and examples. The proof is given in the following sections. In §2 we give majorant functions and estimate the derivatives of solutions. In §3, we give a result concerning functions with asymptotic expansion and by combining it with the estimate obtained, we complete the proof of Theorem 1.4.

§1. Notations and Results

In this section we give notations and definitions and state results more precisely. The coordinates of $\mathbb{C}^{d+1}$ are denoted by $z=(z_0,z_1,\cdots,z_d)=(z_0,z')\in$

$\mathbb{C}\times\mathbb{C}^d$. $|z|=\max\{|z_i|;\ 0\le i\le d\}$ and $|z'|=\max\{|z_i|;\ 1\le i\le d\}$. Its dual variables are $\xi=(\xi_0,\xi')=(\xi_0,\xi_1,\cdots,\xi_d)$. $\mathbb{N}$ is the set of all nonnegative integers, $\mathbb{N}=\{0,1,2,\cdots\}$. For real number a, $[a]$ means the integral part of a. The differentiation is denoted by $\partial_i=\partial/\partial z_i$, and $\partial=(\partial_0,\partial_1,\cdots,\partial_d)=(\partial_0,\partial')$. For a multi-index $\alpha=(\alpha_0,\alpha')\in\mathbb{N}\times\mathbb{N}^d$, $|\alpha|=\alpha_0+|\alpha'|=\sum_{i=0}^d\alpha_i$. Define $\partial^\alpha=\prod_{i=0}^d\partial_i^{\alpha_i}$. We denote $\partial'^{\alpha'}=\prod_{i=1}^d\partial_i^{\alpha_i}$ by $\partial^{\alpha'}$ and $\xi'^{\alpha'}=\prod_{i=1}^d\xi_i^{\alpha_i}$ by $\xi^{\alpha'}$.

Now let $P(z,\partial)$ be an m-th order linear partial differential operator with holomorphic coefficients in a neighborhood of $z=0$,

$$P(z,\partial)=\sum_{|\alpha|\le m}a_\alpha(z)\partial^\alpha. \tag{1.1}$$

Let j_α be the valuation of $a_\alpha(z)$ with respect to z_0. Hence if $a_\alpha(z)\not\equiv 0$, $a_\alpha(z)=z_0^{j_\alpha}b_\alpha(z)$ with $b_\alpha(0,z')\not\equiv 0$ and for $a_\alpha(z)\equiv 0$ put $j_\alpha=+\infty$. Let us proceed to define the characteristic polygon Σ of $P(z,\partial)$ with respect to $K=\{z_0=0\}$. Put

$$e_\alpha=j_\alpha-\alpha_0, \tag{1.2}$$

where $e_\alpha=+\infty$ if $a_\alpha(z)\equiv 0$. We denote by $\Pi(a,b)$ the infinite rectangle $\{(x,y)\in\mathbb{R}^2;x\le a,y\ge b\}$. The characteristic polygon Σ is defined by $\Sigma:=$ *the convex hull of* $\cup_\alpha\Pi(|\alpha|,e_\alpha)$. The boundary of Σ consists of a vertical half line $\Sigma(0)$, a horizontal half line $\Sigma(p)$ and $p-1$ segments $\Sigma(i)$ $(1\le i\le p-1)$ with slope γ_i, $0=\gamma_p<\gamma_{p-1}<\cdots<\gamma_1<\gamma_0=+\infty$.

Let $\{(k_i,e(i))\in\boldsymbol{R}^2;0\le i\le p-1\}$ be vertices of Σ, where $0\le k_{p-1}<k_{p-2}<\cdots<k_i<k_{i-1}<\cdots<k_0=m$. So the endpoints of $\Sigma(i)$ $(1\le i\le p-1)$ are $(k_{i-1},e(i-1))$ and $(k_i,e(i))$.

Definition 1.1. The slope γ_i of $\Sigma(i)$ is called the i-th characteristic index of $P(z,\partial)$ with respect to $K=\{z_0=0\}$.

The definition of the characteristic indices in this paper is slightly different from that in [4], where the characteristic indices were denoted by σ_i, and $\gamma_i=\sigma_i-1$ $(0\le i\le p)$ holds.

Now we notice the vertices of the polygon Σ. So put subsets $\Delta(i)$ of multi-indices and quantities l_i $(0\le i\le p-1)$ as follows:

$$\begin{cases}\Delta(i):=\{\alpha\in\mathbb{N}^{d+1};|\alpha|=k_i,\ j_\alpha-\alpha_0=e(i)\},\\ \quad l_i:=\max\{|\alpha'|:\ \alpha\in\Delta(i)\}.\end{cases} \tag{1.3}$$

Define a subset $\Delta_0(i)$ of $\Delta(i)$

$$\Delta_0(i)=\{\alpha\in\Delta(i);|\alpha'|=l_i\} \tag{1.4}$$

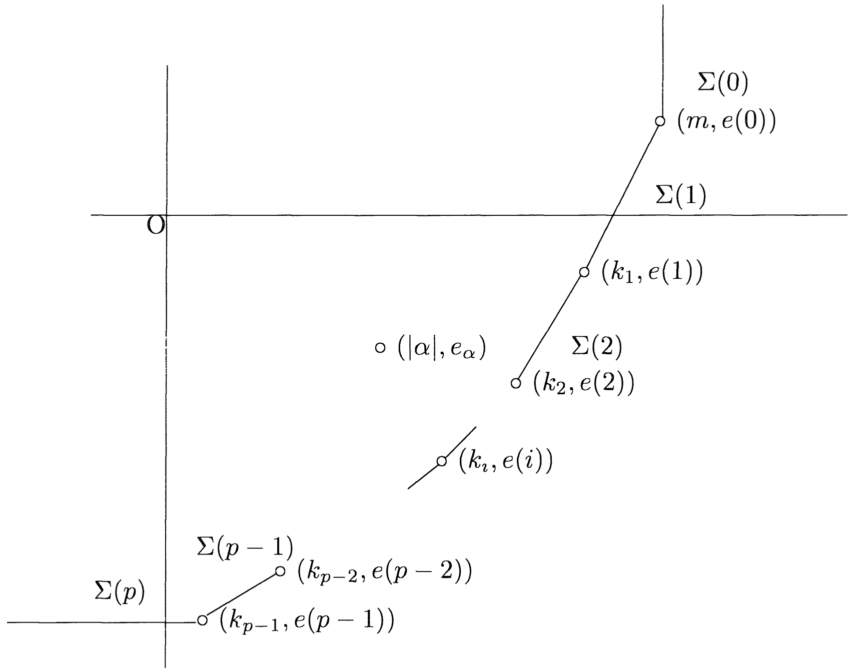

Figure 1. Characteristic polygon

If $\alpha \in \Delta_0(i)$, $j_\alpha = e(i) - k_i + l_i$. So j_α does not depend on α for $\alpha \in \Delta_0(i)$. Hence we can define a polynomial $\chi_{P,i}(z', \xi')$ in ξ' by

$$\chi_{P,i}(z', \xi') = \sum_{\alpha \in \Delta_0(i)} b_\alpha(0, z')\xi^{\alpha'}, \tag{1.5}$$

which is homogeneous in ξ' with degree l_i and plays an important role in the present paper.

Next let us define spaces of holomorphic functions in some regions. Let $\Omega = \Omega_0 \times \Omega'$ be a polydisk with $\Omega_0 = \{z_0 \in \mathbb{C}^1;\ |z_0| < R\}$ and $\Omega' = \{z' \in \mathbb{C}^d; |z'| < R\}$ for some positive constant R. Put $\Omega_0(\theta) = \{z_0 \in \Omega_0 - \{0\};\ |\arg z_0| < \theta\}$ and $\Omega(\theta) = \Omega_0(\theta) \times \Omega'$. $\mathcal{O}(\Omega)$ $(\mathcal{O}(\Omega'),\ \mathcal{O}(\Omega(\theta)))$ is the set of all holomorphic functions on Ω (resp. Ω', $\Omega(\theta)$). $\mathcal{O}(\Omega(\theta))$ contains multi-valued functions, if $\theta > \pi$.

We introduce the subspaces $\mathcal{O}_{(\kappa)}(\Omega(\theta))$ and $Asy_{\{\kappa\}}(\Omega(\theta))$ of $\mathcal{O}(\Omega(\theta))$.

Definition 1.2. $\mathcal{O}_{(\kappa)}(\Omega(\theta))$ $(0 < \kappa \le +\infty)$ is the set of all $u(z) \in$

$\mathcal{O}(\Omega(\theta))$ such that for any $\varepsilon > 0$ and any θ' with $0 < \theta' < \theta$

$$|u(z)| \leq C \exp(\varepsilon |z_0|^{-\kappa}) \quad \text{for } z \in \Omega(\theta') \tag{1.6}$$

holds for a constant $C = C(\varepsilon, \theta')$. We put $\mathcal{O}_{(+\infty)}(\Omega(\theta)) = \mathcal{O}(\Omega(\theta))$ for $\kappa = +\infty$.

Definition 1.3. $Asy_{\{\kappa\}}(\Omega(\theta))$ $(0 < \kappa \leq +\infty)$ is the set of all $u(z) \in \mathcal{O}(\Omega(\theta))$ such that for any θ' with $0 < \theta' < \theta$ and any $N \in \mathbb{N}$

$$|u(z) - \sum_{n=0}^{N-1} u_n(z') z_0^n| \leq AB^N |z_0|^N \Gamma\left(\frac{N}{\kappa} + 1\right) \quad z \in \Omega(\theta'), \tag{1.7}$$

where $u_n(z') \in \mathcal{O}(\Omega')$ $(n \in \mathbb{N})$, holds for constants $A = A(\theta')$ and $B = B(\theta')$.

We say that $u(z) \in Asy_{\{\kappa\}}(\Omega(\theta))$ has an asymptotic expansion with Gevrey exponent (or index) κ. $u(z) \in Asy_{\{+\infty\}}(\Omega(\theta))$ means that $u(z)$ is holomorphic at $z = 0$.

We give a condition on $P(z, \partial)$ considered in the present paper. For fixed i $(0 \leq i \leq p-1)$,

$(C_\imath)$ $\quad j_\alpha = 0$ for all $\alpha \in \Delta_0(i)$.

The main result is the following.

Theorem 1.4. *Suppose that $P(z, \partial)$ satisfies $(C_\imath)$ and $\chi_{P,\imath}(0, \hat{\xi}') \neq 0$, $\hat{\xi}' = (1, 0, \cdots, 0)$. Let $u(z) \in \mathcal{O}_{(\gamma_\imath)}(\Omega(\theta))$ be a solution of*

$$P(z, \partial) u(z) = f(z) \in Asy_{\{\gamma_\imath\}}(\Omega(\theta)) \tag{1.8}$$

such that

$$\partial_1^h u(z_0, 0, z'') \in Asy_{\{\gamma_\imath\}}(\Omega(\theta) \cap \{z_1 = 0\}) \quad \text{for} \quad 0 \leq h \leq l_\imath - 1. \tag{1.9}$$

Then there is a polydisk W centered at $z = 0$ such that $u(z) \in Asy_{\{\gamma_\imath\}}(W(\theta))$.

