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Invariant Sheaves

By

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§0. Introduction

The sheaves of tangent vector fields, differential forms or differential operators are canonical. Namely they are invariant by the coordinate transformations. We call such sheaves *invariant sheaves.*

More precisely for a positive integer n, an invariant sheaf on n-manifold is given by the data: coherent $\mathcal{O}_X$-module F_X for each smooth variety X of dimension n and an isomorphism $\beta(f) : f^*F_Y \xrightarrow{\sim} F_X$ for any étale morphism $f : X \to Y$. We assume that $\beta(f)$ satisfies the chain condition (see §1 for the exact definition).

The purpose of this paper is to study the properties of invariant sheaves on n-manifold.

The first result is that the category $I(n)$ of invariant sheaves is equivalent to the category of modules over a certain group G (with infinite dimension). Let us recall that the category of equivariant sheaves with respect to a transitive action is equivalent to the category of modules over the isotropy subgroup. In our case, manifold may be regarded as a homogeneous space of "the group" of all transformations, and the category of invariant sheaves is regarded as an equivariant sheaf with respect to this action. Let us take an n-dimensional vector space V and let G be the group of (formal) transformations that fix the origin. Hence G is a semi-direct product of GL_n and a projective limit of finite-dimensional unipotent groups. This G plays a role of the isotropy subgroup and we have

Theorem. *The category of invariant sheaves are equivalent to the category of G-modules.*

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The category $I(n)$ of invariant sheaves has other remarkable structure: filtered rigid tensor category. The group G contains $GL(V)$ as a subgroup and it contains $\mathbb{G}_{\mathbf{m}}$ as its center. With respect to $\mathbb{G}_{\mathbf{m}}$, any G-module M has a weight decomposition $M = \oplus M_l$. For any l let us set $W_l(M) = \oplus_{l' \leq l} M_{l'}$. Then it turns out that $W_l(M)$ is a sub-G-module of M. Since the category of G-modules is equivalent to $I(n)$, any object F of $I(n)$ has also a canonical finite filtration W, that we call the *weight filtration.* Thus, $I(n)$ has a structure of filtered category. We say that $F \in I(n)$ is *pure of weight* w if $Gr_l^W F = 0$ for $l \neq w$. Then the category of pure invariant sheaves of weight w is equivalent to the category of $GL(V)$-modules with weight w (with respect to the $\mathbb{G}_{\mathbf{m}}$-action). Hence any pure invariant sheaf is semisimple.

Moreover $I(n)$ has a structure of tensor category by $(F_1 \otimes F_2)_X = F_1 \otimes_{\mathcal{O}_X} F_2$. Thus $I(n)$ is a rigid tensor category.

The weight is preserved by the tensor product: $Gr_l^W(F_1 \otimes F_2) = \oplus_{l=l_1+l_2} Gr_{l_1}^W(F_1) \otimes Gr_{l_2}^W(F_2)$. This structure is very similar to the category of mixed Hodge structures or motives. In particular, we can see easily

(0.1) If F_ν is pure of weight w_ν $(\nu = 1, 2)$, then

$$\operatorname{Ext}^j(F_1, F_2) = 0 \quad \text{for} \quad w_1 - w_2 < j.$$

We conjecture

$$\operatorname{Ext}^j(F_1, F_2) = 0 \quad \text{for} \quad j \neq w_1 - w_2 \quad \text{and} \quad j < n. \tag{0.2}$$

This is translated to a conjecture of Lie algebra cohomology (Conjecture A.8 for Theorem A.3 in [F]. Hence (0.2) is already known for $2j < n$).

The group $\operatorname{Ext}^1(\mathcal{O}, \Omega^1)$ is one-dimensional, and its non-zero element is given by the extension $0 \to \Omega^1 \to \Omega^{n \otimes -1} \otimes \mathcal{P}^{(1)}(\Omega^n) \to \mathcal{O} \to 0$. Here $\mathcal{P}^{(1)}(\Omega^n)_X = p_{1*}((\mathcal{O}_{X \times X}/I^2) \otimes p_2^* \Omega_X^n)$ where I is the defining ideal of the diagonal of $X \times X$, and p_1 and p_2 are the first and the second projection. Note that $\mathcal{O}$ has weight 0 and Ω^1 has weight -1. When $n = 1$, $\operatorname{Ext}^1(\mathcal{O}, \Omega^{1\otimes 2})$ is non zero. Its non-zero element gives an extension

$$0 \to \Omega^{1\otimes 2} \xrightarrow{\varphi_0} K \xrightarrow{\varphi_1} \mathcal{O} \to 0. \tag{0.3}$$

This is connected with the Schwartzian derivative. Namely, if we take a coordinate f of X then the sequence (0.3) splits. Hence there is an element $s(f) \in K$ such that $\varphi_1(s(f)) = 0$. if we take another coordinate g, then there exists $\omega \in \Omega_X^{1\otimes 2}$ such that $\varphi_0(\omega) = s(g) - s(f)$. Then ω is given by $\{g; f\}(df)^{\otimes 2}$. Here $\{g; f\}$

is the Schwartzian derivative $(d^3g/d^3f)/(dg/df) - 3(d^2f/d^2g)^2/2(df/dg)^2$. This explains the cocycle condition of the Schwartzian derivatives:

$$\{h;g\}(dg)^{\otimes 2} + \{g;f\}(df)^{\otimes 2} = \{h;f\}(df)^{\otimes 2}.$$

For any n, the extension group $\oplus_{j=0}^{n} \mathrm{Ext}^j(\mathcal{O}, \Omega^j)$ has a structure of ring by

$$\begin{aligned} \mathrm{Ext}^j(\mathcal{O},\Omega^j) \otimes \mathrm{Ext}^k(\mathcal{O},\Omega^k) &\to \mathrm{Ext}^{j+k}(\mathcal{O},\Omega^j \otimes \Omega^k) \\ &\to \mathrm{Ext}^{j+k}(\mathcal{O},\Omega^{j+k}). \end{aligned} \tag{0.5}$$

There exists a canonical element $c_j \in \mathrm{Ext}^j(\mathcal{O}, \Omega^j)$ such that

$$\oplus \mathrm{Ext}^j(\mathcal{O},\Omega^j) \simeq k[c_1, \cdots, c_n]'.$$

Here $k[c_1, \cdots, c_n]' = k[c_1, \cdots, c_n]/\{\text{degree} > n\}$. This follows from a theorem of Lie algebra cohomologies (cf.[F]). This c_j is connected with the Chern classes. Namely for any n-manifold X, we have the homomorphism

$$\mathrm{Ext}^j_{I(n)}(\mathcal{O},\Omega^j) \to \mathrm{Ext}^j_{\mathcal{O}_X}(\mathcal{O}_X,\Omega^j_X) = H^j(X;\Omega^j_X)$$

and the image of c_j give the j-th Chern class of X.

