

G -surgery on 3-dimensional Manifolds for Homology Equivalences

By

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Abstract

For a finite group G and a G -map $f : X \rightarrow Y$ of degree one, where X and Y are compact, connected, oriented, 3-dimensional, smooth G -manifolds, we give an obstruction element $\sigma(f)$ in a K -theoretic group called the Bak group, with the property: $\sigma(f) = 0$ guarantees that one can perform G -surgery on X so as to convert f to a homology equivalence $f' : X' \rightarrow Y$. Using this obstruction theory, we determine the G -homeomorphism type of the singular set of a smooth action of A_5 on a 3-dimensional homology sphere having exactly one fixed point, where A_5 is the alternating group on five letters.

§1. Introduction

This paper is a continuation of [7]. For a finite group G , we discuss G -equivariant surgery on compact connected oriented 3-dimensional manifolds, and construct an algebraic obstruction to converting a framed G -map of degree one to a homology equivalence by a finite sequence of G -surgeries of free orbit type. The purpose of the current paper is to improve [2, Theorem 1] to Theorem 1.1 below. Moreover, we give a detailed proof of this theorem in the present paper, while [2] omits the details of the proof. In [2], as well as in [7, p.78], we exhibited the importance of 3-dimensional G -surgery theory from the

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viewpoint of smooth actions of G on spheres. In particular, the theory is a key tool constructing smooth actions of A_5 on spheres of dimensions 7 and 8 with exactly one fixed point, and we also apply it in Section 5 to determine the G -homeomorphism type of the singular set of a smooth action of A_5 on a 3-dimensional homology sphere with exactly one fixed point.

Let G be a finite group, $e \in G$ the identity element, and set

$$G(2) = \{g \in G \mid g^2 = e, g \neq e\}.$$

All manifolds are understood to be paracompact smooth manifolds, and G -actions to be smooth, unless otherwise stated.

For a compact G -manifold X , we define the *singular set* X_s (or more precisely $X_{s(G)}$) by

$$X_s = \bigcup_{g \in G \setminus \{e\}} X^g,$$

where X^g is the fixed point set of g in X . In the case $\dim X = 3$, we define

$$G(X) = \{g \in G(2) \mid \dim X^g = 1\}.$$

Here $\dim X^g$ is the maximal dimension of connected components of X^g . We denote by $\mathcal{M}^3(G)$ the family of all compact connected oriented 3-dimensional G -manifolds X (possibly with boundary ∂X) satisfying

$$(\mathcal{M} 1) \quad \dim X_{s(G)} \leq 1.$$

In [2], we assumed the additional condition

$$(\mathcal{M} 2) \quad G(X) = G(2),$$

but this is not necessary in the current paper.

Let $Y \in \mathcal{M}^3(G)$. Then the orientation homomorphism $w = w_Y : G \rightarrow \{\pm 1\}$ is defined by $w(g) = 1$ if $g \in G$ preserves the orientation of Y , and -1 otherwise. For a commutative ring R with identity, the group ring $R[G]$ of G over R is defined to be the set of all formal sums $\sum_{g \in G} r_g g$, with $r_g \in R$. The group ring $R[G]$ has the involution - associated with w_Y which is defined by

$$\left(\sum_{g \in G} r_g g \right)^{-} = \sum_{g \in G} r_g w_Y(g) g^{-1}.$$

We denote by \tilde{Y} the universal covering space of Y . A 1-connected G -map $f : X \rightarrow Y$ (hence $\pi_1(f) : \pi_1(X) \rightarrow \pi_1(Y)$ is surjective) induces the covering

space $\tilde{X} = f^*\tilde{Y}$ of X from \tilde{Y} and the map $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ covering f , giving the pullback diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y. \end{array}$$

The group $\tilde{G} = \pi_1(EG \times_G Y)$ is an extension of G by $\pi_1(Y)$:

$$\{e\} \rightarrow \pi_1(Y) \rightarrow \tilde{G} \rightarrow G \rightarrow G/G \quad (\text{exact})$$

and acts on \tilde{Y} and \tilde{X} (cf. [8]). With respect to the \tilde{G} -actions, \tilde{f} is a \tilde{G} -map.

We say that a finite group H is p -hyperelementary (for p a prime) if H is an extension of a p -group by a cyclic group: $H = C \rtimes P$, where C is cyclic and P is of order p^n for some nonnegative integer n . A hyperelementary group means a p -hyperelementary group for some prime p .

Theorem 1.1. *Let R be a ring such that $\mathbb{Z} \subset R \subset \mathbb{Q}$. Suppose that X and Y in $\mathcal{M}^3(G)$ satisfy*

$$G(X) = G(Y),$$

and $f : (X, \partial X) \rightarrow (Y, \partial Y)$ is a 1-connected G -map of degree one such that

(1.1.1) $\partial f = f|_{\partial X} : \partial X \rightarrow \partial Y$ is a homotopy equivalence, and

(1.1.2) $f_{s(H)} : X_{s(H)} \rightarrow Y_{s(H)}$ is an R -homology equivalence for each hyperelementary subgroup H of G .

Moreover, let $b : T(X) \oplus f^*\eta_- \rightarrow \eta_+$ be an orientation preserving map of G -vector bundles covering f , where η_+ and η_- are oriented real G -vector bundles over Y such that $\eta_+ \supset \varepsilon_Y(\mathbb{R}^4)$. Then there exists an obstruction element $\sigma(f, b)$ in the Bak group $W_3(R[G], \Gamma G(Y); w_Y)$ (cf. [5], [2]), where $\Gamma G(Y)$ is the smallest form parameter on $R[G]$ containing $G(Y)$, with the property that if $\sigma(f, b) = 0$ then one can perform G -surgery on $X \setminus (\partial X \cup X_{s(G)})$ to alter $f : X \rightarrow Y$ a G -map $f' : X' \rightarrow Y$ which is an R -homology equivalence, and $b : T(X) \oplus f^*\eta_- \rightarrow \eta_+$ to a map $b' : T(X') \oplus f'^*\eta_- \rightarrow \eta_+$ of G -vector bundles covering f' .

Remark 1.2. If the reduced projective class group $\tilde{K}_0(\mathbb{Z}[G])$ is trivial, then (1.1.2) can be replaced by the condition that

(1.2.1) $f^P : X^P \rightarrow Y^P$ is mod p homology equivalence for each prime p dividing $|G|$ and every nontrivial p -subgroup P of G .

Theorem 1.1 improves [2, Theorem 1] in two respects. One is that the condition $(\mathcal{M}2)$ is removed, and the other is that Y is not restricted to be simply connected.

In the case without bundle data, we have

Theorem 1.3. *Let R be a ring such that $\mathbb{Z} \subset R \subset \mathbb{Q}$, X and Y in $\mathcal{M}^3(G)$, and $f : (X, \partial X) \rightarrow (Y, \partial Y)$ a 1-connected G -map of degree one satisfying the conditions (1.1.1), (1.1.2) and $(\mathcal{M}2)$ above. Then, there is an obstruction element $\sigma(f)$ in the Bak group $W_3(R[G], \max; \text{triv})$, where \max is the maximal form parameter on $R[G]$ and triv is the trivial homomorphism $G \rightarrow \{1\}$, with the property that if $\sigma(f) = 0$ then one can perform G -surgery on $X \setminus (\partial X \cup X_{s(G)})$ to a G -map $f' : X' \rightarrow Y$ which is an R -homology equivalence.*

For applications of Theorems 1.1 and 1.3, the results of A. Bak, e.g., [2, Theorems 3–5 and Corollary 6], are quite useful, since they guarantee that the G -surgery obstruction vanishes.

The organization of the rest of this paper is as follows. Section 2 treats algebraic preliminaries, including the definition of quadratic forms and G -surgery obstruction groups. The equality in Proposition 2.9 is a key to surgery theory on odd dimensional manifolds. In Section 3, we argue how we assign the G -surgery obstruction $\sigma(f)$ to a G -map $f : X \rightarrow Y$ of degree one satisfying certain conditions. We prove Theorem 1.1 in Section 4. Namely we prove that $\sigma(f) = 0$ guarantees the existence of a finite sequence of G -surgeries converting f to a G -map $f' : X' \rightarrow Y$ which is an R -homology equivalence. In Section 5, we give an application of our G -surgery theory concerned with the singular sets of smooth actions of A_5 on 3-dimensional homology spheres.

§2. Quadratic Modules and the Bak Groups

In the current section we recall the definition of form parameters, quadratic modules and the Bak groups. If the reader is familiar with [2] or [5] then he can skip the section.

Let A be a ring with identity. We always suppose that a finitely generated free A -module has a well-defined rank over A , i.e., if $A^m \cong A^n$ then $m = n$. Let $-$ be an involution on A such that

- (i) $\overline{\overline{a}} = a \quad (a \in A)$,
- (ii) $\overline{a + b} = \overline{a} + \overline{b} \quad (a, b \in A)$,

(iii) $\overline{ab} = \overline{b\bar{a}}$ ($a, b \in A$) and

(iv) $\overline{\bar{1}} = 1$.

Let $s \in \text{Center}(A)$ such that $s\bar{s} = 1$. This element s is called a *symmetry* of A . Then a *form parameter* Γ on A is defined to be an additive subgroup of A such that

($\Gamma 1$) $\{a - s\bar{a} \mid a \in A\} \subset \Gamma \subset \{a \in A \mid a = -s\bar{a}\}$ and

($\Gamma 2$) $a\Gamma\bar{a} \subset \Gamma$ for all $a \in A$.

The maximal and minimal choices are denoted by *max* and *min* respectively, i.e.,

$$\text{max} = \{a \in A \mid a = -s\bar{a}\} \quad \text{and} \quad \text{min} = \{a - s\bar{a} \mid a \in A\}.$$

In the following, quadratic forms and modules are defined depending on the datum

$$\mathbf{A} = (A, -, s, \Gamma)$$

called a *form ring*. Let M be a left A -module. A *sesquilinear form* on M is a biadditive map

$$B : M \times M \rightarrow A$$

such that

$$B(ax, by) = bB(x, y)\bar{a} \quad (a, b \in A, x, y \in M).$$

A sesquilinear form B is called *s-Hermitian* if

$$B(x, y) = s\overline{B(y, x)} \quad (x, y \in M).$$

Definition 2.1. A quadratic \mathbf{A} -module is defined to be a triple (M, B, q) consisting of a finitely generated projective A -module M , an s -Hermitian form $B : M \times M \rightarrow A$ and a map $q : M \rightarrow A/\Gamma$ which satisfy the following conditions (2.1.1)–(2.1.3):

$$(2.1.1) \quad q(ax) = aq(x)\bar{a} \quad (a \in A, x \in M),$$

$$(2.1.2) \quad q(x + y) - q(x) - q(y) \equiv B(x, y) \pmod{\Gamma} \quad (x, y \in M) \text{ and}$$

$$(2.1.3) \quad \tilde{q}(x) + s\overline{\tilde{q}(x)} = B(x, x) \quad (x \in M) \text{ for any lifting } \tilde{q}(x) \in A \text{ of } q(x) \in A/\Gamma.$$

The map $q : M \rightarrow A/\Gamma$ above is called a Γ -*quadratic form*.