We considered in [6] the case $i = p - 1$ and $l_{p-1} = 0$ and obtained the same result for this case. If $(C_\imath)$ does not hold, $u(z)$ does not have asymptotic expansion but the behaviors of solutions become less regular. We studied in [7] the behaviors of solutions under the condition that $i = p - 1$ and $l_{p-1} = 0$ but that (C_{p-1}) does not always hold.

We give an example. Let us consider

$$P(z, \partial) = \partial_1^5 + \partial_1^3 \partial_0 + \partial_0^2, \quad z = (z_0, z_1) \in \mathbb{C}^2. \tag{1.10}$$

We have

$$\begin{cases} \gamma_0 = +\infty, \quad \gamma_1 = 1, \quad \gamma_2 = 1/2, \quad \gamma_3 = 0, \\ \chi_{P.0}(z', \xi_1) = \xi_1^5, \quad \chi_{P,1}(z', \xi_1) = \xi_1^3, \quad \chi_{P,2}(z', \xi_1) = 1. \end{cases}$$

Obviously $P(z, \partial)$ satisfies (C_i) and $\chi_{P,i}(z', 1) \neq 0$ for $i = 0, 1, 2$. Consequently it follows from Theorem 1.4 that there is a polydisk W centered at $z = 0$ such that

$i = 0:$ $u(z) \in \mathcal{O}_{(+\infty)}(\Omega(\theta))$, $\partial_1^h u(z_0, 0) \in Asy_{\{+\infty\}}(\Omega_0(\theta))$ $(0 \leq h \leq 4)$ *and* $f(z) \in Asy_{\{+\infty\}}(\Omega(\theta)) \Rightarrow u(z) \in Asy_{\{+\infty\}}(W(\theta))$,

$i = 1:$ $u(z) \in \mathcal{O}_{(1)}(\Omega(\theta))$, $\partial_1^h u(z_0, 0) \in Asy_{\{1\}}(\Omega_0(\theta))$ $(0 \leq h \leq 2)$ *and* $f(z) \in Asy_{\{1\}}(\Omega(\theta)) \Rightarrow u(z) \in Asy_{\{1\}}(W(\theta))$,

$i = 2:$ $u(z) \in \mathcal{O}_{(1/2)}(\Omega(\theta))$, $f(z) \in Asy_{\{1/2\}}(\Omega(\theta)) \Rightarrow u(z) \in Asy_{\{1/2\}}(W(\theta))$.

§2. Estimate of Derivatives

The purpose of this section is to obtain estimates of the derivatives $\partial_0^N u(z_0, z')$. By using the obtained estimates, we show Theorem1.4 in the following sections. So first let us study majorant functions. For formal power series of one variable t, $A(t) = \sum_{n=0}^{\infty} A_n t^n$ and $B(t) = \sum_{n=0}^{\infty} B_n t^n$, $A(t) \ll B(t)$ means $|A_n| \leq B_n$ for all $n \in \mathbb{N}$. Put

$$(2.1) \qquad \begin{cases} \psi^{(k)}(t) = k!/(r-t)^{k+1} \quad \text{for} \quad k \geq 0 \\ \psi^{(k)}(t) = \displaystyle\int_0^t \psi^{(k+1)}(\tau) d\tau \quad \text{for} \quad k < 0, \end{cases}$$

which satisfy $\dfrac{d\psi^{(k)}}{dt}(t) = \psi^{(k+1)}(t)$ and if $0 < r \leq 1$, $\psi^{(k)}(t) \ll \psi^{(k+1)}(t)$. By modifying $\psi^{(k)}(t)$, define other majorant functions $\Psi_k^{(s)}(t)$ ($k \in \mathbb{Z}$, $s \in \mathbb{N}$)

$$(2.2) \qquad \Psi_k^{(s)}(t) = \left(\frac{d}{dt}\right)^s \left\{\frac{R'}{R'-t}\psi^{(k)}(t)\right\} \quad \text{where} \quad 0 < r < R' < 1.$$

We have

Lemma 2.1. (1) *The following inequalities hold:*

$$(2.3) \quad \begin{cases} (R'-t)\Psi_k^{(s)}(t) \gg 0, \quad \Psi_k^{(s)}(t) \gg \Psi_{k+1}^{(s-1)}(t), \quad \Psi_k^{(s)}(t) \gg \Psi_k^{(s-1)}(t), \\ \dfrac{1}{R-R'}\Psi_k^{(s)}(t) \gg (R-t)^{-1}\Psi_k^{(s)}(t) \quad \textit{for} \ R' < R. \end{cases}$$

(2) *If* $k \geq 0$,

$$\psi^{(s+k)}(t) \ll \Psi_k^{(s)}(t) \ll \frac{R'}{R'-r}\psi^{(s+k)}(t). \tag{2.4}$$

(3) *If* $k < 0$ *and* $R' > 2r$,

$$\psi^{(s+k)}(t) \ll \Psi_k^{(s)}(t) \ll \frac{2^{|k|}R'}{R'-2r}\psi^{(s+k)}(t). \tag{2.5}$$

(4) *Let* $|t| \leq r/2$. *Then*

$$\begin{aligned} |\psi^{(k)}(t)| &\leq \frac{2^{k+1}k!}{r^{k+1}} \quad \textit{for} \quad k \geq 0, \\ |\psi^{(k)}(t)| &\leq \frac{2|t|^{|k|}}{r|k|!} \quad \textit{for} \quad k \leq 0. \end{aligned} \tag{2.6}$$

(5) *Let* $|t| \leq r/2$, $R' > 2r$, $s \geq 0$ *and* $k \geq 0$. *Then there exist constants* C_0 *and* C_1 *such that*

$$|\Psi_{-k}^{(s)}(t)| \leq \frac{C_0 C_1^{k+s} s!}{k!}. \tag{2.7}$$

The proofs are not difficult and we may refer it to [3] (see also [1] and [8]). However for the convenience of the readers we give them.

Proof. (1) From the definition of $\Psi_k^{(0)}(t)$, $(R'-t)\Psi_k^{(0)}(t) \gg 0$. So $(R'-t)\Psi_k^{(1)}(t) \gg \Psi_k^{(0)}(t) \gg 0$. By induction on s we have $(R'-t)\Psi_k^{(s)}(t) \gg 0$. It holds that

$$\Psi_k^{(s)}(t) = R'(d/dt)^{s-1}((R'-t)^{-1}\psi^{(k+1)}(t) + (R'-t)^{-2}\psi^{(k)}(t)).$$

So we have the second inequality and the third one. The fourth inequality follows from $(R-t)^{-1}(R'-t)\Psi_k^{(s)}(t) = (1-(R-R')(R-t)^{-1})\Psi_k^{(s)}(t) \gg 0$.

(2) It is obvious that $\psi^{(s+k)}(t) \ll \Psi_k^{(s)}(t)$ for all $k \in \mathbb{Z}$. From (1) we have $(R'-t)^{-1}\psi^{(k)}(t) \ll (R'-r)^{-1}\psi^{(k)}(t)$. Hence the desired inequality follows.

(3) If $k < 0$, then $R'(R'-t)^{-1}\psi^{(k)}(t) = \sum_{n=0}^{+\infty} C_n t^{n+|k|}/(r^{n+1}(n+1)\ldots(n+|k|))$, where $C_n = \sum_{i=0}^{n} \frac{(n+1)\cdots(n+|k|)}{(n-i+1)\cdots(n-i+|k|)}\left(\frac{r}{R'}\right)^i$. Since $C_0 = 1$

and for $0 < i \leq n$

$$\frac{(n+1)\cdots(n+|k|)}{(n-i+1)\cdots(n-i+|k|)} = \frac{(n+|k|-i+1)\cdots(n+|k|)}{(n+1-i)\cdots n}$$
$$= \left(1+\frac{|k|}{n+1-i}\right)\left(1+\frac{|k|}{n+2-i}\right)\cdots\left(1+\frac{|k|}{n}\right)$$
$$\leq \left(1+\frac{|k|}{1}\right)\left(1+\frac{|k|}{2}\right)\cdots\left(1+\frac{|k|}{i}\right) = \binom{i+|k|}{i} \leq 2^{i+|k|},$$

we have $C_n \leq 2^{|k|}R'/(R'-2r)$. Hence $\Psi_k(t) \ll 2^{|k|}R'/(R'-2r)\psi^{(k)}(t)$, from which the inequality in (3) follows.

(4) Let $|t| \leq r/2$. It is easy to show $|\psi^{(k)}(t)| \leq 2^{k+1}k!/r^{k+1}$ for $k \geq 0$. Let $k < 0$. Then $|\psi^{(k)}(t)| \leq \sum_{l=0}^{+\infty} \frac{|t|^{l+|k|}}{r^{l+1}(l+1)\cdots(l+|k|)} \leq \frac{|t|^{|k|}}{r|k|!}\sum_{l=0}^{+\infty}\frac{|t|^l}{r^l} \leq \frac{2|t|^{|k|}}{r|k|!}$.

(5) It follows from (3) and (4) that $|\Psi_{-k}^{(s)}(t)| \leq 2^k R'|\psi^{(s-k)}(t)|/(R'-2r)$. We have for $s \geq k$ $|\psi^{(s-k)}(t)| \leq 2^{s-k+1}(s-k)!/r^{s-k+1} \leq 2^{s-k+1}s!/(r^{s-k+1}k!)$ and for $k \geq s$ $|\psi^{(s-k)}(t)| \leq 2|t|^{k-s}/(r(k-s)!) \leq 2^{k+1}|t|^{k-s}s!/(rk!)$. Hence there exist constants C_0 and C_1 such that $|\Psi_{-k}^{(s)}(t)| \leq C_0 C_1^{k+s}s!/k!$. □

In order to estimate holomorphic functions in a domain that is sectorial with respect to z_0, let us introduce a series of majorant functions $\{\Psi_k^{(s)}(a;z); k \in \mathbb{Z}, s \in \mathbb{N}\}$,

$$\Psi_k^{(s)}(a;z) = \Psi_k^{(s)}\left(\frac{z_0-a}{ca} + \rho z_1 + z_2 + \cdots + z_d\right), \tag{2.8}$$

where $0 < 2r < R' < 1$, $\rho \geq 1$, $0 < a < R/2$ and $0 < c < 1$. For a domain $\Omega = \{z \in \mathbb{C}^{d+1};\ |z| < R\}$ $(R' < R \leq 1)$, put

$$\Omega(a,c) = \{z = (z_0, z');\ |z_0 - a| \leq caR,\ |z'| < R\}. \tag{2.9}$$

If $|z_0 - a| \leq caR$, then $|z_0| \leq a + caR < 2a < R$ and $\Omega(a,c)$ is a subset of Ω. For formal power series $A(z) = \sum_\alpha A_\alpha(z_0-a)^{\alpha_0}(z')^{\alpha'}$ and $B(z) = \sum_\alpha B_\alpha(z_0-a)^{\alpha_0}(z')^{\alpha'}$ centered at $(a,0,\cdots,0)$, $A(z) \underset{(a,0')}{\ll} B(z)$ means $|A_\alpha| \leq B_\alpha$ for all $\alpha \in \mathbb{N}^{d+1}$. Define an integral operator ∂_1^{-1} for $v(z) \in \mathcal{O}(\Omega(\theta))$ as follows:

$$\partial_1^{-1}v(z) = \int_0^{z_1} v(z_0, \tau, z'')d\tau \tag{2.10}$$

and $\partial_1^{-l}v(z) = (\partial_1^{-1})^l v(z)$ for $l \in \mathbb{N}$.

