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On the Micro-Hyperbolic Boundary Value Problem for Systems of Differential Equations

To the memory of Emmanuel Andronikof

By

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Abstract

We formulate the boundary value problem for systems of linear differential equations which satisfy a certain condition of micro-hyperbolicity at the boundary in the same way as the Kashiwara-Kawai formulation for elliptic systems.

Résumé

Le problème au bord est formulé pour les systèmes des équations aux dérivées partiélles qui satisfont à une condition de micro-hyperbolicité, comme la formulation de Kashiwara-Kawai pour les systèmes elliptiques.

§0. Introduction

In [KK], Kashiwara and Kawai formulated boundary value problems for elliptic systems of differential equations from a microlocal point of view. They described the cohomology groups of $\mathrm{R}\Gamma_{Z_+}\mathrm{R}\mathcal{H}om_{\mathcal{D}_M}(\mathcal{M}, \mathcal{B}_M)|_N$, where Z_+ is a closed domain in a manifold M with boundary N and $\mathcal{M}$ is an elliptic $\mathcal{D}_M$-module, in terms of a system of micro-differential equations induced on the boundary. In this paper we extend the Kashiwara-Kawai formula to systems of differential equations which satisfy a certain condition of micro-hyperbolicity

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at the boundary. In this case, also, we are able to define a coherent $\mathcal{E}_Y$-module $\mathcal{N}^+$ and prove

$$\mathrm{R}\Gamma_{Z_+}\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{B}_M)|_N[1] \cong \mathrm{R}\dot{\pi}_{N*}\mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+,\mathcal{C}_N) \tag{0.1}$$

(see Theorem 1.2). It is also possible to microlocalize formula (0.1) by using the sheaf $\mathcal{C}_{Z_+|X}$ of Kataoka. By this, we can reformulate the main part of the works of Kaneko [Kn] and Oaku [O, Section 3] in the derived category (i.e., for all cohomology groups) of sheaves for systems of differential equations (see Section 3).

This paper is an enlarged version of [U].

Notation. In this paper, we freely use the notation of [KS1] for sheaves and functors. For a complex manifold X, T^*X denotes the cotangent bundle of X, and $\dot{T}^*X$ that with the zero section removed. π denotes the projection $T^*X \to X$. $\mathcal{O}_X$ denotes the sheaf of holomorphic functions on X, $\mathcal{D}_X$ the sheaf of rings of differential operators, and $\mathcal{E}_X$ the sheaf of rings of microdifferential operators (cf. [SKK]). If M is a closed real submanifold of X, T^*_MX denotes the conormal bundle of M. We often denote by M the zero section of T^*_MX, and $\dot{T}^*_MX = T^*_MX \setminus M$. For a holomorphic function f on X and $p \in T^*X$, $df(p)$ denotes $\pi^*(df)$, with $\pi^* : T^*_{\pi(p)}X \hookrightarrow T^*_pT^*X$.

We denote by H the Hamiltonian map $T^*T^*X \to TT^*X$. We identify $TT^*X \cong T(T^*X^{\mathbf{R}})$, $T^*X^{\mathbf{R}}$ being the underlying real manifold of T^*X. For subsets V, W of T^*X, $C(V,W)$ denotes the Whitney normal cone, which is a closed conic subset of TT^*X and considered also as a subset of T^*T^*X by the isomorphism $-H : T^*T^*X \xrightarrow{\sim} TT^*X$.

§1. Main Result

Let M be a real analytic manifold of dimension $n \geq 1$, N a submanifold of M of codimension 1 defined by the equation $f = 0$ for a real-valued analytic function f with $df|_N \neq 0$. (We denote also by f the holomorphic extension of f to a complex neighborhood of M.) Let X be a complex neighborhood of M, Y a closed complex submanifold of X of codimension 1 defined by $f = 0$, and φ the embedding $Y \hookrightarrow X$.

Let Z_+ denote the closed subset $\{f \geq 0\}$ of M; then Z_+ is a closed domain in M with analytic boundary. We set $N^+ = \{k\,df(x) \mid x \in N,\, k > 0\}$. We also set

$$H = \{(x,\xi) \in T^*_MX \mid x \in N, \xi \neq 0\}\,(= N \times_M \dot{T}^*_MX).$$

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. $\mathrm{Ch}(\mathcal{M})$ denotes the characteristic variety of $\mathcal{M}$. We assume the following two conditions:

(a.1) $\varphi : Y \to X$ is non characteristic for $\mathcal{M}$;

