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# Analytic Representation of Generalized Tempered Distributions by Wavelets

By

Byung Keun Sohn* and Dae Hyeon Pahk**

### Abstract

The analytic representation of the generalized tempered distributions of exponential growth, $\mathcal{K}_p^{r\prime}$, is given in terms of series of analytic wavelets. These series converge uniformly on compact subsets of the upper and lower half planes.

## §1. Introduction

The analytic representation of functions or distributions on the real line is usually given by a Cauchy type formula, but in some cases may also be given by an orthogonal series. This is evident for periodic functions and distributions for which trigonometric series may be used [5]. The more natural approch for arbitrary functions on the set of real numbers $R$ seems to be one involving wavelets. G. G. Walter has found an analytic representation of the tempered distributions of polynomial growth in terms of series of analytic wavelets [6].

In the past, the tempered distributions of polynomial growth were extended to various types of generalized tempered distributions of exponential growth [2], [3].

In this paper, we will find an analytic representation of the generalized tempered distributions of exponential growth in terms of series of analytic wavelets.

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*Department of Mathematics, Inje University, Kyungnam, Kimhae, 621-749, (South) Korea.

**Department of Mathematics, Yonsei University, Seoul, 120-749, (South) Korea.

These series converge uniformly on compact subsets of the upper and lower half planes.

## §2. The Generalized Tempered Distributions Space $\mathcal{K}_p'(R)$

We denote by $\mathcal{K}_p(R), p \geq 1$, the space of all functions $\phi \in C^\infty(R)$ such that

$$(1) \qquad \nu_k(\phi) = \sup_{x \in R, \alpha \leq k} e^{k|x|^p} |D^\alpha \phi(x)| < \infty, \ k = 1, 2, \dots ,$$

where $D^\alpha = \frac{d^\alpha}{dx^\alpha}$. The topology in $\mathcal{K}_p(R)$ is defined by the family of the semi-norms $\nu_k$. Then $\mathcal{K}_p(R)$ becomes a Fr$\acute{e}$chet space and the embeddings $\mathcal{D} \hookrightarrow \mathcal{K}_p \hookrightarrow \mathcal{S} \hookrightarrow \mathcal{E}$ are continuous; here $\mathcal{E}$ denotes the space of all $C^\infty$-functions, $\mathcal{S}$ the space of the tempered ditsributions of polynomial growth and $\mathcal{D}$ the space of $C^\infty$-functions with compact supports. By $\mathcal{K}_p'(R)$, we mean the space of continuous linear functionals on $\mathcal{K}_p(R)$. G. Sampson and Z. Zielezny characterized the distributions in $\mathcal{K}_p'(R)$ by the growth at infinity [3]; a distribution $T \in \mathcal{D}'$ is in $\mathcal{K}_p'(R)$ if and only if there exist positive integers $\alpha, k_0$ and a bounded continuous function $f(x)$ on $R$ such that

$$T = D^\alpha [e^{k_0|x|^p} f(x)].$$

**Definition 1.** Let $r$ be a natural number and $p \geq 1$. We denote by $\mathcal{K}_p^r(R)$ the space of all functions $\phi \in C^r(R)$ such that

$$\nu_k^r(\phi) = \sup_{x \in R, \alpha \leq r} e^{k|x|^p} |D^\alpha \phi(x)| < \infty, \ k = 1, 2, 3, \dots .$$

The topology of $\mathcal{K}_p^r(R)$ is defined by the family of semi-norms $\{\nu_k^r\}_{k=1,2,\dots}$. By $\mathcal{K}_p^{r\prime}(R)$, we mean the space of continuous linear functionals on $\mathcal{K}_p^r(R)$. Each $S \in \mathcal{K}_p^{r\prime}(R)$ is characterized by

$$(2) \qquad S = D^r [e^{k_0|x|^p} f(x)],$$

where $f(x)$ is a bounded continuous function on $R$ and $r, k_0 \in N$, the set of natural numbers, by the same method of the above $\mathcal{K}_p'$-case in [3, Theorem 2]. Similarly, we can define

$$\mathcal{S}_r(R) = \{\theta(t) \in C^r(R); |D^k \theta(t)| \leq C_{pk}(1 + |t|)^{-p}, \ p \in N, \ k = 0, 1, \dots , r\}$$

and its dual $\mathcal{S}_r'(R)$. For further details, we refer to [3].

