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Propagation of the Irregularity of a Microdifferential System

By

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Abstract

We construct the functor of microlocal analytic irregularity $I\mu\,\hom(\cdot,\mathcal{O}_X)$ which gives a natural third term of a distinguished triangle associated to the transformation $t\mu\,\hom(\cdot,\mathcal{O}_X)\to\mu\,\hom(\cdot,\mathcal{O}_X)$ of functors on the derived category of $\mathbb{R}$-constructible sheaves. When restricting to $\mathbb{C}$-constructible objects we prove that the microlocal irregularity of a microdifferential system propagates along non 1-microcharacteristic directions, as a consequence of the propagation for $t\mu\,\hom(\cdot,\mathcal{O}_X)$ and $\mu\,\hom(\cdot,\mathcal{O}_X)$.

Introduction

In this paper we treat a problem posed by P. Schapira : to show that the 1-microcharacteristic variety of a microdifferential system $\mathcal{M}$ along an involutive submanifold V contains the microsupport of its solutions in the sheaves $t\mu\,\hom(F,\mathcal{O}_X)$, $\mu\,\hom(F,\mathcal{O}_X)$, whenever the microsupport of the $\mathbb{R}$-constructible complex F is contained in V. Here $\mathcal{O}_X$ denotes the sheaf of holomorphic functions on the complex manifold X. In other words, the microlocal F-irregularity of $\mathcal{M}$ propagates along the non 1-microcharacteristic directions. Let us recall that $\mu\,\hom(\cdot,\mathcal{O}_X)$ and $t\mu\,\hom(\cdot,\mathcal{O}_X)$ are the microlocalized of the functors $\mathbb{R}\mathcal{H}om(\cdot,\mathcal{O}_X)$ and $t\mathcal{H}om(\cdot,\mathcal{O}_X)$, and were respectively introduced by Kashiwara-Schapira (cf. [K-S3]) and Andronikof (cf. [A]).

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At the present status of microlocal analysis, it is not clear if the sheaf $\mathcal{E}_X$ of microdifferential operators acts on $\mu\hom(F,\mathcal{O}_X)$ and $t\mu\hom(F,\mathcal{O}_X)$ but only on its cohomology; however, the work developped by Kashiwara and Schapira ([K-S5]) seems to indicate a positive answer. In this paper, we will treat the $\mathbb{C}$-constructible case, and F is supposed to be perverse.

When $F=\mathbb{C}_Y$ for a complex d-codimensional submanifold, one has

$$C^{\infty}_{Y|X}=\gamma^{-1}\gamma_*\mu\hom(\mathbb{C}_Y,\mathcal{O}_X)[d]=\gamma^{-1}\gamma_* C^{\mathbb{R}}_{Y|X},$$

the holomorphic microfunctions along Y, where γ is the projection of the cotangent bundle minus the zero section on the associated projectif bundle.

Similarly,

$$C_{Y|X}=\gamma^{-1}\gamma_* t\mu\hom(\mathbb{C}_Y,\mathcal{O}_X)[d]=\gamma^{-1}\gamma_* C^{\mathbb{R},f}_{Y|X}$$

is the sheaf of microfunctions of finite order. The propagation in $C^{\infty}_{Y|X}$ was proved by Kashiwara-Schapira in [K-S2], and the propagation in $C_{Y|X}$ was studied by the author in [MF2], Schapira in [S] and Laurent in [L]. When we want to prove the propagation theorem for $t\mu\hom(F,\mathcal{O}_X)$, some of the essential tools developped in the preceding works are no longer available.

However, in the $\mathbb{C}$-constructible case we can use the theory of regular holonomic $\mathcal{D}$-modules, and in particular Kashiwara's theorem which asserts that $t\mathcal{H}om(F,\mathcal{O}_X)$ has regular holonomic cohomology; moreover $\mu\hom(F,\mathcal{O}_X)$ and $t\mu\hom(F,\mathcal{O}_X)$ are obtained from $t\mathcal{H}om(F,\mathcal{O}_X)$ tensorizing respectively by $\mathcal{E}^{\mathbb{R}}_X$, the sheaf of microlocal holomorphic operators, and by $\mathcal{E}^{\mathbb{R},f}_X$, the subsheaf of tempered microlocal operators.

Using the identification of X with the diagonal of $X\times X$, we can reduce the problem to the propagation of the solutions of $\mathcal{M}$ in $C^{\mathbb{R}}_{Y|X}$ and $C^{\mathbb{R},f}_{Y|X}$ respectively, Y an arbitrary complex submanifold, that is, $V=\dot{T}^*_YX$, the conormal bundle to Y minus the zero section.

The second essential tools are Bony's results concerning the propagation for solutions in the sheaf of tempered microfunctions C^f for operators satisfying a Levi condition ([B]) together with a precised Cauchy-Kowalewski theorem of Kashiwara and Schapira for $C^{\mathbb{R}}_{Y|X}$ (cf. [K-S2]).

The paper is organized as follows : in the first section we construct the complex of sheaves of microlocal F-irregularity as the microlocalized of $I\mathcal{H}om(F,\mathcal{O}_X)$ introduced in a previous work ([MF3]). Therefore, $I\mu\hom(F,\mathcal{O}_X)$ provides a natural third term to a distinguished triangle associated to the morphism

$$t\mu\hom(F,\mathcal{O}_X)\to\mu\hom(F,\mathcal{O}_X)$$

and represents the notion of F-irregularity : for example, when $F = \mathbb{C}_Y$ for a submanifold Y of codimension d, $I\mu\hom(\mathbb{C}_Y[-d], \mathcal{O}_X)$ is nothing but the quotient $\frac{C^{\mathbb{R}}_{Y|X}}{C^{\mathbb{R},f}_{Y|X}}$.

The second section is devoted to the propagation theorem in the $\mathbb{C}$-constructible perverse framework, and its successive reductions, for $t\mu\hom\,(F, \mathcal{O}_X)$, $\mu\hom(F, \mathcal{O}_X)$ and hence for $I\mu\hom(F, \mathcal{O}_X)$.

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§1. Construction of $I\mu\hom(F, \mathcal{O}_X)$

We shall recall some properties of the objects $t\mu\hom(F, \mathcal{O}_X)$ and $\mu\hom\,(F, \mathcal{O}_X)$, where X is an n-dimensional complex analytic manifold, $\mathcal{O}_X$ is the sheaf of holomorphic functions and F is an object of $D^b_{\mathbb{R}-c}(X)$, that is a complex of sheaves of $\mathbb{C}$-vector spaces with bounded and $\mathbb{R}$-constructible cohomology. Recall that $D^b_{\mathbb{C}-c}(X)$ denotes the subcategory of $D^b_{\mathbb{R}-c}(X)$ whose objects are the $\mathbb{C}$-constructible complexes. The functor $\mu\hom$ was introduced in the 80^{ties} by Kashiwara and Schapira ([K-S3]), and the tempered version $t\mu\hom$ was introduced by Andronikof ([A]). We also recall some facts about the functor $t\mathcal{H}om$ due to Kashiwara ([K]) which can be recovered by the restriction of $t\mu\hom$ to the base X of the cotangent bundle $T^*X \underset{\pi}{\to} X$.

Let $\mathcal{D}_X$ (resp. $\mathcal{D}^\infty_X$) be the sheaf on X of holomorphic differential operators of finite order (resp. infinite order), $\mathcal{E}_X$ (resp. $\mathcal{E}^\infty_X$) the sheaf of microdifferential operators of finite (resp. infinite) order, $\mathcal{E}^{\mathbb{R},f}_X$ (resp. $\mathcal{E}^{\mathbb{R}}_X$) the sheaf of tempered microlocal operators (resp. microlocal operators).

$\mathcal{E}_X$, $\mathcal{E}^\infty_X$, as well as $\mathcal{E}^{\mathbb{R},f}_X$ and $\mathcal{E}^{\mathbb{R}}_X$ are sheaves on T^*X, satisfying

$$\mathcal{E}^{\mathbb{R},f}_X\mid_X = \mathcal{E}_X\mid_X = \mathcal{D}_X$$
$$\mathcal{E}^{\mathbb{R}}_X\mid_X = \mathcal{E}^\infty_X\mid_X = \mathcal{D}^\infty_X$$

and they are all particular cases of $\mu\hom$ and $t\mu\hom$. One denotes $\mathcal{E}_X(m)$ (resp. $\mathcal{D}_X(m)$) the sheaf of operators of order at most m.

For instance, when $F = \mathbb{C}_Y$, for a smooth complex submanifold of X of codimension d, $\mu\hom(\mathbb{C}_Y, \mathcal{O}_X)$ is the sheaf $C^{\mathbb{R}}_{Y|X}[-d]$ of [S-K-K] and $t\mu\hom(\mathbb{C}_Y, \mathcal{O}_X)$ is the sheaf $C^{\mathbb{R},f}_{Y|X}[-d]$ defined in [A].