(a.2) For all $p \in H$,

$$df(p) \notin C(\mathrm{Ch}(\mathcal{M}), Z_+ \times_M T^*_M X). \tag{1.1}$$

Let $(T^*_N X)^+$ be an open subset of $T^*_N X$ defined by $(T^*_N X)^+ = q^{-1}(N^+)$, with q being the canonical projection $T^*_N X \to T^*_N M$. Let ${}^t\varphi' : T^*X \times_X Y \to T^*Y$ the induced map of φ, and $\rho : T^*_N X \to T^*_N Y$ the restriction of ${}^t\varphi'$ to $T^*_N X$:

$$\begin{array}{ccc} T^*X \times_X Y & \xrightarrow{{}^t\varphi'} & T^*Y \\ \uparrow & & \uparrow \\ T^*_N X & \xrightarrow{\rho} & T^*_N Y \end{array}$$

Let $\widetilde{\mathcal{M}} = \mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{M}$, the $\mathcal{E}_X$-module associated to $\mathcal{M}$. Let $\varphi^*\widetilde{\mathcal{M}}$ denote the induced $\mathcal{E}_Y$-module of $\widetilde{\mathcal{M}}$ on Y. We define the $\mathcal{E}_Y$-module $\mathcal{N}^+$ (on $\dot{T}^*_N Y$) by

$$\mathcal{N}^+ = \rho_*(\mathbf{C}_{(T^*_N X)^+} \otimes \mathcal{E}_{Y\to X} \otimes_{\mathcal{E}_X} \mathcal{M}). \tag{1.2}$$

Then we have

Lemma 1.1. *If we assume* (a.1) *and* (a.2), *$\mathcal{N}^+$ is $\mathcal{E}_Y$-coherent on $\dot{T}^*_N Y$ and there is an $\mathcal{E}_Y$-linear homomorphism $\mathcal{N}^+ \to \varphi^*\widetilde{\mathcal{M}}$ such that*

$$\mathcal{N}^+_q \cong \bigoplus_{p \in (T^*_N X)^+ \cap \mathrm{Supp}(\widetilde{\mathcal{M}}) \cap \rho^{-1}(q)} \mathcal{E}_{Y\to X} \otimes_{\mathcal{E}_X} \widetilde{\mathcal{M}}_p \tag{1.3}$$

for any $q \in T^*_N Y \setminus N$.

Remark. A coherent $\mathcal{E}_Y$-module $\mathcal{N}^+$ with an $\mathcal{E}_Y$-linear map $\alpha : \mathcal{N}^+ \to \varphi^*\widetilde{\mathcal{M}}$ satisfying (1.3) is unique up to isomorphisms. If $(\mathcal{N}^+_1, \alpha_1)$ and $(\mathcal{N}^+_2, \alpha_2)$ are such pairs, we have an isomorphism β such that

$$\begin{array}{ccc} \mathcal{N}^+_1 & \xrightarrow{\alpha_1} & \varphi^*\widetilde{\mathcal{M}} \\ {\scriptstyle\beta}\downarrow & & \| \\ \mathcal{N}^+_2 & \xrightarrow{\alpha_2} & \varphi^*\widetilde{\mathcal{M}} \end{array}$$

is commutative.

Let $\mathcal{B}_M$ be the sheaf of hyperfunctions on M, $\mathcal{C}_N$ the sheaf of microfunctions on N (cf. [SKK]). Let $\mathrm{or}_{N|M}$ be the relative orientation sheaf of N in M.

Theorem 1.2. *Assume* (a.1) *and* (a.2). *There is an isomorphism*

$$\mathrm{R}\Gamma_{Z_+}\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{B}_M)|_N \otimes \mathrm{or}_{N|M}[1] \cong \mathrm{R}\dot{\pi}_{N*}\mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+,\mathcal{C}_N), \tag{1.4}$$

where $\dot{\pi}_N : \dot{T}^*_N Y \to N$.

Corollary 1.3. *Assume* (a.1) *and* (a.2), *and assume also* $\mathcal{N}^+ = 0$. *Letting* $M_+ = Z_+ \setminus N$, *we have the isomorphism*

$$\mathrm{R}\Gamma_{M_+}\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{B}_M)|_N \xrightarrow{\ \sim\ } \mathrm{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y,\mathcal{B}_N),$$

where $\mathcal{M}_Y$ *is the induced* $\mathcal{D}_Y$*-module of* $\mathcal{M}$.

(This follows from Theorem 1.2 and (3.5.8) of [SKK, Ch. II].)

Remark 1. Theorem 1.2 is first proved for elliptic $\mathcal{D}_X$-modules by Kashiwara and Kawai [KK]. Note that (a.1) and (a.2) are automatically satisfied if $\mathcal{M}$ is elliptic.

Remark 2. Condition (1.1) is an analogue of micro-hyperbolicity [KS2] and naturally appears in microlocal study of boundary value problems (cf. [S2, SZ]). It is well known that, if we assume

$$-df(p) \notin C(\mathrm{Ch}(\mathcal{M}), Z_+ \times_M T^*_M X)$$

at $p \in T^*_M X \cap T^*_N X$, this entails propagation of regularity up to the boundary point p from the positive side of N (see [Kt2, S1, S2, SZ]).

