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A Rohlin Property for One-parameter Automorphism Groups of the Hyperfinite II_1 Factor

By

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Abstract

We first define a Rohlin property for one-parameter automorphism groups of the hyperfinite type II_1 factor as an analogue of Kishimoto's definition for one-parameter automorphism groups of unital simple C^*-algebras.

Secondly we prove equivalence between the Rohlin property and the cohomology vanishing in an appropriate central sequence algebra, which is a variation of Kishimoto's theorem in C^*-algebra theory.

§1. Introduction

Classification of group actions on von Neumann algebras was dramatically developed by Connes in the course of proving the uniqueness of aperiodic approximately inner automorphisms of a separable McDuff II_1 factor up to outer conjugacy, in particular, he gave a proof to the uniqueness of aperiodic automorphisms of the hyperfinite type II_1 factor R, which we sometimes call *free integer group actions*, i.e., $\mathbf{Z}$-actions, up to outer conjugacy [3]. Later Jones and Ocneanu classified actions (including non-free ones) of finite groups and discrete amenable groups on R, respectively, up to cocycle conjugacy [5][10]. Then our next interest moves into considering classification of continuous group actions on R, especially one-parameter automorphism groups of R, that is, *real group actions* on R which we shortly call $\mathbf{R}$-actions on R.

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In Connes' classification theory of type III factors [2], the modular automorphism groups have been deeply studied through Tomita-Takesaki theory and their properties play important roles. Therefore, for our interest in one-parameter automorphism groups of R, we could find something common or parallel between the modular automorphism groups and $\mathbf{R}$-actions on R.

Although we have not known the right definition of freeness of $\mathbf{R}$-actions, we propose to say an $\mathbf{R}$-action α is "free" when

$$\text{the Connes spectrum} \quad \Gamma(\alpha) = \mathbf{R}, \qquad \text{and for any } t \neq 0, \quad \alpha_t \notin \mathrm{Int}(R).$$

Later we will see in Example 2.3 these two conditions are independent of each other. However we remark, when M is a factor of type III, the condition $S(M) = [0, \infty)$ implies $T(M) = \{0\}$ i.e., $\Gamma(\sigma^\varphi) = \mathbf{R}$ implies $\sigma_t^\varphi \notin \mathrm{Int}(M)$ for $t \neq 0$ for the modular automorphism groups [2, Théorème 3.4.1], thus it is enough to assume only the full Connes spectrum condition.

So far, classification is completed by Kawahigashi [6] in the case when $\Gamma(\alpha) \neq \mathbf{R}$. As for "free" actions, only the specific actions have been classified. (See [7, Theorem 1.16], [8, Theorem 16].)

In order to deal with "free" $\mathbf{R}$-actions, in a similar way to the method in the case of discrete group actions mentioned above, Connes' strategy would be helpful again. Let us roughly recall what he did. His above result arose on the way to prove the uniqueness of the hyperfinite type III_λ factor for $\lambda \in]0,1[$. Since it was also shown by him that a type III_λ factor is isomorphic to some crossed product algebra of some type II_∞ factor by $\mathbf{Z}$, he decomposed the problem into two parts. One is proving the uniqueness of the injective type II_∞ factor N, and the other is a proof of the uniqueness of trace-scaling $\mathbf{Z}$-actions on N, which is reduced to showing the uniqueness of free $\mathbf{Z}$-actions on R up to outer conjugacy.

His first step is to prove the "non-commutative Rohlin lemma" which is explained in detail in the next section. Secondly, he showed the one-cohomology vanishing of a free $\mathbf{Z}$-action α in the central sequence algebra R_ω, where ω is a free ultrafilter over $\mathbf{N}$. And thirdly, he proved that α is outer conjugate to a model action.

In this paper, we define (in Definition 2.2) an appropriate Rohlin property for one-parameter automorphism groups of R, which is an analogue of Kishimoto's definition for unital simple C*-algebras [9], and show that the cohomology of a one-parameter automorphism group with the Rohlin property vanishes in the α-equicontinuous central sequence algebra $R_{\omega,\mathrm{eq}}$, which is also an analogue of [9, Theorem 2.1].

Though the uniqueness of the hyperfinite factor of type III_1 has been proved by Haagerup [4], if we could classify "free" $\mathbf{R}$-actions on R up to cocycle conjugacy, it is expected that it would give another proof of the uniqueness of the hyperfinite III_1 factor as an analogue of Connes' result for the hyperfinite III_λ factor for $\lambda \in]0,1[$.

§2. A Rohlin Property

We define a Rohlin-type property for one-parameter automorphism groups of the hyperfinite factor of type II_1 as an analogue of Kishimoto's definition of a Rohlin property for one-parameter automorphism groups of unital simple C*-algebras.

Connes proved the following "non-commutative Rohlin Lemma", which has some analogy with the classical lemma of Rohlin in ergodic theory.

Theorem 2.1. **(Connes)** *Let M be a separable II_1 factor and $\alpha \in \mathrm{Aut}(M)$ be an aperiodic automorphism. Suppose that $N \subset M_\omega$ is a separable von Neumann algebra globally fixed by α_ω. Then for any $n \in \mathbf{N}$ there exist projections $e_1, \cdots, e_n \in \mathrm{Proj}(N' \cap M_\omega)$ such that (1) $\alpha(e_j) = e_{j+1}$ (we set $e_{n+1} := e_1$) and (2) $e_1 + \cdots + e_n = 1$.*

Kishimoto considered a certain condition for one-parameter automorphism groups of unital simple C*-algebras as an analogue of the above two conditions and named it a "Rohlin property" [9]. His definition can be translated to our setting in the following way.