When $F = (\mathbb{C}_M)'$, the dual of $\mathbb{C}_M$, for a real analytic submanifold M of X such that X is a complexified of M, $\mu\hom((\mathbb{C}_M)', \mathcal{O}_X)[n]$ is isomorphic to the sheaf of microfunctions on M and $t\mu\hom((\mathbb{C}_M)', \mathcal{O}_X)[n]$ is isomorphic to the tempered microfunctions (cf. [A] and [B]).

When restricting to X one obtains

$$\mu\hom(F,\mathcal{O}_X)\,|_X \simeq \mathbb{R}\mathcal{H}om(F,\mathcal{O}_X),$$

a complex of $\mathcal{D}_X^\infty$-modules, and

$$t\mu\hom(F,\mathcal{O}_X)\,|_X \simeq t\mathcal{H}om(F,\mathcal{O}_X),$$

a complex of $\mathcal{D}_X$-modules.

One also defines $I\mathcal{H}om(F,\mathcal{O}_X)$, the sheaf of analytic local F-irregularity which appears to be a natural third term to a distinguished triangle

$$t\mathcal{H}om(F,\mathcal{O}_X) \to \mathbb{R}\mathcal{H}om(F,\mathcal{O}_X) \to I\mathcal{H}om(F,\mathcal{O}_X) \underset{+1}{\longrightarrow}$$

where the left arrow is the usual one. In particular, when $F = (\mathbb{C}_M)'$ one obtains

$$I\mathcal{H}om(F,\mathcal{O}_X) = \frac{B_M}{Db_M},$$

where B_M is the sheaf of hyperfunctions on M and Db_M is the sheaf of distributions on M; and when $F = \mathbb{C}_Y$, Y a d-codimensional submanifold,

$$I\mathcal{H}om(\mathbb{C}_Y[-d],\mathcal{O}_X) = \frac{B^\infty_{Y|X}}{B_{Y|X}},$$

where $B^\infty_{Y|X}$ (resp. $B_{Y|X}$) denotes the sheaf of holomorphic hyperfuntions (resp. holomorphic hyperfunctions with finite order) along Y.

Let us recall that $I\mathcal{H}om(F,\mathcal{O}_X)$ has $\mathcal{D}_X$-module cohomology.

Let now Δ be the diagonal of $X\times X$ and $\tau : T_\Delta(X\times X)\to\Delta$ be the projection of the normal bundle to Δ. Let $\tilde{X}$ be the complex normal deformation of $X\times X$ along Δ (Δ identified with X by the first projection $p_1 : X\times X\to X$), let $p:\tilde{X}\to X\times X$ be the deformation morphism, let $t:\tilde{X}\to\mathbb{C}$ be the natural projection, let $\Omega = t^{-1}(\mathbb{R}^+)$, and let $\Omega \underset{j}{\hookrightarrow} \tilde{X}$ and $T_\Delta(X\times X)\underset{s}{\hookrightarrow}\tilde{X}$ be the natural inclusions. (For more details we refer [A] (Prop. 3.2.1) and [K-S3].) Let p_2 denote the second projection of $X\times X$ on X. We have a distinguished triangle

$$\begin{aligned}(1)\qquad t\mathcal{H}om((p^!p_2^{-1}F)_\Omega,\mathcal{O}_{\tilde{X}}) &\to \mathbb{R}\mathcal{H}om((p^!p_2^{-1}F)_\Omega,\mathcal{O}_{\tilde{X}})\\ &\to I\mathcal{H}om((p^!p_2^{-1}F)_\Omega,\mathcal{O}_{\tilde{X}}) \underset{+1}{\to}.\end{aligned}$$

Following the constructions in ([A], Lemme 2.1.8) we apply the functor

$$s^{-1}\left(\mathcal{D}_{X\times X\leftarrow\tilde{X}} \overset{\mathbb{L}}{\underset{\mathcal{D}_{\tilde{X}}}{\otimes}}\right).$$

to (1) and set

$$I\nu\text{hom}(F,\mathcal{O}_X)) :=$$
$$\mathcal{D}_{X\underset{p_1}{\leftarrow}X\times X}\otimes_{\tau^{-1}\mathcal{D}_{X\times X}} s^{-1}\left(\mathcal{D}_{X\times X\underset{p}{\leftarrow}\tilde{X}}\overset{\mathbb{L}}{\underset{\mathcal{D}_{\tilde{X}}}{\otimes}} I\mathcal{H}\text{om}((p^!p_2^{-1}F)_\Omega,\mathcal{O}_{\tilde{X}})\right)$$
$$\nu\text{hom}(F,\mathcal{O}_X)) :=$$
$$\mathcal{D}_{X\underset{p_1}{\leftarrow}X\times X}\otimes_{\tau^{-1}\mathcal{D}_{X\times X}} s^{-1}\left(\mathcal{D}_{X\times X\underset{p}{\leftarrow}\tilde{X}}\overset{\mathbb{L}}{\underset{\mathcal{D}_{\tilde{X}}}{\otimes}} \mathbb{R}\mathcal{H}\text{om}((p^!p_2^{-1}F)_\Omega,\mathcal{O}_{\tilde{X}})\right)$$
$$t\nu\text{hom}(F,\mathcal{O}_X)) :=$$
$$\mathcal{D}_{X\underset{p_1}{\leftarrow}X\times X}\otimes_{\tau^{-1}\mathcal{D}_{X\times X}} s^{-1}\left(\mathcal{D}_{X\times X\underset{p}{\leftarrow}\tilde{X}}\overset{\mathbb{L}}{\underset{\mathcal{D}_{\tilde{X}}}{\otimes}} t\mathcal{H}\text{om}((p^!p_2^{-1}F)_\Omega,\mathcal{O}_{\tilde{X}})\right).$$

Then

$$t\nu\text{hom}(F,\mathcal{O}_X)\to\nu\text{hom}(F,\mathcal{O}_X)\to I\nu\text{hom}(F,\mathcal{O}_X)\underset{+1}{\to}$$

is a distinguished triangle in $D^b(\tau^{-1}\mathcal{D}_X)$, the derived category whose objects are complexes of $\tau^{-1}\mathcal{D}_X$-modules with bounded cohomology. Finally, denoting by $\wedge$ the Fourier transform from $D^b_{\mathbb{R}^+}(T_X(X\times X))$ to $D^b_{\mathbb{R}^+}(T^*_X(X\times X))$, we define $I\mu\,\text{hom}(F,\mathcal{O}_X)$ by

$$I\mu\,\text{hom}(F,\mathcal{O}_X) \;:= I\nu\text{hom}(F,\mathcal{O}_X)^\wedge.$$

Here $D^b_{\mathbb{R}^+}(E)$, where E is a real vector bundle on X, denotes the derived category of complexes of sheaves on E of $\mathbb{C}$-vector spaces with bounded and $\mathbb{R}^+$-conic cohomology.

One easily deduces the isomorphisms

$$I\mu\,\text{hom}(F,\mathcal{O}_X)\,|_X\simeq\mathbb{R}\pi_*I\mu\,\text{hom}(F,\mathcal{O}_X)\simeq I\mathcal{H}\text{om}(F,\mathcal{O}_X)$$

from the analogous formulae for $t\mu\,\text{hom}$ and $\mu\,\text{hom}$. Moreover,

$$\mathbb{R}\pi!I\mu\,\text{hom}(F,\mathcal{O}_X)=0$$

by (2.3.2) of [A].

Also, as pointed out above, the correspondence

$$F\to I\mu\,\text{hom}(F,\mathcal{O}_X)$$

defines a contravariant functor from $D^b_{\mathbb{R}-c}(X)$ to $D^b(\tau^{-1}\mathcal{D}_X)$.

On the other hand, since $I\mu\,\text{hom}(F,\mathcal{O}_X)$ is the third term of a distinguished triangle where the other two terms are supported by the microsupport of F, $SS(F)$, $I\mu\,\text{hom}(F,\mathcal{O}_X)$ is also supported by $SS(F)$.

Finally, we observe that the cohomology of $I\mu\,\mathrm{hom}(F,\mathcal{O}_X)$ is obviously provided of a canonical structure of $\mathcal{E}_X$-modules, induced by the structure on $t\mu\,\mathrm{hom}(F,\mathcal{O}_X)$ and $\mu\,\mathrm{hom}(F,\mathcal{O}_X)$ (cf. (TR4) of Proposition 1.4.4 of [K-S3]).