## §3. Multiresolution Analysis of $L^2(R)$ Associated with $\phi \in \mathcal{K}_p^r(R)$

Let $\phi \in \mathcal{K}_p^r(R)$. In order for it to qualify as a scaling function, there must be associated with $\phi$ a multiresolution analysis of $L^2(R)$, i.e., a nested sequence of closed subspaces $\{V_m\}_{m\in Z}$ for the set of integers $Z$ such that

(i) $\{\phi(t-n)\}$ is an orthonormal basis of $V_0$,

(ii) $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L^2(R)$,

(iii) $f \in V_m \Leftrightarrow f(2\cdot) \in V_{m+1}$,

(iv) $\cap_m V_m = \{0\}$, $\overline{\cup_m V_m} = L^2(R)$.

Then $\phi$ has an expansion

$$\phi(t) = \sum\nolimits_n c_n \sqrt{2}\phi(2t-n),\ \{c_n\} \in l^2,\ t \in R, \tag{3}$$

where $l^2 = \{\{c_n\}; \sum_n |c_n|^2 < \infty\}$. Once we have the scaling function $\phi \in \mathcal{K}_p^r(R)$, we can obtain a mother wavelet $\psi$ such that $\{\psi(t-n)\}$ is an orthogonal basis of the space $W_0$, given by the orthogonal complement of $V_0$ in $V_1$. Also, $\psi$ has an expansion

$$\psi(t) = \sum\nolimits_n d_n \sqrt{2}\phi(2t-n),\ \{d_n\} \in l^2, \tag{4}$$

for $d_n$ corresponding to $c_n$ in (3). We will adopt the construction of a mother wavelet defined by $d_n = (-1)^n \overline{c_{1-n}}$. If such a $\psi(t)$ can be found, then $\psi_{mn}(t) = 2^{\frac{m}{2}}\psi(2^m t - n)$ is an orthogonal basis of $W_m$ which is the orthogonal complement of $V_m$ in $V_{m+1}$.

**Example.** In [1], Corollary 5.5.3 states that it is impossible that $\psi$ has exponential decay and that $\psi \in C^\infty$, with all derivatives bounded, unless $\psi = 0$. Hence there is no mother wavelet $\psi \in \mathcal{K}_p(R)$. So we will restrict our attention to $\mathcal{K}_p^r(R)$. Daubechies' compactly supported wavelets are examples of $\mathcal{K}_p^r(R)$, but Battle-Lemarié's wavelets (in the page 152 of [1]) are not $\mathcal{K}_p^r(R)$ wavelets even if they have exponential decay and smoothness.

The reproducing kernel of $V_0$ is given by

$$q(x,t) = \sum\nolimits_n \overline{\phi(x-n)}\phi(t-n),$$

where $\phi(x)$ is the scaling function. The series and its derivatives with respect to $t$ of order $\le r$ converge uniformly for $x \in R$ by the regularity of $\phi \in \mathcal{K}_p^r(R)$, i.e.,

$$|\phi^{(\alpha)}(x)| \le C_{\alpha k} e^{-k|x|^p},\ \alpha = 0,1,\ldots,r; k = 1,2,\ldots. \tag{5}$$

The reproducing kernel for $V_m$ is given by

$$q_m(x,t) = 2^m q(2^m x, 2^m t).$$

Similarly, we can define the reproducing kernel $r_m(x,t)$ for $W_m$ by

$$r_m(x,t) = 2^m \sum\nolimits_n \overline{\psi(2^m x - n)} \psi(2^m t - n),$$

where $\psi(t)$ is the mother wavelet.