Definition 2.2. Let R be the hyperfinite factor of type II_1 and let α be a one-parameter automorphism group of R. We say α has a Rohlin property when it satisfies the condition:

(*) For any $p \in \mathbf{R}$, there exists a unitary $v = \{v_n\} \in R_{\omega,\mathrm{eq}}$ such that $\alpha_t(v) = e^{itp}v$,

where $R_{\omega.\mathrm{eq}}$ is the α-equicontinuous ω-central sequence algebra defined by

$$R_{\omega,\mathrm{eq}} := \left\{ x = \{x_n\} \in R_\omega \;\middle|\; \begin{array}{l} \text{for any} \quad \varepsilon > 0, \quad \text{there exists} \quad \delta > 0 \quad \text{such that} \\ \left\{ n \in \mathbf{N} \;\middle|\; \begin{array}{l} \|\alpha_t(x_n) - x_n\|_2 < \varepsilon \\ \text{for any } t \text{ with } |t| < \delta \end{array} \right\} \in \omega \end{array} \right\}.$$

It is easy to show that the definition of $R_{\omega,\mathrm{eq}}$ is well-defined, i.e., it does not depend on choice of representative sequences of $x \in R_\omega$, and $R_{\omega,\mathrm{eq}}$ is a von Neumann algebra. In general, we do not know whether convergence $\alpha_t(x) \overset{t\to 0}{\longrightarrow} x$ means "ω"-uniform convergence of $\alpha_t(x_n) \overset{t\to 0}{\longrightarrow} x_n$ over $\mathbf{N}$ or not, thus α-equicontinuity is a convenient and natural assumption in our setting.

When we define a Rohlin property for a unital simple C*-algebra A in Kishimoto's way [9], one can replace this α-equicontinuity with usual continuity. Indeed, since he defines it in a central sequence algebra A_∞ instead of the ω-central sequence algebra, it is enough to assume continuity of $t \mapsto \alpha_t(x)$ for $x = \{x_n\} \in A_\infty$ in order to obtain the following equicontinuity

$$\text{for any } \varepsilon > 0 \text{ there exist } \delta > 0 \text{ and } n_0 \in \mathbf{Z} \text{ such that}$$

$$\text{for any } n > n_0 \text{ and } t \in]-\delta, \delta[\ , \quad \|\alpha_t(x_n) - x_n\| < \varepsilon,$$

which corresponds to our α-equicontinuity.

We provide some correspondences between Connes' non-commutative Rohlin lemma and our Rohlin property. Let $v = \int_{\mathbf{T}} e^{i\lambda} dE_\lambda$ be the spectral decomposition in $R_{\omega,\mathrm{eq}}$ of the above v. By the uniqueness of the spectral decomposition we get $\alpha_t(E_\lambda) = E_{\lambda - pt}$. We therefore see the following correspondences.

single automorphism α		one-parameter automorphism group α
$n \in \mathbf{N}$	$\longleftrightarrow$	$1/p \in \mathbf{R}$
projections $e_1, \cdots, e_n$	$\longleftrightarrow$	the spectral projection E_λ of v
$\sum_{j=1}^{n} e_j = 1$	$\longleftrightarrow$	$\int_{\mathbf{T}} dE_\lambda = 1$
$\alpha(e_j) = e_{j+1}$	$\longleftrightarrow$	$\alpha_t(E_\lambda) = E_{\lambda - pt}$

When a one-parameter automorphism group α has the Rohlin property, it is not hard to see that α is a "free" $\mathbf{R}$-action on R. Indeed, it is quite obvious that $\alpha_t \notin \mathrm{Int}(R)$ for $t \neq 0$. For the other condition $\Gamma(\alpha) = \mathbf{R}$, we recall the definition of the Connes spectrum [2, Définition 2.2.1]

$$\Gamma(\alpha) = \bigcap_e \left\{ \mathrm{Sp}(\alpha^e) \middle|\ e \text{ is a central projection of the fixed point algebra } R^\alpha \right\},$$

where α^e is the restriction of α on the reduced von Neumann algebra R_e, and a property of the Arveson spectrum

$$p \in \mathrm{Sp}(\alpha) \Longleftrightarrow \begin{cases} \text{there exists a net } \{x_\iota\}_{\iota \in I} \subset R \text{ with the operator norm one} \\ \text{such that for any compact set } K \subset \mathbf{R} \\ \lim_\iota \|\alpha_t(x_\iota) - e^{ipt} x_\iota\|_2 = 0 \quad \text{uniformly on } K. \end{cases}$$

Though in [2, Lemme 2.3.6], the condition $\lim_\imath \|\alpha_t(x_\imath) - e^{\imath pt} x_\imath\|_2 = 0$ is replaced by $\lim_\imath \|\alpha_t(x_\imath) - e^{\imath pt} x_\imath\|_\infty = 0$, when an algebra is a type II_1 factor there is the unique finite trace, and we notice the above equivalence holds by examining the proof of Connes.

Under the same setting as in Definition 2.2, by the α-equicontinuity of v, for any $\varepsilon > 0$ and e a central projection of R^α there exists $\delta_0 > 0$ such that

$$A := \left\{ n \in \mathbf{N} \;\middle|\; \|\alpha_t^e(ev_n e) - ev_n e\|_2 < \varepsilon \quad \text{for any } t \text{ with } |t| < \delta_0 \right\} \in \omega$$

and

$$\|(e^{\imath tp} - 1) ev_n e\|_2 \le |e^{\imath tp} - 1| < \varepsilon \quad \text{for } t \in [-\delta_0, \delta_0].$$

Since $\lim_{n\to\omega} \|\alpha_t^e(ev_n e) - e^{\imath tp} ev_n e\|_2 = 0$ for any $t \in \mathbf{R}$, for an arbitrary compact set K and $\tilde{K} := [\min K, \max K] \supset K$, setting $K_0 := \left\{ t \in \tilde{K} \;\middle|\; t/\delta_0 \in \mathbf{Z} \right\}$, we have

$$B := \left\{ n \in \mathbf{N} \;\middle|\; \forall t \in K_0 \ \|\alpha_t^e(ev_n e) - e^{\imath tp} ev_n e\|_2 < \varepsilon \right\} \in \omega.$$

Because v is a unitary we may assume each $v_n \in R$ is a unitary, then $\|ev_n e\|_\infty = 1$, and when $n \in A \cap B$ we have

$$\|\alpha_t^e(ev_n e) - e^{\imath tp} ev_n e\|_2 < 3\varepsilon \qquad \text{for} \quad t \in K,$$

in particular, $p \in \mathrm{Sp}(\alpha^e)$.

We close this section with an example.