Lemma 2.2. *Let $v(z) \in \mathcal{O}(\Omega(\theta))$, $0 < \theta' < \min\{\theta, \pi/2\}$ and $c = \sin\theta'$. Suppose that $|v(z)| \leq K_{\theta',a}$ on $\Omega(a,c)$. Then $v(z) \underset{(a,0')}{\ll} K_{\theta',a}(R-t)^{-1} \underset{(a,0')}{\ll} K_{\theta',a}\Psi_0^{(0)}(a;z)$, where $t = \dfrac{z_0-a}{ca} + \rho z_1 + z_2 + \cdots + z_d$.*

Proof. We have by Cauchy's integral formula

$$\partial_0^{\alpha_0} v(z_0, z') = \frac{\alpha_0!}{2\pi i} \oint_{|\zeta - a| = aR\sin\theta'} \frac{v(\zeta, z')}{(\zeta - z_0)^{\alpha_0+1}} d\zeta.$$

Hence $|\partial_0^{\alpha_0} v(a, z')| \le K_{\theta',a}\alpha_0!/(caR)^{\alpha_0}$ and $|\partial^\alpha v(a, 0')| \le K_{\theta',a}\alpha!/((ca)^{\alpha_0} R^{|\alpha|})$. Since $0 < 2r < R' < R \le 1$,

$$v(z) \underset{(a,0')}{\ll} K_{\theta',a}\Big/\left(1 - \frac{z_0 - a}{caR} - \frac{\rho z_1 + z_2 + \cdots + z_d}{R}\right)$$
$$\underset{(a,0')}{\ll} K_{\theta',a}\Big/\left(r - \left(\frac{z_0 - a}{ca} + \rho z_1 + z_2 + \cdots + z_d\right)\right) \underset{(a,0')}{\ll} K_{\theta',a}\Psi_0^{(0)}(a; z).$$

□

Lemma 2.3. *Let $a(z) = z_0^j b(z)$ $(j \in \mathbb{N})$ be a holomorphic function on Ω such that $|b(z)| \le B$. Then $a(z) \underset{(a,0')}{\ll} 2^j BRa^j(R - t)^{-1}$ and*

$$a(z)\partial^\alpha \Psi_k^{(s)}(a; z) \underset{(a,0')}{\ll} \frac{2^j BR\rho^{\alpha_1} a^{j-\alpha_0}}{(R - R')c^{\alpha_0}} \Psi_k^{(s+|\alpha|)}(a; z), \tag{2.11}$$

where $t = \dfrac{z_0 - a}{ca} + \rho z_1 + z_2 + \cdots + z_d$.

Proof. We have $|a(z)| \le ((cR+1)a)^j B \le 2^j Ba^j$ on $\Omega(a, c)$ and by Lemma 2.2 $a(z) \underset{(a,0')}{\ll} 2^j BRa^j(R-t)^{-1}$. Since $\partial^\alpha \Psi_k^{(s)}(a; z) = \rho^{\alpha_1}(ca)^{-\alpha_0}\Psi_k^{(s+|\alpha|)}(a; z)$, we have (2.11) by the last inequality in (2.3) in Lemma 2.1. □

Lemma 2.4. *Let $w(z)$ be a solution of*

$$\begin{cases} \partial_1^l w(z) = f(z), \\ \partial_1^h w(z_0, 0, z'') = 0 \quad \text{for} \quad 0 \le h \le l - 1. \end{cases} \tag{2.12}$$

If $f(z) \underset{(a,0')}{\ll} \Psi_k^{(j+l)}(a; z)$, then $w(z) \underset{(a,0')}{\ll} \rho^{-l}\Psi_k^{(j)}(a; z)$.

Proof. The estimate of $w(z)$ follows from $\partial_1^l \rho^{-l}\Psi_k^{(j)}(a; z) = \Psi_k^{(j+l)}(a; z)$. □

Lemma 2.5. *Let $u(z) \in Asy_{\{\gamma\}}(\Omega(\theta))$ $(0 < \gamma \le +\infty)$. Then $\partial_0^h u(z) \in Asy_{\{\gamma\}}(\Omega(\theta))$ for $h \in \mathbb{N}$ and for any $0 < \theta' < \theta$ there are constants $M = M(\theta')$ and $C = C(\theta')$ such that $|\partial_0^h u(z)| \le MC^h\Gamma(\frac{h}{\delta} + 1)$, $\delta = \gamma/(\gamma + 1)$, in $\Omega(\theta')$.*

Proof. Suppose $u(z)$ has the asymptotic expansion (1.7) with $\kappa = \gamma$. Then it follows from Cauchy's integral formula that $\partial_0^h u(z) \in Asy_{\{\gamma\}}(\Omega(\theta))$ and for any $0 < \theta' < \theta$ there are constants $A_i = A_i(\theta')$ $(i = 1, 2, 3, 4)$ such that $|\partial_0^h u(z) - \sum_{n=0}^{N-1}(n+h)\cdots(n+1)u_{n+h}(z')z_0^n| \leq A_1^{h+1}A_2^N|z_0|^N h!\Gamma(\frac{N+h}{\gamma}+1) \leq A_3^{h+1}\Gamma(\frac{h}{\delta}+1)A_4^N|z_0|^N\Gamma(\frac{N}{\gamma}+1)$ and $|\partial_0^h u(z)| \leq A_3^{h+1}\Gamma(\frac{h}{\delta}+1)$ in $\Omega(\theta')$. □

Now let us return to

$$\text{(Eq)} \qquad P(z, \partial)u(z) = f(z).$$

The coefficients of $P(z, \partial)$ are holomorphic in a domain containing $\overline{\Omega} = \{z \in \mathbb{C}^{d+1}; |z| \leq R\}$. Suppose that $P(z, \partial)$ satisfies (C_i). Then $j_\alpha = 0$ for all $\alpha \in \Delta_0(i)$ and $e(i) = -k_i + l_i$. $\chi_{P,i}(z', \xi') = \sum_{\alpha \in \Delta_0(i)} b_\alpha(0, z')\xi^{\alpha'}$ is a homogeneous polynomial in ξ' with degree l_i. Further suppose that $\chi_{P,i}(0, \hat{\xi}') \neq 0$, $\hat{\xi}' = (1, 0, \cdots, 0)$. Put $\alpha(i) = (-e(i), \alpha'(i)) \in \mathbb{N} \times \mathbb{N}^d$, $\alpha'(i) = (l_i, 0, \cdots, 0)$. Then $b_{\alpha(i)}(0) \neq 0$. Define $m^*(i) = \max_{\{\alpha; e_\alpha < +\infty\}}(e_\alpha - e(i))$.

Proposition 2.6. *Suppose that $P(z, \partial)$ satisfies (C_i) and $\chi_{P,i}(z', \hat{\xi}') \neq 0$ on $\overline{\Omega}' = \{z' \in \mathbb{C}^d; |z'| \leq R\}$, $\hat{\xi}' = (1, 0, \cdots, 0)$. Let $u(z) \in \mathcal{O}(\Omega(\theta))$ be a solution of* (Eq) *with $\partial_1^l u(z_0, 0, z'') = 0$ for $0 \leq l \leq l_i - 1$ and $f(z) \in Asy_{\{\gamma_i\}}(\Omega(\theta))$. Let $0 < \theta' < \min\{\theta, \pi/2\}$ and $c = \sin\theta'$. Put $M(a, c) = \sup\{|\partial_0^n u(z)|;\ z \in \Omega(a, c),\ 0 \leq n < \max\{-e(i), m^*(i)\}\}$.*

Then there are constants $A = A(\theta'), B$ and $\rho = \rho(\theta') \geq 1$ which do not depend on a such that the following estimates hold: if $0 \leq i \leq p-2$,

(2.13)
$$\partial_0^n \partial_1^{l_i} u(z) \underset{(a;0')}{\ll} M(a, c)A^{n+1}\left(\sum_{h=0}^{+\infty} B^h(n+1)^h \left(\sum_{h'=0}^{+\infty} \frac{\Psi_{-h-[h'/\gamma_{i+1}]}^{([n(1+1/\gamma_i)]+m)}(a; z)}{|a|^{h'}}\right)\right),$$

where $1/+\infty = 0$ for $i = 0$, and if $i = p-1$,

(2.14)
$$\partial_0^n \partial_1^{l_{p-1}} u(z) \underset{(a,0')}{\ll} M(a, c)A^{n+1}\left(\sum_{h=0}^{+\infty} B^h(n+1)^h \Psi_{-h}^{([n(1+1/\gamma_i)]+m)}(a; z)\right)$$

for $n \in \mathbb{N}$, where r, R', R are small positive constants with $2r < R' < R < 1$.

Proof. The assumption $\chi_{P,i}(z', \hat{\xi}') \neq 0$ means $a_{\alpha(i)}(0, z') = b_{\alpha(i)}(z') \neq 0$. So we may assume $a_{\alpha(i)}(z) = 1$. Put $v(z) := \partial_1^{l_i} u(z)$ and consider $P(z, \partial)\partial_1^{-l_i} v(z)$.