§1. Definition

We shall fix a positive integer n. Let S be a scheme. Let us first define the category $\mathcal{S}_n(S)$ as follows. The objects of $\mathcal{S}_n(S)$ are smooth morphisms $X \xrightarrow{a} T$ over S with fiber dimension n. A morphism φ from $X \xrightarrow{a} T$ to $X' \xrightarrow{a'} T'$ in $\mathcal{S}_n(S)$ is a pair (φ_s, φ_b) where $\varphi_s : X \to X', \varphi_b : T \to T'$ are such that

$$\begin{array}{ccc} X & \xrightarrow{\varphi_s} & X' \\ a\downarrow & & \downarrow a' \\ T & \xrightarrow[\varphi_b]{} & T' \end{array}$$

commutes and that $X \to X' \times_{T'} T$ is an étale morphism.

An invariant sheaf F is, by definition, given by following data:

(1.1) To any object $X \xrightarrow{a} T$ in $\mathcal{S}_n(S)$,
assign a quasi-coherent $\mathcal{O}_X$-module $F(X \xrightarrow{a} T)$,

(1.2) To any morphism $\varphi : (X \to T) \to (X' \to T')$ in $\mathcal{S}_n(S)$,
assign an isomorphism $\beta(\varphi) : \varphi_s^* F(X' \to T') \xrightarrow{\sim} F(X \to T)$.

We assume that these data satisfy the following associative law:

(1.3) for a chain of morphisms $(X \to T) \xrightarrow{\varphi} (X', T') \xrightarrow{\varphi'} (X'' \to T'')$, the following diagram commutes

$$\begin{array}{ccc} \varphi_s^* \varphi_s'^* F(X'' \to T'') & \xrightarrow{\beta(\varphi')} & \varphi_s^* F(X' \to T') \\ \wr \| & & \beta(\varphi) \downarrow \\ (\varphi_s' \circ \varphi_s)^* F(X'' \to T'') & \xrightarrow{\beta(\varphi' \circ \varphi)} & F(X \to T). \end{array}$$

In the sequel for an object $X \xrightarrow{a} T$ in $\mathcal{S}_n(S)$, we write $F_{X/T}$ for $F(X \to T)$ if there is no afraid of confusion.
The invariant sheaves form an additive category in an evident way. We denote this category by $I(n)_S$. If there is no afraid of confusion we denote it by $I(n)$.

The category $I(n)$ is a commutative tensor category. For objects F_1 and F_2 in $I(n), F_1 \otimes F_2$ that associates $F_{1X/T} \otimes_{\mathcal{O}_X} F_{2X/T}$ for any objects $X \to T$ in $\mathcal{S}_n(S)$ is evidently an object of $I(n)$. Moreover $F_1 \otimes F_2 \cong F_2 \otimes F_1$. Let us give several examples of invariant sheaves.

Example 1.1. The object $\mathcal{O} \in I(n)$. This associates to any $X \to T$ the sheaf $\mathcal{O}_X$.

Example 1.2. The object $\Omega^k \in I(n)$. This associates to any $X \to T$, the sheaf $\Omega^k_{X/T}$ of relative k-forms.

Example 1.3. The object $\Theta \in I(n)$. This associates to any $X \to T$ the sheaf $\Theta_{X/T}$ of relative tangent vectors.

Example 1.4. $S^m(\Omega^k)$. This associates $S^m(\Omega^k_{X/T})$.

Example 1.5. For any object $X \to T$ in $\mathcal{S}_n(S)$, let $\Delta^{(m)}_{X/T}$ be the m-th infinitesimal neighborhood of the diagonal of $X \underset{T}{\times} X$. Namely if we denotes by I the defining ideal of the diagonal $X \hookrightarrow X \underset{T}{\times} X$, then $\Delta^{(m)}_{X/T}$ is the subscheme of $X \underset{T}{\times} X$ defined by I^{m+1}. For $i = 1, 2$ let p_i be the composition $\Delta^{(m)}_{X/T} \hookrightarrow X \underset{T}{\times} X \to X$ where the last arrow is the i-th projection. Then $\mathcal{P}^{(m)}$ associates

$p_{1*}\mathcal{O}_{\Delta^{(m)}_{X/T}}$. More generally, for any invariant sheaf F, $\mathcal{P}^{(m)}(F)$ that assigns $p_{1*}p_2^*F_{X/T}$ is an invariant sheaf. Then there exists an exact sequence

$$0 \to S^m(\Omega^1) \otimes F \to \mathcal{P}^{(m)}(F) \to \mathcal{P}^{(m-1)}(F) \to 0.$$

Example 1.6. $W_m(\mathcal{D})$. This associates the sheaf $W_m(\mathcal{D}_{X/T})$ of the (relative) differential operators of order at most m. We regard this as an $\mathcal{O}_X$-module by the left multiplication.

Example 1.7. $W_m(\mathcal{D}^{op})$. This associates the same sheaf $W_m(\mathcal{D}_{X/T})$ but we regard this as an $\mathcal{O}_X$-module by the right multiplication.

§2. Finiteness and Flat Conditions

§2.1. Finiteness condition

For the sake of simplicity, let us assume that

(2.1.1) S is Noetherian.

We keep this assumption in the rest of paper. An invariant sheaf F is called coherent if $F_{X/T}$ is of locally finite type for any X/T in $\mathcal{S}_n(S)$. Then $F_{X/T}$ is necessarily locally of finite presentation. In fact there exists locally in X and T a morphism X/T to $\mathbf{A}^n \times S/S$ in $\mathcal{S}_n(S)$. Since $\mathbf{A}^n \times S$ is locally Noetherian, $F_{\mathbf{A}^n \times S/S}$ is a coherent $\mathcal{O}_{\mathbf{A}^n \times S}$-module. Hence the pull-back $F_{X/T}$ is locally of finite presentation.

Let us denote by $I_c(n)$ the full subcategory of $I(n)$ consisting of coherent invariant sheaves. Then we can see easily that $I_c(n)$ is an abelian category.

§2.2. Flat condition

An invariant sheaf F is called invariant vector bundle if $F_{X/T}$ is flat over T and locally of finite presentation over $\mathcal{O}_X$ for any X/T in $\mathcal{S}_n(S)$.

