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Theta Constants Associated with the Cyclic Triple Coverings of the Complex Projective Line Branching at Six Points

By

Keiji MATSUMOTO*

Abstract

Let ψ be the period map for a family of the cyclic triple coverings of the complex projective line branching at six points. The symmetric group S_6 acts on this family and on its image under ψ. In this paper, we give an S_6-equivariant expression of ψ^{-1} in terms of fifteen theta constants.

§1. Introduction

Let $C(\lambda)$ be the cyclic triple covering of the complex projective line $\mathbb{P}^1$ branching at six points $\lambda_1, \dots, \lambda_6$:

$$C(\lambda):\ w^3 = \prod_{i=1}^{6}(z-\lambda_i).$$

The moduli space of such curves with a homology marking can be regarded as the configuration space Λ of ordered six distinct points on $\mathbb{P}^1$, which is defined by

$$GL_2(\mathbb{C})\backslash\{\lambda = (\lambda_{ij}) \in M(2,6) \mid \lambda\langle ij\rangle = \begin{vmatrix} \lambda_{1i} & \lambda_{1j} \\ \lambda_{2i} & \lambda_{2j} \end{vmatrix} \neq 0\}/(\mathbb{C}^*)^6.$$

Note that the symmetric group S_6 naturally acts on Λ. It is shown in [15] that the map

$$\iota : \Lambda \ni \lambda \mapsto [\dots, y_{\langle ij;kl;mn\rangle}, \dots] = [\dots, \lambda\langle ij\rangle\lambda\langle kl\rangle\lambda\langle mn\rangle, \dots] \in \mathbb{P}^{14}$$

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*Division of Mathematics, Graduate School of Science, Hokkaido University, Sapporo 060-0810, Japan.

is an S_6-equivariant embedding and that its image is an open subset of Y defined by linear and cubic equations.

The normalized period matrix Ω of $C(\lambda)$ with a homology marking belongs to the Siegel upper half space $\mathbb{S}^4$ of degree 4. By our assignment of the homology marking, Ω can be identified with an element of 3-dimensional complex ball $\mathbb{B}^3 = \{x \in \mathbb{P}^3 \mid {}^t\bar{x}Hx < 0\}$, where $H = \mathrm{diag}(1,1,1,-1)$. In this way, we get a multi-valued map $\psi : \Lambda \to \mathbb{B}^3 \subset \mathbb{S}^4$, which is called the period map. Results in [3], [6] and [13] imply that the image of ψ is an open dense subset of $\mathbb{B}^3$, the monodromy group of ψ is the principal congruence subgroup $\Gamma(1-\omega)$ of level $(1-\omega)$ of $\Gamma = \{g \in GL_4(\mathbb{Z}[\omega]) \mid {}^t\bar{g}Hg = H\}$, and that the inverse of ψ is single valued.

In this paper, we express the inverse of the period map ψ in terms of fifteen theta constants. More precisely, for the two isomorphisms $\psi : \Lambda \to \psi(\Lambda)/\Gamma(1-\omega)$ and $\iota : \Lambda \to \iota(\Lambda) \subset Y \subset \mathbb{P}^{14}$, we present an isomorphism $\Theta : \psi(\Lambda)/\Gamma(1-\omega) \to \iota(\Lambda)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\Lambda & \xrightarrow{\psi} & \psi(\Lambda)/\Gamma(1-\omega) \\
\iota\downarrow & \Theta\swarrow & \\
\iota(\Lambda) \subset Y \subset \mathbb{P}^{14}. & &
\end{array} \tag{1}
$$

The map Θ is given by the ratio of the cubes of the fifteen theta constants on $\mathbb{S}^4$ which are invariant under the action of $\Gamma(1-\omega)$ embedded in $Sp(8,\mathbb{Z})$. Since it is easy to express the inverse of ι^{-1}, the map Θ gives the inverse of ψ. In particular, there are linear and cubic relations among the cubes of fifteen theta constants which coincide with the defining equations of $Y \subset \mathbb{P}^{14}$.