A morphism $(M, B, q) \rightarrow (M', B', q')$ of quadratic A -modules is an A -linear map $M \rightarrow M'$ which preserves both Hermitian and quadratic forms. We say that a quadratic A -module (M, B, q) is *nonsingular* if the Hermitian form B is nonsingular, i.e., the map

$$M \rightarrow \text{Hom}_A(M, A); x \mapsto B(x, \)$$

is bijective.

Definition 2.2. Define the *standard hyperbolic A -module \mathbf{H}_m* of rank $2m$ to be the quadratic A -module (A^{2m}, B, q) such that

$$\begin{aligned} A^{2m} & \text{ is a free } A\text{-module with basis } \{e_1, \dots, e_m, f_1, \dots, f_m\}, \\ B \left(\sum_i (a_i e_i + b_i f_i), \sum_i (a'_i e_i + b'_i f_i) \right) & = \sum_i (b'_i \bar{a}_i + s a'_i \bar{b}_i) \text{ and} \\ q \left(\sum_i (a_i e_i + b_i f_i) \right) & = \left[\sum_i b_i \bar{a}_i \right] \text{ in } A/\Gamma, \end{aligned}$$

where $a_i, b_i, a'_i, b'_i \in A$.

Let M and M' be free A -modules with ordered bases $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_\ell\}$. For an A -homomorphism $f : M \rightarrow M'$, we obtain a matrix $\text{Mat}(f) = (m_{ij})$ (or more precisely $\text{Mat}(f; \{x_i\}, \{y_j\})$) by

$$f(x_i) = \sum_{j=1}^{\ell} m_{ij} y_j$$

for each $i = 1, \dots, n$. We use the notation

$$f \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \text{ for } \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}.$$

Then, we can express the relation between f and $\text{Mat}(f)$ by

$$f \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \text{Mat}(f) \begin{pmatrix} y_1 \\ \vdots \\ y_\ell \end{pmatrix}.$$

Proposition 2.3 (cf. [3, p.37, Beware]). *Let $M = \langle x_1, \dots, x_n \rangle_A$ be as above and $f, g \in \text{End}_A(M)$. Then $\text{Mat}(fg)$ is equal to $\text{Mat}(g)\text{Mat}(f)$.*

Let $A^{2m} = \langle e_1, \dots, e_m, f_1, \dots, f_m \rangle_A$ as before and $f \in \text{End}_A(A^{2m})$. Then the matrix $\text{Mat}(f)$ associated with f is expressed in the form

$$\text{Mat}(f) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

using $m \times m$ -matrices $M_{11} = (a_{ij})$, $M_{12} = (b_{ij})$, $M_{21} = (c_{ij})$, $M_{22} = (d_{ij})$.

Proposition 2.4. *An element $f \in \text{End}_A(A^{2m})$ is an automorphism of \mathbf{H}_m if and only if the following (2.4.1)–(2.4.3) are satisfied:*

$$(2.4.1) \quad \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \in \text{GL}_{2m}(A).$$

$$(2.4.2) \quad \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}^{-1} = \begin{pmatrix} {}^t\overline{M_{22}} & s^t\overline{M_{12}} \\ \overline{s^t M_{21}} & {}^t\overline{M_{11}} \end{pmatrix}.$$

$$(2.4.3) \quad \text{The diagonal coefficients of } M_{12}({}^t\overline{M_{11}}) \text{ and } M_{22}({}^t\overline{M_{21}}) \text{ lie in } \Gamma.$$

Proof. We prove the only if part. Let f be an automorphism of \mathbf{H}_m . Then (2.4.1) clearly holds. Observe the relations:

$$B(f(e_i), f(e_j)) = \sum_k (b_{jk}\overline{a_{ik}} + sa_{jk}\overline{b_{ik}}) = 0,$$

$$B(f(f_i), f(f_j)) = \sum_k (d_{jk}\overline{c_{ik}} + sc_{jk}\overline{d_{ik}}) = 0,$$

$$B(f(e_i), f(f_j)) = \sum_k (d_{jk}\overline{a_{ik}} + sc_{jk}\overline{b_{ik}}) = \delta_{ij} \quad \text{and}$$

$$B(f(f_i), f(e_j)) = \sum_k (b_{jk}\overline{c_{ik}} + sa_{jk}\overline{d_{ik}}) = s\delta_{ij}.$$

It follows that

$$\begin{aligned} & \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} {}^t\overline{M_{22}} & s^t\overline{M_{12}} \\ \overline{s^t M_{21}} & {}^t\overline{M_{11}} \end{pmatrix} \\ &= \begin{pmatrix} (a_{ij}) & (b_{ij}) \\ (c_{ij}) & (d_{ij}) \end{pmatrix} \begin{pmatrix} (\overline{d_{ji}}) & s(\overline{b_{ji}}) \\ \overline{s(c_{ji})} & (\overline{a_{ji}}) \end{pmatrix} \\ &= \begin{pmatrix} \sum_k (a_{ik}\overline{d_{jk}} + \overline{s}b_{ik}\overline{c_{jk}}) & \sum_k (sa_{ik}\overline{b_{jk}} + b_{ik}\overline{a_{jk}}) \\ \sum_k (c_{ik}\overline{d_{jk}} + \overline{s}d_{ik}\overline{c_{jk}}) & \sum_k (sc_{ik}\overline{b_{jk}} + d_{ik}\overline{a_{jk}}) \end{pmatrix} \\ &= \begin{pmatrix} I_m & 0_m \\ 0_m & I_m \end{pmatrix}. \end{aligned}$$

This proves (2.4.2). The final condition (2.4.3) follows from the equalities

$$q(f(e_i)) = \left[\sum_k b_{ik} \overline{a_{ik}} \right] = 0 \text{ in } A/\Gamma \text{ and}$$

$$q(f(f_i)) = \left[\sum_k d_{ik} \overline{c_{ik}} \right] = 0 \text{ in } A/\Gamma.$$

The if part also follows from the equalities above.

Q.E.D.

For a $2m \times 2m$ -matrix $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ and $2m' \times 2m'$ -matrix $M' = \begin{pmatrix} M'_{11} & M'_{12} \\ M'_{21} & M'_{22} \end{pmatrix}$, we define the $2(m+m') \times 2(m+m')$ -matrix $M \oplus M'$ to be

$$\begin{pmatrix} M_{11} & 0 & M_{12} & 0 \\ 0 & M'_{11} & 0 & M'_{12} \\ M_{21} & 0 & M_{22} & 0 \\ 0 & M'_{21} & 0 & M'_{22} \end{pmatrix}.$$

We define several matrix groups which will be needed later. Firstly, define

$$\mathrm{SU}_m(A, \Gamma) = \left\{ \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \in \mathrm{M}_{2m, 2m}(A) \mid (2.4.1)\text{--}(2.4.3) \text{ are satisfied} \right\}.$$

This contains the subgroup

$$\mathrm{TU}_m(A, \Gamma) = \left\{ \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \in \mathrm{SU}_m(A, \Gamma) \mid M_{12} = 0 \right\}.$$

We define the 2×2 -matrix σ by

$$\sigma = \begin{pmatrix} 0 & 1 \\ \bar{s} & 0 \end{pmatrix}.$$

Then σ lies in $\mathrm{SU}_1(A, \Gamma)$. We set $\sigma_i = I_{2(i-1)} \oplus \sigma \oplus I_{2(m-i)}$ and

$$\mathrm{RU}_m(A, \Gamma) = \langle \mathrm{TU}_m(A, \Gamma), \sigma_1, \dots, \sigma_m \rangle \subset \mathrm{SU}_m(A, \Gamma).$$

The stabilization homomorphism

$$j_{m, m+1} : \mathrm{SU}_m(A, \Gamma) \rightarrow \mathrm{SU}_{m+1}(A, \Gamma)$$

is defined by $j_{m, m+1}(M) = M \oplus I_2$.

A matrix having one form among the following $2m \times 2m$ -matrices $\varepsilon_{**}(\)$ and $\mathbf{H}(\varepsilon_{**}(\))$ (for some m) is called a Γ -quadratic elementary matrix.

$$\begin{aligned} \varepsilon_{m+i,j}(a) \ (i \neq j, a \in A) : & \begin{cases} \text{the } (k, k)\text{-entry} = 1 \ (k = 1, \dots, 2m), \\ \text{the } (m+i, j)\text{-entry} = a, \\ \text{the } (m+j, i)\text{-entry} = -\overline{sa}, \\ \text{all other entries} = 0. \end{cases} \\ \varepsilon_{i,m+j}(a) \ (i \neq j, a \in A) : & \begin{cases} \text{the } (k, k)\text{-entry} = 1 \ (k = 1, \dots, 2m), \\ \text{the } (i, m+j)\text{-entry} = a, \\ \text{the } (j, m+i)\text{-entry} = -s\overline{a}, \\ \text{all other entries} = 0. \end{cases} \\ \varepsilon_{m+i,i}(a) \ (a \in \overline{\Gamma}) : & \begin{cases} \text{the } (k, k)\text{-entry} = 1 \ (k = 1, \dots, 2m), \\ \text{the } (m+i, i)\text{-entry} = a, \\ \text{all other entries} = 0. \end{cases} \\ \varepsilon_{i,m+i}(a) \ (a \in \Gamma) : & \begin{cases} \text{the } (k, k)\text{-entry} = 1 \ (k = 1, \dots, 2m), \\ \text{the } (i, m+i)\text{-entry} = a, \\ \text{all other entries} = 0. \end{cases} \\ \mathbf{H}(\varepsilon_{i,j}(a)) \ (i \neq j, a \in A) : & \begin{cases} \text{the } (k, k)\text{-entry} = 1 \ (k = 1, \dots, 2m), \\ \text{the } (i, j)\text{-entry} = a, \\ \text{the } (m+j, m+i)\text{-entry} = -\overline{a}, \\ \text{all other entries} = 0. \end{cases} \end{aligned}$$

Using these Γ -quadratic elementary matrices, we define

$$\begin{aligned} \text{EU}_m(A, \Gamma) &= \langle \text{all elementary matrices} \in \text{SU}_m(A, \Gamma) \rangle, \text{ and} \\ \text{FU}_m(A, \Gamma) &= \langle \text{all elementary matrices} \in \text{SU}_m(A, \Gamma) \text{ of type } \varepsilon_{m+i,j}(a) \\ &\quad (i \neq j, a \in A) \text{ or } \varepsilon_{m+i,i}(a) \ (a \in \Gamma), \text{ and } \sigma_1, \dots, \sigma_m \rangle. \end{aligned}$$

For $L = E, F, R, S, T$, we set

$$\text{LU}(A, \Gamma) = \varinjlim_m \text{LU}_m(A, \Gamma).$$

By [1, Corollary 3.9], we obtain

Lemma 2.5. *It holds that $\text{EU}(A, \Gamma) = [\text{SU}(A, \Gamma), \text{SU}(A, \Gamma)]$ the commutator subgroup of $\text{SU}(A, \Gamma)$.*

It is easy to see

$$\text{RU}(A, \Gamma) \supset \text{EU}(A, \Gamma).$$

Thus, the quotient group $\text{SU}(A, \Gamma)/\text{RU}(A, \Gamma)$ is abelian.