The sequence $\{q_m(x,t)\}$ is a delta sequence in $\mathcal{S}'_r(R) \subset \mathcal{K}^{r\prime}_p(R)$, i.e., $q_m(x,t) \to \delta(x-t)$. This follows from the fact that

$$\int_{-\infty}^{\infty} q_m(x,t)\theta(t)dt \to \theta(x) \text{ as } m \to \infty,$$

for each $\theta \in \mathcal{K}^r_p(R) \subset \mathcal{S}_r(R)$, where the convergence is in the $L^2$-sense. These kernels have a number of interesting properties, some of which come out of the wavelet moment theorem. Since $\mathcal{K}^r_p(R) \subset \mathcal{S}_r(R)$, we have by [1],

**Lemma 2.** *Let $\psi \in \mathcal{K}^r_p(R)$ with $\psi_{mn}(x) = 2^{\frac{m}{2}} \psi(2^m x - n)$ an orthogonal system in $L^2(R)$. Then*

$$\int_{-\infty}^{\infty} x^k \psi(x) dx = 0, \quad k = 0, 1, \dots, r.$$

**Definition 3.** We define the spaces $T_0$ and $U_0$ by $T_0 = \{f; f(t) = \sum_n a_n \phi(t-n)$ for some sequence of complex numbers with $a_n = \mathcal{O}(e^{k_1|n|^p})$ for some $k_1 \in N\}$ and $U_0 = \{g; g(t) = \sum_n a_n \psi(t-n)$ for some sequence of complex numbers with $a_n = \mathcal{O}(e^{k_1|n|^p})$ for some $k_1 \in N\}$. We denote by $T_m$ and $U_m$ their corresponding dilation spaces, i. e., $f \in T_0 \Leftrightarrow f(2^m t) \in T_m$ and $g \in U_0 \Leftrightarrow g(2^m t) \in U_m$.

We may expect that a multiresolution analysis of $\mathcal{K}^{r\prime}_p(R)$ exists, namely,

$$(6) \qquad \cdots \subset T_{-m} \cdots \subset T_{-1} \subset T_0 \subset T_1 \cdots \subset T_m \subset \cdots \subset \mathcal{K}^{r\prime}_p(R)$$

and

$$\overline{\cup_m T_m} = \mathcal{K}^{r\prime}_p(R),$$

where the closure is in the topology of $\mathcal{K}^{r\prime}_p(R)$.

Now in [4], we have found the expansion in orthogonal wavelets from $L^2(R)$ to $\mathcal{K}^{r\prime}_p(R)$.

**Theorem 4.** *Let the scaling function* $\phi \in \mathcal{K}_p^r(R)$ *satisfy the dilation equation* (3) *with* $c_k = \mathcal{O}(e^{-l|k|^p})$ *for all* $l \in N$, *and have an associated multiresolution analysis in* $L^2(R)$; *let* $\psi \in \mathcal{K}_p^r(R)$ *be the mother wavelet given in* (4). *Then there exists a multiresolution analysis* (6) *of closed dilation subspaces* $\{T_m\}$ *whose union is dense in* $\mathcal{K}_p^{r\prime}(R)$; *the closed subspace* $U_m$ *in Definition* 3 *is a complementary subspace of* $T_m$ *in* $T_{m+1}$ *and*

$$T_m = U_0 \oplus U_1 \oplus \cdots \oplus U_m \oplus T_0,$$

*where* $\oplus$ *denotes the nonorthogonal direct sum.*

## §4. Analytic Representation of Distributions of $\mathcal{K}_p^{r\prime}$ by Wavelets

A $quasi-positive\ delta\ sequence$ is a sequence $\{\delta_m(\cdot, y)\}$ of functions in $L^1(R)$ with a parameter $y \in R$ which satisfies the following:

(a) there is a $C > 0$ such that

$$\int_{-\infty}^{\infty} |\delta_m(x,y)|dx \leq C,\ y \in R, \quad m \in N;$$

(b) there is a $c > 0$ such that

$$\int_{y-c}^{y+c} \delta_m(x,y)dx \to 1$$

uniformly on compact subsets of $R$, as $m \to \infty$;