Recall that φ is micro-hyperbolic for $\mathcal{M}$ at $p \in H$ in the sense of [KS2, Definition 2.1.2] if

$$\pm df(p) \notin C(\mathrm{Ch}(\mathcal{M}), T^*_M X)$$

for both $\pm$. Let $\mathcal{A}_M$ be the sheaf of real analytic functions on M.

Theorem 1.4. *Assume that* $\varphi : Y \to X$ *is non characteristic for* $\mathcal{M}$ *and is micro-hyperbolic for* $\mathcal{M}$ *at all* $p \in H$. *Then there is an isomorphism*

$$\mathrm{R}\Gamma_{Z_+}\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{A}_M)|_N \otimes \mathrm{or}_{N|M}[1] \cong \mathrm{R}\dot{\pi}_{N*}\mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+,\mathcal{C}_N) \tag{1.5}$$

as well as isomorphism (1.4), *where* $\mathcal{N}^+$ *is the coherent* $\mathcal{E}_Y$*-module defined by* (1.2).

Example. Let $(x_1, \dots, x_n)$ be a system of local coordinates of M, $Z_+ = \{x_1 \geq 0\}$. Let $D_i = \partial/\partial x_i$, $i = 1, \dots, n$, $D' = (D_2, \dots, D_n)$, and let

$$P(x, D) = D_1^2 I_r - \{A(x, D') + x_1^3 B(x, D')\},$$

where I_r denotes the $r \times r$ identity matrix, $A(x, D')$ is an $r \times r$ matrix of differential operators of order 2 of which the principal symbol $\sigma_2(A)(x, i\eta')$ is a negative semi-definite hermitian matrix on $T_M^* X$ and so is $B(x, D')$. Let $\mathcal{M} = \mathcal{D}_X^r / \mathcal{D}_X^r P$; then $\mathcal{M}$ satisfies (a.1) and (a.2), and $\mathcal{N}^+ = 0$.

§2. Proof of Theorems 1.2 and 1.4

As in [KK], the proof of Theorem 1.2 is divided into two steps. In the first step, we relate the left-hand side of (1.4) to a differential complex with coefficients in $\mathcal{C}_{N|X}$ induced from $\mathcal{M}$. In the second step, proving Lemma 1.1, we complete the proof of Theorem 1.2.

Let us recall the notion of the $\mathcal{E}_X$-module $\mathcal{C}_{Z_+|X}$ due to Kataoka [Kt1] and Schapira [S2]. Following [S2], let

$$\mathcal{C}_{Z_+|X} = \mu\mathrm{hom}(\mathbf{C}_{Z_+}, \mathcal{O}_X) \otimes \mathrm{or}_{M|X}[n].$$

Then all the cohomology groups $H^k(\mathcal{C}_{Z_+|X})$, $k \neq 0$, are zero and $H^0(\mathcal{C}_{Z_+|X})$ is an $\mathcal{E}_X$-module. We identify $\mathcal{C}_{Z_+|X}$ with its zero-th cohomology $H^0(\mathcal{C}_{Z_+|X})$. For the $\mathcal{E}_X$-module $\mathcal{C}_{N|X}$, refer to [KK], [KS2] and also [S1, S2]. (In this paper, we follow the definition of [KK, KS2]: $\mathcal{C}_{N|X} = H^n \mu_N(\mathcal{O}_X) \otimes \mathrm{or}_{N|X}$.) We prepare two lemmas.

Lemma 2.1.

(1) $\mathrm{R}\pi_* \mathcal{C}_{Z_+|X}|_M \cong \mathrm{R}\Gamma_{Z_+} \mathcal{B}_M$.

(2) $\mathrm{supp}(\mathcal{C}_{Z_+|X}) \cap T_N^* X \subset \overline{(T_N^* X)^+}$.

(3) *There is an* $\mathcal{E}_X$*-linear homomorphism* $\mathcal{C}_{N|X} \otimes \mathrm{or}_{N|M} \to \mathcal{C}_{Z_+|X}$*, and this is an isomorphism on* $(T_N^* X)^+$.

For the proof, see [Kt3, Section 4] and [S2, S3].

Lemma 2.2. *If we assume* (1.1) *at a point* p *of* $T_M^* X \cap T_N^* X$*, we have*

$$\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{Z_+|X})|_{T_N^* X} = 0$$

in a neighborhood of p.