Definition 1.1. For any $F \in D^b_{\mathbb{R}-c}(X)$, $I\mu\,\mathrm{hom}(F,\mathcal{O}_X)$ is the complex of microlocal analytic F-irregularity.

§2. Statement of the Main Theorem and Reductions

In this section we start by briefly recalling the essential results on the 1-microcharacteristic variety of a coherent $\mathcal{E}_X$-module $\mathcal{M}$ along a smooth involutive manifold V of $\dot{T}^*X$. Here $\dot{T}^*X$ denotes the complementary of the null section $X \hookrightarrow T^*X$ in T^*X. The 1-microcharacteristic variety $C^1_V(\mathcal{M})$ is a conic involutive closed analytic subset of $T_V(T^*X)$. To build it one needs to introduce the subring $\mathcal{E}_V$ of $\mathcal{E}_X$ generated by the operators of order at most one with a principal symbol vanishing on V; when $P \in \mathcal{E}_V(m) := \mathcal{E}_X(m) \cap \mathcal{E}_V$, modulo $\mathcal{E}_V(m-1)$, the symbol $\sigma^1_V(P)$ is a homogeneous function on $T_V(T^*X)$ and when $\mathcal{S}$ is a coherent ideal of $\mathcal{E}_X$, $C^1_V(\mathcal{E}_X/\mathcal{S})$ is the subset of zeros of $\sigma^1_V(\mathcal{S}\cap\mathcal{E}_V)$. For further details see [MF1], [L] and [S].

Recall that if $\eta \in T_V(T^*X)$, one says that η is non 1-microcharacteristic for $\mathcal{M}$ along V if $\eta \notin C^1_V(\mathcal{M})$. Moreover, the normal cone $C_V(\mathrm{Car}\mathcal{M})$ is contained in $C^1_V(\mathcal{M})$. When $\mathcal{M}$ is a coherent $\mathcal{D}_X$- or $\mathcal{E}_X$-module, $\mathrm{Car}(\mathcal{M})$ will denote its characteristic variety in T^*X. Let $F \in D^b_{\mathbb{R}-c}(X)$, $SS(F)$ its microsupport in T^*X (see [K-S3]); recall that $\mathrm{supp}(F) = SS(F) \cap X$. Moreover, if $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, $\mathrm{Car}(\mathcal{M}) = SS(\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X))$. Let us recall that $C_V(\mathrm{Car}\mathcal{M}) := C_V(\mathcal{M})$ was studied in [K-S1, K-S2]. It is called the microcharacteristic variety of $\mathcal{M}$ along V.

Let us denote by $D^b_{rh}(\mathcal{D}_X)$ the derived category whose objects are the complexes of left $\mathcal{D}_X$-modules with bounded regular holonomic cohomology. Recall that, as proved in [K], when $F \in D^b_{\mathbb{C}-c}(X)$, there exists a unique $\mathcal{N} \in D^b_{rh}(\mathcal{D}_X)$ such that $\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{N},\mathcal{O}_X) \simeq F$ and that correspondence is an equivalence of categories. More precisely, $\mathcal{N} = t\mathcal{H}om(F,\mathcal{O}_X)$ and by ([A], Theorem 4.2.6)

$$\mathcal{E}^{\mathbb{R},f}_X \underset{\pi^{-1}\mathcal{D}_X}{\otimes} \mathcal{N} \simeq t\mu\,\mathrm{hom}(F,\mathcal{O}_X)$$

as well as

$$\mathcal{E}^{\mathbb{R}}_X \underset{\pi^{-1}\mathcal{D}_X}{\otimes} \mathcal{N} \simeq \mu\,\mathrm{hom}(F,\mathcal{O}_X),$$

(cf. [K-S4]).

Let V be a smooth submanifold of a manifold X. One denotes ρ_V the projection

$$V \underset{X}{\times} TX \to T_V X.$$

In most cases studied in this paper, instead of X we consider the cotangent bundle T^*X and V will be an involutive smooth submanifold; we then get

$$\rho_V : V \underset{T^*X}{\times} T^*(T^*X) \to T_V(T^*X).$$

Here we identify $T(T^*X)$ and $T^*(T^*X)$ by $-H$, where H is the Hamiltonian isomorphism. Recall that, if $(x;\xi)$ denotes a system of local canonical coordinates on T^*X in a neighborhood of $p \in T^*X$, if $(x,\xi;\zeta,\eta)$ denotes the associated canonical coordinates on $T^*(T^*X)$, $-H_p(\zeta dx + \eta d\xi) = (\zeta\partial/\partial\xi - \eta\partial/\partial x) \in T_p(T^*X)$

Let Ω_X denote the sheaf of holomorphic differential n-forms on X.

Theorem 2.1. *Let $F \in D^b_{\mathbb{C}-c}(X)$ be perverse and $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module. Let V be smooth involutive in $\dot{T}^*X$ such that $SS(F) \subset V$. Then one has the inclusions*

$$\begin{aligned}
(3) \qquad & \text{a)} \quad \rho_V(SS(\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mu\,\mathrm{hom}(F,\mathcal{O}_X))) \subset C_V(\mathcal{M}),\\
& \text{b)} \quad \rho_V(SS(\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, t\mu\,\mathrm{hom}(F,\mathcal{O}_X))) \subset C^1_V(\mathcal{M}),\\
& \text{c)} \quad \rho_V(SS(\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, I\mu\,\mathrm{hom}(F,\mathcal{O}_X))) \subset C^1_V(\mathcal{M}).
\end{aligned}$$

Proof. c) It derives from a) and b).
a) and b) We may assume that $\mathcal{N} = t\mathcal{H}\mathrm{om}(F,\mathcal{O}_X)$ is concentrated in degree zero. Let us identify $T^*_\Delta(X\times X)$ with T^*X and denote by $j : T^*X \hookrightarrow T^*(X\times X)$ the associated inclusion. Denote by $*$ the duality functor

$$\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_X}(\cdot,\mathcal{E}_X) \underset{\pi^{-1}\mathcal{O}_X}{\otimes} \pi^{-1}\Omega_X^{\otimes -1}[n].$$

Hence it is suffitient to prove that

$$\rho_V\left(SS\left(j^{-1}\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_{X\times X}}\left(\mathcal{M}\underline{\boxtimes}\left(\mathcal{E}_X \underset{\pi^{-1}\mathcal{D}_X}{\otimes} \pi^{-1}\mathcal{N}\right)^*, C^{\mathbb{R}}_{\Delta|X\times X}\right)\right)\right) \subset C_V(\mathcal{M}),$$

as well as with $C^{\mathbb{R}}_{\Delta|X\times X}$ replaced by $C^{\mathbb{R},f}_{\Delta|X\times X}$, and $C_V(\mathcal{M})$ replaced by $C^1_V(\mathcal{M})$. Since $C^{\mathbb{R}}_{\Delta|X\times X}$ is supported by $T^*_\Delta(X\times X)$,

$$\begin{aligned}
& SS(j^{-1}\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_{X\times X}}(\mathcal{M}\underline{\boxtimes}\,\tilde{\mathcal{N}}^*, C^{\mathbb{R}}_{\Delta|X\times X}))\\
& \subset \rho_{T^*_\Delta(X\times X)}(SS(\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_{X\times X}}(\mathcal{M}\,\underline{\boxtimes}\,\tilde{\mathcal{N}}^*, C^{\mathbb{R}}_{\Delta|X\times X})))
\end{aligned}$$

where $\tilde{\mathcal{N}} = \mathcal{E}_X \underset{\pi^{-1}\mathcal{D}_X}{\otimes} \pi^{-1}\mathcal{N}$, and the inclusion holds for $C^{\mathbb{R},f}_{\Delta|X\times X}$ as well. Here we identify $T_{T^*_\Delta(X\times X)}(T^*(X\times X))$ with $T^*(T^*X)$ via the identification of $T_{T^*_\Delta(X\times X)}(T^*(X\times X))$ with $T^*(T^*_\Delta(X\times X))$ and the identification of the last one with $T^*(T^*X)$.