Proposition 2.2.1. *If F is an invariant vector bundle then $F_{X/T}$ is locally free of finite rank for any X/T in $\mathcal{S}_n(S)$.*

Proof. It is enough to show that $F_{\mathbf{A}^n \times S/S}$ is a locally free $\mathcal{O}_{\mathbf{A}^n \times S}$-module. Since this is flat over S, it is enough to show that for any $s \in S$, $F_{\mathbf{A}^n \times s/s}$ is locally free. Thus we may assume that $S = \mathrm{Spec}(k)$ for a field k. Since $F_{\mathbf{A}^n}$ is equivariant over the translation group G and G acts transitively on $\mathbb{A}^n$. Hence F is locally free. □

Let us denote by $I^b(n)$ the category of invariant vector bundles. If S is $\mathrm{Spec}\, k$ for a field k, then $I^b(n)$ and $I_c(n)$ coincides. The functor $\otimes$ is an exact functor on $I^b(n)$, and a right exact functor on $I_c(n)$. For F in $I^b(n)$, let F^* be the invariant sheaf that associates $\mathcal{H}om_{\mathcal{O}_X}(F_{X/T}, \mathcal{O}_X)$ with X/T in $\mathcal{S}_n(S)$. With this, $I^b(n)$ has a structure of rigid tensor category.

§3. Main Results

§3.1. Infinitesimal neighborhood

Let $f : X \hookrightarrow Y$ be an embedding and let I be the defining ideal of $f(X)$. Then for $m \geqq 0$, $\mathrm{Spec}(\mathcal{O}_Y/I^{m+1})$ is called the m-th infinitesimal neighborhood of X (or of $f : X \hookrightarrow Y$).

§3.2. The group G

Let us fix a locally free $\mathcal{O}_S$-module $\mathcal{V}$ of rank n, (e.g. $\mathcal{V} = \mathcal{O}_S^{\oplus n}$). Let V be the associated vector bundle $\mathrm{Spec}\big(S_{\mathcal{O}_S}(\mathcal{V}^*)\big)$. Then $V \to S$ is an object of $\mathcal{S}_n(S)$. Let $i : S \to V$ be the zero section and let us denote by $W_m(V)$ its m-th infinitesimal neighborhood. Then $S = W_0(V) \subset W_1(V) \subset \cdots$ is an increasing sequence of subschemes of V. Let us set

$$G(m) = \big\{g \in \mathrm{Aut}_S\big(W_m(V)\big);\ g \text{ fixes } W_0(V)\big\}.$$

Then $G(m)$ is an affine smooth group scheme over S and we have a canonical smooth surjective morphism $G(m) \to G(m-1)$. Let G be the projective limit of $\{G(m); m \in \mathbb{N}\}$. Then G is an affine group scheme over S. Let $W^m(G)$ be the kernel of $G \to G(m)$. Then

$$W^0(G) = G, \tag{3.2.1}$$

$$G/W^m(G) = G(m), \tag{3.2.2}$$

$$G/W^1(G) = GL(V). \tag{3.2.3}$$

For $m > 0$, $W^m(G)/W^{m+1}(G)$ is an abelian unipotent group scheme corresponding $S^m(\mathcal{V}^*) \otimes \mathcal{V}$ (e.g. $W^m(G)/W^{m+1}(G) = \mathrm{Spec}\big(S((S^m(\mathcal{V}^*) \otimes \mathcal{V})^*)\big)$). Note that G is a semi-direct product of $GL(V)$ and $W_1(G)$.

§3.3. Statement

A G-module M is by definition a quasi-coherent $\mathcal{O}_S$-module with a structure of $\pi_*\mathcal{O}_G$-comodule, where $\pi : G \to S$ is the canonical projection. A G-module M is called coherent if it is coherent over $\mathcal{O}_S$.

If M is a coherent G-module then the action of G on M comes from a $G(m)$-module structure on M for $m \gg 0$. Our main result is the following.

Theorem 3.1. *The category $I_c(n)$ of coherent invariant sheaves is equivalent to the category $\mathrm{Mod}_c(G)$ of coherent G-modules.*

Remark. Let $X \to S$ be a smooth morphism of fiber dimension n and let $i : S \to X$ be its section. Let $W_m(i)$ be the m-th infinitesimal neighborhood of i. Let $G(m)_\imath$ be the group of automorphisms of $W_m(i)$ that fix $W_0(i) = i(S)$. Then $G_\imath = \varprojlim_m G(m)_\imath$ is isomorphic to G locally in S with respect to the Zariski topology. Moreover the category of G-modules is equivalent to the category of $G_\imath$-modules.

§4. The Weight Filtration

§4.1. Definition

The group G contains $\mathbf{G_m}$ as the homothetie subgroup by $\mathbf{G_m} \times V \ni (t, x) \mapsto tx \in V$. Any coherent G-module M has a weight decomposition

$$M = \bigoplus_{\ell \in \mathbf{Z}} M_\ell. \tag{4.1.1}$$

Here $\mathbf{G_m}$ acts on M_ℓ by

$$tu = t^\ell u \quad \text{for} \quad u \in M_\ell, t \in \mathbf{G_m}.$$

We set

$$W_\ell(M) = \bigoplus_{m \le \ell} M_m. \tag{4.1.2}$$

We call this the weight filtration of M.

§4.2. Weight filtration

We shall prove that $W_\ell(M)$ is a sub-G-module of M. We shall embed $\mathbf{G_m}$ into $\mathbb{A}^1$. Let $\mathbf{G_m} \times G \xrightarrow{\varphi} G$ be the modified adjoint action $\varphi(t, g) = t^{-1}gt$. We can see easily the following lemmas.

Lemma 4.2.1. *$\varphi : \mathbf{G_m} \times G \to G$ extends uniquely to a morphism $\tilde{\varphi}$: $\mathbf{A}^1 \times G \to G$.*

Lemma 4.2.2. *For any $\ell \geq 0, \tilde{\varphi} : \mathbb{A}^1 \times W^\ell(G) \to W^\ell(G)$ is equal to the second projection modulo t^ℓ, i.e. the composition $W_{\ell-1}(\mathbb{A}^1) \times W^\ell(G) \to \mathbb{A}^1 \times W^\ell(G) \to W^\ell(G)$ equals the second projection. Here $W_{\ell-1}(\mathbb{A}^1) = \mathrm{Spec}(\mathbf{Z}[t]/t^\ell\mathbf{Z}[t])$.*

These lemmas imply the following result.

Proposition 4.2.3. *Let M be a G-module.*

(i) *$W_\ell(M)$ is a sub-G-module.*

(ii) *For $g \in W^m(G), (g-1)$ sends $W_\ell(M)$ into $W_{\ell-m}(G)$.*

Here $g \in W^m(G)$ means $g \in \mathrm{Hom}_S(T, W^m(G))$ for an S-scheme T. In the sequel, we use the similar abbreviation.