Proof. (Cf. the proof of Corollary 3.3 of [SZ].) Let g be a real-valued smooth function defined on X such that $g|_M = f$. We set $h = g \circ \pi$, with $\pi : T^*X \to X$. From (1.1), we have

$$dh \notin C_p(\mathrm{Ch}(\mathcal{M}), Z_+ \times_M T^*_M X).$$

Hence we can find an open subset U of T^*X so that $U \cap \mathrm{Ch}(\mathcal{M}) = \varnothing$,

$$dh \notin C_p(T^*X \setminus U, Z_+ \times_M T^*_M X),$$

and $dh \notin C_p(T^*X \setminus U, U)$. Let $\mathrm{SS}(Z_+)$ denote, for simplicity, the micro-support $\mathrm{SS}(\mathbf{C}_{Z_+})$ of the sheaf $\mathbf{C}_{Z_+}$ on X (cf. [KS1, Section 5.1]). Since

$$\mathrm{SS}(Z_+) \subset (Z_+ \times_M T^*_M X) \cup U$$

in a neighborhood of p, we have $dh \notin C_p(T^*X \setminus U, \mathrm{SS}(Z_+))$. This yields

$$dh \notin C_p(\mathrm{Ch}(\mathcal{M}), \mathrm{SS}(Z_+)).$$

Since

$$\mathrm{SS}(\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{Z_+|X})) \subset C(\mathrm{Ch}(\mathcal{M}), \mathrm{SS}(Z_+)),$$

it follows from the definition of micro-supports that

$$\mathrm{R}\Gamma_{\{h \geq 0\}} \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{Z_+|X})|_{\{h=0\}} = 0$$

in a neighborhood of p. Since $\mathcal{C}_{Z_+|X}$ is supported on $\mathrm{SS}(Z_+)$ and $\mathrm{SS}(Z_+) \subset \{h \geq 0\}$, we have

$$\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{Z_+|X})|_{\{h=0\}} = 0. \qquad \text{Q.E.D.}$$

Since $\mathbf{C}_{Z_+}$ is cohomologically constructible, if we set

$$F = \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X),$$

it follows from [KS1, Proposition 4.4.2] that

$$\begin{aligned} \mathrm{R}\pi_* \mathrm{R}\Gamma_{T^*_X X} \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{Z_+|X})|_N &\cong \mathrm{R}\pi_* \mathrm{R}\Gamma_{T^*_X X} \mu\mathrm{hom}(\mathbf{C}_{Z_+}, F)|_N[n] \\ &\cong F \otimes \mathrm{R}\mathcal{H}om_{\mathbf{C}}(\mathbf{C}_{Z_+}, \mathbf{C}_X)|_N[n] \\ &\cong F \otimes \mathbf{C}_{M_+}|_N \\ &= 0, \end{aligned}$$

where we set $M_+ = Z_+ \setminus N$. Hence, from Lemma 2.1, we have

$$\begin{aligned} \mathrm{R}\Gamma_{Z_+} \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|_N &\cong \mathrm{R}\pi_* \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{Z_+|X})|_N \\ &\cong \mathrm{R}\pi_* \mathrm{R}\Gamma_{T^*X \setminus X} \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{Z_+|X})|_N \\ &\cong \mathrm{R}\pi'_* (\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{Z_+|X})|_{T^*_N X \setminus N}), \end{aligned}$$

where $\pi' : \dot{T}^*_N X \to N$. It then follows from Lemmas 2.1 (2), (3) and 2.2 that

$$\begin{aligned}\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{Z_+|X})|_{T^*_N X\setminus N} &\cong \mathrm{R}\Gamma_{(T^*_N X)^+}(\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{Z_+|X})|_{T^*_N X\setminus N}) \\ &\cong \mathrm{R}\Gamma_{(T^*_N X)^+}\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{N|X}) \otimes \mathrm{or}_{N|M}.\end{aligned}$$

Thus we have

$$\begin{aligned}\text{(2.1)}\qquad \mathrm{R}\Gamma_{Z_+}\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|_N \otimes \mathrm{or}_{N|M}& \\ \cong \mathrm{R}\pi'_*\mathrm{R}\Gamma_{(T^*_N X)^+}\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{N|X}).&\end{aligned}$$

Since $T^*_Y X \cap \mathrm{Supp}(\widetilde{\mathcal{M}}) \subset T^*_X X$, we have

$$\begin{aligned}\text{RHS of (2.1)} &\cong \mathrm{R}\dot{\pi}_{N*}\mathrm{R}\rho_*\mathrm{R}\Gamma_{(T^*_N X)^+}\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{N|X}) \\ &= \mathrm{R}\dot{\pi}_{N*}\mathrm{R}\rho^+_*(\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{N|X})|_{(T^*_N X)^+}),\end{aligned}$$

where we denote by $\rho^+ : (T^*_N X)^+ \to T^*_N Y$ the restriction of ρ. Hence, in summary, we have

$$\begin{aligned}\text{(2.2)}\quad \mathrm{R}\Gamma_{Z_+}\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|_N \otimes \mathrm{or}_{N|M}& \\ \cong \mathrm{R}\dot{\pi}_{N*}\mathrm{R}\rho^+_*(\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{N|X})|_{(T^*_N X)^+}).&\end{aligned}$$

In the rest of this section, we prove

$$\text{(2.3)}\qquad \mathrm{R}\rho^+_*(\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{N|X})|_{(T^*_N X)^+})[1] \cong \mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+, \mathcal{C}_N)$$

on $T^*_N Y \setminus N$. Combining (2.2) and (2.3), we get isomorphism (1.4).