Suppose we know that for any $\mathcal{E}_X$-coherent module, for any smooth submanifold Y of X,

$$\rho_{\dot{T}^*_Y X}(SS(\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_X}(\mathcal{M}, C^{\mathbb{R}}_{Y|X}))) \subset C_{\dot{T}^*_Y X}(\mathcal{M}),$$
$$\rho_{\dot{T}^*_Y X}(SS(\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_X}(\mathcal{M}, C^{\mathbb{R},f}_{Y|X}))) \subset C^1_{\dot{T}^*_Y X}(\mathcal{M}), \tag{4}$$

where $\rho_{\dot{T}^*_Y X}$ is, as before, the projection

$$\dot{T}^*_Y X \underset{T^*X}{\times} T^*(T^*X) \to T_{\dot{T}^*_Y X}(T^*X).$$

Then, replacing $\mathcal{M}$ by $\mathcal{M} \underline{\boxtimes} \tilde{\mathcal{N}}^*$ we get

$$\begin{aligned}
\rho_{\dot{T}^*_\Delta(X\times X)}(SS(\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_{X\times X}}(\mathcal{M} \underline{\boxtimes} \tilde{\mathcal{N}}^*, C^{\mathbb{R}}_{\Delta|X\times X})))&\\
\subset C_{\dot{T}^*_\Delta(X\times X)}(\mathcal{M} \underline{\boxtimes} \tilde{\mathcal{N}}^*) &:= C(\mathcal{M},\mathcal{N})\\
\rho_{\dot{T}^*_\Delta(X\times X)}(SS(\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_{X\times X}}(\mathcal{M}\underline{\boxtimes}\tilde{\mathcal{N}}^*, C^{\mathbb{R},f}_{\Delta|X\times X})))&\\
\subset C^1_{\dot{T}^*_\Delta(X\times X)}(\mathcal{M}\underline{\boxtimes}\tilde{\mathcal{N}}^*) &:= C^1(\mathcal{M},\mathcal{N})
\end{aligned} \tag{5}$$

where we identify $T^*(X\times X)$ to T^*X by the first projection. □

Since $SS(F)$ equals the characteristic variety of $\mathcal{N}$, we know by [K-O] that $\mathcal{N}$, being regular holonomic, is regular along V. We shall use the following result, which is a slight improvement of the analogous in [MF1].

Lemma 2.2. *Let X be a complex analytic manifold. Let $\mathcal{N}$ be a coherent $\mathcal{E}_X$-module regular along a smooth involutive submanifold $V \subset \dot{T}^*X$. Then, for any coherent $\mathcal{E}_X$-module $\mathcal{M}$,*

$$\rho_V(C(\mathcal{M},\mathcal{N})) \subset C_V(\mathcal{M}),$$
$$\rho_V(C^1(\mathcal{M},\mathcal{N})) \subset C^1_V(\mathcal{M}). \tag{6}$$

Proof. The first inclusion is obvious since $C(\mathcal{M},\mathcal{N}) = C(\mathrm{Car}(\mathcal{M}), \mathrm{Car}(\mathcal{N}))$ and $\mathrm{Car}(\mathcal{N}) \subset V$.

As for the second inclusion, let us start by assuming that V is regular involutive, that is, the canonical 1-form never vanishes on V. By [K-O], we know that $\mathcal{N}$ is locally a quotient of some power N of a coherent $\mathcal{E}_X$-module $\mathcal{L}_V$ supported by V, with simple characteristics. Hence

$$C^1(\mathcal{M},\mathcal{N}) \subset C^1(\mathcal{M},\mathcal{L}_V^N)$$

and then we apply Theorem 1.4.2 in [MF1].

For the general case, we use the dummy variable trick, that is, we consider local canonical coordinates (x,ξ) in T^*X in a neighborhood of V, the regular involutive submanifold of $\dot{T}^*(X\times\mathbb{C})$,

$$\tilde{V} = \{(x,t;\xi,\zeta) \in \dot{T}^*X \times \dot{T}^*\mathbb{C};\ (x;\xi)\in V,\, \zeta \neq 0\}$$

and $\tilde{\mathcal{M}} := \mathcal{M} \boxtimes \mathcal{E}_{\mathbb{C}}$. Since

$$C^1_{\tilde{V}}(\tilde{\mathcal{M}}) = C^1_V(\mathcal{M}) \times \dot{T}^*\mathbb{C},$$

we get

$$\rho_{\tilde{V}}(C^1(\tilde{\mathcal{M}},\tilde{\mathcal{N}})) \subset C^1_{\tilde{V}}(\tilde{\mathcal{M}}) = C^1_V(\mathcal{M}) \times \dot{T}^*\mathbb{C}.$$

On the other hand,

$$\rho_{\tilde{V}}(C^1(\tilde{\mathcal{M}},\tilde{\mathcal{N}})) = \rho_V(C^1(\mathcal{M},\mathcal{N})) \times \dot{T}^*\mathbb{C},$$

hence the result. □

Therefore, a) and b) of Theorem 2.1 hold provided that we prove (4).

Let $f : \tilde{X} \to X$ be a smooth morphism of complex manifolds, let Λ be a smooth involutive submanifold of $\dot{T}^*X$, let $\omega : \tilde{X} \underset{X}{\times} T^*X \to T^*X$ and ${}^tf' : \tilde{X} \underset{X}{\times} T^*X \to T^*\tilde{X}$ be the canonical morphisms. Let $W := {}^tf' \circ \omega^{-1}(\Lambda)$. Then W is a smooth involutive submanifold of $\dot{T}^*\tilde{X}$. In this situation, we have:

Lemma 2.3. *Let $f : \tilde{X} \to X$ be a smooth analytic morphism of finite dimensional complex analytic manifolds. Let Λ be a smooth involutive submanifold of $\dot{T}^*X$ and let $W = {}^tf' \circ \omega^{-1}(\Lambda) \subset \dot{T}^*\tilde{X}$. Set $\omega^N : T_{\omega^{-1}(\Lambda)}(T^*X \underset{X}{\times} \tilde{X}) \to T_\Lambda(T^*X)$ and $\pi^N : T_{\omega^{-1}(\Lambda)}(T^*X \underset{X}{\times} \tilde{X}) \to T_W(T^*\tilde{X})$, the canonical morphisms associated to ω and ${}^tf'$.*

Then:

i) *π^N is injective.*

ii) *Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module. We have the following estimation*:

$$C^1_W(\underline{f}^*\mathcal{M}) = \pi^N(\omega^N)^{-1}C^1_\Lambda(\mathcal{M}).$$

Proof. Since the statement is of local nature, we may assume that $\tilde{X} \simeq X \times Y$, that $f\colon \tilde{X} \times Y \to X$ is the projection, and consider local coordinates on $X \times Y$, (x, x'), such that x are local coordinates on X, x' are local coordinates on Y and $f(x, x') = x$. Consider the associated canonical coordinates $(x, x'; \xi, \xi')$ on $T^*\tilde{X}$. We get

$$\begin{gathered}
\tilde{X} \underset{X}{\times} T^*X = Y \times T^*X, \\
W = \{(x', \xi') \in T^*Y,\ \xi' = 0\} \times \Lambda = Y \times \Lambda, \\
T_W(T^*\tilde{X}) \simeq T_Y(T^*Y) \times T_\Lambda(T^*X), \\
T_{\omega^{-1}(\Lambda)}\left(\tilde{X} \underset{X}{\times} T^*X\right) \simeq Y \times T_\Lambda(T^*X),
\end{gathered}$$

and, for $x' \in Y$ and $p \in T_\Lambda(T^*X)$, $\pi^N(x', p) = ((x'; 0), p)$. Here we identify Y to the zero sections of TY, of T^*Y and of $T_Y(T^*Y)$. This proves i).

As for ii), it will be enough to consider the case where $\mathcal{M}$ is of the form $\mathcal{M} = \frac{\mathcal{E}_X}{\mathcal{J}}$ with $\mathcal{J}$ a coherent ideal of $\mathcal{E}_X$. In that case, $\underline{f}^*\mathcal{M}$ is isomorphic to $\frac{\mathcal{E}_{\tilde{X}}}{\mathcal{E}_{\tilde{X}}\mathcal{J} + \mathcal{E}_{\tilde{X}} D_{x'}}$, where $\mathcal{E}_{\tilde{X}} D_{x'}$ denotes the ideal generated by the derivations in the x' variables. Hence,

$$C^1_W(\underline{f}^*\mathcal{M}) = \{(q,\ p) \in T_Y(T^*Y) \times T_\Lambda(T^*X),\ q \in Y,\ p \in C^1_\Lambda(\mathcal{M})\},$$

hence the result. $\square$

We shall now return to (4).

Lemma 2.4. *Let Y be a smooth submanifold of X. Then, for any coherent $\mathcal{E}_X$-module $\mathcal{M}$,*

$$\text{(7)} \qquad \begin{aligned}
&\text{a')} \quad \rho_{\dot{T}^*_Y X}(SS(\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, C^{\mathbb{R}}_{Y|X}))) \subset C_{\dot{T}^*_Y X}(\mathcal{M}), \\
&\text{b')} \quad \rho_{\dot{T}^*_Y X}(SS(\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, C^{\mathbb{R},f}_{Y|X}))) \subset C^1_{\dot{T}^*_Y X}(\mathcal{M}).
\end{aligned}$$

Proof. a') It is a form of Theorem 8.2.1 of [K-S1]. This proves a) of Theorem 2.1.
b') Since the statements are local and invariant by canonical transformation, we may assume that Y is a hypersurface.