Proof. For any $g \in G$, $b \in \mathbf{Z}$ and $u_b \in M_b$ let us write $gu_b = \Sigma g_{ab}u_b$ with $g_{ab}u_b \in M_a$. Then $\varphi(t,g)u_b = \Sigma t^{b-a}g_{ab}u_b$. Since this is a polynomial in $t, g_{ab}u_b = 0$ for $a > b$. This implies (i). If $g \in W^m(G)$, then the coefficients of t^c in $\Sigma t^{b-a}g_{ab}u_b$ $(0 < c < m)$ vanishes. Hence $g_{ab}u_b = 0$ for $b > a > b-m$. Thus $gu_b - u_b \in \bigoplus_{a \leq b-m} M_a$. This shows (ii). □

Since M is coherent, $W(M)$ is a finite filtration of M. For $a, b \in \mathbf{Z}$ with $a \leq b$, we say that M has weights in $[a,b]$ if $W_b(M) = M$ and $W_{a-1}(M) = 0$. For $w \in \mathbf{Z}$, we say that M is pure of weight w if M has weights in $[w,w]$.

Corollary 4.2.4. *If M has weights in $[a,b]$, then the G-module structure of M comes from a unique $G(b-a)$-module structure on M.*

§5. Functor Φ

§5.1. Definition

Let F be a coherent invariant sheaf in $I_c(n)$. Let $i : S \to V$ be the zero section of the vector bundle $V \to S$ (c.f. §3). Set $\Phi(F) = i^*F_{V/S}$. Then $\Phi(F)$ is a coherent $\mathcal{O}_S$-module. In the sequel we shall endow a G-module structure on $\Phi(F)$.

§5.2. Weight decomposition

The group $GL(V)$ acts on V and hence on $i^*F_{V/S}$. Therefore $\Phi(F)$ is evidently a $GL(V)$-module. Since $\mathbf{G_m}$ is contained in $GL(V)$ as the center, $\Phi(F)$ has a weight decomposition

$$\Phi(F) = \bigoplus_{l \in \mathbb{Z}} \Phi(F)_l \tag{5.2.1}$$

where $t \in \mathbf{G_m}$ acts on $\Phi(F)_l$ by t^l.

As in §4, we set

$$W_l(\Phi(F)) = \bigoplus_{l' \leq l} \Phi(F)_{l'} \tag{5.2.2}$$

Then W is a finite filtration on $\Phi(F)$. We call it the weight filtration of $\Phi(F)$.

Similarly to the G-module case, we say that for $a \leq b$, F is with weight in $[a, b]$ if $W_b(\Phi(F)) = \Phi(F)$ and $W_{a-1}(\Phi(F)) = 0$.

Let $X \to T$ be an object in $\mathcal{S}_n(S)$ and $i : T \to X$ its section.

Proposition 5.2.1. *Let f and g be morphisms in $\mathcal{S}_n(S)$ from $X \to T$ to $X' \to T'$. Let $i : T \to X$ be a section and let $T^{(m)}$ be its m-th infinitesimal neighborhood.*

Let F be a coherent invariant sheaf with weights in $[a, b]$. We assume

(5.2.3) *The diagram*
$$\begin{array}{ccc} T^{(m)} & \longrightarrow & X \\ \downarrow & & \downarrow f_s \\ X & \xrightarrow{g_s} & X' \end{array}$$
commutes.

$$m > b - a. \tag{5.2.4}$$

Then the following diagram commutes:

$$\begin{array}{ccccc} (f_s \circ i)^* F_{X'/T'} & = & i^* f_s^* F_{X'/T'} & & \\ & & & \searrow^{\beta(f)} & \\ \downarrow & & & & i^* F_{X/T}. \\ & & & \nearrow_{\beta(g)} & \\ (g_s \circ i)^* F_{X'/T'} & = & i^* g_s^* F_{X'/T'} & & \end{array}$$

The proof will be given in §5.4.

Admitting this proposition for a while, we shall give its corollary.

Let T be an S-scheme and $T^{(m)}$ a T-scheme. We assume that locally in $T, T^{(m)}$ is isomorphic to the m-th infinitesimal neighborhood of a section $T \to X$ of a smooth T-scheme $X \to T$ with fiber dimension n.

Corollary 5.2.2. *Let F be a coherent invariant sheaf with weights in $[a,b]$ and $m > b-a$. Then there exists a $\mathcal{O}_{T^{(m)}}$-module F_0 satisfying the following properties* (5.2.5) *and* (5.2.6).

(5.2.5) *For $g : T' \to T$, let $X' \to T'$ be an object of $\mathcal{S}_n(S)$ and let $j' : T'^{(m)} = T' \underset{T}{\times} T^{(m)} \hookrightarrow X'$ be an embedding by which $T'^{(m)}$ is the m-th infinitesimal neighborhood of $i' : T' \hookrightarrow T'^{(m)} \hookrightarrow X'$. Then there is an isomorphism $\gamma(j') : i'^* F_{X'/T'} \xrightarrow{\sim} g^* F_0$.*

(5.2.6) *$\gamma(j')$ satisfies the chain condition. Namely let $f : (X'' \to T'') \to (X' \to T')$ be a morphism in $\mathcal{S}_n(S)$, $j'' : T'' \underset{T}{\times} T^{(m)} \hookrightarrow X''$ a morphism over j' and i'' the composition of $T'' \hookrightarrow T'' \underset{T}{\times} T^m$ and j''. Then the diagram*

$$\begin{array}{ccc} i'^* F_{X'/T'} & \simeq & i''^* f_s^* F_{X'/T'} \\ \gamma(j')\big\downarrow & & \big\downarrow \beta(f) \\ f_b^* g^* F_0 & \underset{\gamma(j'')}{\longleftarrow} & i''^* F_{X''/T''} \end{array}$$

commutes.

Since the proof is straightforward we omit the proof.

§5.3. Deformation of Normal cone

In order to prove Proposition 5.2.1, we use the deformation of normal cone. Let us recall its definition. Let X be a scheme and $Y \subset X$ a subscheme defined by an ideal I.

Let t be an indeterminate and consider the ring

$$\underset{n \in \mathbb{Z}}{\oplus} I^n t^{-n} \subset \mathcal{O}_X[t, t^{-1}].$$

Here we understand $I^n = \mathcal{O}_X$ for $n \leq 0$.

Set $\tilde{C}_{Y/X} = \mathrm{Spec}(\oplus I^n t^{-n})$ and let $q : \tilde{C}_{Y/X} \to X$ be the projection. This is called the deformation of normal cone. Then t gives a morphism $\tilde{C}_{Y/X} \to \mathbb{A}^1$.