(c) for each $\gamma > 0$,

$$\sup_{|x-y|\leq\gamma} |\delta_m(x,y)| \to 0 \quad \text{as} \quad m \to \infty.$$

Then since $\mathcal{K}_p^r(R) \subset \mathcal{S}_r(R)$, we have the following important lemmas as in [7]:

**Lemma 5.** *Let* $\{\delta_m(x,y)\}$ *be a quasi-positive delta sequence and let* $f \in L^1(R)$ *be continuous on* $(a,b)$. *Then*

$$f_m(y) = \int_{-\infty}^{\infty} \delta_m(x,y)f(x)dx \to f(y) \quad as \quad m \to \infty$$

*uniformly on compact subsets of* $(a,b)$.

**Lemma 6.** *If the scaling function* $\phi \in \mathcal{K}_p^r(R)$, *then the reproducing kernel* $q_m(x,y)$ *and* $K_m(x,t) = \frac{(x-t)}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha} q_m(x,t)$ *for* $\alpha \in N, 0 \leq \alpha \leq r$, *are quasi-positive delta sequences on* $R$.

In order to represent an element of $\mathcal{K}_p^{r\prime}(R)$ by series of analytic wavelets, we impose conditions on the scaling function $\phi$ again. Since $\mathcal{K}_p^r(R) \subset L^2(R)$, an analytic represetation of $\phi$ is given by

$$\phi^{\pm}(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\phi(x)}{x-z}dx, \;\; Imz \gtrless 0,$$

where $\phi^{\pm}$ are analytic in the upper half-plane and the lower half-plane, respectively. An analytic representation of the mother wavelet is also given by

$$\psi^{\pm}(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\psi(x)}{x-z}dx, \;\; Imz \gtrless 0,$$

and the analytic wavelets $\psi_{mn}^{\pm}$ are obtained by dilation and translation of $\psi^{\pm}$. Now, we define $T_0^{\pm} = \{f(z) = \sum_n a_n \phi^{\pm}(z-n); a_n = \mathcal{O}(e^{l_0|n|^p}) \; for \; some \; l_0 \in N\}$ and we denote by the subspaces $T_m^{\pm}$ of $T_0^{\pm}$ the corresponding dilation spaces. Then the spaces $T_m^{+}$ and $T_m^{-}$ are composed of analytic functions in the upper and the lower half-planes, respectively, whose boundary functions are continuous functions of exponential growth. Since $\overline{\cup T_m} = \mathcal{K}_p^{r\prime}(R)$, we might expect to obtain an analytic representation of $f \in \mathcal{K}_p^{r\prime}(R)$ in terms of wavelets,

$$f^{+}(z) = \sum_{n=-\infty}^{\infty} a_n \phi^{+}(z-n) + \sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} b_{mn} 2^{\frac{m}{2}} \psi^{+}(2^m z - n),$$

where the first series may not converge. Since an analytic representation is a continuous map from $\mathcal{K}_p^{r\prime}(R)$ to a corresponding space of analytic functions and $f_m(x) = (f, q_m(x,t)) \to f(x) = D^r F(x)$ in $\mathcal{K}_p^{r\prime}(R)$ for a continuous function of exponential growth $F(x)$ [cf. (2)] by Lemmas 5 and 6, $f_m^{+}(z) \to f^{+}(z)$ uniformly on bounded subsets of the upper half-plane. Moreover, $f^{+}(z) = D_z^r F^{+}(z)$, where $F^{+}(z)$ is an analytic representation of $F(z)$, and is given by

$$F^{+}(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F(x)}{x-z} e^{-k|x|^p} e^{k|z|^p} dx,$$

for a sufficiently large $k$ such that $F(x)e^{-k|x|^p} \in L^2(R)$.