Definition 2.6. The *Bak group* $W_1^s(A, \Gamma)$, where s is the symmetry of A , is defined to be the quotient group $\mathrm{SU}(A, \Gamma)/\mathrm{RU}(A, \Gamma)$. For a commutative ring R with identity, $W_3(R[G], \Gamma; w)$ stands for $W_1^{-1}(R[G], \Gamma)$, where the involution $-$ on $R[G]$ is one induced by the orientation homomorphism $w: G \rightarrow \{\pm 1\}$.

We obtain the next proposition by straightforward calculation.

Proposition 2.7. *If $\tau(\) = \varepsilon_{**}(\)$ is a Γ -quadratic elementary matrix, then*

$$(2.7.1) \quad \tau(a)\tau(b) = \tau(a + b).$$

If i, j and k are distinct, then

$$(2.7.2) \quad \sigma_i^{-1}\sigma_j^{-1}\varepsilon_{m+i,j}(a)\sigma_i\sigma_j = \varepsilon_{i,m+j}(\bar{s}a) \quad (a \in A),$$

$$(2.7.3) \quad \sigma_i^{-1}\varepsilon_{m+i,i}(a)\sigma_i = \varepsilon_{i,m+i}(sa) \quad (a \in \bar{\Gamma}),$$

$$(2.7.4) \quad [\varepsilon_{i,m+j}(1), \varepsilon_{m+i,k}(a)] = \mathbf{H}(\varepsilon_{jk}(-sa)) \quad (a \in A),$$

$$(2.7.5) \quad \sigma_j^{-1}\varepsilon_{m+i,j}(a)\sigma_j = \mathbf{H}(\varepsilon_{ji}(-\bar{a})) \quad (a \in A),$$

where $[x, y] = x^{-1}y^{-1}xy$.

This proposition clearly implies the following two.

Proposition 2.8. *Provided $m \geq 2$, it holds that $\mathrm{FU}_m(A, \Gamma) \supset \mathrm{EU}_m(A, \Gamma)$.*

Proposition 2.9. *For each element $x \in \mathrm{SU}(A, \Gamma)$, it holds that*

$$\mathrm{TU}(A, \Gamma)x\mathrm{FU}(A, \Gamma) = x\mathrm{RU}(A, \Gamma) = \mathrm{RU}(A, \Gamma)x$$

as subsets of $\mathrm{SU}(A, \Gamma)$.

Let (M, B, q) be a quadratic \mathbf{A} -module. We say that a submodule N of M is *totally isotropic* if $B(x, y) = 0$ for all $x, y \in N$ and if $q(x) = 0$ for all $x \in N$.

Proposition 2.10. *Let (A^{2m}, B, q) be a nonsingular quadratic \mathbf{A} -module. If a direct summand N of $M = A^{2m}$ is a free A -module with basis $\{x_1, \dots, x_m\}$ and is totally isotropic, then there exists a totally isotropic complementary direct summand L with A -basis $\{y_1, \dots, y_m\}$ such that $B(x_i, y_j) = \delta_{ij}$ for all i and j .*

Proof. Let L' be a direct summand complementary to N in M , i.e., $N \oplus L' = M$. We denote by p the projection from M to L' , and by f the inclusion of N to M . The Hermitian form B induces an A -homomorphism

$$\Psi: M \rightarrow M^\# = \text{Hom}_A(M, A); \quad x \mapsto B(x, \cdot).$$

Since B is nonsingular, Ψ is an isomorphism. It is easy to see that $f^\# \Psi: L' \rightarrow M^\#$ is an isomorphism, because the rank over A is well-defined. There exist $z_i \in M$, $i = 1, \dots, m$, such that $B(z_i, x_j) = s\delta_{ij}$ (equally $B(x_i, z_j) = \delta_{ij}$). Here we note that $\{p(z_1), \dots, p(z_m)\}$ is a basis of L' . Let us make an inductive assumption that $B(z_i, z_j) = 0$ and $q(z_i) = 0$ for all $i, j \leq k$. Set

$$z'_{k+1} = z_{k+1} - \bar{s} \left(\sum_{i=1}^k B(z_i, z_{k+1})x_i + \tilde{q}(z_{k+1})x_{k+1} \right),$$

where $\tilde{q}(z_{k+1}) \in A$ is a lifting of $q(z_{k+1}) \in A/\Gamma$. If $i \leq k$, then

$$B(z_i, z'_{k+1}) = B(z_i, z_{k+1}) - \bar{s}B(z_i, z_{k+1})s = 0.$$

Furthermore,

$$\begin{aligned} q(z'_{k+1}) &= q(z_{k+1}) + q \left(-\bar{s} \left(\sum_{i=1}^k B(z_i, z_{k+1})x_i + \tilde{q}(z_{k+1})x_{k+1} \right) \right) \\ &\quad + B \left(z_{k+1}, -\bar{s} \left(\sum_{i=1}^k B(z_i, z_{k+1})x_i + \tilde{q}(z_{k+1})x_{k+1} \right) \right) \\ &= q(z_{k+1}) - \bar{s}\tilde{q}(z_{k+1})B(z_{k+1}, x_{k+1}) \\ &= q(z_{k+1}) - \tilde{q}(z_{k+1}) \\ &= 0 \quad \text{in } A/\Gamma \end{aligned}$$

and

$$B(z'_{k+1}, z'_{k+1}) = \tilde{q}(z'_{k+1}) + \overline{s\tilde{q}(z'_{k+1})} = 0$$

because of the condition $(\Gamma 2)$ and $\tilde{q}(z'_{k+1}) \in \Gamma$. Thus, there exist $y_i \in M$, $i = 1, \dots, m$, such that $B(x_i, y_j) = \delta_{ij}$, $B(y_i, y_j) = 0$ and $q(y_i) = 0$. We set $L = \langle y_1, \dots, y_m \rangle_A$. Since $\{p(y_1), \dots, p(y_m)\}$ is a basis of L' , M is the direct sum of N and L . Q.E.D.

We close this section by remarking

$$\text{RU}_m(A, \Gamma) = \langle \text{TU}_m(A, \Gamma), \sigma_1 \rangle$$

as subsets of $\text{SU}_m(A, \Gamma)$.

§3. Definition of G -surgery Obstructions

Let X and Y be elements in $\mathcal{M}^3(G)$, $f : (X, \partial X) \rightarrow (Y, \partial Y)$ a 1-connected G -map of degree one, and R a ring such that $\mathbb{Z} \subset R \subset \mathbb{Q}$. In this section we choose solid tori ($\cong S^1 \times D^2$) in X and disks in Y so that after G -homotopically deforming f , it becomes a prenormal map over R in the sense of [7, Definition 7.1].

First we note that the orientation homomorphism $G \rightarrow \{\pm 1\}$ induced by the G -action on X coincides with that for Y , namely $w_X = w_Y$.

Fix a point y_0 in $\text{Int}(Y_{r(G)})$, where

$$Y_{r(G)} = Y \setminus Y_{s(G)}.$$

After G -homotopically deforming f if necessary, we may assume that f is transverse regular to the point y_0 . We can choose a tiny disk neighborhood D_Y^3 of y_0 in $\text{Int}(Y_{r(G)})$ and tiny disk neighborhoods D_z of $z \in f^{-1}(y_0)$ in $\text{Int}(X_{r(G)})$ such that $D_Y^3 \cap gD_Y^3 = \emptyset$ if $g \neq e$ ($g \in G$), $D_{z_1} \cap gD_{z_2} = \emptyset$ if $z_1 \neq z_2$ or $g \neq e$ ($z_1, z_2 \in f^{-1}(y_0)$, $g \in G$), and

$$f^{-1}(\text{Int}(D_Y)) = \coprod_{z \in f^{-1}(y_0)} \text{Int}(D_z).$$

Fix a reference point y_1 in ∂D_Y^3 . Arbitrarily choose and fix $z_1 \in f^{-1}(y_0)$ and take connecting tubes ($\cong I \times D^2$) between ∂D_{z_1} and the other ∂D_z ($z \in f^{-1}(y_0) \setminus \{z_1\}$) in general position of

$$\text{Int}(X_{r(G)}) \setminus \bigcup_{w \in f^{-1}(y_0)} (G\text{Int}(D_w)).$$

Since $\pi_1(f) : \pi_1(X) \rightarrow \pi_1(Y)$ is surjective, we can choose the connecting tubes so that f maps them to $\{y_1\}$ after G -homotopical deformation of f . Let D_X^3 be the union of all D_z and the connecting tubes. Smoothing corners, D_X^3 becomes diffeomorphic to a 3-dimensional disk. Moreover it holds that $D_X^3 \cap gD_X^3 = \emptyset$ if $g \neq e$ ($g \in G$), and $f(D_X^3) = D_Y^3$.