We decompose $P(z,\partial)\partial_1^{-l_\imath}$ into two parts,

$$
(2.15)\qquad \begin{cases} P(z,\partial)\partial_1^{-l_\imath} = Q(z,\partial) + R(z,\partial), \\ Q(z,\partial) = \displaystyle\sum_{\{\alpha;e_\alpha > e(\imath)\}} a_\alpha(z)\partial^\alpha \partial_1^{-l_\imath}, \\ R(z,\partial) = \displaystyle\sum_{\{\alpha;e_\alpha \le e(\imath)\}} a_\alpha(z)\partial^\alpha \partial_1^{-l_\imath}. \end{cases}
$$

We show (2.13) by induction on n. We may assume by Lemma 2.2 that (2.13) holds for $0 \le n \le N-1$ with $N \ge \max\{-e(i), m^*(i)\}$. By differentiating $a_\alpha(z)\partial^\alpha \partial_1^{-l_\imath} v(z)$ $(N+e(i))$- times with respect to z_0,

(2.16)

$$
\partial_0^{N+e(\imath)}(a_\alpha(z)\partial^\alpha\partial_1^{-l_\imath}v(z)) = \sum_{r=0}^{N+e(\imath)} \binom{N+e(i)}{r} \partial_0^r a_\alpha(z)\partial^{\alpha'}\partial_1^{-l_\imath}\partial_0^{N+\alpha_0+e(\imath)-r}v(z).
$$

First let us estimate $\partial_0^{N+e(\imath)}Q(z,\partial)v(z)$. So let α be a multi-index with $e_\alpha > e(i)$. Suppose $r \ge j_\alpha$. Then $N+\alpha_0+e(i)-r \le N+e(i)-e_\alpha \le N-1$. Hence by the inductive hypothesis, Lemma 2.3 and Lemma 2.4 there is a constant C_0 such that

$$
\begin{aligned}
&\partial_0^r a_\alpha(z)\partial^{\alpha'}\partial_1^{-l_\imath}\partial_0^{N+\alpha_0+e(\imath)-r}v(z) \\
&\underset{(a,0')}{\ll} \quad M(a,c)A^N\rho^{\alpha_1-l_\imath}C_0^{r+1}r!\left(\sum_{h=0}^{+\infty}B^hN^h\right. \\
&\qquad\qquad \times\left.\left(\sum_{h'=0}^{+\infty}\frac{\Psi^{([(N+\alpha_0+e(\imath)-r)(1+1/\gamma_\imath)]+|\alpha'|-l_\imath+m)}_{-h-[h'/\gamma_{\imath+1}]}(a;z)}{|a|^{h'}}\right)\right).
\end{aligned}
$$

We have, by $(e_\alpha - e(i))/\gamma_\imath \ge |\alpha| - k_\imath$,

$$
\begin{aligned}
&[(N+\alpha_0+e(i)-r)(1+1/\gamma_\imath)] + |\alpha'| - l_\imath \\
=&[N(1+1/\gamma_\imath) + (e(i)-e_\alpha - r + j_\alpha)(1+1/\gamma_\imath)] + |\alpha'| - l_\imath \\
=&[N(1+1/\gamma_\imath) + (e(i)-e_\alpha - r + j_\alpha)/\gamma_\imath] + |\alpha'| + e(i) - e_\alpha - r + j_\alpha - l_\imath \\
=&[N(1+1/\gamma_\imath) + (e(i)-e_\alpha - r + j_\alpha)/\gamma_\imath] + |\alpha| - k_\imath - r \\
\le&[N(1+1/\gamma_\imath) - (r-j_\alpha)/\gamma_\imath] - r \le [N(1+1/\gamma_\imath)] - r.
\end{aligned}
$$

Hence by Lemma 2.1 we have for $r \geq j_\alpha$

(2.17)
$$\begin{aligned}
&\partial_0^r a_\alpha(z)\partial^{\alpha'}\partial_1^{-l_\imath}\partial_0^{N+\alpha_0+e(\imath)-r}v(z) \\
&\underset{(a,0')}{\ll} M(a,c)A^N\rho^{\alpha_1-l_\imath}C_0^{r+1}r!\left(\sum_{h=0}^{+\infty}B^hN^h\left(\sum_{h'=0}^{+\infty}\frac{\Psi_{-h-[h'/\gamma_{\imath+1}]}^{([N(1+1/\gamma_\imath)]-r+m)}(a;z)}{|a|^{h'}}\right)\right) \\
&\underset{(a,0')}{\ll} M(a,c)A^N\rho^{\alpha_1-l_\imath}C_0^{r+1}r!\left(\sum_{h=0}^{+\infty}B^hN^h\left(\sum_{h'=0}^{+\infty}\frac{\Psi_{-h-r-[h'/\gamma_{\imath+1}]}^{([N(1+1/\gamma_\imath)]+m)}(a;z)}{|a|^{h'}}\right)\right).
\end{aligned}$$

Suppose $0 \leq r < j_\alpha$. Then $\partial_0^r a_\alpha(z) = O(|z_0|^{j_\alpha - r})$. By the assumption $N \geq m^*(i)$ and $e_\alpha - e(i) > 0$, we have $0 \leq N + e(i) - e_\alpha \leq N-1$ and $\partial_0^{N+\alpha_0+e(\imath)-r} = \partial_0^{j_\alpha - r}\partial_0^{N+e(\imath)-e_\alpha}$. So by Lemmas 2.3 and 2.4

$$\begin{aligned}
&\partial_0^r a_\alpha(z)\partial^{\alpha'}\partial_1^{-l_\imath}\partial_0^{N+\alpha_0+e(\imath)-r}v(z) \\
&\underset{(a,0')}{\ll} M(a,c)A^N\rho^{\alpha_1-l_\imath}C_0^{r+1}r!c^{r-j_\alpha}\left(\sum_{h=0}^{+\infty}B^hN^h\right. \\
&\left.\times\left(\sum_{h'=0}^{+\infty}\frac{\Psi_{-h-[h'/\gamma_{\imath+1}]}^{([(N+e(\imath)-e_\alpha)(1+1/\gamma_\imath)]+|\alpha'|+j_\alpha-r-l_\imath+m)}(a;z)}{|a|^{h'}}\right)\right).
\end{aligned}$$

We have, by the relation $(e_\alpha - e(i))/\gamma_\imath \geq |\alpha| - k_\imath$,

$$\begin{aligned}
&[(N+e(i)-e_\alpha)(1+1/\gamma_\imath)] + |\alpha'| + j_\alpha - r - l_\imath \\
\leq&[N(1+1/\gamma_\imath)] + (e(i)-e_\alpha) + k_\imath - \alpha_0 + j_\alpha - r - l_\imath \\
\leq&[N(1+1/\gamma_\imath)] - r
\end{aligned}$$

and

$$\begin{aligned}
&\Psi_{-h-[h'/\gamma_{\imath+1}]}^{([(N+e(\imath)-e_\alpha)(1+1/\gamma_\imath)]+|\alpha'|+j_\alpha-r-l_\imath+m)}(a;z) \\
\ll&\Psi_{-h-[h'/\gamma_{\imath+1}]}^{([N(1+1/\gamma_\imath)]-r+m)}(a;z) \ll \Psi_{-h-r-[h'/\gamma_{\imath+1}]}^{([N(1+1/\gamma_\imath)]+m)}(a;z).
\end{aligned}$$

Thus we have for $0 \leq r < j_\alpha$

$$\begin{aligned}
&\partial_0^r a_\alpha(z)\partial^{\alpha'}\partial_1^{-l_\imath}\partial_0^{N+\alpha_0+e(\imath)-r}v(z) \underset{(a,0')}{\ll} M(a,c)A^N\rho^{\alpha_1-l_\imath}C_0^{r+1}r!c^{r-j_\alpha} \\
&\times\left(\sum_{h=0}^{+\infty}B^hN^h\left(\sum_{h'=0}^{+\infty}\frac{\Psi_{-h-r-[h'/\gamma_{\imath+1}]}^{([N(1+1/\gamma_\imath)]+m)}(a;z)}{|a|^{h'}}\right)\right).
\end{aligned} \tag{2.18}$$

Hence, by choosing B such that $B > 2C_0$ and Lemma 2.1, there are constants $A_0 = A_0(\rho, c)$ and $A_1 = A_1(\rho, c)$ such that

$$
\begin{aligned}
&\partial_0^{N+e(i)} Q(z,\partial)v(z)\\
&\underset{(a,0')}{\ll} M(a,c)A_0A^N\left(\sum_{r=0}^{N+e(i)}\binom{N+e(i)}{r}C_0^{r+1}r!\right.\\
&\qquad\qquad \left.\times\left(\sum_{l=0}^{+\infty}B^lN^l\left(\sum_{h'=0}^{+\infty}\frac{\Psi^{([N(1+1/\gamma_i)]+m)}_{-l-r-[h'/\gamma_{i+1}]}(a;z)}{|a|^{h'}}\right)\right)\right)\\
&= M(a,c)A_0A^NC_0\left(\sum_{r=0}^{N+e(i)}\frac{(N+e(i))!C_0^r}{(N+e(i)-r)!}\right.\\
&\qquad\qquad \left.\times\left(\sum_{l=0}^{+\infty}B^lN^l\left(\sum_{h'=0}^{+\infty}\frac{\Psi^{([N(1+1/\gamma_i)]+m)}_{-l-r-[h'/\gamma_{i+1}]}(a;z)}{|a|^{h'}}\right)\right)\right)\\
&\underset{(a,0')}{\ll} M(a,c)A_0A^NC_0\\
&\qquad\qquad \left(\sum_{r=0}^{N+e(i)}(NC_0)^r\left(\sum_{l=0}^{+\infty}B^lN^l\left(\sum_{h'=0}^{+\infty}\frac{\Psi^{([N(1+1/\gamma_i)]+m)}_{-l-r-[h'/\gamma_{i+1}]}(a;z)}{|a|^{h'}}\right)\right)\right)\\
&\underset{(a,0')}{\ll} M(a,c)A_0A^NC_0\left(\sum_{h=0}^{+\infty}N^h\left(\sum_{l+r=h}C_0^rB^l\right)\left(\sum_{h'=0}^{+\infty}\frac{\Psi^{([N(1+1/\gamma_i)]+m)}_{-h-[h'/\gamma_{i+1}]}(a;z)}{|a|^{h'}}\right)\right)\\
&= M(a,c)A_0A^NC_0\left(\sum_{h=0}^{+\infty}B^hN^h\left(\sum_{r=0}^{h}\Big(\frac{C_0}{B}\Big)^r\right)\left(\sum_{h'=0}^{+\infty}\frac{\Psi^{([N(1+1/\gamma_i)]+m)}_{-h-[h'/\gamma_{i+1}]}(a;z)}{|a|^{h'}}\right)\right)\\
&\underset{(a,0')}{\ll} M(a,c)A_1A^N\left(\sum_{h=0}^{+\infty}B^hN^h\left(\sum_{h'=0}^{+\infty}\frac{\Psi^{([N(1+1/\gamma_i)]+m)}_{-h-[h'/\gamma_{i+1}]}(a;z)}{|a|^{h'}}\right)\right).
\end{aligned}
$$

Next let us consider $\partial_0^{N+e(i)}R(z,\partial)v(z)$. We divide it into two parts, $\partial_0^{N+e(i)}R(z,\partial)v(z) = R_0^N(z,\partial)v(z) + R_1^N(z,\partial)v(z)$,