We may express $f_m$ as

$$f_m = f_0 + f_m - f_0 = f_0 + \sum_{k=0}^{m-1}\sum_{n=-\infty}^{\infty} b_{kn}\psi_{kn},$$

and if the inner sum converges,

$$f_m^{+}(z) - f_0^{+}(z) = \sum_{k=0}^{m-1}\sum_{n=-\infty}^{\infty} b_{kn}\psi_{kn}^{+}(z) + g_m(z), \tag{7}$$

where $g_m(z)$ is an entire function.

**Lemma 7.** *Let* $\psi \in \mathcal{K}_p^r(R)$ *and* $b_n = \mathcal{O}(e^{k|n|^{p-\epsilon}})$ *for any* $k \in N$ *and some* $\epsilon > 0$. *Then*

$$\sum\nolimits_{n=-\infty}^{\infty} b_n \psi^+(z-n)$$

*converges uniformly on compact subsets of the upper half-plane.*

*Proof.* The proof is based on the moment property, Lemma 2,

$$\int_{-\infty}^{\infty} x^l \psi(x)dx = 0,\ l = 0, 1, \dots, r.$$

Hence, for any $k \in N$ and a natural number $p \leq r+1$,

$$
\begin{aligned}
(8)\qquad & e^{k|z|^p}\psi^+(z) \\
&= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{k|z|^p}}{z^p}\cdot\frac{z^p}{x-z}\psi(x)dx \\
&= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{k|z|^p}}{z^p}\cdot\frac{z^p-x^p}{x-z}\psi(x)dx \\
&\qquad + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{k|z|^p}}{z^p}\cdot\frac{x^p}{x-z}\psi(x)dx \\
&= -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{k|z|^p}}{z^p}\cdot(x^{p-1}+zx^{p-2}+\cdots+z^{p-2}x+z^{p-1})\psi(x)dx \\
&\qquad + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{k|z|^p}}{z^p}\cdot\frac{x^p}{x-z}\psi(x)dx \\
&= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{k|z|^p}}{z^p}\cdot\frac{x^p}{x-z}\psi(x)dx
\end{aligned}
$$

holds. By the growth condition of $\psi \in \mathcal{K}_p^r(R)$, $\left|e^{k|z|^p}\psi^+(z)\right|$ is uniformly bounded on compact subsets of the half-plane $Imz \geq \epsilon > 0$ for any $k \in N$ and a natural number $p \leq r+1$. Hence, the preceding fact holds for any $k \in N$ and any $p \leq r+1$. Thus the conclusion follows. □

**Theorem 8.** *Let* $f \in \mathcal{K}_p^{s\prime}(R)$, $\phi, \psi \in \mathcal{K}_p^r(R)$, $s < r$ *and let* $b_{mn} = \langle f, \psi_{mn}\rangle$, $m = 0, 1, 2, \dots; n = 0, \pm 1, \pm 2, \dots$ *be the wavelet coefficients of* $f$. *Then an analytic representation of* $f$ *is given by*

$$f^+(z) = f_0^+(z) + \sum\nolimits_{m=0}^{\infty}\sum\nolimits_{n=-\infty}^{\infty} b_{mn}\psi_{mn}^+(z),$$

*where the series converges uniformly on compact subsets of the half-plane* $Im\ z \geq 1$ *and* $f_0^+(z)$ *is an analytic representation of* $f_0$, *the projection of* $f$ *on* $T_0$.

*Proof.* First, we will estimate $|b_{mn}|$. Each $f \in \mathcal{K}_p^{s\prime}(R)$ is characterized by

$$f = D^s[e^{k_0|x|^p}\mu]$$

for some integer $k_0$ and finite measure $\mu$ on $R$. Each $\psi \in \mathcal{K}_p^r(R)$ satisfies

$$|\psi^{(l)}(x)| \leq C_j e^{-j|x|^p}, \; l = 1, 2, \ldots, r; j \geq 0.$$

If we use integration by parts $s$-times and the inequality $(a+b)^p \leq 2^p(a^p + b^p)$, $a, b > 0$, we have, for $m > 1$,