Next take solid tori T_1, \dots, T_m ($\cong S^1 \times D^2$) in general position in

$$\text{Int}(X_{r(G)} \setminus GD_X^3)$$

such that their cores ($\cong S^1$) generate $\text{Ker}(\pi_1(f)) \otimes_{\mathbb{Z}} R$. Let $\beta_j : S^1 \times D^2 \rightarrow T_j$ be orientation preserving diffeomorphisms and let e_j and f_j be the meridians and longitudes of β_j respectively. Take a reference point $x_1 \in \partial D_X^3$ and a tiny

disk neighborhood D_X^2 of x_1 in ∂D_X^3 . Take connecting tubes between D_X^2 and ∂T_j in general position of

$$\text{Int}(X_{r(G)}) \setminus \text{Int}(GD_X^3) \setminus \prod_{j=1}^m \text{Int}(GT_j).$$

We may suppose that f maps all T_j and connecting tubes to $\{y_1\}$. Let $T(m)$ be the union of D_X^3 , T_1, \dots, T_m and the connecting tubes. Then $T(m)$ is a solid torus of genus m . We may suppose that all e_j and f_j lie on $\partial T(m)$.

We define

$$U = GT(m), \quad V = GD_Y^3, \quad X_0 = X \setminus \text{Int}(U) \quad \text{and} \quad Y_0 = Y \setminus \text{Int}(V).$$

By (1.1.1), $\partial f : \partial X \rightarrow \partial Y$ is not only an R -homology equivalence but also an $R[\pi_1(Y)]$ -homology equivalence. Since f is a map of degree one, f is a prenormal map over R as well as over $R[\pi_1(Y)]$ in the sense of [7, Definition 7.1].

Let Λ denote R or $R[\pi_1(Y)]$. If $\partial f : \partial X \rightarrow \partial Y$ is a Λ -homology equivalence then we obtain the associated butterfly diagram over Λ .

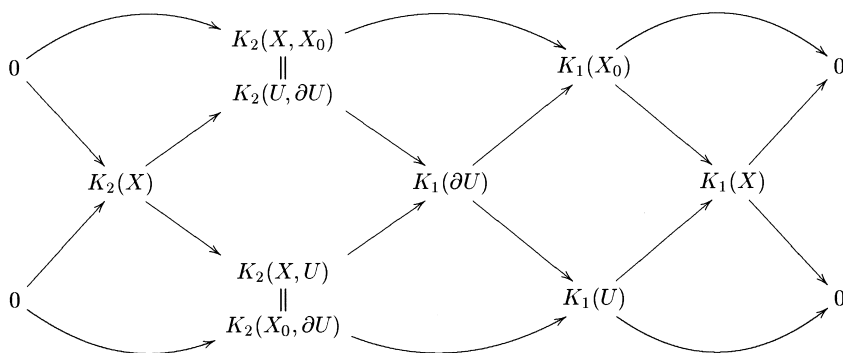


Diagram 3.1

We note that $K_1(\partial U; \Lambda)$ and $K_2(U, \partial U; \Lambda)$ are Λ -free modules of rank $2m$ and m respectively. If (1.1.2) is fulfilled then $K_1(X_0; R)$ and $K_2(X, U; R)$ are stably free $R[G]$ -modules. In such a case, taking sufficiently large m , we may assume that these two modules are free $R[G]$ -modules. Set

$$G(2)_+ = \{g \in G(2) \mid w_Y(g) = 1\} \quad \text{and} \\ G(2)_- = \{g \in G(2) \mid w_Y(g) = -1\}.$$

We adopt -1 as the symmetry of $R[G]$. Let Q be a subset of $\{e\} \cup G(2)_+$ and $\Gamma = \Gamma Q$ the smallest form parameter that includes Q . Using the $R[G]$ -basis

$$\{e_1, \dots, e_m, f_1, \dots, f_m\}$$

of $K = K_1(\partial U; R)$, we algebraically define the sesquilinear form $B : K \times K \rightarrow R[G]$ over the ring $R[G]$ and the Γ -quadratic form $q : K \rightarrow R[G]/\Gamma$ by

$$B \left(\sum_i (a_i e_i + b_i f_i), \sum_j (c_j e_j + d_j f_j) \right) = \left(\sum_i d_i \bar{a}_i - c_i \bar{b}_i \right) \quad \text{and}$$

$$q \left(\sum_i (a_i e_i + b_i f_i) \right) = \sum_i [b_i \bar{a}_i]$$

where a_i, b_i, c_j and d_j are elements in $R[G]$. Then, B coincides with the geometric equivariant intersection form on K . Thus,

$$(3.2) \quad B(K_2(X, U; R), K_2(X, U; R)) = \{0\}.$$

If an automorphism α on the quadratic module $(K_1(\partial U; R), B, q)$ satisfies

$$\alpha(K_2(U, \partial U; R)) = K_2(X_0, \partial U; R)$$

then α is said to be *preferable*. For a preferable automorphism α , the matrix $\text{Mat}(\alpha)$ associated with α with respect to the basis $\{e_1, \dots, e_m, f_1, \dots, f_m\}$ is called a *surgery matrix*. In the case where a preferable automorphism exists, the based quadratic module $(K_1(\partial U; R), B, q)$ determines the surgery matrix uniquely up to the left action of TU_m .

Decompose G into the disjoint union of the form

$$G = \{e\} \amalg G(2) \amalg C \amalg C^{-1},$$

where $C^{-1} = \{g^{-1} \mid g \in C\}$. Then, the map q can be regarded as the collection of maps q_g , where $g \in (\{e\} \cup G(2) \cup C) \setminus Q$, such that

$$q_g : K \rightarrow R/2R \quad (g \in \{e\} \cup G(2)_+ \setminus Q),$$

$$q_g : K \rightarrow R \quad (g \in G(2)_- \text{ and } g \in C),$$

$$q_g : K \rightarrow R \quad (g \in C),$$

via the relation

$$q(x) = \sum_{g \in (\{e\} \cup G(2) \cup C) \setminus Q} [q_g(x)g] \quad (x \in K).$$

Note that

$$B(x, x) = \widetilde{q(x)} - \overline{q(x)},$$

where $\widetilde{q(x)}$ is a lifting of $q(x)$. Thus, it follows from (3.2) that

$$(3.3) \quad q_g(K_2(X, U; R)) = \{0\} \quad (g \in G(2)_- \cup C),$$

and we conclude

Lemma 3.4. *Let $f : (X, \partial X) \rightarrow (Y, \partial Y)$ be as in Theorem 1.3. Then, after a G -homotopical deformation of (f, b) , one obtains Diagram 3.1 and the quadratic module (K, B, q) over $R[G]$, where $K = K_1(\partial U; R)$, $B : K \times K \rightarrow R[G]$ and $q : K \rightarrow R[G]/max$, for which $K_2(X, U; R)$ is totally isotropic.*

Definition 3.5. Let $f : (X, \partial X) \rightarrow (Y, \partial Y)$ be as in Theorem 1.3. We define $\sigma(f)$ by

$$\sigma(f) = [\text{Mat}(\alpha)] \in W_3(R[G], max; triv),$$

after choosing an arbitrary preferable automorphism α of the quadratic module (K, B, q) over $R[G]$, where $K = K_1(\partial U; R)$, $B : K \times K \rightarrow R[G]$ and $q : K \rightarrow R[G]/max$.

Remark 3.6. The algebraic element $\sigma(f)$ above is not necessarily uniquely determined by the originally given G -map f . If the reader likes to obtain a unique algebraic object, then he can adopt

$$\sigma(f) = \left\{ \sigma(f') \in W_3(R[G], max; triv) \left| \begin{array}{l} \text{possible } \{T_1, \dots, T_m\}, \{\beta_1, \dots, \beta_m\}, \text{ and } f' \text{ which} \\ \text{is } G\text{-homotopic to } f \text{ and for which } \sigma(f') \text{ can be} \\ \text{defined with respect to } \{\beta_1, \dots, \beta_m\} \end{array} \right. \right\}$$

instead and read the condition $\sigma(f) = 0$ in Theorem 1.3 as $\sigma(f) \ni 0$.

In the remainder of the current section, we discuss the triviality of q_g for

$$g \in \{e\} \cup G(2)_+ \setminus G(Y).$$

Let η_+ and η_- be oriented real G -vector bundles over Y such that $\eta_+ \supset \varepsilon_Y(\mathbb{R}^4)$. Let $b : T(X) \oplus f^*\eta_- \rightarrow \eta_+$ be an orientation preserving map of G -vector bundles covering $f : X \rightarrow Y$. Let ω_+ and ω_- be G -frames over V of

the oriented G -vector bundles η_+ and η_- respectively. Let $b_U^* \omega_+$ denote the G -frame over U induced by

$$b_U = b| : T(X)_U \oplus (f^* \eta_-)_U \rightarrow (\eta_+)_V$$

from ω_+ , and let $f|_U^* \omega_-$ denote the G -frame over U of $f^* \eta_-$ induced by the canonical map

$$f|_U^* \eta_- \rightarrow (\eta_-)_V$$

covering $f|_U$ from ω_- . Then, applying [7, Proposition 2.2] to X replaced by $T(m)$ above, we obtain a G -frame κ over U of the oriented G -vector bundle $T(U)$ such that $\kappa + f|_U^* \omega_-$ is homotopic to $b_U^* \omega_+$.

Hypothesis 3.7. In the case with bundle data as above, we assume that all β_1, \dots, β_m are preferable in the sense of [7, Definition 7.4].

If $\beta_j : S^1 \times D^2 \rightarrow T_j$ is not preferable then we can replace β_j by a preferable one. For example, adopt $\beta'_j : S^1 \times D^2 \rightarrow T_j$ defined by

$$\beta'_j(z_1, z_2) = \beta_j(z_1, z_1 z_2) \quad (z_1, z_2 \in \mathbb{C} \text{ with } |z_1| = 1 \text{ and } |z_2| \leq 1)$$

instead of β_j . Thus, the arguments developed so far do not lose generality by the hypothesis.

By [7, Theorem 8.1], the quadratic form $q_e : K_1(\partial U; R) \rightarrow R/2R$ vanishes on $K_2(X, U; R)$.

Let g be an element in $G(2)_+ \setminus G(Y)$. Then, g acts freely on X and Y . Thus, $X/\langle g \rangle$ and $Y/\langle g \rangle$ are oriented manifolds and the induced map

$$f/\langle g \rangle : (X/\langle g \rangle, \partial X/\langle g \rangle) \rightarrow (Y/\langle g \rangle, \partial Y/\langle g \rangle)$$

is a map of degree one.