We prepare two lemmas for the second part of the proof. Lemma 1.1 follows from the following Lemma 2.3 with $\Omega = T^*_N Y \setminus N$.

Lemma 2.3. *Let Ω be a conic open subset of $T^*_N Y \setminus N$. Let $\rho : T^*_N X \to T^*_N Y$, and let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module on a conic neighborhood of $\rho^{-1}(\Omega)$. Assume*

(1) *$\varphi : Y \to X$ is non characteristic for $\mathcal{M}$ on a neighborhood of Ω in the sense of* [SKK, Ch. II, Def. 3.5.4].

(2) *For a conic neighborhood U of $\rho^{-1}(\Omega) \cap T^*_M X$, $U \cap (T^*_N X)^+ \cap \mathrm{Supp}(\mathcal{M}) = \varnothing$.*

Then ρ is finite on $\rho^{-1}(\Omega) \cap (T^*_N X)^+ \cap \mathrm{Supp}(\mathcal{M})$, and, if we set

$$\mathcal{N}^+ = \rho_*(\mathbf{C}_{(T^*_N X)^+} \otimes \mathcal{E}_{Y\to X} \otimes_{\mathcal{E}_X} \mathcal{M}),$$

$\mathcal{N}^+$ is a coherent $\mathcal{E}_Y$-module over Ω.

Proof. By hypothesis (1), we may assume that $\mathcal{M}$ is a coherent $\mathcal{E}_X$-module defined on ${}^t\varphi'^{-1}(V)$ for an open conic neighborhood V of Ω in $\dot{T}^*Y$. In order to prove the coherency of $\mathcal{N}^+$ locally on Ω, by taking a local coordinate system (z_1, z') of X which is real on M and such that $Z_+ = M \cap \{z_1 \geq 0\}$, we identify $T^*X \times_X Y = T^*Y \times \mathbf{C}_\tau$ and ${}^t\varphi'$ with the projection to T^*Y. Then

$$
\begin{aligned}
{}^t\varphi'^{-1}(V) &= V \times \mathbf{C}_\tau, \\
{}^t\varphi'^{-1}(V) \cap (T_N^*X)^+ &= \Omega \times \{\operatorname{Re}\tau > 0\}.
\end{aligned}
$$

It then follows from the assumption (1) and (2) that, for any $q \in \Omega$, there exist a neighborhood V_q of q in V and $\delta > 0$ such that $\mathcal{M} = 0$ on $(V_q \cap \Omega) \times \{\tau \mid 0 < \operatorname{Re}\tau < \delta\}$. Again by (1), by making V_q smaller enough if necessary, we may assume also that $\mathcal{M} = 0$ on $V_q \times \{\tau \mid \operatorname{Re}\tau = \delta/2\}$. Let us set

$$
W = V_q \times \mathbf{C}_\tau \quad \text{and} \quad W' = V_q \times \{\tau \mid \operatorname{Re}\tau > \delta/2\};
$$

they are open subsets of $T^*X \times_X Y$. Then ${}^t\varphi'$ is proper on $W' \cap \operatorname{Supp}(\mathcal{M})$, and

$$
\operatorname{Supp}(\mathcal{M}) \cap W \cap (T_N^*X)^+ = \operatorname{Supp}(\mathcal{M}) \cap W' \cap \rho^{-1}(\Omega).
$$

Hence $\mathcal{N}^+ = {}^t\varphi'_*(\mathcal{E}_{Y\to X} \otimes_{\mathcal{E}_X} \mathcal{M}|_{W'})$ on $V_q \cap \Omega$, and it follows from Theorem 3.5.3 of [SKK, Ch. II] that the right-hand side is a coherent $\mathcal{E}_Y$-module. Q.E.D.

Lemma 2.4. *Let $\mathcal{M}$ be as in Lemma 2.3. Then there exists a commutative diagram on Ω*

$$
\begin{array}{ccc}
\mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+, \mathcal{E}_Y) & \xrightarrow{\sim} & \mathrm{R}\rho^+_*(\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{E}_{X\leftarrow Y})|_{(T_N^*X)^+})[1] \\
\uparrow & & \uparrow \\
\mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\varphi^*\mathcal{M}, \mathcal{E}_Y) & \xrightarrow{\sim} & \mathrm{R}\rho_*(\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{E}_{X\leftarrow Y})|_{T_N^*X})[1]
\end{array}
$$

and every horizontal arrow is an isomorphism, where $\rho^+ = \rho|_{(T_N^*X)^+}$ and $\mathcal{N}^+$ is the $\mathcal{E}_Y$-module defined in Lemma 2.3.

Proof. This follows from the definition of $\mathcal{N}^+$ and Theorem 3.5.6 of [SKK, Ch. II]. Q.E.D.