Then $p^{-1}(0)$ is isomorphic to the normal cone $N_{Y/X} = \mathrm{Spec}(\bigoplus_{n\geq 0} I^n/I^{n+1})$ and $p^{-1}(\mathbf{A}^1\backslash\{0\}) \xrightarrow{\sim} X \times (\mathbf{A}^1\backslash\{0\})$. The homomorphism $\bigoplus_n I^n t^{-n} \to \bigoplus_{n\geq 0} \mathcal{O}_X t^n \to \bigoplus_{n\geq 0} \mathcal{O}_Y t^n$ gives the embedding $Y \times \mathbf{A}^1 \subset \tilde{C}_{Y/X}$.

If X and Y are smooth over T, then $\tilde{C}_{Y/X}$ is also smooth over T. If there is a smooth morphism $X' \xrightarrow{f} X$ and $f^{-1}Y \cong Y'$, then there is a Cartesian diagram

$$\begin{array}{ccc} \tilde{C}_{Y'/X'} & \longrightarrow & \tilde{C}_{Y/X} \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X. \end{array}$$

If X is a vector bundle over T and if Y is the zero section of $X \to T$, then there is a unique isomorphism $X \times \mathbf{A}^1 \xrightarrow{\sim} \tilde{C}_{Y/X}$ such that $X \times \mathbf{A}^1 \xrightarrow{\sim} \tilde{C}_{Y/X} \to X$ is given $(x,t) \to tx$ and $X \times \mathbf{A}^1 \cong \tilde{C}_{Y/X} \xrightarrow{p} \mathbf{A}^1$ is the second projection.

§5.4. Proof of Proposition 5.2.1

Let us prove Proposition 5.2.1. By [EGA], we may assume T to be Noetherian. By replacing T with S we may assume $T = S$. Locally in Y, there exists a morphism from $Y \to S$ to $V \to S$ in $\mathcal{S}_n(S)$ such that the composition $S \to X \to Y \to V$ coincides with the zero section. Hence replacing $Y \to S$ with $V \to S$ we may assume from the beginning that

(5.4.1) $\quad Y = V$

(5.4.2) $\quad S \to X \to Y$ coincides with the zero section.

Hence $\tilde{C}_{S/Y} \cong Y \times \mathbf{A}^1$ as seen in the preceding section. Thus we obtain a diagram of schemes over $S \times \mathbf{A}^1$.

$$\begin{array}{ccccc} & & \tilde{C}_{S/X} & \longrightarrow & X \times \mathbf{A}^1 \\ & & \tilde{f}_s \Big\downarrow\Big\downarrow \tilde{g}_s & & f_s \times id \Big\downarrow\Big\downarrow g_s \times id \\ Y \times \mathbf{A}^1 & \cong & \tilde{C}_{S/Y} & \longrightarrow & Y \times \mathbf{A}^1 \end{array}$$

Note that $\tilde{f}_s$ and $\tilde{g}_s$ are étale and hence $\tilde{f}_s$ and $\tilde{g}_s$ give morphisms $\tilde{f}$ and $\tilde{g}$ from $(\tilde{C}_{S/X} \to S \times \mathbf{A}^1)$ to $(\tilde{C}_{S/Y} \to S \times \mathbf{A}^1)$ in $\mathcal{S}_n(S)$.

Lemma 5.4.1. *$\tilde{f}_s$ and $\tilde{g}_s$ are equal modulo t^m, i.e.*

$$\mathrm{Spec}(\mathcal{O}_{\tilde{C}_{S/X}}/t^m\mathcal{O}_{\tilde{C}_{S/X}}) \to \tilde{C}_{S/X} \overset{\tilde{f}_s}{\underset{\tilde{g}_s}{\rightrightarrows}} \tilde{C}_{S/Y}$$

commutes (i.e. the two possible compositions are equal).

Proof. Let $I_X \subset \mathcal{O}_X$ and $I_Y \subset \mathcal{O}_Y$ be the defining ideal of $S \subset X$ and $S \subset Y$. Then by (5.2.3), $\mathcal{O}_Y \overset{f^*}{\underset{g^*}{\rightrightarrows}} \mathcal{O}_X \to \mathcal{O}_X/I_X^{1+m}$ commutes. Hence $I_Y \overset{f^*}{\underset{g^*}{\rightrightarrows}} I_X \to I_X/I_X^{1+m}$ commutes. Thus $I_Y^l \overset{f^*}{\underset{g^*}{\rightrightarrows}} I_X^l \to I_X^l/I_X^{l+m}$ commutes for $l \geq 1$.
Hence $\underset{l}{\oplus} I_Y t^{-l} \overset{f^*}{\underset{g^*}{\rightrightarrows}} \underset{l}{\oplus} I_X^l t^{-l} \to \mathcal{O}_{\tilde{C}_{S/X}}/t\mathcal{O}_{\tilde{C}_{S/X}} = \underset{0\leq l\leq m}{\oplus}(\mathcal{O}_X/I_x^l)t^{m-l} \oplus \underset{l\geq 1}{\oplus}(I_X^l/I_X^{l+m})t^{-l}$ commutes. □

Now let $\tilde{j} : S \times \mathbf{A}^1 \to \tilde{C}_{S/X}$ be the canonical embedding. Let $\tilde{j}_Y$ be the composition $\tilde{f}_s \circ \tilde{j} = \tilde{g}_s \circ \tilde{j}$.

Then we obtain the homomorphism $\tilde{\varphi}$:

$$\begin{aligned}
&\tilde{j}_Y^* F(Y \times \mathbf{A}^1 \to S \times \mathbf{A}^1) \to \tilde{j}_Y^* F(\tilde{C}_{S/Y} \to S \times \mathbf{A}^1)\\
&\cong \tilde{j}^* \tilde{f}_s^* F(\tilde{C}_{S/Y} \to S \times \mathbf{A}^1) \xrightarrow{\beta(\tilde{f})} \tilde{f}^* F(\tilde{C}_{S/X} \to S \times \mathbf{A}^1)\\
&\underset{\beta(\tilde{g})}{\xleftarrow{\sim}} \tilde{j}^* \tilde{g}_s^* F(\tilde{C}_{S/Y} \to S \times \mathbf{A}^1) \cong \tilde{j}_Y^* F(\tilde{C}_{S/Y} \to S \times \mathbf{A}^1)\\
&\xrightarrow{\sim} j_Y^* F(Y \times \mathbf{A}^1 \to S \times \mathbf{A}^1).
\end{aligned}$$

Let us denote by φ the composition

$$\Phi(F) \xrightarrow{\sim} j^* F_Y \xrightarrow{\sim} j^* f^* F_Y \xrightarrow{\beta(F)} j^* F_X \underset{\beta(g)}{\xleftarrow{\sim}} j^* g^* F_Y \xrightarrow{\sim} i^* F_Y \xrightarrow{\sim} \Phi(F).$$