Example 2.3 [8], [9, Proposition 2.5]. Let α be a one-parameter automorphism group of the irrational rotation C*-algebra A_θ such that $\alpha_t(u) = e^{2\pi\imath\lambda t} u$ and $\alpha_t(v) = e^{2\pi\imath\mu t} v$ with $uv = e^{2\pi\theta\imath} vu$ where u, v are the standard generators of A_θ. We assume $\lambda/\mu \notin \mathbf{Q}$ and λ/μ is not in the $\mathrm{GL}(2, \mathbf{Q})$ orbit of θ. Extend this α to the weak closure R of A_θ with respect to the trace. Then α has the Rohlin property.

It is easy to see that $\dfrac{\lambda}{\mu} \notin \mathbf{Q}$ is equivalent to $\Gamma(\alpha) = \mathbf{R}$. Moreover, the conditions $\dfrac{\lambda}{\mu} \notin \mathbf{Q}$ and $\dfrac{\lambda}{\mu} \notin \left\{ \dfrac{a\theta + b}{c\theta + d} \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbf{Q}) \right\}$ hold if and only if $\alpha_t \notin \mathrm{Int(R)}$ for any $t \neq 0$. Although, in this example, outerness of α_t for $t \neq 0$ implies $\Gamma(\alpha) = \mathbf{R}$, it is known in general that the outerness does not indicate the full Connes-spectrum condition, indeed, when

$$\alpha_t = \bigotimes_{n=1}^{\infty} \mathrm{Ad}\left(\exp it \begin{pmatrix} \lambda_n & 0 \\ 0 & 0 \end{pmatrix} \right) \qquad \text{with } \lambda_n = (2\sqrt{3})^n,$$

we have $\Gamma(\alpha) = \{0\}$ and each α_t for $t \neq 0$ is outer. These explain, as we mentioned in the introduction, that the two conditions of "freeness" are independent of each other.

It is known that this α is cocycle conjugate to an infinite tensor product type $\mathbb{R}$-action with the full Connes spectrum [8, Theorem 16], which is unique up to cocycle conjugacy. Thus when an $\mathbb{R}$-action is of infinite tensor product type, our Rohlin property is equivalent to the full Connes spectrum condition since the Rohlin property is invariant under cocycle conjugacy. However any one-parameter automorphism group of a UHF C*-algebra does not have the Rohlin property in Kishimoto's sense since K_1 is trivial [9].

§3. Cohomology Vanishing

Here is our main theorem which corresponds to the second step of Connes' classification strategy as we mentioned in the first section.

Theorem 3.1. *Let R be the hyperfinite factor of type II_1 and let α be its one-parameter automorphism group. Then the following two conditions are equivalent.*

(1) *A one-parameter automorphism group α has the Rohlin property. i.e., for any $p \in \mathbb{R}$ there exists a unitary $v = \{v_n\} \in R_{\omega,\mathrm{eq}}$ such that $\alpha_t(v) = e^{\imath tp}v$.*

(2) *For any unitary α-cocycle $u(t) = \{u_n(t)\} \in R_{\omega,\mathrm{eq}}$, there exists a unitary $w \in R_{\omega,\mathrm{eq}}$ with $u(t) = w\alpha_t(w^*)$.*

Lemma 3.2. *Let $v = \{v_n\} \in R_\omega$ be a unitary with $\alpha_t(v) = e^{2\pi\imath tp}v$ for some $p \in \mathbb{R}$. Define a linear map Φ from the algebraic tensor product algebra $R \odot L^\infty(\mathbb{T})$ into the ultraproduct algebra R^ω by*

$$\Phi(a \otimes f) := af(v).$$

Then the following conditions hold.

(i) *The map Φ is a $*$-homomorphism, i.e.,*
$\Phi(a \otimes f)\Phi(b \otimes g) = \Phi(ab \otimes fg)$ *and* $\Phi(a^* \otimes f^*) = \Phi(a \otimes f)^*$.

(ii) $\|\Phi(\sum_{n=1}^k a_n \otimes f_n)\|_2 = \|\sum_{n=1}^k a_n \otimes f_n\|_2$.

(iii) $\|\Phi(\sum_{n=1}^k a_n \otimes f_n)\|_\infty \leq \|\sum_{n=1}^k a_n \otimes f_n\|_\infty$.

(iv) *We can uniquely extend Φ so that it is defined on the weak closure of $R \odot L^\infty(\mathbb{T})$, i.e., $R \otimes L^\infty(\mathbb{T})$.*

Proof. **(Lemma 3.2)** It is easy to show (i) because $f(v) \in R_\omega$.

For (ii), we recall that $\|af(v)\|_2 = \|a\|_2\|f(v)\|_2$ when $v \in R_\omega = R' \cap R^\omega$, $a \in R$, by the uniqueness of the trace for a type II_1 factor. Let $v = \int_{\mathbf{T}} e^{2\pi i\lambda} dE_\lambda$ be the spectral decomposition of v in R_ω. Then it follows that for any $p \in \mathbf{R}$,

$$\int_{\mathbf{T}} e^{2\pi i(pt+\lambda)} dE_\lambda = e^{2\pi i pt} v = \alpha_t(v) = \int_{\mathbf{T}} e^{2\pi i\lambda} d(\alpha_t(E_\lambda)).$$

By the uniqueness of the spectral decomposition and the uniqueness of the trace it follows that $\mathrm{tr}(E_\lambda) = \mathrm{tr}(E_{\lambda-pt})$ for $\lambda \in \mathbf{T}$. Define a measure $\mu : (\mathbf{T}, \mathcal{B}) \to [0,1]$ by $\mu(A) = \mathrm{tr}(E_A)$. We remark that μ is a rotation invariant measure and coincides with the Lebesgue measure dt on $\mathbf{T}$. Therefore, we see that

$$\|f(v)\|_2^2 = \int_{\mathbf{T}} |f(t)|^2 d\mu(t) = \int_{\mathbf{T}} |f(t)|^2 dt = \|f\|_2^2.$$

With the above preparations we obtain

$$\begin{aligned}\|\Phi(a \otimes f)\|_2 &= \|af(v)\|_2 = \|a\|_2\|f(v)\|_2 \\ &= \|a\|_2\|f\|_2 = \|a \otimes f\|_2.\end{aligned}$$

When $k \geq 2$, it is easy to prove the equality.