It is known that $\Gamma/\langle\Gamma(1-\omega), -I_4\rangle$ is isomorphic to S_6, which naturally acts on $\psi(\Lambda)/\Gamma(1-\omega)$. The period map ψ is S_6-equivariant. By considering the action $S_6 \simeq \Gamma/\langle\Gamma(1-\omega), -I_4\rangle$ on the fifteen theta characteristics, we label fifteen theta constants as $(ij;kl;mn)$, where $\{i,j,k,l,m,n\} = \{1,\dots,6\}$. Then it turns out that the diagram (1) is S_6-equivariant.

An explicit expression of ψ^{-1} is given in [5]. We want to know the combinatorial structure of ψ^{-1} in order to study the inverse of the period map from a family of smooth cubic surfaces to the 4-dimensional complex ball $\mathbb{B}^4$ in [1]. This inverse map is constructed in [9].

For a 2-dimensional subfamily of ours defined by $\lambda_5 = \lambda_6$, the period map and its inverse are studied in [11] and [12].

§2. Configuration Space Λ of Six Points on $\mathbb{P}^1$

Let $M(m,n)$ be the set of complex $(m \times n)$ matrices. We define the configuration space Λ of ordered six distinct points on the complex projective line $\mathbb{P}^1$ as

$$\Lambda = GL_2(\mathbb{C}) \backslash M'(2,6)/(\mathbb{C}^*)^6,$$

where

$$M'(2,6) = \{\lambda = (\lambda_{ij}) \in M(2,6) \mid \lambda\langle kl\rangle = \begin{vmatrix} \lambda_{1k} & \lambda_{1l} \\ \lambda_{2k} & \lambda_{2l} \end{vmatrix} \neq 0 \ (1 \le k \neq l \le 6)\},$$

and $GL_2(\mathbb{C})$ and $(\mathbb{C}^*)^6$ (regarding as the group of (6×6) diagonal matrices) act naturally on $M'(2,6)$ from the left and right, respectively. Note that we regard the column vectors of $\lambda \in M'(2,6)$ as the homogeneous coordinates of six points on $\mathbb{P}^1$ and the action of $GL_2(\mathbb{C})$ as the projective transformation. Six distinct points $\lambda_1, \ldots, \lambda_6$ on $\mathbb{C}$ are expressed by an element of Λ by (2×6) matrix

$$\lambda = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \end{pmatrix}.$$

By normalizing $(\lambda_1, \lambda_2, \lambda_3)$ as $(\infty, 0, 1)$, matrices of the form

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \ell_1 & \ell_2 & \ell_3 \end{pmatrix}, \quad \ell_i \neq 0, 1, \ell_j \ (1 \le i < j \le 3)$$

represent Λ.

We define a map ι from Λ to the 14-dimensional projective space $\mathbb{P}^{14}$ by

$$\iota : \Lambda \ni \lambda \mapsto [\ldots, y_{\langle ij;kl;mn\rangle}, \ldots] = [\ldots, \lambda\langle ij\rangle\lambda\langle kl\rangle\lambda\langle mn\rangle, \ldots] \in \mathbb{P}^{14},$$

where λ is a (2×6) matrix represent of an element of Λ and projective coordinates of $\mathbb{P}^{14}$ are labeled by $I = \langle ij;kl;mn\rangle$ $(\{i,j,k,l,m,n\} = \{1,\ldots,6\},\ i < j,\ k < l,\ m < n)$. Since the image $\iota(\lambda)$ is invariant under the actions $GL_2(\mathbb{C})$ and $(\mathbb{C}^*)^6$, this map is well defined. We use the following convention

$$y_{\langle ij;kl;mn\rangle} = y_{\langle kl;ij;mn\rangle} = y_{\langle ij;mn;kl\rangle} = -y_{\langle ji;kl;mn\rangle}.$$

The image $\iota(\Lambda)$ is studied in [15], it is described as the following.