Lemma 3.8. For each $g \in G(2)_+ \setminus G(Y)$, one has $q_g(K_2(X, U; R)) = \{0\}$.

Proof. For the proof, by the definition of q , we may suppose

$$G = \langle g \rangle.$$

Note that

$$\varepsilon(q(x)) = \varepsilon(q_e(x) + q_g(x)g) = q_e(x) + q_g(x),$$

where $\varepsilon : (R/2R)[G] \rightarrow R/2R$ is the augmentation homomorphism. If $x \in K_2(X, U; R)$ then $q_e(x) = 0$ and hence $\varepsilon(q(x)) = q_g(x)$. But $\varepsilon \circ q$ coincides with $q' \circ \pi$, where $q' : K_1(\partial U/G; R) \rightarrow R/2R$ is the algebraic quadratic form associated with the compositions of $\beta_i, i = 1, \dots, m$, with the projection map $X \rightarrow X/G$, and $\pi : K_1(\partial U; R) \rightarrow K_1(\partial U/G; R)$ is the canonical homomorphism. By [7, Theorem 8.1], $q'(z) = 0$ holds for any $z \in K_2(X/G, U/G; R)$. Thus we obtain

$$q_g(x) = \varepsilon(q(x)) = q'(\pi(x)) = 0$$

for $x \in K_2(X, U; R)$.

Q.E.D.

Putting all together, we obtain

Lemma 3.9. *Let (f, b) be as in Theorem 1.1. Then, after a G -homotopical deformation of (f, b) , one obtains Diagram 3.1 and the quadratic module (K, B, q) over $R[G]$, where $K = K_1(\partial U; R)$, $B : K \times K \rightarrow R[G]$ and $q : K \rightarrow R[G]/\Gamma G(Y)$, for which $K_2(X, U; R)$ is totally isotropic.*

Definition 3.10. Let (f, b) be as in Theorem 1.1. We define $\sigma(f, b)$ by

$$\sigma(f, b) = [\text{Mat}(\alpha)] \in W_3(R[G], \Gamma G(Y); w),$$

after choosing an arbitrary preferable automorphism α on the quadratic module (K, B, q) over $R[G]$, where $K = K_1(\partial U; R)$, $B : K \times K \rightarrow R[G]$ and $q : K \rightarrow R[G]/\Gamma G(Y)$.

§4. Proof of Theorems 1.1 and 1.3

Let (f, b) be as in Theorem 1.1 and $(K_1(\partial U; R), B, q)$ as in Definition 3.10.

G -surgery along β_k . We observe how the surgery matrix $\text{Mat}(\alpha)$ in Definition 3.10 is influenced by the G -surgery along $\beta_k : S^1 \times D^2 \rightarrow X$, where k is a fixed integer with $1 \leq k \leq m$. Let $f' : X' \rightarrow Y$ be the G -map resulting from the G -surgery along β_k , let $\beta'_i = \beta_i$ for all $i \neq k$ and let β'_k be the dual to β_k . Since $\partial \text{Im}(\beta'_i) = \partial \text{Im}(\beta_i)$ we can use the same connecting tubes. We obtain $X'_0, T(m)'$ and U' for $\{\beta'_i\}$ instead of $X_0, T(m)$ and U obtained for $\{\beta_i\}$, respectively. The meridian and longitude of β'_i are denoted by e'_i and f'_i respectively for each i . Clearly, $X'_0 = X_0$ and $\partial U' = \partial U$. However the new

basis of $K_1(\partial U'; R)$ is $\{e'_1, \dots, e'_m, f'_1, \dots, f'_m\}$. Define an $R[G]$ -endomorphism α' of $K_1(\partial GU'; R)$ by

$$\alpha' \begin{pmatrix} e'_1 \\ \vdots \\ e'_m \\ f'_1 \\ \vdots \\ f'_m \end{pmatrix} = \alpha \begin{pmatrix} e_1 \\ \vdots \\ e_m \\ f_1 \\ \vdots \\ f_m \end{pmatrix}.$$

Then, noting that $e_k = f'_k$ and $f_k = -e'_k$, we obtain

$$\alpha' \begin{pmatrix} e'_1 \\ \vdots \\ e'_m \\ f'_1 \\ \vdots \\ f'_m \end{pmatrix} = \text{Mat}(\alpha) \begin{pmatrix} e_1 \\ \vdots \\ e_m \\ f_1 \\ \vdots \\ f_m \end{pmatrix} = \text{Mat}(\alpha)\sigma_k \begin{pmatrix} e'_1 \\ \vdots \\ e'_m \\ f'_1 \\ \vdots \\ f'_m \end{pmatrix}.$$

Thus, α' is an automorphism of $(K_1(\partial U'; R), B', q')$, where $B' = B$ and $q' = q$. Furthermore it holds that

$$\begin{aligned} \alpha'(K_2(U', \partial U'; R)) &= \langle \alpha'(e'_1), \dots, \alpha'(e'_m) \rangle_{R[G]} = \langle \alpha(e_1), \dots, \alpha(e_m) \rangle_{R[G]} \\ &= K_2(X_0, \partial U; R) = K_2(X'_0, \partial U'; R). \end{aligned}$$

This shows that α' is preferable for defining $\sigma(f', b')$. By definition, the surgery matrix $\text{Mat}(\alpha')$ associated with α' is $\text{Mat}(\alpha)\sigma_k$. We have proved

Proposition 4.1. *Let $\text{Mat}(\alpha)$ be a surgery matrix for $\{\beta_1, \dots, \beta_m\}$. Then the G -surgery along β_k alters the matrix $\text{Mat}(\alpha)$ to $\text{Mat}(\alpha)\sigma_k$.*

Reversing β_k . Let k be an integer with $1 \leq k \leq m$. We observe how the surgery matrix $\text{Mat}(\alpha)$ in Definition 3.10 is influenced by replacing β_k with $\beta'_k : S^1 \times D^2 \rightarrow T_j$ defined by

$$\beta'_k(x, y) = \beta(\bar{x}, \bar{y}) \text{ for } x \in S^1 \text{ and } y \in D^2,$$

where S^1 and D^2 are regarded as the unit circle and disk of \mathbb{C} respectively, and \bar{x} and \bar{y} stand for the complex conjugates of x and y respectively. For $i \neq k$, we set $\beta'_i = \beta_i$. Moreover, all new data derived from $\{\beta'_1, \dots, \beta'_m\}$ are denoted here by the initial notation with prime ', e.g. e'_i, f'_i, U' etc. Then,

we may adopt $U' = U$ in order to obtain a surgery matrix for $\{\beta'_1, \dots, \beta'_m\}$. Obviously, $e'_k = -e_k$, $f'_k = -f_k$, $e'_j = e_j$ and $f'_j = f_j$, $j \neq k$, hold. Define $\alpha' : K_1(\partial GU'; R) \rightarrow K_1(\partial GU'; R)$ by

$$\alpha' \begin{pmatrix} e'_1 \\ \vdots \\ e'_k \\ \vdots \\ e'_m \\ f'_1 \\ \vdots \\ f'_k \\ \vdots \\ f'_m \end{pmatrix} = \alpha \begin{pmatrix} e_1 \\ \vdots \\ e_k \\ \vdots \\ e_m \\ f_1 \\ \vdots \\ f_k \\ \vdots \\ f_m \end{pmatrix}.$$

This α' is a preferable automorphism, i.e. $\alpha'(K_2(U', \partial U'; R)) = K_2(X'_0, \partial U'; R)$. It is easy to see that the associated surgery matrix $\text{Mat}(\alpha')$ is $\text{Mat}(\alpha)\sigma_k^2$. Thus we obtain

Proposition 4.2. *Let $\text{Mat}(\alpha)$ be a surgery matrix with respect to $\{\beta_1, \dots, \beta_m\}$. Then reversing β_k alters the matrix $\text{Mat}(\alpha)$ to $\text{Mat}(\alpha)\sigma_k^2$.*

For $a \in R^\times$ we define a 2×2 -matrix $\iota(a)$ by

$$\iota(a) = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}.$$

For an integer k with $1 \leq k \leq m$ we define a $2m \times 2m$ -matrix $\iota_k(a)$ by

$$\iota_k(a) = I_{2k-2} \oplus \iota(a) \oplus I_{2m-2k}.$$

Multiplication of β_k . Let a be a natural number such that $1/a \in R$. Then, we can take an orientation preserving embedding $\beta'_k : S^1 \times D^2 \rightarrow \text{Int}(T_k)$, where $T_k = \text{Im}(\beta_k)$, such that $e'_k = (1/a)e_k$ and $f'_k = af_k$ in $H_1(T_k \setminus \text{Int}(T'_k); R)$, where $T'_k = \text{Im}(\beta'_k)$. We call β'_k an a -multiplication of β_k . For $i \neq k$, we adopt $\beta'_i = \beta_i$. Then, subsets U' and X'_0 of X are obtained from $\{\beta'_1, \dots, \beta'_m\}$. Observe the canonical exact sequence

$$K_2(U, U'; R) \rightarrow K_2(X, U'; R) \rightarrow K_2(X, U; R) \rightarrow K_1(U, U'; R).$$

Since $K_2(U, U'; R) = 0$ and $K_1(U, U'; R) = 0$, $K_2(X, U'; R) = K_2(X, U; R)$ holds via the canonical map. Let us observe how replacing β_k with β'_k influences the surgery matrix $\text{Mat}(\alpha)$. Formally using the identities $e_i = e'_i$ and $f_i = f'_i$ ($i \neq k$) together with $e_k = ae'_k$ and $f_k = (1/a)f'_k$, we define $\alpha' : K_1(\partial U'; R) \rightarrow K_1(\partial U'; R)$ by

$$\alpha' \begin{pmatrix} e'_1 \\ \vdots \\ e'_k \\ \vdots \\ e'_m \\ f'_1 \\ \vdots \\ f'_k \\ \vdots \\ f'_m \end{pmatrix} = \alpha \begin{pmatrix} e_1 \\ \vdots \\ e_k \\ \vdots \\ e_m \\ f_1 \\ \vdots \\ f_k \\ \vdots \\ f_m \end{pmatrix}.$$

(The elements $\alpha(e_i)$ and $\alpha(f_i)$ of the right hand side are linear combinations of e_s and f_t , $1 \leq s, t \leq m$. Thus they are linear combinations of e'_s and f'_t .) It holds that $\alpha'(K_2(U', \partial U'; R)) = K_2(X'_0, \partial U'; R)$, namely α' is preferable. The matrix $\text{Mat}(\alpha')$ associated with α' with respect to $\{e'_1, \dots, e'_m, f'_1, \dots, f'_m\}$ is $\text{Mat}(\alpha)\iota_k(a)$. This concludes

Proposition 4.3. *Let $\text{Mat}(\alpha)$ be a surgery matrix with respect to $\{\beta_1, \dots, \beta_m\}$ and a natural number invertible in R . Then taking a-multiplication of β_k converts the matrix $\text{Mat}(\alpha)$ to $\text{Mat}(\alpha)\iota_k(a)$.*

Stabilization. We discuss a stabilization of the surgery matrix $\text{Mat}(\alpha)$.