The inequality (iii) follows from

$$\begin{aligned}\left\|\Phi\left(\sum_{n=1}^{k} a_n \otimes f_n\right)\right\|_\infty &= \left\|\sum_{n=1}^{k} a_n f_n(v)\right\|_\infty = \left\|\int_{\mathbf{T}} \Big(\sum_{n=1}^{k} a_n f_n(\lambda)\Big) dE_\lambda\right\|_\infty \\ &\leq \operatorname*{ess.\,sup}_{\lambda} \left\|\sum_{n=1}^{k} a_n f_n(\lambda)\right\|_\infty = \left\|\sum_{n=1}^{k} a_n \otimes f_n\right\|_\infty .\end{aligned}$$

It is clear to see (iv). □

Now we are ready for proving the main theorem. The outline of this proof is roughly the same as the Kishimoto's for unital simple C*-algebras [9, Theorem 2.1]. Furthermore, as we mention at the end of this section, this proof is essentially based on Connes' technique for cohomology vanishing theorem for aperiodic automorphisms.

Proof. **(Theorem 3.1)** The direction from (2) to (1) follows by putting $u(t) = e^{itp}$.

We shall prove the other direction. Remark that we can assume the unitarity of each $u_n(t)$ from an easy argument of the polar decomposition.

For each $k \in \mathbf{N}$, we fix $N \in \mathbf{N}$ larger than πk. Because the spectral decomposition of $u(t)$ can be done in the algebra $R_{\omega,\mathrm{eq}}$ and one can take rectifiable paths from $u(t)$ to 1 in the unitary group of $R_{\omega,\mathrm{eq}}$, we denote by $l(u(t))$ the infimum of its lengths in the $\|\cdot\|_2$-norm with respect to the trace, which is less than π. Then we have $l(u(t)) \leq \pi < N/k$. By the continuity of $u(t)$, there is $n_k \in \mathbf{N}$ such that for $s,t \in [0,N]$ with $|s-t| < 1/2^{n_k}$, $\|u(s)-u(t)\|_2 < 1/k$. We set

$$\mathcal{P}_k := \left\{ s \in [0,N] \quad \middle| \quad 2^{n_k} s \in \mathbf{N} \right\}.$$

Since $\mathcal{P}_k$ is a finite set,

$$A_k := \left\{ n \in \mathbf{N} \middle| \begin{array}{l} s_0, t_0 \in \mathcal{P}_k \\ \|u_n(s_0) - u_n(t_0)\|_2 < \|u(s_0) - u(t_0)\|_2 + 1/k \end{array} \right\} \cap [k,\infty) \cap A_{k-1} \in \omega.$$

Again by the finiteness of $\mathcal{P}_k$, there exists $\delta_k > 0$ such that

$$B_k := \left\{ n \in \mathbf{N} \middle| \begin{array}{l} \text{for any } s \text{ with } |s| < \delta_k \text{ and } t_0 \in \mathcal{P}_k \\ \|\alpha_s(u_n(t_0)) - u_n(t_0)\|_2 < 1/k \end{array} \right\} \cap [k,\infty) \cap B_{k-1} \cap A_k \in \omega.$$

Here we introduce a modified path $\tilde{u}_n(t)$ from $1 = u_n(0)$ to $u_n(N)$ satisfying

$$\tilde{u}_n(t) := \begin{cases} u_n(t), & \text{for} \quad t \in \mathcal{P}_k, \\ e^{ih(t-t_0)2^{n_k}} u_n(t_0), & t \in [0,N] \setminus \mathcal{P}_k, \end{cases}$$

where $t_0 \leq t \leq t_1$, $t_0, t_1 \in \mathcal{P}_k$, $t_1 - t_0 = 1/2^{n_k}$ and $h \in R$ is a self-adjoint element with $\|h\|_\infty \leq \pi$ and $u_n(t_1)u_n(t_0)^{-1} = e^{ih}$.

We remark that for $s,t \in [0,N]$ and $n \in A_k$,

$$
\begin{aligned}
&\|\tilde{u}_n(s) - \tilde{u}_n(t)\|_2 \\
&\quad \le \|\tilde{u}_n(s) - \tilde{u}_n(s_0)\|_2 + \|\tilde{u}_n(s_0) - \tilde{u}_n(t_0)\|_2 + \|\tilde{u}_n(t_0) - \tilde{u}_n(t)\|_2 \\
&\quad \le \left\|u_n\left(s_0 + \frac{1}{2^{n_k}}\right) - u_n(s_0)\right\|_2 + \left(\|u(s_0) - u(t_0)\|_2 + \frac{1}{k}\right) \\
&\qquad + \left\|u_n\left(t_0 + \frac{1}{2^{n_k}}\right) - u_n(t_0)\right\|_2 \\
&\quad \le \left\|u\left(s_0 + \frac{1}{2^{n_k}}\right) - u(s_0)\right\|_2 + \frac{1}{k} \\
&\qquad + \left(\|u(s) - u(t)\|_2 + \|u(s) - u(s_0)\|_2 + \|u(t) - u(t_0)\|_2 + \frac{1}{k}\right) \\
&\qquad + \left\|u\left(t_0 + \frac{1}{2^{n_k}}\right) - u(t_0)\right\|_2 + \frac{1}{k} \\
&\quad \le \frac{2}{k} + \left(\|u(s) - u(t)\|_2 + \frac{3}{k}\right) + \frac{2}{k} = \|u(s) - u(t)\|_2 + \frac{7}{k}
\end{aligned}
$$

where $s_0 \le s, t_0 \le t$ and $s_0, t_0 \in \mathcal{P}_k$ with $|s - s_0| < 1/2^{n_k}, |t - t_0| < 1/2^{n_k}$.