$$\begin{aligned}
|b_{mn}| &\leq \int_{-\infty}^{\infty} |D^s[e^{k_0|x|^p}]\psi_{mn}(x)|d|\mu| \leq \int_{-\infty}^{\infty} e^{k_0|x|^p}|\psi_{mn}^{(s)}(x)|d|\mu| \\
&\leq \int_{-\infty}^{\infty} e^{k_0|x|^p} c_{k_0} 2^{\frac{m}{2}+sm} e^{-k_0|2^m x - n|^p} d|\mu| \\
&\leq \int_{-\infty}^{\infty} e^{2^p k_0 |x - n2^{-m}|^p} e^{2^p k_0 |n2^{-m}|^p} c_{k_0} 2^{\frac{m}{2}+sm} e^{-k_0(2^m)^p|x-n2^{-m}|^p} d|\mu| \\
&\leq c'_{k_0} 2^{\frac{m}{2}+sm} e^{2^p k_0 |n2^{-m}|^p}.
\end{aligned}$$

By the fact in the proof of Lemma 7, on every compact subset $K$ of the half-plane $Imz \geq 1$, there exists a constant $c$ such as $|\psi^+(z)| \leq ce^{-k|z|^p}$ for any $k \in N$. Let $k > k_0$ and $M = max(7, s + \frac{3}{2})$. Then we have, for every $z \in K$,

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} |b_{mn}\psi_{mn}^+(z)| \\
&\leq \sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} c'_{k_0} e^{(\frac{1}{2}+s)m} e^{2^p k_0|n2^{-m}|^p} c 2^{\frac{m}{2}} e^{-k|2^m z - n|^p} \\
&\leq \sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} c'_{k_0} e^{(\frac{3}{2}+s)m} e^{2^p k_0|n2^{-m}|^p} c e^{-k(2^m)^p|(n2^{-m} - Re\; z)^2 + 1|^{\frac{p}{2}}} 2^{-\frac{m}{2}} \\
&\leq \left\{ \sum_{m=0}^{M}\sum_{n=-\infty}^{\infty} + \sum_{m=M+1}^{\infty}\sum_{n=-\infty}^{\infty} \right\} cc'_{k_0} e^{(\frac{3}{2}+s)m} e^{2^p k_0|n2^{-m}|^p} \\
&\qquad \times e^{-k2^p|(n2^{-m} - Re\; z)^2+1|^{\frac{p}{2}}} e^{-km^{2p}|(n2^{-m} - Re\; z)^2+1|^{\frac{p}{2}}} 2^{-\frac{m}{2}} \\
&\leq \sum_{m=M+1}^{\infty} C_{k_0,z} 2^{-\frac{m}{2}} < \infty,
\end{aligned}$$

where we use $(2^m)^p \geq 2^p + (2^{m-1})^p \geq 2^p + m^{2p}$ for $m \geq 7$. Hence the series $\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} b_{mn}\psi_{mn}^+(z)$ converges uniformly on compact subsets of the half-plane $Im\; z \geq 1$.

Now, by taking the limit in (7) as $m \to \infty$, we have

$$f^+(z) = f_0^+(z) + \sum_{k=0}^{\infty}\sum_{n=-\infty}^{\infty} b_{kn}\psi_{kn}^+(z) + g_\infty(z),$$

where $g_\infty(z) = \lim_{m\to\infty} g_m(z)$ is an entire function. Since an analytic representation plus an entire function is an analytic representation, we can drop $g_\infty$ in (7). □

*Remark.* We have only worked out the convergence for $f^+$ but proof for $f^-$ is parallel. Then by the same method as in the proof of Theorem 8, an analytic representation of $f$ is given by

$$f^-(z) = f_0^-(z) + \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} b_{mn} \psi_{mn}^-(z),$$

where the series converges uniformly on compact subsets of the half-plane $Im\ z \leq -1$ and $f_0^-(z)$ is an analytic representation of $f_0$, the projection of $f$ on $T_0$.

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