Proposition 4.4. *Let $\text{Mat}(\alpha)$ be a surgery matrix with respect to $\{\beta_1, \dots, \beta_m\}$. Let $\beta_{m+1} : S^1 \times D^2 \rightarrow X$ be an additional orientation preserving trivial embedding. Perform the G -surgery along β_{m+1} , and let $\beta'_{m+1} : S^1 \times D^2 \rightarrow X'$ be the dual to β_{m+1} . Furthermore let $\beta''_{m+1} : S^1 \times D^2 \rightarrow X'$ be the embedding defined by $\beta''(x, y) = \beta'(\bar{x}, \bar{y})$ for $x \in S^1$ and $y \in D^2$. Then $j_{m, m+1}(\text{Mat}(\alpha)) (= \text{Mat}(\alpha) \oplus I_2)$ is a new surgery matrix for $\{\beta_1, \dots, \beta_m, \beta''_{m+1}\}$.*

Proof. The matrix $\text{Mat}(\alpha) \oplus \sigma$ is a surgery matrix for $\{\beta_1, \dots, \beta_m, \beta_{m+1}\}$. By Proposition 4.1, $(\text{Mat}(\alpha) \oplus \sigma)\sigma_{m+1}$ is a surgery matrix with respect to

$\{\beta_1, \dots, \beta_m, \beta'_{m+1}\}$. Clearly,

$$(\text{Mat}(\alpha) \oplus \sigma)\sigma_{m+1} = \text{Mat}(\alpha) \oplus (-I_2).$$

Thus, by Proposition 4.2, $\text{Mat}(\alpha) \oplus I_2$ is a surgery matrix with respect to $\{\beta_1, \dots, \beta_m, \beta'_{m+1}\}$. Q.E.D.

This proposition allows us to treat surgery matrices in stable range.

Let $\Sigma_m = \sigma \oplus \sigma \oplus \dots \oplus \sigma$ (m -fold sum). By definition, Σ_m lies in $\text{FU}_m(R[G], \Gamma G(Y))$. It is remarkable that

$$\begin{aligned} \sigma(f) = 0 &\Rightarrow \text{Mat}(\alpha) \in \text{RU}_m \quad (\text{for large } m) \\ &\Rightarrow \text{Mat}(\alpha) \in \text{TU}_m \Sigma_m \text{FU}_m \\ &\Rightarrow \text{TU}_m \text{Mat}(\alpha)F_1 \dots F_\ell \ni \Sigma_m \quad \text{for some } F_1, \dots, F_\ell \in \text{FU}_m. \end{aligned}$$

Here we may suppose that each F_i above has one of the forms in the list

$$(4.5) \quad \begin{cases} (4.5.1) & \varepsilon_{m+k,k}(g + \bar{g}), \quad \varepsilon_{m+k,k}(-(g + \bar{g})) \quad (g \in G), \\ (4.5.2) & \varepsilon_{m+h,k}(g), \quad \varepsilon_{m+h,k}(-g) \quad (g \in G \text{ and } h \neq k), \\ (4.5.3) & \varepsilon_{m+k,k}(g), \quad \varepsilon_{m+k,k}(-g) \quad (g \in G(Y)), \\ (4.5.4) & \iota_k(a), \quad \iota_k(-a) \quad (a \in \mathbb{N} \text{ with } 1/a \in R), \\ (4.5.5) & \sigma_k, \end{cases}$$

where $1 \leq h, k \leq m$.

Lemma 4.6. *If $\text{TU}_m \text{Mat}(\alpha)$ contains Σ_m then*

$$K_1(\partial U; R) = K_2(U, \partial U; R) + K_2(X_0, \partial U; R),$$

and $f : X \rightarrow Y$ is an R -homology equivalence.

Proof. This follows from Diagram 3.1. Q.E.D.

For proving Theorem 1.1, it suffices, for each matrix F in (4.5), to find an operation which alters a surgery matrix $\text{Mat}(\alpha)$ to $\text{Mat}(\alpha)F$. We have already found operations for F of types (4.5.4) and (4.5.5). Thus, it suffices to prove

Proposition 4.7. *There exists a G -surgery operation which alters the surgery matrix $\text{Mat}(\alpha)$ to $\text{Mat}(\alpha)F$ for each matrix*

$$(4.7.1) \quad F = \varepsilon_{m+k,k}(\pm(g + \bar{g})) \quad (g \in G),$$

$$(4.7.2) \quad F = \varepsilon_{m+h,k}(\pm g) \quad (g \in G \text{ and } h \neq k),$$

$$(4.7.3) \quad F = \varepsilon_{m+k,k}(\pm g) \quad (g \in G(Y)).$$

Lemma 4.8 (cf. (4.7.1)). *Let $g \in G$ and $\epsilon = 1$ (resp. -1). Perform an isotopical deformation in $X_{r(G)}$ of a small portion of β_k and link it to $g\beta_k$ in the negative (resp. positive) direction to ge_k . This alters the surgery matrix $\text{Mat}(\alpha)$ with respect to $\{\beta_1, \dots, \beta_m\}$ to a surgery matrix $\text{Mat}(\alpha')$ for $\{\beta'_1, \dots, \beta'_m\}$ such that $\text{Mat}(\alpha') = \text{Mat}(\alpha)\epsilon_{m+k,k}(g + \bar{g})$ (resp. $\text{Mat}(\alpha)\epsilon_{m+k,k}(-(g + \bar{g}))$), where $\beta'_i = \beta_i$ if $i \neq k$, and β'_k is the embedding obtained after the deformation of β_k .*

The domain of β_k is $S^1 \times D^2$. A small portion above means the part

$$(\exp((\pi - \delta)\sqrt{-1}), \exp((\pi + \delta)\sqrt{-1})) \times D^2$$

of $S^1 \times D^2$ for small $\delta > 0$.

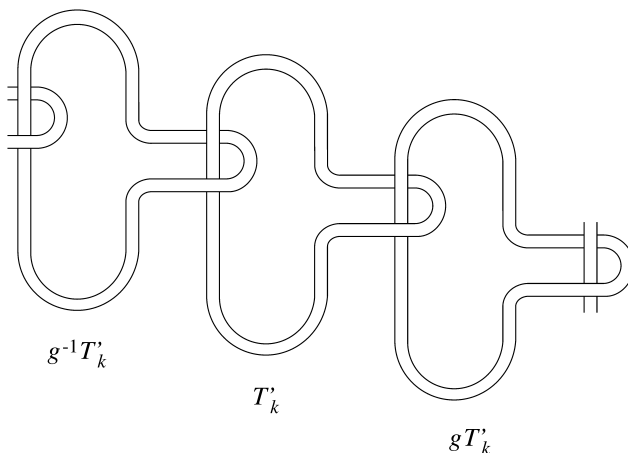


Figure 4.9

Proof. Without loss of generality, we can assume $k = 1$. Define an $R[G]$ -homomorphism $\beta : K_1(\partial GU; R) \rightarrow K_1(\partial GU'; R)$ by

$$\beta \begin{pmatrix} e_1 \\ \vdots \\ e_m \\ f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix} = \begin{pmatrix} e'_1 \\ \vdots \\ e'_m \\ \epsilon(g + \bar{g})e'_1 + f'_1 \\ f'_2 \\ \vdots \\ f'_m \end{pmatrix} \begin{pmatrix} \\ \\ \\ = \epsilon_{m+1,1}(\epsilon(g + \bar{g})) \\ \\ \\ \end{pmatrix} \begin{pmatrix} e'_1 \\ \vdots \\ e'_m \\ f'_1 \\ \vdots \\ f'_m \end{pmatrix}.$$

Clearly, β is an isomorphism $(K_1(\partial GU; R), \lambda, \mu) \rightarrow (K_1(\partial GU'; R), \lambda', \mu')$. Moreover it holds that

$$\beta(K_2(X_0, \partial GU; R)) = K_2(X'_0, \partial GU'; R).$$

Define an endomorphism α' of $K_1(\partial GU'; R)$ by

$$\alpha' \begin{pmatrix} e'_1 \\ \vdots \\ e'_m \\ f'_1 \\ \vdots \\ f'_m \end{pmatrix} = \text{Mat}(\alpha)\varepsilon_{m+1,1}(\epsilon(g + \bar{g})) \begin{pmatrix} e'_1 \\ \vdots \\ e'_m \\ f'_1 \\ \vdots \\ f'_m \end{pmatrix}.$$

Then α' is an automorphism of $(K_1(\partial GU'; R), \lambda', \mu')$. Since $\alpha'(e'_i) = \beta\alpha(e_i)$ for all $i = 1, \dots, m$, we have

$$\alpha'(K_2(GU', \partial GU'; R)) = \beta\alpha(K_2(GU, \partial GU; R)) = K_2(X'_0, \partial GU'; R).$$

Hence, α' is preferable. The associated surgery matrix $\text{Mat}(\alpha')$ is

$$\text{Mat}(\alpha)\varepsilon_{m+1,1}(\epsilon(g + \bar{g})).$$

Q.E.D.