For $t \in [0, N]$, $n \in B_k$ and $|s| < \delta_k$,

$$
\begin{aligned}
&\|\alpha_s(\tilde{u}_n(t)) - \tilde{u}_n(t)\|_2 \\
&\le \|\alpha_s(\tilde{u}_n(t) - \tilde{u}_n(t_0))\|_2 + \|\alpha_s(\tilde{u}_n(t_0)) - \tilde{u}_n(t_0)\|_2 + \|\tilde{u}_n(t_0) - \tilde{u}_n(t)\|_2 \\
&= 2\|\tilde{u}_n(t) - \tilde{u}_n(t_0)\|_2 + \|\alpha_s(\tilde{u}_n(t_0)) - \tilde{u}_n(t_0)\|_2 \\
&< 2 \times 2\frac{1}{k} + \frac{1}{k} = \frac{5}{k},
\end{aligned}
$$

where $t_0 \in \mathcal{P}_k$, $t_0 \le t$, $t - t_0 < 1/2^{n_k}$. Define

$$
\mathcal{Q}_k := \left\{ t \in [0, N] \;\middle|\; t/\delta'_k \in \mathbf{N} \right\}
$$

for $\delta'_k := \min\{1/2^{n_k}, \delta_k\}$. Then from the finiteness of $\mathcal{Q}_k$ and the α-cocycle condition of $u(t)$, we have

$$
\begin{aligned}
&C_k \\
&:= \left\{ n \in \mathbf{N} \;\middle|\; \begin{array}{l} \|u_n(s_0)\alpha_{s_0}(u_n(t_0)) - u_n(s_0 + t_0)\|_2 < 1/k \\ s_0,\ t_0 \in \mathcal{Q}_k \end{array} \right\} \cap [k, \infty) \cap C_{k-1} \cap B_k \in \omega.
\end{aligned}
$$

For $s, t \in [0, N]$ and $n \in C_k$, we choose $s_0, t_0 \in \mathcal{Q}_k$ such that $|s - s_0| < \delta'_k$ and $|t - t_0| < \delta'_k$. We obtain

$$
\begin{aligned}
&\|\tilde{u}_n(s)\alpha_s(\tilde{u}_n(t)) - \tilde{u}_n(s+t)\|_2 \\
&\le \|\tilde{u}_n(s)\alpha_s(\tilde{u}_n(t) - \tilde{u}_n(t_0))\|_2 + \|\tilde{u}_n(s)\{\alpha_s(\tilde{u}_n(t_0)) - \alpha_{s_0}(\tilde{u}_n(t_0))\}\|_2 \\
&\quad + \|\{\tilde{u}_n(s) - \tilde{u}_n(s_0)\}\alpha_{s_0}(\tilde{u}_n(t_0))\|_2 + \|\tilde{u}_n(s_0)\alpha_{s_0}(\tilde{u}_n(t_0)) - \tilde{u}_n(s_0+t_0)\|_2 \\
&\quad + \|\tilde{u}_n(s_0+t_0) - \tilde{u}_n(s+t)\|_2 \\
&\le \|\tilde{u}_n(t) - \tilde{u}_n(t_0)\|_2 + \|\alpha_{s-s_0}(\tilde{u}_n(t_0)) - \tilde{u}_n(t_0)\|_2 + \|\tilde{u}_n(s) - \tilde{u}_n(s_0)\|_2 \\
&\quad + \|\tilde{u}_n(s_0)\alpha_{s_0}(\tilde{u}_n(t_0)) - \tilde{u}_n(s_0+t_0)\|_2 + \|\tilde{u}_n(s_0+t_0) - \tilde{u}_n(s+t)\|_2 \\
&\le \frac{2}{k} + \frac{5}{k} + \frac{2}{k} + \frac{1}{k} + \frac{4}{k} = \frac{14}{k}.
\end{aligned}
$$

From the above properties of $\tilde{u}_n(t)$, we may regard $\{\tilde{u}_n(t)\}_n$ as $u(t)$ on $t \in [0, N]$.

To construct a coboundary, we introduce a path $\chi_n(t)$ in the following way. Let

$$u_n(N) = \int_{[-\pi,\pi[} e^{\imath\lambda} dE_\lambda^{(n)}$$

be the spectral decomposition of $u_n(N)$ and define a unitary by

$$\chi_n(t) := \int_{[-\pi,\pi[} e^{\imath t\lambda/N} dE_\lambda^{(n)} \qquad \text{for} \quad 0 \le t \le N.$$

Put $\chi(t) := \{\chi_n(t)\}_n$, then we have by the centrality of $u(N)$

$$\chi(t) = \{\chi_n(t)\}_n = \{f_t(u_n(N))\}_n = f_t(u(N)) \in R_{\omega,\mathrm{eq}}$$

where $f : \mathbf{T} \to \mathbf{T}$ is defined by

$$f_t(e^{\imath\lambda}) = e^{\imath\lambda t/N} \qquad \text{with} \quad -\pi \le \lambda < \pi.$$

We also have

$$
\begin{aligned}
&\|\chi_n(s) - \chi_n(t)\|_2 \\
&= \left\| \int_{[-\pi,\pi[} (e^{\imath(s-t)\lambda/N} - 1)e^{\imath t\lambda/N} dE_\lambda^{(n)} \right\|_2 \le \int_{[-\pi,\pi[} |e^{\imath(s-t)\lambda/N} - 1| d(\mathrm{tr}(E_\lambda^{(n)})) \\
&\le \int_{[-\pi,\pi[} \frac{1}{N}|s-t||\lambda| d(\mathrm{tr}(E_\lambda^{(n)})) = \frac{\pi}{N}|s-t| \le \frac{|s-t|}{k}.
\end{aligned}
$$

We define a unitary w in $R \otimes L^\infty(\mathbf{T})$ which will give the desired coboundary for $u(t)$ with some deformation afterward,

$$w(s) := u_n(Ns)\alpha_{Ns-N}(\chi_n(Ns)^*) \qquad \text{for } s \in [0,1] \simeq \mathbf{T}.$$

Define a one-parameter automorphism group γ on $L^\infty(\mathbf{T})$ by $(\gamma_t f)(s) = f(s-t)$. By the assumption of (1), there exists a unitary $v \in R_{\omega,\mathrm{eq}}$ such that $\alpha_t(v) = e^{-2\pi\imath t/N}v$. For the map Φ defined in Definition 3.2, note that

$$\begin{aligned}\Phi \cdot (\alpha_t \otimes \gamma_{t/N})(a \otimes f) &= \alpha_t(a)\int(\gamma_{t/N}f)(\lambda)dE_\lambda = \alpha_t(a)f(e^{-2\pi\imath t/N}v)\\ &= \alpha_t(a)f(\alpha_t(v)) = \alpha_t(a)\alpha_t(f(v)) = \alpha_t(af(v)) = \alpha_t \cdot \Phi(a \otimes f).\end{aligned}$$