Then outside $t \neq 0, \tilde{\varphi}$ coinsides with $t^{-1}\varphi t$. Thus $t^{-1}\varphi t$ extends to $t = 0$, and equals to the identity modulo t^m by Lemma 5.4.1. Now let us write

$$\varphi(u) = \sum_{\nu} \varphi_{\nu\mu}(u) \quad \text{for } u \in \Phi(F)_\mu$$
$$\text{with } \varphi_{\nu\mu}(u) \in \Phi(F)_\nu.$$

Then $\tilde{\varphi}(u) = \sum t^{-\nu}\varphi_{\nu\mu}(tu) = \sum t^{\mu-\nu}\varphi_{\nu\mu}(u)$. We have $\tilde{\varphi}(u) \equiv u \bmod t^m$. Hence $\varphi_{\nu\mu}(u) = 0$ for $\mu - \nu < 0$ and $\varphi_{\nu\mu}(u) = 0$ for $m > \mu - \nu > 0, \varphi_{\mu\mu}(u) = u$. They imply that $\varphi(u) - u \in W_{\mu-m}(\Phi(F))$. Therefore we obtain $\varphi = id$ by (5.2.4). This completes the proof of Proposition 5.2.1

§5.5. The G-module structure on $\Phi(F)$

Let F be a coherent invariant sheaf and let us take $b \geq a$ such that

$$\Phi(F) = \bigoplus_{a \leq l \leq b} \Phi(F)_l.$$

Let us take $m > b - a$. We shall endow the structure of $G(m)$-module on $\Phi(F)$ as follows. For $g \in G(m)$, locally on S, there exist a morphism $f : V \to V$ such that the diagram

$$\begin{array}{ccc} W_m(S) & \xrightarrow{g} & W_m(S) \\ \downarrow & & \downarrow \\ V & \xrightarrow{f} & V \end{array}$$

commutes. Hence f is étale on a neighborhood of $i(S)$. We define the action of g on $\Phi(F) = i^*F$ as the inverse of the composition

$$i^*F_V = (f \circ i)^*F_V \xrightarrow{\sim} i^*f^*F_V \xrightarrow{\beta(f)} i^*F_V.$$

This definition does not depend on the choice of f by Proposition 5.2.1. This gives evidently the structure of $G(m)$-module and hence the structure of G-module via $G \to G(m)$. Thus we obtain the functor Φ from $I_c(n)$ to the category of coherent G-modules. Evidently Φ commutes with the tensor product.

§6. The Functor $\mathcal{B}$

§6.1. Jet bundle

Let us construct a quasi-inverse $\mathcal{B}$ of Φ. We shall use a standard technique that uses jet bundles. Let us recall the definition of a jet bundle. Let $X \to T$ be a smooth morphism with fiber dimension n. Let $\triangle^{(m)}_{X/T}$ be the m-th infinitesimal neighborhood of the diagonal X in $X \times_T X$. Let $p_1 : \triangle^{(m)}_{X/T} \to X \times_T X \to X$ be the first projection and $p_2 : \triangle^{(m)}_{X/T} \to X \times_T X \to X$ the second projection. The jet bundle $J^{(m)}_{X/T}$ of order m is the scheme over X that represents the functor

$$X' \mapsto \{\varphi; \varphi \text{ is an isomorphism from } X' \times W^m(\mathbf{A}^n) \text{ to } X' \times_X \triangle^{(m)}_{X/T}\}.$$

Here $X' \times_X \triangle^{(m)}_{X/T}$ is the fiber product via $\triangle^{(m)}_{X/T} \xrightarrow{p_1} X$. Hence there exists a canonical isomorphism

$$J^{(m)}_{X/T} \times W^m(\mathbf{A}^n) \xrightarrow{\sim} J^{(m)}_{X/T} \times_X \triangle^{(m)}_{X/T}.$$

Moreover the action of $G(m)$ on $W_m(\mathbb{A}^n)$ induces the action on $J^{(m)}_{X/T}$ and $\pi : J^{(m)}_{X/T} \to X$ is a principal $G(m)$ bundle. Note that $J^{(m)}_{X/T} \to X$ is locally trivial with respect to the Zariski topology of X.

§6.2. Construction of the functor $\mathcal{B}$

Let M be a coherent G-module. Let us take $m \gg 0$ such that the G-action on M comes from a $G(m)$-action on M.

For a morphism $X \to T$, let $\mathcal{B}(M)_X$ be the associated bundle of M with respect to $J^{(m)}_{X/T}$. Namely let $q : J^{(m)}_{X/T} \to S$ and $\pi : J^{(m)}_{X/T} \to X$ be the projections. Then $\mathcal{B}(M)_X$ is the subsheaf of $\pi_* q^* M$ consisting of the sections invariant under the action of $G(m)$. Here the action of $G(m)$ on $\pi_* q^* M$ is induced by its action on M and the one on $J^{(m)}_{X/T}$. This definition does not depend on m. In fact for $m' \geq m$, there is a canonical G-equivariant morphism $J^{(m')}_{X/T} \to J^{(m)}_{X/T}$. Then $X \mapsto \mathcal{B}(M)_X$ is evidently an invariant sheaf and we shall denote it by $\mathcal{B}(M)$. This definition does not depend on the choice of m and it gives an exact functor from $\mathrm{Mod}_c(G)$ to $I_c(n)$.

§6.3. $\mathcal{B}$ and Φ

We shall prove that $\mathcal{B}$ and Φ are quasi-inverse to each other. We can see easily that $\Phi\mathcal{B}(M) \cong M$ for $M \in \mathrm{Mod}_c(G)$. In the sequel we shall show $\mathcal{B}\Phi(F) \cong F$ for $F \in I_c(n)$. Let us set $M = \Phi(F)$ and let us take $b \geq a$ such that $W_b(M) = M$ and $W_{a-1}(M) = 0$. Then for $m > b - a$, $G(m)$ acts on M. Let us take $X \to T$ in $\mathcal{S}_n(S)$ and let us consider the diagram

$$\begin{array}{ccccccc} & & \multicolumn{3}{c}{\overset{j'}{\overbrace{\hspace{6cm}}}} & & \\ J^{(m)}_{X/T} \times W_m(\mathbb{A}^n) & \xrightarrow{\sim} & J^{(m)}_{X/T} \underset{X}{\times} \triangle^{(m)}_{X/T} & \hookrightarrow & J^{(m)}_{X/T} \underset{T}{\times} X & \xrightarrow{f_s} & X \\ & & & & \downarrow & & \downarrow \\ & & & & J^{(m)}_{X/T} & \xrightarrow[f_b = \pi]{} & T \end{array}$$