Let $(x, t) = (x_1, \cdots, x_{n-1}, t)$ be local coordinates on X such that $t = 0$ defines $Y, (x, t; \xi, \zeta)$ the associated canonical coordinates in T^*X and γ be the section

$$Y \underset{\gamma}{\hookrightarrow} \dot{T}^*_Y X, \quad \gamma(y) = (y; 1),$$

in the neighborhood of $0 \in Y$.

We shall regard $T^*Y(\simeq T^*(\gamma(Y)))$ as a submanifold of $T^*(\dot{T}^*_Y X)$ via the composition ℓ_Y of the morphisms:

$$T^*Y \hookrightarrow T^*_Y X \underset{Y}{\times} T^*Y \hookrightarrow T^*(\dot{T}^*_Y X)$$

where the left arrow derives from the section γ and the right arrow is the immersion associated to $\dot{T}^*_Y X \to Y$.

More precisely, if $(x;\xi) \in T^*Y$, $\ell_Y(x;\xi) = (x,1;\xi,0) \in T^*(\dot{T}^*_Y X)$. Moreover, if H denotes the Hamiltonian isomorphism, $-H$ induces an isomorphism:

$$T^*(\dot{T}^*_Y X) \simeq T_{\dot{T}^*_Y X}(T^*X)$$

(see (6.2.2) and (6.2.3) of [K-S 3]). We still denote by $-H$ this isomorphism for the sake of simplicity. Explicitely,

$$-H \circ \ell_Y(x;\xi) = -H(x,1;\xi,0) = (-\xi,1;x,0).$$

Therefore we may regard T^*Y as a submanifold of $T_{\dot{T}^*_Y X}(T^*X)$ by the immersion $\tilde{\rho}_Y := -H \circ \ell_Y$.

Let $V = \{\xi_1 = \cdots = \xi_{n-1} = 0; \zeta \neq 0\}$. The composition of the natural morphism of vector bundles $s_Y : T_{\dot{T}^*_Y X}(T^*X) \to T_V(T^*X)$ with $\tilde{\rho}_Y$ is injective ; more precisely, if $(x;\xi) \in T^*Y$ then

$$s_Y\, \tilde{\rho}_Y(x;\xi) = (-\xi,0,1;x) \in T_V(T^*X).$$

We set $\phi_Y := s_Y\, \tilde{\rho}_Y$. By means of ϕ_Y we identify T^*Y to a submanifold of $T_V(T^*X)$. Moreover, if $\mathcal{M}$ is an arbitrary $\mathcal{E}_X$-coherent module we have an inclusion

$$C^1_V(\mathcal{M}) \cap T^*Y \subset s_Y\left(C^1_{\dot{T}^*_Y X}(\mathcal{M}) \cap T^*Y\right)$$

since the sheaf $\mathcal{E}_V$ is a subsheaf of $\mathcal{E}_{\dot{T}^*_Y X}$ and, by the above identification, for any $P \in \mathcal{E}_V$, $\sigma^1_V(P)|_{T^*Y} = \sigma^1_{\dot{T}^*_Y X}(P)|_{T^*Y}$.

For any coherent $\mathcal{E}_X$-module $\mathcal{M}$ we regard $\mathcal{L} := \mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, C^{\mathbb{R},f}_{Y|X})$ as a complex on $\dot{T}^*_Y X$. Then, to get b'), it is enough to prove the inclusion

$$\text{b")} \ -H(SS(\mathcal{L})) \subset C^1_{\dot{T}^*_Y X}(\mathcal{M}).$$

Assume that, for any coherent $\mathcal{E}_X$-module $\mathcal{M}$, the following inclusion holds :

$$SS(\mathcal{L}\,|_{\gamma(Y)}) \subset C^1_V(\mathcal{M}) \cap T^*Y. \tag{8}$$

Then, to get b"), we may use (8) by adjunction of a new variable following a suggestion of M. Kashiwara. Let $\tilde{X} = X \times \mathbb{C}$ with the coordinates (x, t', s) and let $f : \tilde{X} \to X$ be the smooth morphism

$$f(x, t', s) = (x, se^{t'}).$$

Let $\tilde{\mathcal{M}} = \underline{f}^*\mathcal{M}$ be the inverse image of $\mathcal{M}$, a coherent $\mathcal{E}_{\tilde{X}}$-module. Let $Y' \subset \tilde{X}$ be defined by $s = 0$ and let

$$V' = \{(x, t', s; \xi, \zeta', \eta); \quad \xi = 0, \quad \zeta' = 0, \quad \eta \neq 0\} \subset \dot{T}^*\tilde{X}.$$

Let $f' : T\tilde{X} \to TX \underset{X}{\times} \tilde{X}$, ${}^tf' : T^*X \underset{X}{\times} \tilde{X} \to T^*\tilde{X}$ and $\omega\colon T^*X \underset{X}{\times} \tilde{X} \to T^*X$ be the canonical morphisms:

$$\dot{T}^*X \underset{\omega}{\leftarrow} \dot{T}^*X \underset{X}{\times} \tilde{X} \underset{{}^tf'}{\rightarrow} \dot{T}^*\tilde{X}.$$

Explicitely

$$f'_{(x,t',s)}(\xi', \zeta', \eta) = (\xi', \zeta' se^{t'} + \eta e^{t'}),$$

and

$${}^tf'_{(x,t',s)}(\xi, \zeta) = (\xi, \zeta se^{t'}, \zeta e^{t'}).$$

Restricting to $\eta = 1$ and fixing a determination of $\log(\frac{1}{\zeta})$ we get a section h of ω

$$h\colon \dot{T}^*X \to \dot{T}^*X \underset{X}{\times} \tilde{X}$$

and $\overline{h} := {}^tf' \circ h$ gives an immersion of the corresponding open subdomain of $\{(x, t; \xi, \zeta) \in T^*X;\ \zeta \neq 0\}$ in $\{(x, t', s; \xi', \zeta', \eta) \in T^*\tilde{X};\ \eta = 1\}$. The image $\overline{h}(\dot{T}^*_YX)$ is an open subset of $\dot{T}^*_{Y'}\tilde{X} \cap \{\eta = 1\}$. More precisely,

$$h(x, t; \xi, \zeta) = ((x, t; \xi, \zeta), \zeta t,\ \log\,(1/\zeta))$$

and

$$\overline{h}(x, t; \xi, \zeta) = (x,\ \log\,(1/\zeta), \zeta t; \xi, \zeta t, 1).$$

Remark that we can cover $\{p \in \dot{T}^*X;\ \zeta \neq 0\}$ by two such open domains, Ω_1 and Ω_2. Since the notion of microsupport is of local nature, it is enough to prove b") for the restriction of F to each of $\Omega_1 \cap \dot{T}^*_YX$, $\Omega_2 \cap \dot{T}^*_YX$. In that situation, $\overline{h}\colon \dot{T}^*X \to \dot{T}^*\tilde{X}$ induces an analytic isomorphism

$$h_Y\colon \dot{T}^*_YX \simeq \dot{T}^*_{\tilde{Y}'}\tilde{X} \cap \{\eta = 1\}.$$

More precisely, setting $i_Y : \dot{T}^*_YX \hookrightarrow \dot{T}^*X$ and $i'_{Y'}\colon \dot{T}^*_{\tilde{Y}'}\tilde{X} \cap \{\eta = 1\} \hookrightarrow \dot{T}^*\tilde{X}$, we have

$$\overline{h} \circ i_Y = i'_{Y'} \circ h_Y.$$

Let h^c be the canonical isomorphism

$$T^*(\dot{T}^*_Y X) \to T^*(T^*_{Y'}\tilde{X} \cap \{\eta = 1\})(\simeq T^*Y')$$

induced by h_Y. Composing with the immersion

$$\phi_{Y'} \colon T^*Y' \hookrightarrow T_{V'}(T^*\tilde{X}),$$

we get an isomorphism

$$T^*(T^*_Y X) \simeq T_{V'}(T^*\tilde{X}) \mid_{\{s=0,\eta=1\}} .$$

We set

$$\overline{h}^N := \phi_{Y'} \circ h^c \circ (-H)^{-1} \colon T_{\dot{T}^*_Y X}(T^*X) \to T_{V'}(T^*\tilde{X}).$$