We obtain

$$\|w\Big(\alpha_t \otimes \gamma_t\Big)(w^*) - u_n(t) \otimes 1\|_2 \le \frac{t+28}{k}$$

by the following arguments. Suppose that $s \in [0,1]$ and $0 \le t \le N$. When $t \le Ns$, since

$$w\Big(\alpha_t \otimes \gamma_{t/N}\Big)(w^*)(s) = u_n(Ns)\alpha_{N(s-1)}\Big(\chi_n(Ns)^*\chi_n(Ns-t)\Big)\alpha_t\Big(u_n(Ns-t)\Big)^*,$$

$$\|\chi(Ns)^*\chi(Ns-t) - 1\|_2 < \frac{t}{k} \quad \text{and} \quad \|u_n(Ns) - u_n(t)\alpha_t(u_n(Ns-t))\|_2 < \frac{14}{k},$$

we have

$$\begin{aligned}&\|w\Big(\alpha_t \otimes \gamma_{t/N}\Big)(w^*)(s) - u_n(t)\|_2\\ &= \|u_n(Ns)\alpha_{N(s-1)}\Big(\chi_n(Ns)^*\chi_n(Ns-t)\Big) - u_n(t)\alpha_t\Big(u_n(Ns-t)\Big)\|_2\\ &\le \|\chi(Ns)^*\chi(Ns-t) - 1\|_2 + \|u_n(Ns) - u_n(t)\alpha_t(U_n(Ns-t))\|_2 \le \frac{t+14}{k}.\end{aligned}$$

When $Ns < t$, we identify $s - t/N$ with $s - t/N + 1$ modulo $\mathbf{Z}$, then

$$\begin{aligned}&w\Big(\alpha_t \otimes \gamma_{t/N}\Big)(w^*)(s)\\ &= u_n(Ns)\alpha_{N(s-1)}\Big(\chi_n(Ns)^*\Big)\alpha_{Ns}(\chi_n(Ns-t+N))\alpha_t\Big(u_n(Ns-t+N)^*\Big).\end{aligned}$$

Thus we obtain

$$
\begin{aligned}
&\|w\Big(\alpha_t \otimes \gamma_{t/N}\Big)(w^*)(s) - u_n(t)\|_2 \\
&\leq \|u_n(Ns)\alpha_{N(s-1)}(\chi_n(Ns)^*) \\
&\qquad - u_n(t)\alpha_t(u_n(N+Ns-t))\alpha_{Ns}(\chi_n(N+Ns-t)^*)\|_2 \\
&\leq \|u_n(Ns)\alpha_{N(s-1)}(\chi_n(Ns)^*) - u_n(Ns)\|_2 \\
&\qquad + \|u_n(Ns) - u_n(t)\alpha_t(u_n(N+Ns-t))\alpha_{Ns}(\chi_n(N+Ns-t)^*)\|_2 \\
&\leq \|\chi_n(Ns) - 1\|_2 + \|u_n(Ns)\alpha_{Ns}(\chi_n(N+Ns-t)) - u_n(Ns)\alpha_{Ns}(u_n(N))\|_2 \\
&\qquad + \|u_n(Ns)\alpha_{Ns}(u_n(N)) - u_n(t)\alpha_t(u_n(N+Ns-t))\|_2 \\
&\leq \frac{Ns}{k} + \|\chi_n(N+Ns-t) - \chi_n(N)\|_2 \\
&\qquad + \|u_n(Ns)\alpha_{Ns}(u_n(N)) - u_n(N+Ns)\|_2 \\
&\qquad + \|u_n(N+Ns) - u_n(t)\alpha_t(u_n(N+Ns-t))\|_2 \\
&\leq \frac{Ns}{k} + \frac{t-Ns}{k} + \frac{14}{k} + \frac{14}{k} = \frac{t+28}{k}.
\end{aligned}
$$

By the Kaplansky density theorem, for any $\varepsilon = 1/k > 0$, there exists $w_1 = \sum_{\text{finite}} a_n \otimes f_n \in R \odot L^\infty(\mathbb{T})$ such that $\|w_1\|_\infty \leq \|w\|_\infty = 1$ and $\|w_1 - w\|_2 < 1/k$. Then $\Phi(w_1) = \{W_n\}$ is almost a unitary because

$$
\begin{aligned}
&\|\Phi(w_1)\Phi(w_1^*) - 1\|_2 = \|w_1 w_1^* - 1\|_2 \\
&\leq \|(w_1 - w)w_1^*\|_2 + \|w(w_1^* - w)\|_2 \leq 2\|w_1 - w\|_2 < 2/k,
\end{aligned}
$$

where we use the property of Φ as we have seen in Lemma 3.2. We also have $\|\Phi(w_1^*)\Phi(w_1) - 1\|_2 < 2/k$ in the same way. Thus

$$
\begin{aligned}
&D_k := \\
&\left\{ n \in \mathbb{N} \middle| \|W_n^* W_n - 1\|_2 < \frac{3}{k},\ \|W_n W_n^* - 1\|_2 < \frac{3}{k} \right\} \cap [k, \infty) \cap D_{k-1} \cap C_k \in \omega.
\end{aligned}
$$

We have

$$
\begin{aligned}
&\|\alpha_s(\Phi(w_1)) - \Phi(w_1)\|_2 \\
&\leq \sum_{\text{finite}} \|\alpha_s(a_n) f_n(\alpha_s(v)) - a_n f_n(v)\|_2 \\
&\leq \sum_{\text{finite}} \Big(\|\alpha_s(a_n)\|_\infty \|f_n(a_s(v)) - f_n(v)\|_2 + \|\alpha_s(a_n) - \alpha_n\|_2 \|f_n(v)\|_\infty \Big) \\
&\leq \sum_{\text{finite}} \Big(\|a_n\|_\infty \|f_n\|_\infty \|\alpha_s(v) - v\|_2 + \|\alpha_s(a_n) - a_n\|_2 \|f_n\|_\infty \Big).
\end{aligned}
$$

Since $v \in R_{\omega,\mathrm{eq}}$ is α-equicontinuous, the above computations show that $\Phi(w_1)$ is α-equicontinuous, which means for any $\varepsilon = 1/k$, there exists $\delta > 0$ such that

$$E_k := \left\{ n \in \mathbf{N} \,\middle|\, \begin{array}{l} \|\alpha_s(W_n) - W_n\|_2 < 1/k \\ |s| < \delta \end{array} \right\} \cap [k, \infty) \cap E_{k-1} \cap D_k \in \omega.$$