Since $\mathcal{N}^+$ is coherent over $\mathcal{E}_Y$ and ρ^+ is finite on $\operatorname{Supp}(\widetilde{\mathcal{M}}) \cap (T_N^*X)^+$, by Lemma 2.4, we have

$$
\begin{aligned}
\mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+, \mathcal{C}_N) &\cong \mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+, \mathcal{E}_Y) \otimes^{\mathbb{L}}_{\mathcal{E}_Y} \mathcal{C}_N \\
&\cong \rho^+_*[\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{E}_{X\leftarrow Y})|_{(T_N^*X)^+}] \otimes^{\mathbb{L}}_{\mathcal{E}_Y} \mathcal{C}_N[1] \\
&\cong \rho^+_*[\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{E}_{X\leftarrow Y})|_{(T_N^*X)^+} \otimes^{\mathbb{L}}_{\rho^{-1}\mathcal{E}_Y} \rho^{-1}\mathcal{C}_N][1].
\end{aligned}
$$

Using the $\mathcal{E}_X$-linear map $\mathcal{E}_{X\leftarrow Y}\otimes_{\rho^{-1}\mathcal{E}_Y}\rho^{-1}\mathcal{C}_N\to\mathcal{C}_{N|X}$ [KK, II], we have

$$\begin{aligned}(2.4)\qquad &\mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+,\mathcal{C}_N)\\ &\cong\rho_*^+[\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}},\mathcal{E}_{X\leftarrow Y}\otimes^{\mathbb{L}}_{\rho^{-1}\mathcal{E}_Y}\rho^{-1}\mathcal{C}_N)|_{(T_N^*X)^+}][1]\\ &\to\rho_*^+[\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}},\mathcal{C}_{N|X})|_{(T_N^*X)^+}][1].\end{aligned}$$

Let $q\in T_N^*Y\setminus N$. For $k\in\mathbf{Z}$, looking at the stalk at q, we have from (2.4)

$$\mathrm{Ext}^k_{\mathcal{E}_Y}(\mathcal{N}_q^+,\mathcal{C}_{Nq})\to\bigoplus_{p\in(T_N^*X)^+\cap\mathrm{Supp}(\mathcal{M})\cap\rho^{-1}(q)}\mathrm{Ext}^{k+1}_{\mathcal{E}_X}(\widetilde{\mathcal{M}}_p,(\mathcal{C}_{N|X})_p).$$

It follows from the division theorem for the $\mathcal{E}_X$-module $\mathcal{C}_{N|X}$ [KK, II, Proposition 3; KS2, 6.3.1] and the definition of $\mathcal{N}^+$ that this is an isomorphism for any $k\in\mathbf{Z}$; therefore (2.4) is an isomorphism in $D^b(\dot{T}_N^*Y)$. This completes the proof of Theorem 1.2.

Proof of Theorem 1.4. Let $p\in H$. If df is micro-hyperbolic for $\mathcal{M}$ at p, we have

$$\mathrm{R}\Gamma_{\pi^{-1}Z_+}\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{C}_M)_p=0$$

([KS2, Theorem 2.2.1]). Since this holds at all $p\in H$ by hypothesis, we have an isomorphism

$$\mathrm{R}\Gamma_{Z_+}\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{A}_M)|_N\xrightarrow{\ \sim\ }\mathrm{R}\Gamma_{Z_+}\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{B}_M)|_N.$$

Combining this and (1.4), we get (1.5). Q.E.D.

§3. Microlocal Boundary Value Problem

We follow the notation of Section 1. By the proof given in Section 2, we have the following microlocal formula.

Theorem 3.1. *Let Ω be an open conic subset of $T_N^*Y\setminus N$. Let $\rho: T_N^*X\to T_N^*Y$, and let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module defined on $\rho^{-1}(\Omega)$. We assume*

(1) *$\varphi:Y\to X$ is non characteristic for $\mathcal{M}$ on a neighborhood of Ω in the sense of* [SKK, Ch. II, Def. 3.5.4]

(2) *$df(p)\notin C(\mathrm{Supp}(\mathcal{M}),Z_+\times_M T_M^*X)$, at any point p of $\rho^{-1}(\Omega)\cap T_M^*X$.*

We then have the isomorphism

$$(3.1)\qquad \mathrm{R}\rho_*\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M},\mathcal{C}_{Z_+|X})\otimes\mathrm{or}_{N|M}[1]\cong\mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+,\mathcal{C}_N)$$

on Ω, where $\mathcal{N}^+$ is the coherent $\mathcal{E}_Y$-module given by Lemma 2.3.