(2.19)
$$
\begin{cases}
R_0^N(z,\partial)v(z)\\
\qquad = \displaystyle\sum_{\{\alpha;e_\alpha\le e(i)\}}\sum_{r=0}^{\min\{j_\alpha,N+e(i)\}}\binom{N+e(i)}{r}\partial_0^r a_\alpha(z)\partial^{\alpha'}\partial_1^{-l_i}\partial_0^{N+\alpha_0+e(i)-r}v(z),\\
R_1^N(z,\partial)v(z)\\
\qquad = \displaystyle\sum_{\{\alpha;e_\alpha\le e(i)\}}\sum_{j_\alpha<r\le N+e(i)}\binom{N+e(i)}{r}\partial_0^r a_\alpha(z)\partial^{\alpha'}\partial_1^{-l_i}\partial_0^{N+\alpha_0+e(i)-r}v(z),
\end{cases}
$$

where if $j_\alpha > N + e(i)$, $R_1^N(z,\partial)v(z) = 0$. Let us estimate $R_1^N(z,\partial)v(z)$. We have

$$\begin{aligned}
&\partial_0^r a_\alpha(z)\partial^{\alpha'}\partial_1^{-l_\imath}\partial_0^{\alpha_0+e(\imath)-j_\alpha}\partial_0^{N-r+j_\alpha}v(z)\\
&\underset{(a,0')}{\ll} M(a,c)A^N C_0^{r+1}r!\rho^{\alpha_1-l_\imath}\\
&\qquad\left(\sum_{h=0}^{+\infty}B^hN^h\partial_0^{\alpha_0+e(\imath)-j_\alpha}\left(\sum_{h'=0}^{+\infty}\frac{\Psi_{-h-[h'/\gamma_{\imath+1}]}^{([(N-r+j_\alpha)(1+1/\gamma_\imath)]+|\alpha'|-l_\imath+m)}(a;z)}{|a|^{h'}}\right)\right)\\
&\underset{(a,0')}{\ll} M(a,c)A^N C_0^{r+1}r!\rho^{\alpha_1-l_\imath}c^{-(e(\imath)-e_\alpha)}\\
&\qquad\left(\sum_{h=0}^{+\infty}B^hN^h\left(\sum_{h'=0}^{+\infty}\frac{\Psi_{-h-[h'/\gamma_{\imath+1}]}^{([(N-r+j_\alpha)(1+1/\gamma_\imath)]+|\alpha|-k_\imath-j_\alpha+m)}(a;z)}{|a|^{h'+e(\imath)-e_\alpha}}\right)\right).
\end{aligned}$$

Since

$$\begin{aligned}
&[(N-r+j_\alpha)(1+1/\gamma_\imath)]+|\alpha|-k_\imath-j_\alpha\\
\le&[N(1+1/\gamma_\imath)]-r+|\alpha|-k_\imath\\
\le&[N(1+1/\gamma_\imath)]-r-(e(i)-e_\alpha)/\gamma_{\imath+1},
\end{aligned}$$

we have

$$\Psi_{-h-[h'/\gamma_{\imath+1}]}^{([(N-r+j_\alpha)(1+1/\gamma_\imath)]+|\alpha|-k_\imath-j_\alpha+m)}(a;z)\underset{(a,0')}{\ll}\Psi_{-r-h-[(h'+e(\imath)-e_\alpha)/\gamma_{\imath+1}]}^{([N(1+1/\gamma_\imath)]+m)}(a;z).$$

Hence we have, by choosing B with $B > 2C_0$,

$$\begin{aligned}
&R_1^N(z,\partial)v(z)\\
&\quad\underset{(a,0')}{\ll} M(a,c)A_0A^N\left(\sum_{\{\alpha;\ e_\alpha\le e(\imath)\}}\sum_{r=j_\alpha+1}^{N+e(\imath)}\frac{(N+e(i))!}{(N+e(i)-r)!}C_0^{r+1}\right.\\
&\qquad\left.\left(\sum_{h=0}^{+\infty}B^hN^h\left(\sum_{h'=0}^{+\infty}\frac{\Psi_{-r-h-[(h'+e(\imath)-e_\alpha)/\gamma_{\imath+1}]}^{([N(1+1/\gamma_\imath)]+m)}(a;z)}{|a|^{h'+e(\imath)-e_\alpha}}\right)\right)\right)\\
&\quad\underset{(a,0')}{\ll} M(a,c)A_0A^NC_0\left(\sum_{\{\alpha;\ e_\alpha\le e(\imath)\}}\sum_{r=j_\alpha+1}^{N+e(\imath)}N^rC_0^r\right.\\
&\qquad\left.\left(\sum_{h=0}^{+\infty}B^hN^h\left(\sum_{h'=0}^{+\infty}\frac{\Psi_{-r-h-[(h'+e(\imath)-e_\alpha)/\gamma_{\imath+1}]}^{([N(1+1/\gamma_\imath)]+m)}(a;z)}{|a|^{h'+e(\imath)-e_\alpha}}\right)\right)\right)
\end{aligned}$$

$$\begin{aligned}&\underset{(a,0')}{\ll} M(a,c)A_0A^NC_0\left(\sum_{\{\alpha;\ e_\alpha\le e(\imath)\}}\left(\sum_{r>0}\left(\frac{C_0}{B}\right)^r\right)\right.\\&\left.\left(\sum_{h''=0}^{+\infty}\sum_{h'=0}^{+\infty}B^{h''}N^{h''}\frac{\Psi^{([N(1+1/\gamma_\imath)]+m)}_{-h''-[(h'+e(\imath)-e_\alpha)/\gamma_{\imath+1}]}(a;z)}{|a|^{h'+e(\imath)-e_\alpha}}\right)\right)\\&\underset{(a,0')}{\ll} M(a,c)A_1A^N\left(\sum_{h=0}^{+\infty}B^hN^h\left(\sum_{h'=0}^{+\infty}\frac{\Psi^{([N(1+1/\gamma_\imath)]+m)}_{-h-[h'/\gamma_{\imath+1}]}(a;z)}{|a|^{h'}}\right)\right)\end{aligned}$$

for some constants $A_0 = A_0(\rho, c)$ and $A_1 = A_1(\rho, c)$. As for the estimate of the derivatives of $f(z)$ it follows from Lemma 2.5 that for any $0 < \theta' < \theta$

$$|\partial_0^h f(z)| \le MC^h\Gamma(h(1+\frac{1}{\gamma_\imath})+1)$$

holds for $z \in \Omega(\theta')$. Therefore we have

$$\text{(2.20)}\qquad \begin{cases} R_0^N(z,\partial)v(z) = F_N(z),\\ F_N(z) = -(\partial_0^{N+e(\imath)}Q(z,\partial)+R_1^N(z,\partial))v(z)+\partial_0^{N+e(\imath)}f(z),\end{cases}$$

where there is a constant $A_1 = A_1(\rho, c)$ such that

$$\text{(2.21)}\qquad \begin{aligned}&F_N(z)\underset{(a,0')}{\ll} M(a,c)A_1A^N\\&\left(\sum_{h=0}^{+\infty}B^hN^h\left(\sum_{h'=0}^{+\infty}\frac{\Psi^{([N(1+1/\gamma_\imath)]+m)}_{-h-[h'/\gamma_{\imath+1}]}(a;z)}{|a|^{h'}}\right)\right).\end{aligned}$$

Finally let us obtain the estimate of $\partial_0^N v(z)$ by using the equation (2.20). We have, by the assumption $a_{\alpha(\imath)}(z) = 1$,

(2.22)

$$\begin{aligned}&R_0^N(z,\partial)v(z)\\&=\sum_{\{\alpha;e_\alpha\le e(\imath)\}}\sum_{r=0}^{\min\{j_\alpha,N+e(\imath)\}}\binom{N+e(i)}{r}\partial_0^r a_\alpha(z)\partial^{\alpha'}\partial_1^{-l_\imath}\partial_0^{N+\alpha_0+e(\imath)-r}v(z)\\&=(I+K_N(z,\partial))\partial_0^N v(z),\end{aligned}$$

where I is the identity operator and

$$\text{(2.23)}\qquad K_N(z,\partial)={\sum}'\binom{N+e(i)}{r}\partial_0^r a_\alpha(z)\partial^{\alpha'}\partial_1^{-l_\imath}\partial_0^{\alpha_0+e(\imath)-r}.$$

Here $\sum'$ means the sum with respect to $(\alpha, r) \neq (\alpha(i), 0)$ with $e_\alpha \leq e(i)$ and $0 \leq r \leq \min\{j_\alpha, N + e(i)\}$. Let α be a multi-index appearing in $\sum'$. Then $e(i) - e_\alpha = e(i) - j_\alpha + \alpha_0 \geq 0$, $r \leq j_\alpha$ and $|\alpha| \leq k_i$. Hence $\alpha_0 + e(i) - r \geq 0$. Let us show $l_i > \alpha_1$. It holds that

$$l_i = k_i + e(i) \geq k_i + e_\alpha = k_i + j_\alpha - \alpha_0 = k_i + j_\alpha - |\alpha| + |\alpha'| \geq |\alpha'|.$$

So if $e(i) > e_\alpha$ or $k_i > |\alpha|$ or $j_\alpha > 0$, we have $l_i > |\alpha'| \geq \alpha_1$. Let α be a multi-index with $e(i) = e_\alpha$, $|\alpha| = k_i$ and $j_\alpha = 0$. Then $\alpha = (-e(i), \alpha') \in \Delta_0(i)$ with $|\alpha'| = l_i$ and $\alpha \neq \alpha(i)$. Hence $|\alpha'| > \alpha_1$. Thus we have $l_i > \alpha_1$ for α appearing in the sum $\sum'$.

It follows from (2.20) and (2.22) that

$$\partial_0^N v(z) = \sum_{s=0}^{+\infty} (-K_N(z, \partial))^s F_N(z). \tag{2.24}$$

Let us estimate $K_N(z, \partial) F_N(z)$. We have from (2.21)

$$\begin{aligned}
&\sum{}' \binom{N + e(i)}{r} \partial_0^r a_\alpha(z) \partial^{\alpha'} \partial_1^{-l_i} \partial_0^{\alpha_0 + e(i) - r} F_N(z) \\
&\underset{(a,0')}{\ll} M(a,c) A_1 A^N \sum{}' \Bigg(\rho^{\alpha_1 - l_i} C_0^{r+1} c^{-e(i) - \alpha_0 + r} \\
&\quad \Bigg(\sum_{h=0}^{+\infty} B^h N^{h+r} \Bigg(\sum_{h'=0}^{+\infty} \frac{\Psi^{([N(1+1/\gamma_i)] + |\alpha| - k_i - r + m)}_{-h - [h'/\gamma_{i+1}]}(a; z)}{|a|^{h' + e(i) - e_\alpha}} \Bigg) \Bigg) \Bigg) \\
&\underset{(a,0')}{\ll} M(a,c) A_1 A^N \rho^{-1} \sum{}' \Bigg(C_0^{r+1} c^{-e(i) - \alpha_0 + r} \\
&\quad \Bigg(\sum_{h=0}^{+\infty} B^h N^{h+r} \Bigg(\sum_{h'=0}^{+\infty} \frac{\Psi^{([N(1+1/\gamma_i)] + |\alpha| - k_i + m)}_{-h - r - [(e(i) - e_\alpha + h')/\gamma_{i+1}]}(a; z)}{|a|^{h' + e(i) - e_\alpha}} \Bigg) \Bigg) \Bigg),
\end{aligned}$$

where $A_1 = A_1(\rho, c)$ in (2.21). Hence, by choosing $B > 2C_0$, there is a constant $C = C(c)$ which is independent of ρ such that

$$\begin{aligned}
&K_N(z, \partial) F_N(z) \\
&\underset{(a,0')}{\ll} M(a,c) \rho^{-1} A_1(\rho, c) A^N \Bigg(\sum{}' c^{-e(i) - \alpha_0 + r} C_0^{r+1} \\
&\quad \Bigg(\sum_{h=0}^{+\infty} B^h N^{h+r} \Bigg(\sum_{h'=0}^{+\infty} \frac{\Psi^{([N(1+1/\gamma_i)] + m)}_{-h - r - [(e(i) - e_\alpha + h')/\gamma_{i+1}]}(a; z)}{|a|^{h' + e(i) - e_\alpha}} \Bigg) \Bigg) \Bigg)
\end{aligned}$$

$$\underset{(a,0')}{\ll} M(a,c)C\rho^{-1}A_1A^N\left(\sum_{h=0}^{+\infty}B^hN^h\left(\sum_{h'=0}^{+\infty}\frac{\Psi^{([N(1+1/\gamma_\imath)]+m)}_{-h-[h'/\gamma_{\imath+1}]}(a;z)}{|a|^{h'}}\right)\right).$$