We embed $u_n(t)$ canonically into R^ω, (i.e., $u_n(t) \hookrightarrow \{u_n(t)\} \in R^\omega$), and have

$$\begin{aligned}
&\|\Phi(w_1)\alpha_t(\Phi(w_1)^*) - u_n(t)\|_2 \\
&\qquad = \|\Phi(w_1)\Phi(\alpha_t \otimes \gamma_{t/N})(w_1^*) - \Phi(u_n(t) \otimes 1)\|_2 \\
&\qquad = \|w_1(\alpha_t \otimes \gamma_{t/N}(w_1^*)) - u_n(t) \otimes 1\|_2 \\
&\qquad \le 2\|w_1 - w\|_2 + \|w(\alpha_t \otimes \gamma_{t/N})(w^*) - u_n(t) \otimes 1\|_2 \\
&\qquad \le \frac{2}{k} + \frac{t+28}{k} = \frac{t+30}{k}.
\end{aligned}$$

For the above δ, there exists m_k such that $n_k \le m_k$, $1/2^{m_k} \le \delta$, and for

$$\mathcal{R}_k := \left\{ t \in [0, N] \;\middle|\; 2^{m_k} t \in \mathbf{N} \right\},$$

$$F_k := \left\{ m \in \mathbf{N} \,\middle|\, \begin{array}{l} \|W_m \alpha_{t_0}(W_m^*) - u_n(t_0)\|_2 \le \dfrac{t+31}{k} \\ t_0 \in \mathcal{R}_k \end{array} \right\} \cap [k, \infty) \cap F_{k-1} \cap E_k \in \omega.$$

We may choose W_m's such that $\|W_m\|_\infty \le \|\Phi(w_1)\|_\infty \le \|w_1\|_\infty \le 1$. Let $W_m = V_m|W_m|$ be the polar decomposition of W_m. Adding an appropriate partial isometry V_m' to V_m, we can make $V_m + V_m'$ a unitary and denote it by V_m again. Then when $m \in F_k \setminus F_{k+1}$, we see that $V_m\alpha_t(V_m^*)$ is an approximate coboundary for $u_n(t)$ by the following observations. For $t \in [0, N]$, we will show that $\|V_m\alpha_t(V_m^*) - u_n(t)\|_2$ is small. Since we have

$$\begin{aligned}
&\|V_m\alpha_t(V_m^*) - u_n(t)\|_2 \\
&\le \|V_m\alpha_t(V_m^*) - W_m\alpha_t(W_m^*)\|_2 + \|W_m\alpha_t(W_m^*) - u_n(t)\|_2,
\end{aligned}$$

we compute the two terms separately. We have

$$\begin{aligned}
(\text{the first term}) &= \|1 - V_m^* W_m \alpha_t((V_m^* W_m)^*)\|_2 = \|1 - |W_m|\alpha_t(|W_m|)\|_2 \\
&= \|1 - (|W_m| - 1)\alpha_t(|W_m|) - \alpha_t(|W_m| - 1) - \alpha_t(1)\|_2 \\
&\le (\|W_m\|_\infty + 1)\||W_m| - 1\|_2 \le 2\||W_m|^2 - 1\|_2 \le 2 \times 3/k,
\end{aligned}$$

$$\begin{aligned}
&(\text{the second term}) \\
&\qquad \le \|W_m\Big(\alpha_t(W_m^*) - \alpha_{t_0}(W_m^*)\Big)\|_2 + \|W_m\alpha_{t_0}(W_m^*) - u_n(t_0)\|_2 \\
&\qquad\qquad + \|u_n(t_0) - u_n(t)\|_2 \\
&\qquad \le \|W_m\|_\infty \|\alpha_{t-t_0}(W_m^*) - W_m^*\|_2 + \frac{31+t}{k} + \frac{2}{k} \le \frac{34+t}{k},
\end{aligned}$$

where t_0 is the nearest point of t in $\mathcal{R}_k$. Thus we have $\|V_m\alpha_t(V_m^*) - u_n(t)\|_2 \leq (40+t)/k$.

For $n \in F_k \setminus F_{k-1}$, there exists a unitary $\mathcal{W}_n^{(k)} \in \mathcal{U}(R)$ such that for $t \in [0,N]$ $\|\mathcal{W}_n^{(k)}\alpha_t(\mathcal{W}_n^{(k)*}) - u_n(t)\|_2 \leq (40+t)/k$. Define $W := \{\mathcal{W}_n^{(k)}\}_n$, then it satisfies $W\alpha_t(W^*) = u(t)$ in R^ω for every $t \in \mathbf{R}$. Note that this holds even if t is negative, because for $t \geq 0$ we have $u(t)\alpha_t(u(-t)) = 1$, therefore $u(-t) = \alpha_{-t}\Big(u(t)^*\Big) = \alpha_{-t}\Big(\alpha_t(W)W^*\Big) = W\alpha_{-t}(W^*)$.

Now it remains for us to show that W is a central sequence and α-equicontinuous, i.e., $W \in R_{\omega,\mathrm{eq}}$.

Choose a sequence $\{x_l\}$ which is $\|\cdot\|_2$-dense in the unit ball of R. Since $u(t)$ is a central sequence

$$G_k := \left\{ n \in \mathbf{N} \,\middle|\, \begin{array}{l} \text{for } s_0 \in \mathcal{P}_k \quad \text{and} \quad l = 1,\cdots,k \\ \|u_n(s_0)x_l - x_l u_n(s_0)\|_2 < 1/k \end{array} \right\} \cap [k,\infty) \cap G_{k-1} \cap F_k \in \omega.$$

Then for $n \in G_k$, $l = 1,\cdots,k$ and $s \in [0,N]$

$$\begin{aligned} \Big\|\Big[u_n(s), x_l\Big]\Big\|_2 &\leq 2\|x_l\|_\infty \|u_n(s) - u_n(s_0)\|_2 + \Big\|\Big[u_n(s_0), z_l\Big]\Big\|_2 \\ &\leq \frac{4}{k} + \frac{1}{k} = \frac{5}{k}. \end{aligned}$$