Remark that, by functoriality, $\overline{h}^N$ is also the quotient morphism associated to $\overline{h}$ and $\overline{h} \circ i_Y$. On the other hand, the sequence of morphisms of vector bundles :

$$\dot{T}^*X \underset{h}{\to} \dot{T}^*X \underset{X}{\times} \tilde{X} \underset{{}^t f'}{\to} \dot{T}^*\tilde{X}$$

induces a sequence

$$\dot{T}^*_Y X \underset{e'}{\to} \dot{T}^*_Y X \underset{X}{\times} \tilde{X} \underset{\pi'}{\to} V'$$

and π' is injective since ${}^t f'$ is injective. Setting $j_{Y'} \colon \dot{T}^*_{Y'}\tilde{X} \cap \{\eta = 1\} \hookrightarrow V'$, by construction

$$j_{Y'} \circ h_Y = \pi' \circ e'.$$

Set $\tilde{V}' := \dot{T}^*_Y X \underset{X}{\times} \tilde{X}$. Then

$$\tilde{V}' = T^*X \underset{X}{\times} \tilde{X} \cap \omega^{-1}(\dot{T}^*_Y X),$$

$$\pi'(\tilde{V}') = \dot{T}^*_{Y'}\tilde{X},$$

$$\pi'\, e'(\dot{T}^*_Y X) = h_Y(\dot{T}^*_Y X) = \dot{T}^*_{Y'}\tilde{X} \cap \{\eta = 1\}.$$

Let

$$e^N \,:\, T_{\dot{T}^*_Y X}T^*X \to T_{\tilde{V}'}(T^*X \underset{X}{\times} \tilde{X}),$$

$$\omega^N \,:\, T_{\tilde{V}'}(T^*X \underset{X}{\times} \tilde{X}) \to T_{\dot{T}^*_Y X}T^*X,$$

$$\pi'^N \,:\, T_{\tilde{V}'}(T^*X \underset{X}{\times} \tilde{X}) \to T_{V'}(T^*\tilde{X}),$$

be the canonical morphisms respectively associated to h and e', to ω and $\omega|_{\tilde{V}'}$, to ${}^t f'$ and π'. Consider the diagram of morphisms below:

$$T^*(\dot{T}^*_Y X) \underset{-H}{\simeq} T_{\dot{T}^*_Y X} T^*X \underset{e^N}{\to} T_{\tilde{V}'}(T^*X \underset{X}{\times} \tilde{X}) \underset{\pi'^N}{\to} T_{V'}(T^*\tilde{X}).$$

(Here $-H$ denotes the isomorphism induced by the Hamiltonian isomorphism as before.) Remark that ${\pi'}^N$ is the composition of the morphisms:

$$T_{\tilde{V}'}\left(T^*X \underset{X}{\times} \tilde{X}\right) \underset{\pi^N}{\to} T_{\dot{T}^*_{Y'}\tilde{X}} T^*\tilde{X} \underset{s_{Y'}}{\to} T_{V'}(T^*\tilde{X}).$$

Furthermore, $s_{Y'}|_{\dot{T}^*Y'}$ is injective, π^N is injective (cf. Lemma 2.3) and ${\pi'}^N \circ e^N = \overline{h}^N = \phi_{Y'} \circ h^c \circ (-H)^{-1}$. Since h is a section of ω, that is, $\omega \circ h$ =Id, by functoriality, e^N is a section of ω^N.

Recall we assumed (8), hence we have the inclusion

$$SS(\tilde{\mathcal{L}}\,|_{\{\eta=1\}}) \subset C^1_{V'}(\tilde{\mathcal{M}}) \cap T^*Y'$$

where $\tilde{\mathcal{L}}$ denotes

$$\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_{\tilde{X}}}(\tilde{\mathcal{M}}, C^{\mathbb{R},f}_{Y'|\tilde{X}}).$$

On the other hand, for any coherent $\mathcal{E}_X$-module $\mathcal{M}$,

$$h_Y^{-1}(\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_{\tilde{X}}}(\tilde{\mathcal{M}}, C^{\mathbb{R},f}_{Y'|\tilde{X}})\,|_{\eta=1}) \simeq \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_X}(\mathcal{M}, C^{\mathbb{R},f}_{Y|X}) = \mathcal{L}.$$

To prove this, it is enough to consider $\mathcal{M} \simeq \mathcal{E}_X$, hence

$$\tilde{\mathcal{M}} = \frac{\mathcal{E}_{\tilde{X}}}{\mathcal{E}_{\tilde{X}}(sD_s - D_{t'})}$$

in which case the isomorphism above is clear.

Therefore,

$$\begin{aligned} SS(\mathcal{L}) &= (h^c)^{-1}(SS(\tilde{\mathcal{L}}\,|_{\{\eta=1\}})) \\ &\subset (h^c)^{-1}(C^1_{V'}(\tilde{\mathcal{M}}) \cap T^*Y') \\ &\subset (h^c)^{-1}\left(s_{Y'}\left(C^1_{\dot{T}^*_{Y'}\tilde{X}}(\tilde{\mathcal{M}}) \cap T^*Y'\right)\right). \end{aligned}$$

In order to get b") we shall prove the inclusion

$$-H(h^c)^{-1}(s_{Y'}(C^1_{\dot{T}^*_{Y'}\tilde{X}}(\tilde{\mathcal{M}}) \cap T^*Y')) \subset C^1_{\dot{T}^*_Y X}(\mathcal{M}).$$

Using Lemma 2.3 with $\Lambda = T_Y^* X$, we have

$$\begin{aligned}
&-H(h^c)^{-1}\left(s_{Y'}\left(C^1_{\dot{T}^*_{Y'},\tilde{X}}(\tilde{\mathcal{M}}) \cap T^*Y'\right)\right)\\
&\quad = -H(h^c)^{-1}\phi_{Y'}^{-1}\left(s_{Y'}\left(\pi^N(\omega^N)^{-1}\left(C^1_{\dot{T}^*_Y X}(\mathcal{M})\right) \cap T^*Y'\right)\right)\\
&\quad = (\overline{h}^N)^{-1}\left(s_{Y'}\left(\pi^N(\omega^N)^{-1}\left(C^1_{\dot{T}^*_Y X}(\mathcal{M})\right) \cap T^*Y'\right)\right)\\
&\quad = (e^N)^{-1}(\pi'^N)^{-1}\left(s_{Y'}\left(\pi^N(\omega^N)^{-1}\left(C^1_{\dot{T}^*_Y X}(\mathcal{M})\right) \cap T^*Y'\right)\right)\\
&\quad = (e^N)^{-1}(\pi^N)^{-1}s_{Y'}^{-1}\left(s_{Y'}\left(\pi^N(\omega^N)^{-1}\left(C^1_{\dot{T}^*_Y X}(\mathcal{M})\right) \cap T^*Y'\right)\right)\\
&\quad \subset (e^N)^{-1}(\omega^N)^{-1}\left(C^1_{\dot{T}^*_Y X}(\mathcal{M})\right)\\
&\quad \subset C^1_{\dot{T}^*_Y X}(\mathcal{M}).
\end{aligned}$$

Therefore b") holds provided that we prove (8).

Let $\theta \in T_V(T^*X)$ such that $\theta \notin C^1_V(\mathcal{M})$. Considering the local coordinates in $T_V(T^*X)$, $((x,t;\eta);\tilde{\xi}_1,\cdots,\tilde{\xi}_{n-1})$ and using the technique of [MF2] or [S], we may assume $\theta = ((0,0;1);1,\cdots,0)$. We shall identify $T_Y^*X \cap \{\zeta = 1\}$ to Y. Furthermore, we may assume by classical arguments that $\mathcal{M}$ is of the form $\mathcal{E}_X/\mathcal{E}_X P$ with

$$P(x,t,D_x,D_t) = D_{x_1}^m + \sum_{0\le j\le m-1} A_j(x,t,D_{x'},D_t)D_{x_1}^j \tag{9}$$

where $D_{x'} = (D_{x_2},\cdots,D_{x_{n-1}})$ and $A_j \in \mathcal{E}_V(m-j)$. We shall prove that $(0,dx_1) \notin SS(R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, C^{\mathbb{R},f}_{Y|X})\mid_{\{\zeta=1\}})$. For that purpose we need Lemmas 2.5, 2.6 and 2.7 below.

Lemma 2.5. *The sheaf* $C^{\mathbb{R},f}_{Y|X}$ *satisfies*:

1) **The analytic continuation principle:** *Let* $\omega \subset \Omega$ *be two open subsets of* Y, Ω *connected and* $\omega \neq \phi$, *and assume that* $u \in \Gamma(\Omega, C^{\mathbb{R},f}_{Y|X})_{\{\zeta=1\}}$ *vanishes in* ω. *Then* $u \equiv 0$.