By repeating this process, we obtain

$$(2.25)\qquad \begin{aligned}(K_N(z,\partial))^sF_N(z) &\underset{(a,0')}{\ll} M(a,c)(C\rho^{-1})^sA_1A^N\\ &\left(\sum_{h=0}^{+\infty}B^hN^h\left(\sum_{h'=0}^{+\infty}\frac{\Psi^{([N(1+1/\gamma_\imath)]+m)}_{-h-[h'/\gamma_{\imath+1}]}(a;z)}{|a|^{h'}}\right)\right).\end{aligned}$$

By choosing ρ, A so that $C\rho^{-1}<1/2$ and $A>2A_1(\rho,c)$, we have

$$\begin{aligned}\partial_0^Nv(z) &= \sum_{s=0}^{\infty}(K_N(z,\partial))^sF_N(z)\\ &\underset{(a,0')}{\ll} M(a,c)A^{N+1}\left(\sum_{h=0}^{+\infty}B^h(N+1)^h\left(\sum_{h'=0}^{+\infty}\frac{\Psi^{([N(1+1/\gamma_\imath)]+m)}_{-h-[h'/\gamma_{\imath+1}]}(a;z)}{|a|^{h'}}\right)\right),\end{aligned}$$

which means (2.13) for $n=N$.

If $i=p-1$, then $e_\alpha\geq e(p-1)$ for all α. So

$$R_0^N(z,\partial)v(z)=\sum_{\{\alpha;e_\alpha=e(p-1)\}}\sum_{0\leq r\leq \jmath_\alpha}\binom{N+e(i)}{r}\partial_0^ra_\alpha(z)\partial^{\alpha'}\partial_1^{-l_\imath}\partial_0^{N+\alpha_0+e(\imath)-r}v(z)$$

and we have the estimate (2.14) by modifying a little the above arguments and noting that the sum is taken for only α with $e_\alpha=e(p-1)$ in $R_0(z,\partial)$ and $R_1(z,\partial)$. □

Corollary 2.7. *Suppose the assumptions in Proposition* 2.6 *hold. Further assume* $u(z)\in\mathcal{O}_{(\gamma_\imath)}(\Omega(\theta))$. *Then if* $i=0$, $u(z)$ *is holomorphic at* $z=0$ *and if* $i>0$, *there is a polydisk* W *centered at* $z=0$ *such that for any* $\varepsilon>0$ *and any* θ' *with* $0<\theta'<\theta$ *there are constants* $C=C(\theta')$ *and* $M_\varepsilon=M_\varepsilon(\theta')$ *such that*

(2.26)
$$|\partial_0^nu(z_0,z')|\leq M_\varepsilon C^n\exp(\varepsilon|z_0|^{-\gamma_\imath})\Gamma\left(n\left(\frac{\gamma_\imath+1}{\gamma_\imath}\right)+1\right)\quad \textit{for}\quad z\in W(\theta')$$

and $n=0,1,\cdots$.

Proof. Let $|t| \leq r/2$. Then it follows from Lemma 2.1 that there are constants C_0 and C_1 such that if $0 \leq i \leq p-2$,

$$|\Psi^{([n(1+1/\gamma_i)]+m)}_{-h-[h'/\gamma_{i+1}]}(t)| \leq C_0 C_1^{n+h+h'} \frac{\Gamma(n(1+1/\gamma_i)+1)}{\Gamma(h+1)\Gamma([h'/\gamma_{i+1}]+1)}$$

and if $i = p-1$,

$$|\Psi^{([n(1+1/\gamma_i)]+m)}_{-h}(t)| \leq C_0 C_1^{n+h} \frac{\Gamma(n(1+1/\gamma_i)+1)}{\Gamma(h+1)}.$$

Let $W_0 = \{z_0 \in \mathbb{C}; |z_0| < R/2\}$, $W' = \{z' \in \mathbb{C}^d; |z_1| < r/2\rho(d+1),\ |z_i| < r/2(d+1),\ 2 \leq i \leq d\}$ and $W = W_0 \times W'$. Let $0 < a < R/2$, $0 < \theta_0 < \min\{\theta, \pi/2\}$, $c = \sin\theta_0$ and $|(z_0-a)/(ca)| < r/2(d+1)$. Then $|\frac{z_0-a}{ca}| + |\rho z_1| + |z_2| + \cdots + |z_d| < r/2$ for $z' \in W'$. Hence it follows from Proposition 2.6 that for $z \in \{z_0; |\frac{z_0-a}{ca}| < r/2(d+1)\} \times W'$

$$\begin{aligned}|\partial_0^n \partial_1^{l_i} u(z)| &\leq M(a,c) C_0 (AC_1)^{n+1} \Gamma\left(n\left(1+\frac{1}{\gamma_i}\right)+1\right) \\ &\left(\sum_{h=0}^{+\infty} \frac{(BC_1)^h (n+1)^h}{\Gamma(h+1)}\right)\left(\sum_{h'=0}^{+\infty} \frac{C_1^{h'}}{|a|^{h'}\Gamma([h'/\gamma_{i+1}]+1)}\right) \\ &\leq M(a,c) C_0 (AC_1)^{n+1} C_2^{n+1} \exp(C^*|a|^{-\gamma_{i+1}}) \Gamma\left(n\left(1+\frac{1}{\gamma_i}\right)+1\right)\end{aligned}$$

for some constants $C_2 > 0$ and $C^* \geq 0$, where if $i = p-1$, $C^* = 0$. So if $i = 0$, then $\gamma_0 = +\infty$ and

$$|\partial_0^n \partial_1^{l_i} u(z)| \leq M(a,c) \exp(C^*|a|^{-\gamma_{i+1}}) C_0 {C_3}^{n+1} n!,$$

which means the holomorphy of $u(z)$ at $z = 0$, by considering the conditions on the traces $\partial_1^l u(z) (0 \leq l < l_i)$ on $z_1 = 0$. Suppose that $i > 0$ and $u(z) \in \mathcal{O}_{(\gamma_i)}(\Omega(\theta))$. Then for any $\varepsilon > 0$ and θ'' with $\theta_0 < \theta'' < \theta$, there is a constant $K_\varepsilon = K_\varepsilon(\theta'')$ such that $M(a,c) \leq K_\varepsilon \exp(\varepsilon|a|^{-\gamma_i}/2)$. Hence

$$|\partial_0^n \partial_1^{l_i} u(z)| \leq K_\varepsilon \exp(\varepsilon|a|^{-\gamma_i}/2 + C^*|a|^{-\gamma_{i+1}}) C_0 {C_3}^{n+1} \Gamma\left(n\left(1+\frac{1}{\gamma_i}\right)+1\right).$$

Therefore we have from the conditions of the traces of $\partial_l^l u(z)$ $(0 \leq l < l_i)$ on $z_1 = 0$

$$|\partial_0^n u(z)| \leq M_\varepsilon \exp(\varepsilon|a|^{-\gamma_i}) C^n \Gamma\left(n\left(1+\frac{1}{\gamma_i}\right)+1\right)$$

and, by putting $z_0 = a$,

$$|\partial_0^n u(a, z')| \le M_\varepsilon \exp(\varepsilon |a|^{-\gamma_\imath}) C^n \Gamma\left(n\left(1 + \frac{1}{\gamma_\imath}\right) + 1\right).$$

In the above we assume $a > 0$. If $(a, z') \in W(\theta')$, by the transformation $u_\phi(z_0, z') = u(z_0 e^{\imath\phi}, z')$ $(\phi = \arg a)$, we can reduce this case to the preceding. Hence we have (2.26). □

§3. Asymptotic Expansion

In order to complete the proof of Theorem 1.4 we require the following theorem, which was given in [6],

Theorem 3.1. *Let U be a polydisk in $\mathbb{C}^{d+1}$ with center $z = 0$ and $0 < \gamma \le +\infty$. Suppose that $u(z) \in \mathcal{O}(U(\theta))$ satisfies the following estimate: for any $\varepsilon > 0$ and $0 < \theta' < \theta$ there exist constants $M_\varepsilon = M(\varepsilon, \theta')$ and $C = C(\theta')$ such that*

(3.1)
$$\left|\left(\frac{\partial}{\partial z_0}\right)^n u(z)\right| \le M_\varepsilon \exp(\varepsilon |z_0|^{-\gamma}) C^n \Gamma\left(n\left(\frac{\gamma+1}{\gamma}\right) + 1\right) \quad \text{for} \quad z \in U(\theta')$$

for all $n \in \mathbb{N}$. Then $u(z) \in Asy_{\{\gamma\}}(W(\theta))$ for some polydisk W with center $z = 0$.

Theorem 1.4 immediately follows from Corollary 2.7 and Theorem 3.1. In [6] we showed Theorem 3.1 for rational γ, where we reduced the proof to the analysis of some simple partial differential equation of Fuchsian type. In this paper we show it for real γ and give a different proof which Prof. Honda (Hokkaido Univ.) suggested to the author. Here we use a differential operator with infinite order and its inverse operator, which are similar to those in [2].

Only the variable z_0 is essential in Theorem 3.1. So in the following discussion we treat functions in one variable and regard other variables as parameters. First we give

Lemma 3.2. *Let $g(t)$ be a continuous function on $[0, T]$ $(T > 0)$ and κ be a positive constant. Suppose that there exist positive constants M and c such that for any $m \in \mathbb{N}$*

$$|g(t)| \le M c^m t^m \Gamma\left(\frac{m}{\kappa} + 1\right) \quad \text{on} \quad [0, T]. \tag{3.2}$$

Then $|g(t)| \le C_0 M (ct)^{-\kappa/2} e^{-(ct)^{-\kappa}}$ holds for a constant C_0 that is independent of M and c.