To obtain the formula above by the proof of Section 2, we have to prove Lemma 2.2 for $\mathcal{E}_X$-modules, and we need to use the micro-support estimate

$$\mathrm{SS}(\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mu\mathrm{hom}(\mathbf{C}_{Z_+}, \mathcal{O}_X)) \subset C(\mathrm{Supp}(\mathcal{M}), \mathrm{SS}(Z_+)) \tag{3.2}$$

for coherent $\mathcal{E}_X$-modules. (Note that $\mu\mathrm{hom}(\mathbf{C}_{Z_+}, \mathcal{O}_X)$ is an object of $D^b(\mathcal{E}_X)$.) This estimate can be obtained from the argument of 10.5.1 of [KS3] and Theorem 10.4.2. (We sketch the proof below.) If once we get the micro-support estimate above, we are able to prove Theorem 3.1 in the same way as in Section 2.

Sketch of Proof of (3.2). (See Sections 10.4 and 10.5 of [KS3] for the details.) We may assume that $\mathcal{M}$ is quasi-isomorphic to $\mathcal{E}_X \otimes_{\mathcal{E}(G;D)} M$ for a bounded complex M of free $\mathcal{E}(G;D)$ modules of finite rank. For the ring $\mathcal{E}(G;D)$, see Section 10.4.1 of [KS3] and [KS2]. Let $x \in D$, and U a sufficiently small, G-round neighborhood of x. Let Ω_0 and Ω_1 be G-open sets in X with $\Omega_0 \subset \Omega_1$ and $\Omega_1 \setminus \Omega_0 \subset U$. We set

$$\mathcal{F} = \phi_G^{-1} \mathrm{R}\Gamma_{\Omega_1 \setminus \Omega_0} \mathrm{R}\phi_{G*} \mathcal{O}_X|_{\Omega_1},$$

with $\phi_G : X \to X_G$ the G-topology map. Then $\mathcal{F} \in D^b(\mathcal{E}(G;D))$, and we have a canonical morphism $\mathcal{F} \to \mathcal{O}_X$ such that $\mathcal{F} \cong \mathcal{O}_X$ in $D^b(\Omega_1; V)$, where

$$V = \mathrm{Int}(\Omega_1 \setminus \Omega_0) \times \mathrm{Int}(G^{\circ a}).$$

We then have

$$\begin{aligned} \mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mu\mathrm{hom}(\mathbf{C}_{Z_+}, \mathcal{O}_X))|_V &\cong \mathrm{R}\mathcal{H}om_{\mathcal{E}(G;D)}(M, \mu\mathrm{hom}(\mathbf{C}_{Z_+}, \mathcal{F}))|_V \\ &\cong \mu\mathrm{hom}(\mathbf{C}_{Z_+}, \mathrm{R}\mathcal{H}om_{\mathcal{E}(G;D)}(M, \mathcal{F}))|_V. \end{aligned}$$

By applying Theorem 10.4.2, we have

$$\mathrm{SS}(\mathrm{R}\mathcal{H}om_{\mathcal{E}(G;D)}(M, \mathcal{F})) \subset \mathrm{Supp}(\mathcal{M}).$$

Use the micro-support estimate for μhom, and we get (3.2). Q.E.D.

We say that $\mathcal{M}$ is locally semi-hyperbolic on Ω if $\mathcal{M}$ satisfies conditions (1) and (2) in Theorem 3.1. This can be described (in the case of $\mathcal{M} = \mathcal{E}_X/\mathcal{E}_X P$) in local coordinates as follows. Let $(x_1, \dots, x_n)$ be a system of local coordinates of M, $Z_+ = \{x_1 \geq 0\}$. Let

$$P = D_1^m + A_1(x, D')D_1^{m-1} + \cdots + A_m(x, D') \tag{3.3}$$

be a micro-differential operator of order m, where $D' = (D_2, \dots, D_n)$, and let $\mathcal{M} = \mathcal{E}_X/\mathcal{E}_X P$. Let (z, ζ) denote the coordinates of T^*X, $z = x + iy$, and $\zeta = \xi + i\eta$. We denote by $\tau = \tau_1(z, \zeta'), \dots, \tau_m(z, \zeta')$ the roots of $\sigma_m(P)(z; \tau, \zeta') = 0$. Then $\mathcal{M}$ is locally semi-hyperbolic on Ω if and only if, for every compact subset K of Ω, there are positive numbers δ and C (depending on K) such that

$$\operatorname{Re} \tau_j(z, \zeta') \le C\{|\xi'| + (|y| + (-x_1)_+)|\eta'|\} \tag{3.4}$$

holds for all $j = 1, \dots, m$ if $(x', i\eta') \in K$, $|x_1| \le \delta$, $|y| \le \delta$, $|\xi'| \le \delta|\eta'|$, and $|\operatorname{Re} \tau_j(z, \zeta')| \le \delta|\eta'|$. Cf. [O, Section 3].

Now let $M_+ = Z_+ \setminus N$, and set

$$\mathcal{C}_{M_+|X} = \mu\mathrm{hom}(\mathbf{C}_{M_+}, \mathcal{O}_X) \otimes \mathrm{or}_{M|X}[n]$$

(cf. Schapira [S2]). This is an object of $D^b(\mathcal{E}_X)$.