Lemma 4.10 (cf. (4.7.2)). *Let $g \in G$ and $\epsilon = 1$ (resp. -1). Perform an isotopical deformation in $X_{r(G)}$ of a small portion of β_h and link it to $g\beta_k$ in the negative (resp. positive) direction to ge_k . This alters the surgery matrix $\text{Mat}(\alpha)$ for $\{\beta_1, \dots, \beta_m\}$ to a surgery matrix $\text{Mat}(\alpha')$ for $\{\beta'_1, \dots, \beta'_m\}$ such that $\text{Mat}(\alpha') = \text{Mat}(\alpha)\varepsilon_{m+h,k}(g)$ (resp. $\text{Mat}(\alpha)\varepsilon_{m+h,k}(-g)$), where $\beta'_i = \beta_i$ if $i \neq h$, and β'_h is the embedding obtained after the deformation of β_k .*

Proof. We may suppose that $h = 1$ and $k = 2$ without loss of generality. Define a $R[G]$ -homomorphism $\beta : K_1(\partial GU; R) \rightarrow K_1(\partial GU'; R)$ by

$$\beta \begin{pmatrix} e_1 \\ \vdots \\ e_m \\ f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_m \end{pmatrix} = \begin{pmatrix} e'_1 \\ \vdots \\ e'_m \\ \epsilon ge'_2 + f'_1 \\ \epsilon \bar{g}e'_1 + f'_2 \\ f'_3 \\ \vdots \\ f'_m \end{pmatrix} \left(= \varepsilon_{m+1,2}(\epsilon g) \begin{pmatrix} e'_1 \\ \vdots \\ e'_m \\ f'_1 \\ \vdots \\ f'_m \end{pmatrix} \right).$$

Clearly, β is an isomorphism $(K_1(\partial GU; R), \lambda, \mu) \rightarrow (K_1(\partial GU'; R), \lambda', \mu')$. It holds that

$$\beta(K_2(X_0, \partial GU; R)) = K_2(X'_0, \partial GU'; R).$$

Define an $R[G]$ -endomorphism α' of $K_1(\partial GU'; R)$ by

$$\alpha' \begin{pmatrix} e'_1 \\ \vdots \\ e'_m \\ f'_1 \\ \vdots \\ f'_m \end{pmatrix} = \text{Mat}(\alpha)\varepsilon_{m+1,2}(\varepsilon g) \begin{pmatrix} e'_1 \\ \vdots \\ e'_m \\ f'_1 \\ \vdots \\ f'_m \end{pmatrix}.$$

Then α' is an automorphism of $(K_1(\partial GU'; R), \lambda', \mu')$. Since $\alpha'(e'_i) = \beta\alpha(e_i)$ for all $i = 1, \dots, m$, we have

$$\alpha'(K_2(GU', \partial GU'; R)) = \beta\alpha(K_2(GU, \partial GU; R)) = K_2(X'_0, \partial GU'; R).$$

Hence, α' is preferable and the surgery matrix $\text{Mat}(\alpha')$ associated with α' is

$$\text{Mat}(\alpha)\varepsilon_{m+1,2}(\varepsilon g). \quad \text{Q.E.D.}$$

Next we treat the final case (4.7.3) for $g \in G(Y)$. Choose and fix a connected component X^g_γ of X^g such that $\dim X^g_\gamma = 1$. Let H be the principal isotropy subgroup of G on GX^g_γ such that $g \in H$. Set $L = (X^g_\gamma)^H \setminus (X^g_\gamma)^{>H}$. Then, $\dim L = 1$, and $\text{codim } L = 2$. Recall $\dim X^{g'} \leq 1$ for all $g' \neq g$, because of (M1). This implies that H acts freely on the normal fiber of L except the origin. Thus, H must be a cyclic group. We denote a generator of H by h . Clearly we get $g = h^{|H|/2}$. Perform an isotopical deformation of a small portion of β_k and link it to L and set it again in general position. Then β_k is linked to $h^1\beta_k, h^2\beta_k, \dots, h^{|H|-1}\beta_k$. Consider an element $a = h^i$ with $1 \leq i \leq |H|/2 - 1$. Since $\dim X^a = 1$, the action of a on X preserves the orientation and hence $\varepsilon(a + \bar{a}) = \varepsilon(a + a^{-1})$. Using the isotopical deformation in Lemma 4.8, we can eliminate the linking of β_k with $h^i\beta_k$ and the linking of β_k with $h^{|H|-i}\beta_k$ in a pair. Regarding $\{h^1, h^2, \dots, h^{|H|-1}\}$ as $\{(h^1, h^{|H|-1}), (h^2, h^{|H|-2}), \dots, (h^{|H|/2-1}, h^{|H|/2+1}), h^{|H|/2}\}$, we use the technique of pairwise elimination and remove the linking of β_k with $h^i\beta_k$, where $1 \leq i \leq |H| - 1$ and $i \neq |H|/2$. Thus, there exists an isotopical deformation of a small portion of β_k such that β_k is linked only to $g\beta_k$.

Lemma 4.11 (cf. (4.7.3)). *Let $g \in G(Y)$ and $\epsilon = 1$ (resp. -1). Choose and fix a connected component X_γ^g of X^g such that $\dim X_\gamma^g = 1$. Perform an isotopical deformation of a small portion of β_k and link it only to $g\beta_k$ in the negative (resp. positive) direction to ge_k . This alters the surgery matrix $\text{Mat}(\alpha)$ for $\{\beta_1, \dots, \beta_m\}$ to a surgery matrix $\text{Mat}(\alpha')$ for $\{\beta'_1, \dots, \beta'_m\}$ such that $\text{Mat}(\alpha') = \text{Mat}(\alpha)\epsilon_{m+k,k}(g)$ (resp. $\text{Mat}(\alpha)\epsilon_{m+k,k}(-g)$), where $\beta'_i = \beta_i$ if $i \neq k$, and β'_k is the embedding obtained after the deformation of β_k .*

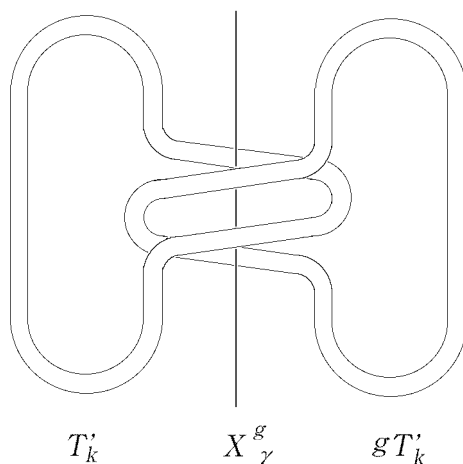


Figure 4.12

Proof. We may suppose $k = 1$. We define an $R[G]$ -homomorphism $\beta : K_1(\partial GU; R) \rightarrow K_1(\partial GU'; R)$ by

$$\beta \begin{pmatrix} e_1 \\ \vdots \\ e_m \\ f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix} = \begin{pmatrix} e'_1 \\ \vdots \\ e'_m \\ \epsilon g e'_1 + f'_1 \\ f'_2 \\ \vdots \\ f'_m \end{pmatrix} \left(= \epsilon_{m+1,1}(\epsilon g) \begin{pmatrix} e'_1 \\ \vdots \\ e'_m \\ f'_1 \\ \vdots \\ f'_m \end{pmatrix} \right).$$

Then β is an isomorphism $(K_1(\partial GU; R), \lambda, \mu) \rightarrow (K_1(\partial GU'; R), \lambda', \mu')$. It holds that

$$\beta(K_2(X_0, \partial GU; R)) = K_2(X'_0, \partial GU'; R).$$

Define an endomorphism α' of $K_1(\partial GU'; R)$ by

$$\alpha' \begin{pmatrix} e'_1 \\ \vdots \\ e'_m \\ f'_1 \\ \vdots \\ f'_m \end{pmatrix} = \text{Mat}(\alpha)\varepsilon_{m+1,1}(\varepsilon g) \begin{pmatrix} e'_1 \\ \vdots \\ e'_m \\ f'_1 \\ \vdots \\ f'_m \end{pmatrix}.$$

Then α' is an automorphism of $(K_1(\partial GU'; R), \lambda', \mu')$. Since $\alpha'(e'_i) = \beta\alpha(e_i)$ for all $i = 1, \dots, m$, we have

$$\alpha'(K_2(GU', \partial GU'; R)) = \beta\alpha(K_2(GU, \partial GU; R)) = K_2(X'_0, \partial GU'; R).$$

Hence, α' is preferable and the surgery matrix $\text{Mat}(\alpha')$ associated with α' is $\text{Mat}(\alpha)\varepsilon_{m+1,1}(\varepsilon g)$. Q.E.D.

Putting all together, we have proved Theorem 1.1.

One can prove Theorem 1.3 by a similar argument using the lemma below.

Let a and k be natural numbers satisfying $1 \leq k \leq m$. Let $\beta_k : S^1 \times D^2 \rightarrow X$ be an embedding as before. Then, $\text{Twist}_a(\beta_k) : S^1 \times D^2 \rightarrow X$ is defined by

$$\text{Twist}_a(\beta_k)(x, y) = \beta_k(x, x^a y) \quad (x \in S^1 \text{ and } y \in D^2).$$

We call $\text{Twist}_a(\beta_k)$ the a -times twisted embedding of β_k .

Lemma 4.13. *Replacing β_k by the $(-a)$ -times twisted embedding $\text{Twist}_{(-a)}(\beta_k)$ alters the surgery matrix $\text{Mat}(\alpha)$ for $\{\beta_1, \dots, \beta_m\}$ to a surgery matrix $\text{Mat}(\alpha')$ for $\{\beta'_1, \dots, \beta'_m\}$ such that $\text{Mat}(\alpha') = \text{Mat}(\alpha)\varepsilon_{m+k,k}(a)$, where $\beta'_i = \beta_i$ if $i \neq k$ and $\beta'_k = \text{Twist}_{(-a)}(\beta_k)$.*

Proof. Without loss of generality, we may assume that $k = 1$. Since $\text{Im } \beta'_1 = \text{Im } \beta_1$, we may use $U' = U$ and $X'_0 = X_0$ to obtain a surgery matrix. Define an $R[G]$ -endomorphism α' of $K_1(\partial GU'; R)$ by

$$\alpha' \begin{pmatrix} e'_1 \\ \vdots \\ e'_m \\ f'_1 \\ \vdots \\ f'_m \end{pmatrix} = \alpha \begin{pmatrix} e_1 \\ \vdots \\ e_m \\ f_1 \\ \vdots \\ f_m \end{pmatrix}.$$

Then we obtain

$$\alpha' \begin{pmatrix} e'_1 \\ \vdots \\ e'_m \\ f'_1 \\ \vdots \\ f'_m \end{pmatrix} = \text{Mat}(\alpha) \begin{pmatrix} e'_1 \\ \vdots \\ e'_m \\ ae'_1 + f'_1 \\ \vdots \\ f'_m \end{pmatrix} = \text{Mat}(\alpha) \varepsilon_{m+1,1}(a) \begin{pmatrix} e'_1 \\ \vdots \\ e'_m \\ f'_1 \\ \vdots \\ f'_m \end{pmatrix},$$

because $e'_1 = e_1$ and $f'_1 = f_1 + (-a)e_1$ in $K_1(\partial U'; R) = K_1(\partial U; R)$. The endomorphism α' is an automorphism of $(K_1(\partial GU'; R), \lambda', \mu')$. Furthermore

$$\alpha'(K_2(GU', \partial GU'; R)) = K_2(X'_0, \partial GU'; R).$$

Thus this α' is preferable for $\{\beta'_1, \dots, \beta'_m\}$. The surgery matrix $\text{Mat}(\alpha')$ satisfies $\text{Mat}(\alpha') = \text{Mat}(\alpha) \varepsilon_{m+1,1}(a)$. Q.E.D.