Because we have taken the path $\chi(t)$ for $0 \leq t \leq N$ in $R_{\omega,\mathrm{eq}}$, by the same arguments as for $u_n(t)$ (or $\tilde{u}_n(t)$), for each k there exists an element H_k of ω and $0 < \rho_k \leq 1$ (we may assume $H_k \subset [k,\infty) \cap H_{k-1} \cap G_k$) such that when $t \in [0,N], n \in H_k$ and $|s| < \rho_k$,

$$\|\alpha_s(\chi_n(t)) - \chi_n(t)\|_2 < 5/k.$$

Let

$$\mathcal{S}_k := \Big\{ t \in [0,N] \quad \Big| \quad t/\rho_k \in \mathbf{N} \Big\},$$

then

$$I_k := \left\{ n \in \mathbf{N} \,\middle|\, \begin{array}{l} \text{for } s_0 \in \mathcal{S}_k, \quad \text{and} \quad l = 1,\cdots,k \\ \Big\|\Big[\alpha_{s_0 - N}\left(\chi_n(s_0)^*\right), x_l\Big]\Big\|_2 < 1/k \end{array} \right\} \cap [k,\infty) \cap I_{k-1} \cap H_k \in \omega$$

by the ω-centrality of $\chi(t)$. For $n \in I_k, l = 1, \cdots, k$ and $s \in [0, N]$ we have

$$\begin{aligned}
&\left\| \left[\alpha_{s-N}(\chi_n(s)^*), x_l\right] \right\|_2 \\
&\leq \left\| \left(\alpha_{s-N}(\chi_n(s)^*) - \alpha_{s_0-N}(\chi_n(s_0)^*)\right) x_l \right\|_2 + \left\| \left[\alpha_{s_0-N}(\chi_n(s_0)^*), x_l\right] \right\|_2 \\
&\quad + \left\| x_l \left(\alpha_{s_0-N}(\chi_n(s_0)^*) - \alpha_{s-N}(\chi_n(s)^*)\right) \right\|_2 \\
&\leq 2\|x_l\|_\infty \|\alpha_{s-N}(\chi_n(s)^*) - \alpha_{s_0-N}(\chi_n(s_0)^*)\|_2 + \frac{1}{k} \\
&\leq 2\Big(\|\alpha_{s-N}(\chi_n(s)^*) - \alpha_{s-N}(\chi_n(s_0)^*)\|_2 \\
&\quad + \|\alpha_{s-N}(\chi_n(s_0)^*) - \alpha_{s_0-N}(\chi_n(s_0)^*)\|_2 \Big) + \frac{1}{k} \\
&= 2\|\chi_n(s) - \chi_n(s_0)\|_2 + 2\|\alpha_{s-s_0}(\chi_n(s_0)^*) - \chi_n(s_0)^*\|_2 + \frac{1}{k} \\
&\leq 2 \times \frac{|s - s_0|}{k} + 2 \times \frac{5}{k} + \frac{1}{k} = \frac{13}{k},
\end{aligned}$$

where $s_0 \in \mathcal{S}_k$ is the nearest points of those s. Combining the above arguments, we have for $n \in I_k, l = 1, \cdots, k$ and $s \in [0, N]$

$$\left\| \left[u_n(s)\alpha_{s-N}(\chi_n(s)^*),\ x_l\right] \right\|_2 \leq \frac{5}{k} + \frac{13}{k} = \frac{18}{k},$$

which means W is a central sequence.

Moreover, since

$$\begin{aligned}
\|\alpha_t(\mathcal{W}_n^{(k)}) - \mathcal{W}_n^{(k)}\|_2 &= \|\alpha_t(\mathcal{W}_n^{(k)}) - u_n(t)^*\mathcal{W}_n^{(k)}\|_2 + \|u_n(t)^*\mathcal{W}_n^{(k)} - \mathcal{W}_n^{(k)}\|_2 \\
&= \|\mathcal{W}_n^{(k)}\alpha_t(\mathcal{W}_n^{(k)*}) - u_n(t)\|_2 + \|u_n(t) - 1\|_2 \\
&\leq \frac{40 + t}{k} + \frac{2}{k} = \frac{42 + t}{k}
\end{aligned}$$

for $n \in F_k$ and $t \in [0, 1/2^{n_k}]$, it is obvious that W is α-equicontinuous. Now we have proved that $W \in R_{\omega,\mathrm{eq}}$. □

In the above proof, we take a finite number of points (cf. $\mathcal{P}_k, \mathcal{Q}_k, \mathcal{R}_k$) from the interval $[0, N]$ and consider approximate arguments such as continuity, the cocycle condition, the coboundary condition, the central sequence condition, on those points. Concerning the rest of the points in $[0, N]$, by continuity we can regard them as the nearest point lives in the finite number of points. After all, a problem of continuous parameter is reduced to a finite number of single automorphisms, and we can find some similarity between Connes' and our constructions of coboundaries in the following way.

Under the same notations as in Theorem 2.1, Connes constructs a coboundary $w\alpha(w^*)$ of a given unitary u with

$$w = \sum_{m=1}^{J} U_m e_m,$$

where $e_1, \cdots, e_J$ is the projections obtained from the non-commutative Rohlin lemma and $U_m := u\alpha(u)\alpha^2(u)\cdots\alpha^{m-1}(u)$ is an α-cocycle.

Recall, in our proof, that we have defined our coboundary as

$$\Phi(w_1(s)) \approx \Phi(w(s)) = \Phi\Big(u_n(Ns)\alpha_{Ns-N}(\chi_n(Ns)^*)\Big).$$

Looking at the both coboundaries, we notice that the α-cocycle $u_n(s)$ corresponds to the α-cocycle U_m, and since Φ is a functional calculus of v, we may consider it as a function calculus of the spectral projection E_λ's, therefore, Φ corresponds to $\sum_{m=1}^{J} \cdot e_m$. As for $\alpha_{Ns-N}(\chi_n(Ns)^*)$, it makes w well-defined, i.e., $(w(0) = w(1))$. Though it causes an error t/k to the estimate of $\|w\,(\alpha_t \otimes \gamma_t)\,(w^*)(s) - u_n(t)\|_2 \leq (t+28)/k$, it does not affect by a great deal of amount if we consider the equality $W\alpha_t(W^*) = u(t)$ in the ultraproduct algebra.

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