Fact 2.1. *The closure $Y = \overline{\iota(\Lambda)}$ of $\iota(\Lambda)$ is a subvariety of $\mathbb{P}^{14}$ defined by the linear and cubic equations*

$$y_{\langle ij;kl;mn\rangle} - y_{\langle ij;km;ln\rangle} + y_{\langle ij;kn;lm\rangle} = 0$$

$$y_{\langle ij;kl;mn\rangle} y_{\langle ik;jn;lm\rangle} y_{\langle im;jl;kn\rangle} = y_{\langle ij;kn;lm\rangle} y_{\langle ik;jl;mn\rangle} y_{\langle im;jn;kl\rangle}.$$

We define $\hat{\Lambda}$ as the compactification of Λ isomorphic to Y.

§3. Period Matrix of C

Let $C = C(\lambda)$ be the cyclic triple covering of $\mathbb{P}^1$ branching at six distinct points λ_i's:

$$C(\lambda):\ w^3 = \prod_{i=1}^{6}(z-\lambda_i);$$

this curve is of genus 4. Let ρ be the automorphism of C defined by

$$\rho : C \ni (z,w) \mapsto (z, \omega w) \in C,$$

where $\omega = (-1+\sqrt{-3})/2$. We give a basis of the vector space of holomorphic 1-forms on C as follows

$$\varphi_1 = \frac{dz}{w},\quad \varphi_2 = \frac{dz}{w^2},\quad \varphi_3 = \frac{zdz}{w^2},\quad \varphi_4 = \frac{z^2dz}{w^2}. \tag{2}$$

For a fixed λ such that $\lambda_i \in \mathbb{R},\ \lambda_1 < \ldots < \lambda_6$, we take a symplectic basis $\{A_1,\ldots,A_4,B_1,\ldots,B_4\}$ of $H_1(C,\mathbb{Z})$ (i.e., $A_i\cdot A_j = B_i\cdot B_j = 0,\ B_i\cdot A_j = \delta_{ij}$) such that

$$\rho(B_i) = A_i\ (i=1,2,3),\quad \rho(B_4) = -A_4, \tag{3}$$

see Figure 1.

Put

$$\begin{pmatrix} \int_{A_i}\varphi_j \\ \int_{B_i}\varphi_j \end{pmatrix}_{i,j} = \begin{pmatrix} \Omega_A \\ \Omega_B \end{pmatrix}.$$

Let φ be the normalized basis of vector space of holomorphic 1-forms so that Ω_B becomes I_4. Note that the normalized period $\Omega = \Omega_A\Omega_B^{-1}$ belongs to the Siegel upper half space $\mathbb{S}^4$ of degree 4. The next proposition shows that Ω can be expressed in terms of

$$x = {}^t(x_1,\ldots,x_4) = {}^t\left(\int_{A_1}\varphi_1,\ldots,\int_{A_4}\varphi_1\right).$$

Proposition 3.1. *We have*

$$\Omega = \omega[I_4 - (1-\omega)(x\,{}^txH)/({}^txHx)]H = \omega[H - (1-\omega)(x\,{}^tx)/({}^txHx)]$$
$$= \begin{pmatrix} \omega &&& \\ & \omega && \\ && \omega & \\ &&& -\omega \end{pmatrix} - \frac{\sqrt{-3}}{x_1^2+x_2^2+x_3^2-x_4^2}\begin{pmatrix} x_1x_1 & x_1x_2 & x_1x_3 & x_1x_4 \\ x_2x_1 & x_2x_2 & x_2x_3 & x_2x_4 \\ x_3x_1 & x_3x_2 & x_3x_3 & x_3x_4 \\ x_4x_1 & x_4x_2 & x_4x_3 & x_4x_4 \end{pmatrix},$$

where $H = \mathrm{diag}(1,1,1,-1)$ *and* ${}^t\bar{x}Hx < 0$.