§5. Singular Sets of Actions of A_5 on Homology 3-spheres

Let A_5 be the alternating group on five letters, and $\text{SO}(3)$ the special orthogonal group of degree three. For a nontrivial homomorphism $\rho : A_5 \rightarrow \text{SO}(3)$, the Poincaré sphere $\Sigma = \Sigma(\rho)$ is defined to be the space of left cosets, $\text{SO}(3)/\rho(A_5)$. A smooth action of A_5 on the Poincaré sphere Σ is naturally given. That is,

$$A_5 \times \Sigma \rightarrow \Sigma; (g, h\rho(A_5)) \mapsto \rho(g)h\rho(A_5).$$

We call this action the *standard action* of A_5 on Σ . This standard action on the Poincaré sphere is investigated in [9]. In particular, there are two A_5 -diffeomorphism (or A_5 -homotopy) types of the Poincaré spheres. They are decided by the characters of homomorphisms $A_5 \rightarrow \text{SO}(3)$. The purpose of this section is to study the G -homeomorphism types of the singular sets of smooth actions of A_5 on homology spheres of dimension 3 with exactly one fixed point, and to prove Theorem 5.5 below.

We denote the cyclic group of order m by C_m and the dihedral group of order $2m$ by D_{2m} . We also denote by A_4 the alternating group on four letters, which is isomorphic to the tetrahedral group. For elements g_1, g_2, \dots, g_n of A_5 , the subgroup of A_5 generated by g_1, g_2, \dots, g_n is denoted by $\langle g_1, g_2, \dots, g_n \rangle$.

Definition 5.1. We set $x = (1, 2)(3, 4)$, $y = (3, 5, 4)$, $z = (1, 2, 3, 5, 4)$ and $u = (1, 3)(2, 4)$ in A_5 . The subgroup $\langle x \rangle$ is properly contained in the

following seven subgroups.

$$\langle x, u \rangle (\cong D_4), \quad \langle x, y \rangle (\cong D_6), \quad \langle x, yu \rangle (\cong D_6), \\ \langle x, z \rangle (\cong D_{10}), \quad \langle x, zu \rangle (\cong D_{10}), \quad \langle x, z^2uz \rangle (\cong A_4) \quad \text{and} \quad A_5$$

The subgroup $\langle x, z^2uz \rangle$ contains $\langle x, u \rangle$. Throughout this section, unless otherwise stated, the above elements x, y, z and u are fixed as above and we use the notation

$$C_2 = \langle x \rangle, C_3 = \langle y \rangle, C_5 = \langle z \rangle$$

$$D_4 = \langle x, u \rangle, D_6 = \langle x, y \rangle, D_{10} = \langle x, z \rangle \quad \text{and} \quad A_4 = \langle x, z^2uz \rangle.$$

Any subgroup of A_5 is conjugate to one of the groups in the next figure (cf. [4, p.10]).

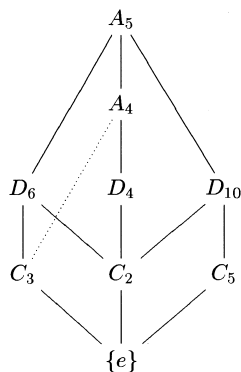


Figure 5.2

Definition 5.3. We denote by \mathfrak{M}^3 the family of closed, oriented, 3-dimensional, smooth A_5 -manifolds X satisfying the following conditions (5.3.1)–(5.3.4).

$$(5.3.1) \quad |X^{A_5}| = 1.$$

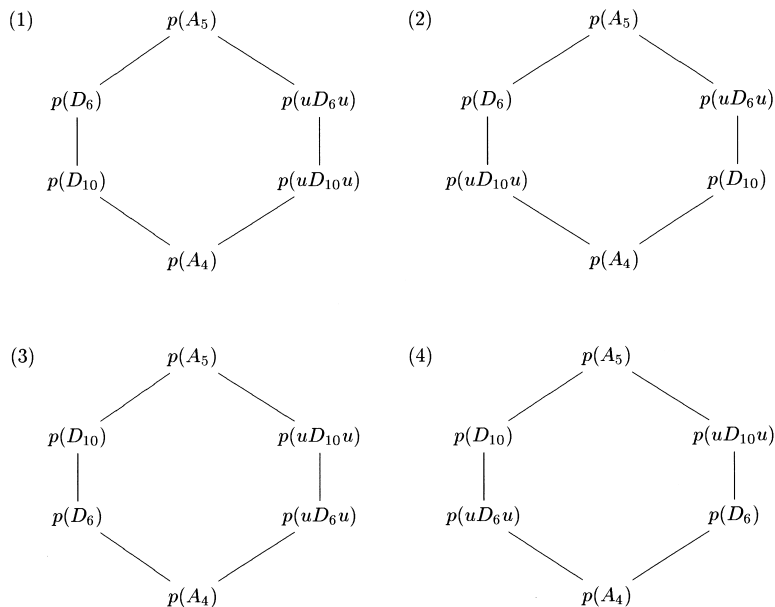
$$(5.3.2) \quad X^H = X^K \quad \text{whenever} \quad H \subset K \subset A_5, H \cong D_4 \quad \text{and} \quad K \cong A_4.$$

$$(5.3.3) \quad |X^H| = 2 \quad \text{whenever} \quad H \subset A_5 \quad \text{and} \quad H \cong D_{2m} \quad \text{for some} \quad m = 2, 3 \quad \text{or} \quad 5.$$

$$(5.3.4) \quad X^H \quad \text{is diffeomorphic to} \quad S^1 \quad \text{whenever} \quad H \subset A_5 \quad \text{and} \quad H \cong C_m \quad \text{for some} \\ m = 2, 3 \quad \text{or} \quad 5.$$

It is well-known that if X is a 3-dimensional homology sphere having a smooth action of A_5 with exactly one fixed point, then X lies in \mathfrak{M}^3 . The Poincaré sphere Σ is a homology sphere and the standard action of A_5 on it has exactly one fixed point. Thus, Σ belongs to \mathfrak{M}^3 .

Let $X \in \mathfrak{M}^3$. For each subgroup H of A_5 such that $H \cong A_5, A_4, D_{10}$ or D_6 , there exists a unique point $p(H) \in X$ with isotropy subgroup H . Imagine that we walk on the circle X^{C_2} starting from and ending at $p(A_5)$. Since $ux = xu$, the action of u gives a diffeomorphism of X^{C_2} fixing $p(A_5)$ and $p(A_4)$ and interchanging $p(D_{2m})$ and $p(uD_{2m}u)$ for $m = 3$ and 5 . Hence, on X^{C_2} , we must meet the intersection points $p(H)$ in one of the following order. (Note: In each case, we do not specify a direction.)



Definition 5.4. Let $C_2, A_4, D_6, D_{10}, uD_6u$ be the specified subgroups of A_5 as in Definition 5.1. We say that $X \in \mathfrak{M}^3$ is of *type* $(A_5 - D_6 - D_{10} - A_4)$, $(A_5 - D_6 - uD_{10}u - A_4)$, $(A_5 - D_{10} - D_6 - A_4)$ or $(A_5 - D_{10} - uD_6u - A_4)$ according as the figure of X^{C_2} is (1), (2), (3) or (4) above.

The type of the Poincaré sphere with standard action is determined in [9, Theorem 1.13]. Let $\rho : A_5 \rightarrow \text{SO}(3)$ be a nontrivial representation. Then the Poincaré sphere $\Sigma(\rho)$ with standard action is of type $(A_5 - D_6 - uD_{10}u - A_4)$

if $\chi_\rho(z) = (1 + \sqrt{5})/2$, and $(A_5 - D_6 - D_{10} - A_4)$ if $(1 - \sqrt{5})/2$, where χ_ρ is the character associated with ρ .

Theorem 5.5. *For any nontrivial real representation $\rho : A_5 \rightarrow \text{SO}(3)$ of A_5 and any type γ of the singular set, there exists a smooth action X of A_5 on a 3-dimensional homology sphere with exactly one fixed point such that the tangential representation at the unique fixed point is isomorphic to $V(\rho)$, the type of the singular set of X is γ , and X is A_5 -cobordant to $\Sigma(\rho)$, where $V(\rho)$ is the A_5 -module associated with ρ .*

Proof. The tangential representation of $\Sigma(\rho)$ at the unique A_5 -fixed point $p(A_5)$ is isomorphic to $V(\rho)$. Take a closed A_5 -disk neighborhood $D(V(\rho))$ around $p(A_5)$ in $\Sigma(\rho)$. Pinching the outside of $\text{Int } D(V(\rho))$, we obtain an A_5 -map $f'' : \Sigma(\rho) \rightarrow Y$ of degree one, where $Y = S(\mathbb{R} \oplus V(\rho))$ and $f''(p(A_5)) = (1, 0)$. It is easy to see that f'' can be converted by A_5 -surgeries of isotropy type (C_2) to an A_5 -map $f : X \rightarrow Y$ such that $X \in \mathfrak{M}^3$ is of type γ (by the same argument as [9, Proof of Lemma 2.4]). Since the employed surgeries are of isotropy type (C_2) , they do not change the fixed point sets of A_5 , A_4 , (D_4) , D_6 and D_{10} . By Theorem 1.3, the A_5 -surgery obstruction $\sigma(f)$ to converting $f : X \rightarrow Y$ to a homology equivalence keeping $f_s : X_s \rightarrow Y_s$ fixed, lies in the group $W_3(\mathbb{Z}[A_5], \text{max}; \text{triv})$. Since by [2, Corollary 6] this group is trivial, we can perform A_5 -surgery, and obtain a homology equivalence $f' : X' \rightarrow Y$. Here $X' \in \mathfrak{M}^3$ is the A_5 -manifold required in Theorem 5.5. Q.E.D.

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