Then π gives a morphism f from $\left(J^{(m)}_{X/T} \underset{T}{\times} X \to J^{(m)}_{X/T}\right)$ to $(X \to T)$ in $\mathcal{S}_n(S)$ and hence an isomorphism

$$\beta(f) : f_s^* F_{X/T} \xrightarrow{\sim} F_{J^{(m)}_{X/T} \times X / J^{(m)}_{X/T}}.$$

Let $i : J^{(m)}_{X/T} \hookrightarrow J^{(m)}_{X/T} \times \mathbb{A}^n$ and $i' : J^{(m)}_{X/T} \hookrightarrow J^{(m)}_{X/T} \underset{T}{\times} X$ denote the embeddings. Then by Corollary 5.2.2 we have a canonical isomorphism

$$i^* F_{J^{(m)}_{X/T} \times \mathbb{A}^n / J^{(m)}_{X/T}} \simeq i'^* F_{J^{(m)}_{X/T} \underset{T}{\times} X / J^{(m)}_{X/T}}. \tag{6.3.1}$$

We have $i^*F_{J^{(m)}_{X/T}\times\mathbf{A}^n/J^{(m)}_{X/T}} = q^*M$ where $q : J^{(m)}_{X/T} \to S$ is the canonical projection and $i'^*F_{J^{(m)}_{X/T}\underset{T}{\times}X/J^{(m)}_{X/T}} = f_s^*F_{X/T}$. We can see easily that the isomorphism $q^*M \simeq f_s^*F_{X/T}$ is $G(m)$-equivariant and hence $\mathcal{B}(M) \cong F_{X/T}$. This completes the proof of $\mathcal{B}\circ\Phi \cong id$.

§7. The Weight Filtration

We established the equivalence $\mathrm{Mod}_c(G)$ and $I_c(n)$. Since any object of $\mathrm{Mod}_c(G)$ has a weight filtration W, any object $I_c(n)$ has a weight filtration W.

The corresponding properties of W for $\mathrm{Mod}_c(G)$ imply the following properties.

(7.1) $F \mapsto W_l(F)$ and $F \mapsto Gr_l^W(F)$ are exact functors from $I_c(n)$ to $I_c(n)$.
(7.2) For invariant sheaves $F_1, F_2 \in I_c(n)$, we have

$$W_{l_1+l_2}(W_{l_1}(F_1)\otimes W_{l_2}(F_2)) = W_{l_1}(F_1)\otimes W_{l_2}(F_2).$$

(7.3) For $F_1, F_2 \in I_c(n)$ and $l \in \mathbf{Z}$, the above isomorphism induces an isomorphism

$$\oplus_{l=l_1+l_2} Gr^W_{l_1}(F_1)\otimes Gr^W_{l_2}(F_2) \xrightarrow{\sim} Gr^W_l(F_1\otimes F_2).$$

(7.4) For $F \in I^b(n)$, $W_{-l-1}(W_l(F)^*) = 0$ and $Gr_l^W(F^*) \cong (Gr^W_{-l}(F))^*$.

Thus $I^b(n)$ has a structure of a filtered rigid tensor category.

Example 7.1. $\mathcal{O}$ is pure of weight 0. Θ is pure of weight 1 and Ω^k is pure of weight $-k$.

Example 7.2. $\mathcal{P}^{(m)}$ is of weight $[-m, 0]$ (c.f. Example 1.5) and $\mathcal{P}^{(m)}/W_{-1-l}(\mathcal{P}^{(m)}) = \mathcal{P}^{(l)}$ for $0 \le l \le m$.

Example 7.3. $W_m(\mathcal{D})$ is of weight $[0, m]$ (c.f. Example 1.6) and $W_l(W_m(\mathcal{D})) = W_l(\mathcal{D})$ for $0 \le l \le m$. We have $W_m(\mathcal{D}) = (\mathcal{P}^{(m)})^*$.

§8. Lie Derivative

§8.1. Definition

Let F be a coherent invariant sheaf, $X \to T$ an object in $\mathcal{S}_n(S)$ and v a relative tangent vector on X/T. Then we can define a Lie derivative $L(v)$:

$F_{X/T} \to F_{X/T}$ that satisfies

$$L(v)(au) = aL(v)u + v(a)u \quad \text{for } a \in \mathcal{O}_X \text{ and } u \in F_{X/T}. \tag{8.1.1}$$

Let us set $T' = T \times \operatorname{Spec}(\mathbb{Z}[\varepsilon]/\varepsilon^2\mathbb{Z}[\varepsilon])$ and $X' = X \times_T T'$ and define an automorphism $f : X' \to X'$ over T' by $x \mapsto x + \varepsilon v(x)$. Let p be the projection $(X' \to T')$ to $(X \to T)$. Then we have a homomorphism

$$\psi : p_s^* F_{X/T} \simeq F_{X'/T'} \xrightarrow{\beta(f)} F_{X'/T'} = p_s^* F_{X/T}.$$

Since $p_{s*}p_s^* F_{X/T} = F_{X/T} \oplus \varepsilon F_{X/T}$, we define $\psi(v)$ by $\psi(u) = u \oplus \varepsilon L(v)u$. Then $L(v)$ satisfies the relation (7.1.1). Moreover we have

$$[L(v_1), L(v_2)] = L([v_1, v_2]) \quad \text{for } v_1, v_2 \in \Theta_{X/T}. \tag{8.1.2}$$

Note that for any $s \in F_{X/T}$, $v \mapsto L(v)s$ is a differential operator from $\Theta_{X/T}$ to $F_{X/T}$.

This definition coincides with the usual definition of the Lie derivative on $\Omega^k_{X/T}$. The Lie derivative acts on $W_m(\mathcal{D})$ by the adjoint action.

§8.2. The infinitesimal action

Let $\mathfrak{g}$ be the subsheaf of $p_*(\Theta_{V/S})$ consisting of tangent vectors that vanishes at the zero section. Here $p : V \to S$ is the projection. Then we have

$$\mathfrak{g} = S_+(\mathcal{V}^*) \otimes_{\mathcal{O}_S} \mathcal{V} \tag{8.2.1}$$

where $S_+(\mathcal{V}^*) = \oplus_{l>0} S^l(\mathcal{V}^*)$. Set $W_l(\mathfrak{g}) = \oplus_{1-l' \le l} S_{l'}(\mathcal{V}*) \otimes \mathcal{V}$. Then $W_0(\mathfrak{g}) = \mathfrak{g}$ and $\mathfrak{g}/W_{-m-1}(\mathfrak{g})$ is the Lie algebra of $G(m)$. Hence for $F \in I(n)$, $\mathfrak{g}$ acts on $\Phi(F)$ as its infinitesimal action. This action coincides with the action through the Lie derivative.