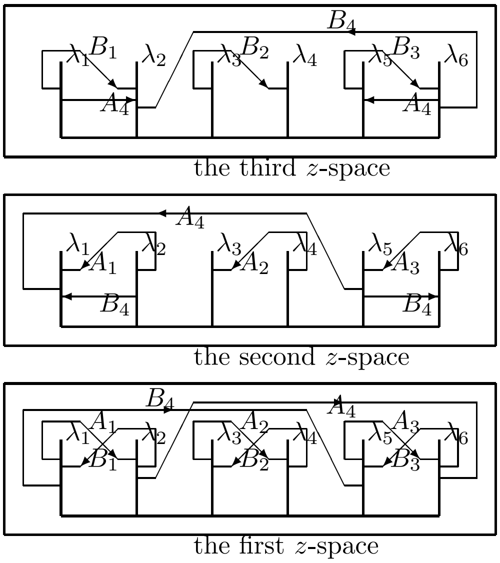

Figure 1. Basis of $H_1(C, \mathbb{Z})$

Proof. Put $\Omega_A = (x, b, c, d)$; by (2) and (3), Ω_B can be expressed as

$$\Omega_B = (\omega H x, \omega^2 H b, \omega^2 H c, \omega^2 H d) = \omega^2 H \Omega_A + (\omega - \omega^2) H(x, O).$$

We have

$$\Omega^{-1} = \Omega_B \Omega_A^{-1} = \omega^2 H + (\omega - \omega^2) H(x, O) \Omega_A^{-1}.$$

Put

$$\Omega_A^{-1} = \begin{pmatrix} \xi \\ * \end{pmatrix}, \quad \xi = (\xi_1, \ldots, \xi_4);$$

note that

$$\xi x = \sum_{i=1}^{4} \xi_i x_i = 1.$$

We have

$$H(x, O)\Omega_A^{-1} = Hx\xi = \frac{1}{\xi x} \begin{pmatrix} x_1\xi_1 & x_1\xi_2 & x_1\xi_3 & x_1\xi_4 \\ x_2\xi_1 & x_2\xi_2 & x_2\xi_3 & x_2\xi_4 \\ x_3\xi_1 & x_3\xi_2 & x_3\xi_3 & x_3\xi_4 \\ -x_4\xi_1 & -x_4\xi_2 & -x_4\xi_3 & -x_4\xi_4 \end{pmatrix}, \tag{4}$$

which must be symmetric. Thus we have

$$x_i\xi_j = x_j\xi_i \ (1 \le i < j \le 3), \quad x_i\xi_4 = -x_4\xi_i \ (i = 1, 2, 3).$$

By eliminating ξ_i in (4), we have

$$H(x, O)\Omega_A^{-1} = (Hx\,{}^t xH)/({}^t xHx).$$

Then

$$\Omega^{-1} = \omega^2 H[I_4 - (1-\omega^2)(x\,{}^t xH)/({}^t xHx)].$$

It is easy to see that

$$[I_4 - (1-\omega^2)(x\,{}^t xH)/({}^t xHx)]^{-1} = I_4 - (1-\omega)(x\,{}^t xH)/({}^t xHx),$$

we have

$$\Omega = \omega[I_4 - (1-\omega)(x\,{}^t xH)/({}^t xHx)]H.$$

The imaginary part of Ω is $\sqrt{3}/2$ times

$$H - x\,{}^t x/({}^t xHx) - \bar{x}\,{}^t\bar{x}/({}^t\bar{x}H\bar{x}), \tag{5}$$

which must be positive definite. If $x_4 = 0$ then the $(4, 4)$ component of (5) is -1, which implies that (5) can not be positive definite. Thus we have $x_4 \neq 0$. Put

$$\eta = \begin{pmatrix} x_4 & 0 & 0 \\ 0 & x_4 & 0 \\ 0 & 0 & x_4 \\ x_1 & x_2 & x_3 \end{pmatrix};$$

note that $(\eta, x) \in GL_4(\mathbb{C})$ and that ${}^t xH\eta = (0, 0, 0)$. We have

$${}^t\overline{(\eta, x)H}\left(H - \frac{x\,{}^t x}{{}^t xHx} - \frac{\bar{x}\,{}^t\bar{x}}{{}^t\bar{x}H\bar{x}}\right)H(\eta, x) = \begin{pmatrix} {}^t\bar{\eta}H\eta & 0 \\ {}^t 0 & -\,{}^t\bar{x}Hx \end{pmatrix}.$$