Proof. First we assume $c = 1$. Let $m \in \mathbb{N}$ with $\kappa/m \le T^\kappa$. Suppose that $\kappa/(m+1) \le t^\kappa \le \kappa/m$. Then, by (3.2) and Stiring's formula $\Gamma(m/\kappa + 1) \sim (\frac{m}{\kappa})^{m/\kappa}\sqrt{2\pi\frac{m}{\kappa}}e^{-m/\kappa}$, there is a constant C_0 such that

$$|g(t)| \le M\left(\frac{\kappa}{m}\right)^{\frac{m}{\kappa}}\Gamma\left(\frac{m}{\kappa}+1\right) \le C_0 M\sqrt{\frac{m}{\kappa}}e^{-(m+1)/\kappa} \le C_0 M\frac{e^{-t^{-\kappa}}}{t^{\kappa/2}}.$$

This means $|g(t)| \le C_0 M t^{-\kappa/2}\exp(-t^{-\kappa})$ for $t \in [\kappa/(m+1), \kappa/m]$ and for all $m \in \mathbb{N}$ with $\kappa/m \le T^\kappa$. So the assertion holds for $c = 1$. By considering $g(t/c)$, we have the estimate of $g(t)$ for general $c > 0$. □

Now put

$$A(\lambda) = \prod_{n=1}^{+\infty}\left(1 + \frac{\lambda}{n^{1+\frac{1}{\gamma}}}\right). \tag{3.3}$$

Lemma 3.3. (i) *$A(\lambda)$ is an entire function with estimate*

$$|A(\lambda)| \le C_0\exp(c_0|\lambda|^{\gamma/(\gamma+1)}). \tag{3.4}$$

(ii) *$1/A(\lambda)$ is holomorphic in $\mathbb{C} - (-\infty, -1]$ and there are positive constants C_1 and c_1 such that*

$$|A(\lambda)|^{-1} \le C_1\exp(-c_1|\lambda|^{\gamma/(\gamma+1)}) \quad \textit{for} \quad \Re\lambda \ge 0 \tag{3.5}$$

and for $0 < \theta' < \pi$ there are positive constants $C_2 = C(\theta')$ and $c_2 = c_2(\theta')$ such that

$$|A(\lambda)|^{-1} \le C_2\exp(c_2|\lambda|^{\gamma/(\gamma+1)}) \quad \textit{for} \quad |\arg\lambda| \le \theta'. \tag{3.6}$$

Proof. (i) Let $q^{1+1/\gamma} \le |\lambda| \le (q+1)^{1+1/\gamma}$ $(q \in \mathbb{N})$. Then

$$\begin{aligned}
&\prod_{n=1}^{q}\left(1+\frac{|\lambda|}{n^{1+1/\gamma}}\right)\prod_{n=q+1}^{+\infty}\left(1+\frac{|\lambda|}{n^{1+1/\gamma}}\right) \\
&\le \exp\left(\sum_{n=q+1}^{+\infty}\frac{|\lambda|}{n^{1+1/\gamma}}\right)\prod_{n=1}^{q}\left(1+\frac{|\lambda|}{n^{1+1/\gamma}}\right) \\
&\le \exp\left(\sum_{n=q+1}^{+\infty}\left(\frac{q+1}{n}\right)^{1+1/\gamma}\right)\prod_{n=1}^{q}\left(1+\frac{|\lambda|}{n^{1+1/\gamma}}\right) \\
&\le \exp(C'(q+1))\prod_{n=1}^{q}\left(1+\frac{|\lambda|}{n^{1+1/\gamma}}\right) \\
&\qquad \left(\text{by } \sum_{n=q+1}^{+\infty}\frac{1}{n^{1+1/\gamma}} \le C'(q+1)^{-1/\gamma}\right)
\end{aligned}$$

$$\leq \quad \exp(C'(q+1))\frac{2^q|\lambda|^q}{(q!)^{1+1/\gamma}} \leq AB^q\frac{|\lambda|^q}{(q!)^{1+1/\gamma}} \leq C_0\exp(c_0|\lambda|^{\gamma/(\gamma+1)}).$$

(ii) Let $\Re\lambda \geq 0$. Then $|1+\frac{\lambda}{n^{1+1/\gamma}}| \geq 1$. Hence

$$\prod_{n=1}^{+\infty}\left|1+\frac{\lambda}{n^{1+1/\gamma}}\right| \geq \sup_l \prod_{n=1}^{l}\left|1+\frac{\lambda}{n^{1+1/\gamma}}\right| \geq \sup_l \frac{|\lambda|^l}{(l!)^{1+1/\gamma}}.$$

If $q^{1+1/\gamma} \leq |\lambda| \leq (q+1)^{1+1/\gamma}$ ($q \in \mathbb{N}$), then we have by the Stiring's formula $q! \sim (2\pi q)^{1/2}q^q e^{-q}$

$$\frac{|\lambda|^q}{(q!)^{1+1/\gamma}} \geq \frac{q^{q(1+1/\gamma)}}{(q!)^{1+1/\gamma}} \geq C'\exp(c'q) \geq C_1^{-1}\exp(c_1|\lambda|^{\gamma/(\gamma+1)})$$

for some positive constants C', C_1, c and c', from which (3.5) follows.

Finally let us show the estimate (3.6). Let $n(x) = [x^{\gamma/(\gamma+1)}]$ for $x \geq 0$. Then, by integration by parts, we have

$$\log A(\lambda) = \int_0^{+\infty}\log\left(1+\frac{\lambda}{x}\right)dn(x) = \lambda\int_0^{+\infty}\frac{n(x)dx}{(\lambda+x)x} = \lambda\int_0^{+\infty}\frac{N(x)}{(x+\lambda)^2}dx$$

where $N(x) = \displaystyle\int_{(0,x]}\frac{n(t)}{t}dt$. Since $N(x) \leq Cx^{\gamma/(\gamma+1)}$, there is a constant $c_2 = c_2(\theta')$ such that for λ with $|\arg\lambda| < \theta' < \pi$

$$\begin{aligned}|\log A(\lambda)| &\leq C|\lambda|\int_0^{+\infty}\frac{x^{\gamma/(\gamma+1)}}{|x+\lambda|^2}dx \leq C|\lambda|^{2+\gamma/(\gamma+1)}\int_0^{+\infty}\frac{t^{\gamma/(\gamma+1)}}{||\lambda|t+\lambda|^2}dt \\ &= C|\lambda|^{\gamma/(\gamma+1)}\int_0^{+\infty}\frac{t^{\gamma/(\gamma+1)}}{|t+e^{i\arg\lambda}|^2}dt \leq c_2|\lambda|^{\gamma/(\gamma+1)},\end{aligned}$$

which means (3.6). □

It follows from (3.4) that $A(\lambda)$ is an entire function with exponential order $\gamma/(\gamma+1)$. So by the theory of entire functions we have

$$A(\lambda) = \sum_{n=0}^{+\infty}a_n\lambda^n, \quad |a_n| \leq \frac{AB^n}{\Gamma(n(\frac{\gamma+1}{\gamma})+1)} \tag{3.7}$$

for some constants A and B. Hence we can define a differential operator $A(d/dt)$ with infinite order that operates on holomorphic functions: $A(d/dt)u(t) = \sum_{n=0}^{+\infty}a_n(d/dt)^n u(t)$.

Define

$$K(t)=\frac{1}{2\pi i}\int_0^{\infty e^{i\varphi}}\frac{\exp(\lambda t)}{A(\lambda)}d\lambda, \tag{3.8}$$

where $|\arg t+\varphi-\pi|<\pi/2$ and $|\varphi|<\pi$.

Lemma 3.4. (i) *$K(t)$ is holomorphic in $\mathbb{C}_{(-\pi/2,5\pi/2)}=\{t\neq 0; -\pi/2<\arg t<5\pi/2\}$ and*

$$A(d/dt)K(t)=\frac{-1}{2\pi i t}. \tag{3.9}$$

(ii) *Let $0\le\arg t\le 2\pi$. Then there are constants A and D such that*

$$\left|\left(\frac{d}{dt}\right)^n K(t)\right|\le AD^n\Gamma\left((n+1)\left(\frac{\gamma+1}{\gamma}\right)\right). \tag{3.10}$$

(iii) *There exist positive constants A, D_0 and c such that*

(3.11)

$$\left|\left(\frac{d}{dt}\right)^n (K(t)-K(te^{2\pi i}))\right|\le AD_0^n\exp(-ct^{-\gamma})\Gamma\left((n+1)\left(\frac{\gamma+1}{\gamma}\right)\right) \text{ for } t\ge 0.$$

Proof. (i) It follows from (3.6) that $K(t)$ is holomorphic in $\mathbb{C}_{(-\pi/2,5\pi/2)}$. It is obvious that $K(t)$ satisfies (3.9).

(ii) For t with $0\le\arg t\le 2\pi$ we can choose $|\varphi|\le\pi/2$ such that $|\arg t+\varphi-\pi|\le\pi/2$. Hence, by Lemma 3.3-(ii), we have

$$\begin{aligned}\left|\left(\frac{d}{dt}\right)^n K(t)\right| &= \frac{1}{2\pi}\left|\int_0^{\infty e^{i\varphi}}\frac{\lambda^n\exp(\lambda t)}{A(\lambda)}d\lambda\right| \\ &\le A_1\int_0^{+\infty} r^n\exp(-c_1 r^{\gamma/(\gamma+1)})dr\le AD^n\Gamma\left((n+1)\left(\frac{\gamma+1}{\gamma}\right)\right).\end{aligned}$$

(iii) Let $t>0$. Then we have

$$\begin{aligned}\left(\frac{d}{dt}\right)^l (K(t)-K(te^{2\pi i})) &= \frac{1}{2\pi i}\left(\int_0^{\infty e^{i\pi/2}}-\int_0^{\infty e^{-i\pi/2}}\right)\frac{\lambda^l\exp(\lambda t)}{A(\lambda)}d\lambda \\ &= \frac{1}{2\pi i}\int_{\infty e^{-i\pi/2}}^{\infty e^{i\pi/2}}\frac{\lambda^l\exp(\lambda t)}{A(\lambda)}d\lambda.\end{aligned}$$

So

$$\lim_{t\to+0}\left(\frac{d}{dt}\right)^l (K(t)-K(te^{2\pi i})) = \frac{1}{2\pi i}\int_{\infty e^{-i\pi/2}}^{\infty e^{i\pi/2}} \lambda^l A(\lambda)^{-1} d\lambda = 0$$

and

$$\begin{aligned}\left|\left(\frac{d}{dt}\right)^l (K(t)-K(te^{2\pi i}))\right| &\le A_1\int_{-\infty}^{+\infty} |r|^l \exp(-c_1|r|^{\gamma/(\gamma+1)})dr \\ &\le A_2 D^l \Gamma\left((l+1)\left(\frac{\gamma+1}{\gamma}\right)\right).\end{aligned}$$

Hence it follows from Taylor's expansion of $(\frac{d}{dt})^n(K(t)-K(te^{2\pi i})$ that there are constants A_3 and D_0 such that

$$\begin{aligned}&\left|\left(\frac{d}{dt}\right)^n (K(t)-K(te^{2\pi i})\right| \\ &\le \frac{|t|^m}{m!}\left|\left(\frac{d}{dt}\right)^{n+m}(K(\theta t)-K(\theta te^{2\pi i}))\right| \quad (0<\theta<1) \\ &\le A_2 D^{m+n}\Gamma\left((m+n+1)\left(\frac{\gamma+1}{\gamma}\right)\right) t^m/m! \\ &\le A_3 D_0^{m+n}\Gamma\left((n+1)\left(\frac{\gamma+1}{\gamma}\right)\right)\Gamma\left(\frac{m}{\gamma}+1\right)t^m\end{aligned}$$

for any $m\in\mathbb{N}$. So by Lemma 3.2 there exist positive constants A and c such that $|(\frac{d}{dt})^n(K(t)-K(te^{2\pi i}))| \le AD_0^n\Gamma((n+1)(\frac{\gamma+1}{\gamma}))\exp(-ct^{-\gamma})$ for $t\ge 0$. □