If we identify $\mathrm{or}_{N|M}$ with the constant sheaf $\mathbf{C}_N$ by the choice of an orientation, we have

Corollary 3.2. *Assume* (1) *and* (2) *of Theorem* 3.1. *Let* $M_+ = Z_+ \setminus N$. *We have the isomorphism*

$$\mathrm{R}\rho_* \mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{M_+|X}) \cong \mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{M}_Y/\mathcal{N}^+, \mathcal{C}_N) \tag{3.5}$$

on Ω, *where* $\mathcal{M}_Y$ *is the induced* $\mathcal{E}_Y$*-module of* $\mathcal{M}$, *and* $\mathcal{N}^+$ *the coherent* $\mathcal{E}_Y$*-submodule given by Lemma* 2.3.

Proof. This follows from (3.1) and Proposition 3 of [KK, II]. Q.E.D.

Remark. In getting (3.5), it is better to choose the isomorphism $\mathrm{or}_{N|M} \cong \mathbf{C}_N$ so that

$$\begin{array}{ccccc} \mathbf{C}_{Z_+} & \longrightarrow & \mathbf{C}_N & \longrightarrow & \mathbf{C}_{M_+}[1] \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{H}om(\mathbf{C}_{M_+}, \mathbf{C}_M) & \longrightarrow & \mathrm{or}_{N|M} & \longrightarrow & \mathcal{H}om(\mathbf{C}_{Z_+}, \mathbf{C}_M)[1] \end{array}$$

becomes commutative. (This corresponds to choosing a non-degenerate section df of $T^*_N M$ as positive orientation.) By this orientation, the isomorphism (3.5) becomes compatible with the boundary value morphism

$$\mathrm{R}\rho_* \mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{M_+|X}) \to \mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{M}_Y, \mathcal{C}_N)$$

constructed by Schapira [S2].

It will be useful in application to state Corollary 3.2 in the following form.

Corollary 3.3. *Let* Ω *be as in Theorem* 3.1. *Let* $\mathcal{M}$ *be a coherent* $\mathcal{E}_X$*-module defined on* $\rho^{-1}(\Omega)$ *and satisfying condition* (1) *in Theorem* 3.1. *Let* $\mathcal{L}$ *be a coherent quotient* $\mathcal{E}_X$*-module of* $\mathcal{M}$. *Assume that* $\mathcal{L}$ *satisfies condition* (2) *in Theorem* 3.1. *Let* Z *be a closed conic subset of* Ω. *We then have a commutative diagram*

$$\begin{array}{ccc} \mathrm{R}\rho_*\mathrm{R}\Gamma_{\rho^{-1}(Z)}\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{L},\mathcal{C}_{M_+|X}) & \xrightarrow{\alpha} & \mathrm{R}\Gamma_Z\mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{L}_Y/\mathcal{N}^+(\mathcal{L}),\mathcal{C}_N) \\ \downarrow & & \downarrow \\ \mathrm{R}\rho_*\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M},\mathcal{C}_{M_+|X}) & \xrightarrow{\beta} & \mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{M}_Y,\mathcal{C}_N), \end{array}$$

where $\mathcal{N}^+(\mathcal{L})$ *denotes the coherent* $\mathcal{E}_Y$*-module induced from* $\mathcal{L}$ *by Lemma* 2.3, *and* α *is an isomorphism.* (β *is the microlocal boundary value morphism* [S2].)

Let $\mathcal{M}$, $\mathcal{L}$ be as in Corollary 3.3. Assume also that there is a coherent $\mathcal{E}_Y$-module $\mathcal{N}$, and an $\mathcal{E}_Y$-linear map $\mathcal{N} \to \mathcal{M}_Y$ such that the composite $\mathcal{N} \to \mathcal{M}_Y \to \mathcal{L}_Y/\mathcal{N}^+(\mathcal{L})$ is an isomorphism on Ω. Then, by Corollary 3.3, there exists a morphism γ such that

$$\begin{array}{ccc} \mathrm{R}\rho_*\mathrm{R}\Gamma_{\rho^{-1}(Z)}\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M},\mathcal{C}_{M_+|X}) & \xleftarrow{\gamma} & \mathrm{R}\Gamma_Z\mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N},\mathcal{C}_N) \\ \| & & \uparrow \\ \text{the same as above} & \longrightarrow & \mathrm{R}\Gamma_Z\mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{M}_Y,\mathcal{C}_N) \end{array}$$

is commutative (if we start from $\mathrm{R}\Gamma_Z\mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N},\mathcal{C}_N)$). This recovers a result of Oaku [O, Theorem 3] in the case where $\mathcal{M} = \mathcal{E}_X/\mathcal{E}_X P$, with P being a micro-differential operator of (3.3) (except that condition (3.4) is a little more restrictive than condition (C.1) of [O]).

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