§9. Characteristic Zero Case

In this section 9, let us take $\operatorname{Spec}(k)$ as S for a field k of characteristic 0. Then V may be regarded as an n-dimensional vector space over k. In this case, the Lie algebra $\mathfrak{g}$ in §8.2 coincides with $S_+(V^*) \otimes V$ where $S_+(V^*) = \oplus_{l>0} S^l(V^*)$. It contains the Lie algebra $V^* \otimes V$ of $GL(V)$. Therefore the category of G-modules coincides with the category of $(\mathfrak{g}, GL(V))$-modules.

Set $W_{-l}(\mathfrak{g}) = \oplus_{1-l' \leq -l} S^{l'}(V^*) \otimes V$. The action homomorphism $\mathfrak{g} \otimes M \to M$ preserves the weight filtration W for a $(\mathfrak{g}, GL(V))$-module M. Hence if M is a pure module, $W_{-1}(\mathfrak{g})$ annihilates M and hence M is a $GL(V)$-module. Thus we have

Proposition 9.1. *Any pure invariant sheaf is semisimple.*

This implies the following result by a standard argument.

Proposition 9.2. *Let F_ν be a pure invariant sheaf of weight w_ν ($\nu = 1, 2$). Then we have*

(9.2.1) $$\mathrm{Ext}^k_{I^b(n)}(F_1, F_2) = 0 \quad \textit{for} \quad w_1 - w_2 < k.$$

As stated in the introduction, we conjecture

Conjecture $\mathrm{Ext}^k_{I^b(n)}(F_1, F_2) = 0$ for $w_1 - w_2 \neq k$ and $k < n$.

Since the category of G-modules coincides with the category of $(\mathfrak{g}, GL(V))$-modules, we can translate results in the Lie algebra cohomology (e.g. in [F]) in our framework. For example by the result of Goncharova([G]), we have when $n = 1$

$$\mathrm{Ext}^i_{I(1)}(\mathcal{O}, \Omega^{1 \otimes j}) = \begin{cases} k & \text{for } i = 0 \text{ and } j = 0, \\ k & \text{for } i \geq 1 \text{ and } j = (3i^2 - i)/2 \text{ or } (3i^2 + i)/2, \\ 0 & \text{otherwise.} \end{cases}$$

§10. Variants

§10.1. Complex analytic case

We can perform the same construction for the complex analytic case. Namely we take $\mathcal{S}_n$ the category of smooth morphisms $X \to T$ of fiber dimension n of complex analytic spaces. A morphism f from $X \xrightarrow{a} T$ to $X' \xrightarrow{a'} T'$ is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f_\sharp} & X' \\ a' \downarrow & & a \downarrow \\ T' & \xrightarrow{f_b} & T \end{array}$$

such that $X \to X' \times_T T'$ is a local isomorphism. Then the invariant sheaves are defined similarly to the algebraic case. The category of invariant sheaves (in the complex analytic case) is equivalent to the category of G-modules with $S = \mathrm{Spec}(\mathbf{C})$.

Hence it is equivalent to $I(n)_{\mathrm{Spec}(\mathbf{C})}$. In another word invariant sheaves are same in the complex analytic case and algebraic case.

§10.2. Multiple case

Instead of working on the sheaves on X, we can work on the sheaves on $X \times_T X$. More precisely we can consider the following category $I(n;2)$. An object of $I(n;2)$ is the data:
(10.2.1) To any object $X \to T$ in $\mathcal{S}_n(S)$, assign a quasi-coherent $\mathcal{O}_{X\times_T X}$ modules $F_{X/T}$ whose support is contained in the diagonal set.
(10.2.2) To any morphism $\varphi = (\varphi_s, \varphi_b) : (X \to T) \to (X' \to T')$ in $\mathcal{S}_n(S)$, assign an isomorphism

$$\beta(\varphi) : (\varphi_s \times \varphi_s)^* F_{X'/T'} \xrightarrow{\sim} F_{X/T}.$$

Here $\varphi_s \times \varphi_s$ is the morphism $X' \times_{T'} X' \to X \times_T X$ induced by φ.

We assume the similar associative law to the invariant sheaf case. We call an object of $I(n;2)$ a *double invariant* sheaf. Similarly to the invariant sheaf case we define $I_c(n;2)$ to be the category of double invariant sheaves F such that $F_{X/T}$ are locally of finite presentation. For an object $X \to T$ in $\mathcal{S}_n(S)$, let $p_1 : X \times_T X \to X$ be the projection. Then for a double invariant sheaf $F_{X/T}$, $X/T \mapsto p_{1*}F_{X/T}$ is an invariant sheaf. Thus we obtain the functor

$$p_{1*} : I(n;2) \to I(n).$$

Let us denote by $\mathcal{O}_{\Delta^{(m)}}$ the double invariant sheaf that associates $\mathcal{O}_{\Delta^{(m)}_{X/T}}$ to $X \to T$ in $\mathcal{S}_n(S)$. Here $\Delta^{(m)}_{X/T}$ is the m-th infinitesimal neighborhood of the diagonal embedding $X \hookrightarrow X \times_T X$. Then for a double invariant sheaf F, there is an action $\mathcal{O}_{\Delta^{(m)}_{X/T}} \otimes_{\mathcal{O}_{X\times_T X}} F_{X/T} \to F_{X/T}$ if we take m sufficiently large. It induces $p_{1*}(\mathcal{O}_{\Delta^{(m)}_{X/T}}) \otimes p_{1*}(F_{X/T}) \to p_{1*}(F_{X/T})$. Thus we obtain a homomorphism in $I(n)$

$$p_{1*}\mathcal{O}_{\Delta^{(m)}} \otimes p_{1*}F \to p_{1*}F.$$

We can see easily

$$\Phi(p_{1*}\mathcal{O}_{\Delta^{(m)}}) = p_*\mathcal{O}_{W^m(V)}.$$

Here $p : W^m(V) \to S$ is the projection. We have $p_*\mathcal{O}_{W^m(V)} = S(\mathcal{V}^*)/W_{-m-1} S(\mathcal{V}^*)$. Here $W_{-l}(S(\mathcal{V}^*)) = \oplus_{l' \geq l} S^{l'}(\mathcal{V}^*)$. Thus we obtain

Proposition 10.2.1. *$I_c(n;2)$ is equivalent to a category of G-modules with the structure of $S(\mathcal{V}^*)$-modules M such that $S(\mathcal{V}^*) \otimes M \to M$ is G-equivariant (more precisely $W_{-l}(S(\mathcal{V}^*))M = 0$ for $l \gg 0$ and $S(\mathcal{V}^*)/W_{-l}(S(\mathcal{V}^*)) \otimes M \to M$ is G-equivariant).*

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