Now let $U_0=\{w_0; |w_0|<R\}$, $V_0=\{z_0; |z_0|<R/2\}$ and $0<\theta<\pi/2$. Let $0<\theta_0<\theta$, $u(w_0)\in\mathcal{O}(U_0(\theta))$ and $z_0\in V_0(\theta_0)$. Define

$$(3.12)\qquad (Ku)(z_0) = \frac{1}{2\pi i}\int_{\mathcal{C}} K(w_0-z_0)u(w_0)dw_0,$$

where $\mathcal{C}$ is a closed path in $U_0(\theta)$ in w_0-space, which starts at a fixed point $w_0=a$, $3R/4\le a<R$, encloses once the point $w_0=z_0$, $z_0=x_0+iy_0$, anticlockwise and ends at point $w_0=a$. We may take $\mathcal{C}$ so that $-\pi/2<\arg(w_0-z_0)<5\pi/2$ for $w_0\in\mathcal{C}$. We have

Proposition 3.5. (i) $(Ku)(z_0)\in\mathcal{O}(V_0(\theta_0))$ *and*

$$(3.13)\qquad \begin{cases} A(-\partial_{z_0})(Ku)(z_0) = -u(z_0), \\ A(-\partial_{z_0})(Ku)(z_0) = \dfrac{1}{2\pi i}\displaystyle\int_{\mathcal{C}} K(w_0-z_0)A(-\partial_{w_0})u(w_0)dw_0 + \tilde{u}(z_0),\end{cases}$$

where $\tilde{u}(z_0)$ *is holomorphically extensible to a neighborhood of the origin.*
(ii) *If* $u(z_0) \in \mathcal{O}_{(\gamma)}(U_0(\theta))$, *then* $(Ku)(z_0) \in Asy_{\{\gamma\}}(V_0(\theta_0))$.

Proof. (i) It is obvious that $(Ku)(z_0) \in \mathcal{O}(V_0(\theta_0))$. By Lemma 3.4-(i),

$$A(-\partial_{z_0})(Ku(z_0)) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{-u_0(w_0)dw_0}{w_0 - z_0} = -u(z_0).$$

On the other hand we have

$$\begin{aligned}(-\partial_{z_0})^n(Ku)(z_0) &= \frac{1}{2\pi i}\int_{\mathcal{C}}((\partial_{w_0})^n K(w_0 - z_0))u(w_0)dw_0 \\ &= \frac{1}{2\pi i}\int_{\mathcal{C}} K(w_0 - z_0)(-\partial_{w_0})^n u(w_0)dw_0 + \tilde{u}_n(z),\end{aligned}$$

where

$$\tilde{u}_n(z) = \sum_{\imath=0}^{n-1}(\partial_{w_0})^{n-1-\imath}K(w_0 - z_0)(-\partial_{w_0})^{\imath}u(w_0)|_{w_0\in\partial\mathcal{C}}$$

and $\partial\mathcal{C}$ means the boundary of $\mathcal{C}$, that is, the starting point and the endpoint. Since $A(\lambda) = \sum_{n=0}^{+\infty} a_n\lambda^n$ with $|a_n| \le AB^n/\Gamma(n(\gamma+1)/\gamma)+1)$, by putting $\tilde{u}(z) = \sum_{n=0}^{+\infty} a_n\tilde{u}_n(z)$, we have (3.13).
(ii) We have

$$\partial_{z_0}^n(Ku)(z_0) = \frac{(-1)^n}{2\pi i}\int_{\mathcal{C}} K^{(n)}(w_0 - z_0)u(w_0)dw_0.$$

Here we choose path $\mathcal{C}$ as follows: let $\hat{z}_0 = \sqrt{a^2 - y_0^2} + iy_0$ and $\mathcal{C} = \mathcal{C}_0 + \mathcal{C}_1(\epsilon) + \mathcal{C}_2(\epsilon) - \mathcal{C}_1(\epsilon) - \mathcal{C}_0$,

$$\begin{aligned}\mathcal{C}_0 &= \{w_0 = s\hat{z}_0 + (1-s)a;\ 0 \le s \le 1\}, \\ \mathcal{C}_1(\epsilon) &= \{w_0 = s(z_0+\epsilon) + (1-s)\hat{z}_0;\ 0 \le s \le 1\}, \\ \mathcal{C}_2(\epsilon) &= \{w_0 = z_0 + \epsilon e^{2\pi\imath s};\ 0 \le s \le 1\},\end{aligned}$$

where $\epsilon > 0$ is sufficiently small constant.

We have $0 \le \arg(w_0 - z_0) \le 2\pi$ for $w_0 \in \mathcal{C}_1(\epsilon)\cup\mathcal{C}_2(\epsilon)$. Hence $K^{(n)}(w_0 - z_0)$ is bounded on $\mathcal{C}_1(\epsilon)\cup\mathcal{C}_2(\epsilon)$ by Lemma 3.4-(ii). By letting $\epsilon \to 0$,

(3.14)
$$\begin{aligned}&\partial_{z_0}^n(Ku)(z_0) = (K_0^n u)(z_0) + (K_1^n u)(z_0), \\ &(K_0^n u)(z_0) = \frac{(-1)^n}{2\pi i}\left(\int_{\mathcal{C}_0} + \int_{-\mathcal{C}_0}\right) K^{(n)}(w_0 - z_0)u(w_0)dw_0. \\ &(K_1^n u)(z_0) = \frac{(-1)^n}{2\pi i}\int_{\mathcal{C}_1(0)} \{K^{(n)}(w_0 - z_0) - K^{(n)}((w_0 - z_0)e^{2\pi\imath})\}u(w_0)dw_0.\end{aligned}$$

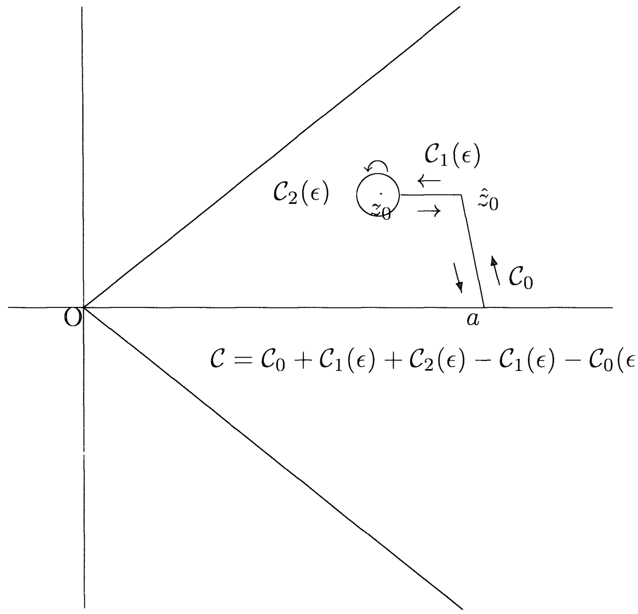

Figure 2. Path $\mathcal{C}$

We have $|(K_0^n u)(z_0)| \leq AD^n n!$ for $z_0 \in V_0(\theta_0)$. As for $(K_1^n u)(z_0)$ it follows from Lemma 3.4-(iii) that for small $\varepsilon > 0$

$$\begin{aligned} &|(K_1^n u)(z_0)| \\ \leq\ & A_\varepsilon D_0^n \Gamma\left((n+1)\left(\frac{\gamma+1}{\gamma}\right)\right) \int_{\mathcal{C}_1(0)} \exp(-c|w_0 - z_0|^{-\gamma} + \varepsilon|w_0|^{-\gamma})|dw_0| \\ \leq\ & A'_\varepsilon D_1^n \Gamma\left((n+1)\left(\frac{\gamma+1}{\gamma}\right)\right). \end{aligned}$$

Hence $|\partial_{z_0}^n (Ku)(z_0)| \leq A_1 D_2^n \Gamma((n+1)(\frac{\gamma+1}{\gamma}))$ for $z_0 \in V_0(\theta_0)$. Thus $\partial_{z_0}^n (Ku)(z_0)$ is bounded in $V_0(\theta_0)$ for any $n \in \mathbb{N}$ and has Gevrey type estimate, which means that $\lim_{z_0 \to 0}(K^n u)(z_0)$ exists in $V_0(\theta_0)$ and $(Ku)(z_0) \in Asy_{\{\gamma\}}(V_0(\theta_0))$. $\square$

Let $c > 0$ and $A_c(\lambda) = A(c\lambda)$ and define

$$K_c(t) = \frac{1}{2\pi i} \int_0^{\infty e^{i\varphi}} \frac{\exp(\lambda t)}{A_c(\lambda)} d\lambda.$$

Then the similar results are valid if we replace $A(\lambda)$ by $A_c(\lambda)$ and $K(\lambda)$ by $K_c(\lambda)$ respectively. We note that $A_c(\lambda) = \sum_{n=0}^{+\infty} a_n(c)\lambda^n$ with $a_n(c) = c^n a_n$

and $|a_n(c)| \leq A(cB)^n/\Gamma(n(\gamma+1)/\gamma+1)$, where the constants A and B are those in (3.7)

Proof of Theorem 3.1. If $\gamma = +\infty$, $(\gamma+1)/\gamma = 1$ and it is easy to show that $u(z)$ is holomorphic at $z = 0$. Suppose that $0 < \gamma < +\infty$ and $u(z_0)$ satisfies (3.1):

$$\left|\left(\frac{\partial}{\partial z_0}\right)^n u(z)\right| \leq M_\varepsilon \exp(\varepsilon|z_0|^{-\gamma})C^n\Gamma\left(n\left(\frac{\gamma+1}{\gamma}\right)+1\right) \quad \textit{for } z \in U(\theta')$$

holds for all $n \in \mathbb{N}$. Let $A_c(\lambda) = A(c\lambda)$ and choose $c > 0$ so that $cCB < 1$. Let $0 < \theta' < \min\{\pi/2, \theta\}$ and $0 < \theta_0 < \theta'$. Then, by the above estimate of $\partial_0^n u(z)$, $v(z_0) = A_c(-\partial_{z_0})u(z_0)$ converges and $v(z_0) \in U_{(\gamma)}(\theta')$. It follows from Proposition 3.5 that

$$\begin{aligned} -u(z_0) &= A_c(-\partial_{z_0})(K_c u)(z_0) \\ &= \frac{1}{2\pi i}\int_C K_c(w_0 - z_0)A_c(-\partial_{w_0})u(w_0)dw_0 + \tilde{u}(z_0) \\ &= (K_c v)(z_0) + \tilde{u}(z_0). \end{aligned}$$

Since $(K_c v)(z_0) \in Asy_{\{\gamma\}}(V_0(\theta_0))$ and $\tilde{u}(z_0)$ is holomorphic at the origin, we have $u(z_0) \in Asy_{\{\gamma\}}(V_0(\theta_0))$. If $\theta > \pi/2$, we have the conclusion of Theorem 3.1 by the rotation with respect to z_0. □

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