2) *If* V *is a conic subset of* $\dot{T}_Y^*X$ *of the form* $W \times \Gamma$, *where* W *is compact Stein in* Y *and* Γ *a convex cone in* $\mathbb{C}^*$ *containing* 1, *such that* $\Gamma \cap S^1$ *is closed, then,*

$$\forall j \ge 1, \qquad H^j(V, C^{\mathbb{R},f}_{Y|X}) = 0.$$

In particular, if W *is compact Stein in* Y, $H^j(W, C^{\mathbb{R},f}_{Y|X}\mid_{\{\zeta=1\}}) = 0$, $\forall j \ge 1$.

Proof. 1) Since $C^{\mathbb{R},f}_{Y|X}$ is a subsheaf of $C^{\mathbb{R}}_{Y|X}$, it is enough to prove that u vanishes as a section of $C^{\mathbb{R}}_{Y|X}\mid_{\{\zeta=1\}}$ but this is a consequence of the analytic

continuation principle for $C^{\mathbb{R}}_{Y|X}$ which is well known (cf. [K-S1]).
2) We have

$$\forall j, \quad H^j(W\times\Gamma, C^{\mathbb{R},f}_{Y|X}) = \varinjlim_{W',\Gamma'} H^j(W'\times\Gamma', C^{\mathbb{R},f}_{Y|X}),$$

where W' runs through a neighborhood system of W formed by open Stein relatively compact subsets of Y and Γ' through a neighborhood system of Γ formed by convex open cones.

Let $V' = W'\times\Gamma'$. Then

$$H^j(V', C^{\mathbb{R},f}_{Y|X}) \simeq H^{j+1}_{V'^0}(\tau^{-1}\pi(V'), {}^t\nu_Y(\mathcal{O}_X))$$

where $\tau : T_YX \to Y$ is the projection and ${}^t\nu_Y(\mathcal{O}_X)$ denotes the tempered specialisation of the sheaf $\mathcal{O}_X$ along Y. We denote by $\nu_Y(\mathcal{O}_X)$ the usual specialisation of $(\mathcal{O}_X)$ along Y. Let us study the long exact sequence

$$\begin{aligned}(10)\quad \cdots &\to H^i_{V'^\circ}(\tau^{-1}\pi(V'), {}^t\nu_Y(\mathcal{O}_X)) \to H^i(\tau^{-1}\pi(V'), {}^t\nu_Y(\mathcal{O}_X)) \to \\ &\to H^i(\tau^{-1}\pi(V')\setminus V'^\circ, {}^t\nu_Y(\mathcal{O}_X)) \to H^{i+1}_{V'^\circ}(\tau^{-1}\pi(V'), {}^t\nu_Y(\mathcal{O}_X)) \to \cdots\end{aligned}$$

On one hand, we have, $\forall i \geq 1$:

$$\begin{aligned}H^i(\tau^{-1}\pi(V'), {}^t\nu_Y(\mathcal{O}_X)) &= H^i(\tau^{-1}\pi(V'), \nu_Y(\mathcal{O}_X)) \\ &= H^i(W', \mathcal{O}_X) \\ &= 0.\end{aligned}$$

On the other hand, we have $V'^\circ = W'\times\Gamma'^\circ$, therefore $\tau^{-1}\pi(V')\setminus V'^\circ = W'\times(\mathbb{C}\setminus\Gamma'^\circ)$.

Hence it remains to prove that, $\forall i \geq 1$,

$$(11)\qquad \varinjlim_{W',\Gamma'} H^i(W'\times(\mathbb{C}\setminus\Gamma'^\circ), {}^t\nu_Y(\mathcal{O}_X)) = 0.$$

This direct limit equals

$$H^i(W\times H, {}^t\nu_Y(\mathcal{O}_X)) = 0,$$

where H is the cone generated by $\overline{\mathbb{C}\setminus\Gamma^\circ}\cap S^1$. Let now V denote $W\times H$.

We have

$$H^i(V, {}^t\nu_Y(\mathcal{O}_X)) = \varinjlim_{U} H^i(\mathbb{R}\Gamma(X, t\mathcal{H}om(\mathbb{C}_U, \mathcal{O}_X)),$$

with U running through the open subanalytic sets in X such that $V \cap C_Y(X \setminus U) = \phi$, by (3.1.2) of [A].

By Proposition 4.1.3 of [K-S3], U may be taken to range through the family $p(U' \cap t^{-1}(\mathbb{R}^+))$ where U' ranges through a neighborhood system of V in the real normal deformation of X along Y, $\tilde{X}_Y$, $t : \tilde{X}_Y \to \mathbb{R}$ being the canonical projection and $p : \tilde{X}_Y \to X$ the deformation morphism. Since $V = W \times H$ we may assume that U is of the form $(W' \times \Gamma'') \cap B$ where W' is Stein subanalytic, Γ'' is an open cone in $\mathbb{C}$ and B is an open polydisc in $\mathbb{C}^n$. Hence U is Stein subanalytic relatively compact and we may apply Hörmander's results in [H]. More precisely, $t\mathcal{H}\mathrm{om}(\underline{\mathbb{C}}_U, \mathcal{O}_X)$ is concentrated in degree zero and $H^i(X, t\mathcal{H}\mathrm{om}(\underline{\mathbb{C}}_U, \mathcal{O}_X)) = 0$, $\forall i > 0$. (Cf. also Lemma 2.6 and Lemma 2.16 of [Be].) □

Proof of Lemma 2.4 (continued). We shall now use some concepts introduced by J.M. Bony and P. Schapira (cf. [B-S]).

Let Ω be an open convex subset of Y. Let us note H_h and Z, respectively, the hyperplane of $\mathbb{C}^n$ of equation $x_1 = h$ and $x_1 = 0$, hence $Z = H_0$. Let δ be a real positif number. We say that Ω is δ-$Y \cap H_h$-flat if, whenever $x \in \Omega$ and $\tilde{x} \in H_h$, satisfy

$$| x_1 - h | \geq \delta \mid x_j - \tilde{x}_j |_{j=2,\cdots,n-1}$$

entails $\tilde{x} \in \Omega$.

If Ω is δ -$Y \cap Z$-flat, then for any $\rho > 0$, $\rho\Omega$ is still δ-$Z \cap Y$-flat, and for any $\omega = (\varepsilon, 0, \cdots, 0)$, $\Omega + \omega$ is δ-$(Z + \omega) \cap Y$-flat.

Let $P(x, t, D_x, D_t)$ be of the form (9), that is, P is Weierstrass with respect to D_{x_1}, and belongs to $\mathcal{E}_V$ in a suitable neighborhood of $(0; 1, \cdots, 0)$. □

Lemma 2.6 (Precised Cauchy Problem). *There exists an open neighborhood Ω_1 of $0 \in Y$ and $\delta > 0$ such that, for any convex open subset $\Omega \subset \Omega_1$ which is δ-$Y \cap Z$-flat, the Cauchy problem*

$$P_f = g, \quad \gamma(f) = (h)$$

where $\gamma(f) = (f\mid_Z, \cdots, D_{x_1}^{m-1} f\mid_Z)$, $g \in C_{Y|X}^{\mathbb{R},f}\mid_{\{\zeta=1\}}(\Omega)$ *and*

$$(h) \in C_{Y \cap Z|Z}^{\mathbb{R},f}\mid_{\{\zeta=1\}}(\Omega \cap Z)^m$$

admits a unique solution

$$f \in C_{Y|X}^{\mathbb{R},f}\mid_{\{\zeta=1\}}(\Omega).$$

Moreover, there exists $h_0 > 0$ depending only on P, Ω_1 and δ, such that, if Ω is δ-$H_h \cap Y$-flat, and the Cauchy data is given on H_h for $| h |< h_0$, the same result holds in Ω.

We shall not give here the detailed proof of this lemma since the unicity is an immediate consequence of Lemma 5.2 of [K-S2] and, as for the existence, one follows step by step the proof of this same lemma, using Theorem 2.4.3 of [B] to be sure that the unique solution is in fact in $C^{\mathbb{R},f}_{Y|X}$. □

We give here a version, in our framework, of Zerner's classical result on the propagation at the boundary for holomorphic solutions of partial differential equations ([Z]).

Lemma 2.7. *Let φ be a C^∞ function in a neighborhood of $0 \in Y$ such that $\varphi(0) = 0$ and $d\varphi(0) = dx_1$. Let $\Omega = \{x, \varphi(x) < 0\}$. Let $u \in \Gamma(\Omega, C^{\mathbb{R},f}_{Y|X} |_{\{\zeta=1\}})$ and assume that Pu extends as a section of $C^{\mathbb{R},f}_{Y|X} |_{\{\zeta=1\}}$ to a neighborhood of 0. Then u extends to a neighborhood of 0.*

Proof. Let Ω_1, h_0 and δ be as in Lemma 2.6. We may assume φ is defined in Ω_1, and that Pu extends to Ω_1. We have $\varphi(x) = \mathrm{Re} x_1 - \psi(\mathrm{Im}\, x_1, x_2, \cdots, x_{n-1})$ with $d\psi(0) = 0$. Let $0 < \varepsilon \ll 1$: there exists $R > 0$ such that $||x'|| < R$ entails $-\varepsilon < \psi(0, x')$, that is, denoting as before $H_{-\varepsilon} = \{x_1 = -\varepsilon\}$,

$$H_{-\varepsilon} \cap \{(x, x') \in \Omega_1, | x' |< R\} \subset \Omega.$$

Since $\psi(0, x') = 0(|x'|)$, we may assume that $\varepsilon < \delta R$ and that the open polydisc centered in $(-\varepsilon, 0, 0)$ with radius $\max(R, \delta R)$ is contained in Ω_1.

Then $W_\varepsilon = \{(x_1, x'); | x_1 + \varepsilon |< \delta(R - ||x'||), ||x'|| < R\} \subset \Omega_1$ will be δ-$H_{-\varepsilon} \cap Y$-flat and is a neighborhood of zero.

Let us now consider the Cauchy problem :

$$Pu_\varepsilon = Pu, \quad \gamma_\varepsilon(u_\varepsilon) = \gamma_\varepsilon(u),$$

where γ_ε denotes the traces along $H_{-\varepsilon}$. By Lemma 2.6, the solution $u_\varepsilon = u$ is defined in W_ε, which achieves the proof. □

Conclusion of the proof of Lemma 2.4 *and of Theorem* 2.1. We shall use Proposition 5.1.1. 3) of [K-S3] which gives a characterization of the microsupport of a complex of sheaves better adapted to our situation.

Let $p_0 = (0; dx_1) \in T^*Y$. Let Ω_1, δ and h_0 be given by Lemma 2.6. Let $0 < \varepsilon \ll h_0$ and $0 < R \ll 1$ be small enough and satisfying :

i) $\varepsilon < R$,

ii) the open polydisc $B_R(-\varepsilon)$ centered in $(-\varepsilon, 0, \cdots 0)$ and radius R is contained in Ω_1.

Let $U = B_R(-\varepsilon)$. Let γ be the proper closed convex cone of $\mathbb{C}^{n-1}$ defined by

$$\gamma = \{(x_1, \cdots, x_{n-1}), \mathrm{Re} x_1 \leq -\delta(||x'|| + \mid \mathrm{Im} x_1 \mid)\}.$$

In particular, $(1, 0 \cdots 0) \in \mathrm{int}\ \gamma^{\circ a}$, $\underline{a}$ denoting the antipodal map.

It is easy to check that $(U + \gamma) \cap H$, where H denotes the real half space $\{(x_1, x'), \mathrm{Re}\ x_1 \geq -\varepsilon\}$, is bounded. Hence, for any $x \in U$, $H \cap (x + \gamma)$ is a compact.

Up to a suitable choice of ε, R and δ, we may assume that $(U+\gamma) \cap H \subset \Omega_1$ and

$$\{(U + \gamma) \cap H\} \times \mathrm{Int}\ \gamma^{\circ a} \cap C^1_V\left(\frac{\mathcal{E}_X}{\mathcal{E}_X P}\right) = \phi.$$

Let $L = \{x, \mathrm{Re}\ x_1 = -\varepsilon\}$. We shall prove that the natural morphism of complexes :

$$\begin{array}{ccc} R\Gamma(H \cap (x + \gamma), C^{\mathbb{R},f}_{Y|X} \mid_{\{\zeta=1\}}) & \xrightarrow{P} & R\Gamma(H \cap (x + \gamma), C^{\mathbb{R},f}_{Y|X} \mid_{\{\zeta=1\}}) \\ \downarrow & & \downarrow \\ R\Gamma(L \cap (x + \gamma), C^{\mathbb{R},f}_{Y|X} \mid_{\{\zeta=1\}}) & \xrightarrow{P} & R\Gamma(L \cap (x + \gamma), C^{\mathbb{R},f}_{Y|X} \mid_{\{\zeta=1\}}) \end{array}$$

is a quasi isomorphism.

Since $H \cap (x + \gamma)$ and $L \cap (x + \gamma)$ are compact convex subsets, hence Stein, by Lemma 2.5 it remains to prove that P induces an isomorphism in the quotient

$$\frac{\Gamma(L \cap (x + \gamma), C^{\mathbb{R},f}_{Y|X} \mid_{\{\zeta=1\}})}{\Gamma(H \cap (x + \gamma), C^{\mathbb{R},f}_{Y|X} \mid_{\{\zeta=1\}})}.$$

Here, we used the analytic continuation principle to identify $\Gamma(H \cap (x + \gamma), C^{\mathbb{R},f}_{Y|X} \mid_{\{\zeta=1\}})$ with a submodule of $\Gamma(L \cap (x + \gamma), C^{\mathbb{R},f}_{Y|X} \mid_{\{\zeta=1\}})$.

To prove the surjectivity of P, we apply the analogous of Lemma 2.4.7 of [MF1] or Lemma 3.1.5 of [S] which is proved by the same method thanks to Lemmas 2.6 and 2.7. Let $v \in \Gamma(L \cap (x + \gamma), C^{\mathbb{R},f}_{Y|X} \mid_{\{\zeta=1\}})$. We solve the equation $Pu = v$ in a neighborhood of $H_{-\varepsilon} \cap (x+\gamma)$ using Lemma 2.6 and then extend u to a neighborhood of $L \cap (x + \gamma)$ since any real hyperplane through x, with a 1-microcharacteristic normal, which intersects $L \cap (x + \gamma)$ intersects $H_{-\varepsilon} \cap (x + \gamma)$ (see for example page 152 of [S]).

As for the injectivity, we shall use the construction of [K-S3, Proposition 5.1.5].

For each $a \in U \cap (H \setminus L)$, we construct a family of open subsets $\{\Omega_t(a)\}_{t\in\mathbb{R}^+}$, such that :

i) $\Omega_t(a) \subset a + \text{ int } \gamma$,

ii) $\Omega_t(a) \cap L = (a + \text{ int } \gamma) \cap L$,

iii) $\Omega_t(a) = \bigcup_{r<t} \Omega_r(a)$,

iv) $\delta\Omega_t(a)$ is smooth real analytic,

v) $Z_t(a) = (\bigcap_{s>t} \overline{\Omega_s(a) \setminus \Omega_t(a)}) \cap H \subset \delta\Omega_t(a)$, and the conormal of $\Omega_t(a)$ at $Z_t(a)$ is non 1-microcharacteristic for the operator P everywhere in $Z_t(a)$,

vi) $\left(\bigcup_{t>0} \Omega_t(a)\right) \cap H = (a + \text{ int } \gamma) \cap H$,

vii) $\left(\bigcap_{t>0} \Omega_t(a)\right) \cap H = (a + \text{ int } \gamma) \cap L$.

We recall that if $v = (1, 0, \cdots 0)$, then the family

$$\{(x + \rho v + \text{ int } \gamma) \cap H\}_{\rho>0}$$

forms a neighborhood system of $(x + \gamma) \cap H$, and the family $\{\Omega_t(x + \rho v) \cap H\}_{\rho>0,t>0}$ forms a neighborhood system of $(x + \gamma) \cap L$.

Let $u \in \Gamma(L \cap (x + \gamma), C^{\mathbb{R},f}_{Y|X} \mid_{\{\zeta=1\}})$ such that $Pu = w$ extends to a neighborhood of $H \cap (x + \gamma)$.

Let $\rho > 0$ and $t_0 > 0$ such that u is defined in $\Omega_{t_0}(x + \rho v) \cap H$ and w in $(x + \rho v + \text{ int } \gamma) \cap H$. By v) and Lemma 2.7, u extends to a neighborhood of $\delta\Omega_{t_0}(x + \rho v) \cap H$ hence to $\Omega_{t'}(x + \rho v) \cap H$ for some $t' > t_0$, and the definition of the family Ω_t entails that this procedure leads to an extension of u in a neighborhood of $(x + \gamma) \cap H$. □

Remark. By the functorial properties of $\mu\hom(\cdot, \mathcal{O}_X)$, assuming that $\mathcal{M}$ is generated by a coherent $\mathcal{D}_X$-module, one easily deduces that if F is an object of $D^b_{\mathbb{R}-c}(X)$ such that $SS(F) \subset V$ as in Theorem 2.1,

$$SS(\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_X}(\mathcal{M}, \mu\hom(F, \mathcal{O}_X))) \subset C_V(\mathcal{M}),$$

and this inclusion may still be improved to the microdifferential framework using other tools.

Our conjecture is that Theorem 2.1 may be generalized to the case where $F \in D^b_{\mathbb{R}-c}(X)$.

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