If

$$-\bar{x}Hx = -|x_1|^2 - |x_2|^2 - |x_3|^2 + |x_4|^2 > 0$$

then the 3×3 matrix

$${}^t\bar{\eta}H\eta = |x_4|^2 I_3 - \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix}(x_1, x_2, x_3)$$

is positive definite. Hence the matrix (5) is positive definite if and only if

$${}^t\bar{x}Hx = |x_1|^2 + |x_2|^2 + |x_3|^2 - |x_4|^2 < 0. \qquad \square$$

We embedded the domain $\mathbb{B}^3 = \{x \in \mathbb{P}^3 \mid {}^t\bar{x}Hx < 0\}$ in $\mathbb{S}^4$ by the map

$$\jmath : \mathbb{B}^3 \ni x \mapsto \Omega = \omega[I_4 - (1-\omega)(x\,{}^txH)/({}^txHx)]H \in \mathbb{S}^4.$$

§4. Monodromy

Let $(\lambda_1, \ldots, \lambda_6)$ vary as an element in Λ, we have two multi-valued map

$$\begin{array}{lccc} \psi : & \Lambda & \to & \mathbb{B}^3 \\ & \lambda & \mapsto & x = {}^t\left(\int_{A_1}\varphi_1, \ldots, \int_{A_4}\varphi_1\right), \\ \tilde{\psi} = \jmath \circ \psi : & \Lambda & \to & \mathbb{S}^4 \\ & \lambda & \mapsto & \Omega = \jmath(\psi(\lambda)). \end{array}$$

We call them period maps. The map ψ and its monodromy group were studied in [3], [13], [14] and [15], the results are as follows.

Fact 4.1. *The image of ψ is open dense in $\mathbb{B}^3$. The monodromy group of ψ is conjugate to the congruence subgroup*

$$\Gamma(1-\omega) = \{g \in \Gamma \mid g \equiv I_4 \mod (1-\omega)\}$$

of the modular group

$$\Gamma = \{g \in GL_4(\mathbb{Z}[\omega]) \mid {}^t\bar{g}Hg = H\}.$$

The Satake compactification $\hat{\mathbb{B}}^3/\Gamma(1-\omega)$ of $\mathbb{B}^3/\Gamma(1-\omega)$ is isomorphic to Y.

For a column vector $v \in \mathbb{C}^4$ such that ${}^t\bar{v}Hv \neq 0$, we define reflections $R^\omega(v)$ and $R^\zeta(v)$ with root v and exponent ω and $\zeta = -\omega^2$, respectively, as

$$\begin{aligned} R^\omega(v) &= I_4 - (1-\omega)v({}^t\bar{v}Hv)^{-1}\,{}^t\bar{v}H, \\ R^\zeta(v) &= I_4 - (1-\zeta)v({}^t\bar{v}Hv)^{-1}\,{}^t\bar{v}H. \end{aligned}$$

It is shown in [2] that $\Gamma(1-\omega)$ can be generated by fifteen reflections $R^\omega_{ij} = R^\omega(v_{ij})$ $(1 \le i < j \le 6)$ and that Γ by $-I_4$ and five reflections $R^\zeta_{i,i+1} = R^\zeta(v_{i,i+1})$ $(1 \le i \le 5)$, where

$$\begin{array}{lll} v_{12} = {}^t(1,0,0,0), & v_{13} = {}^t(-1,1,0,1), & v_{14} = {}^t(-1,-\omega^2,0,1), \\ v_{15} = {}^t(\omega^2,0,-\omega^2,1), & v_{16} = {}^t(\omega^2,0,\omega,1), & v_{23} = {}^t(\omega^2,1,0,1), \\ v_{24} = {}^t(\omega^2,-\omega^2,0,1), & v_{25} = {}^t(-\omega,0,-\omega^2,1), & v_{26} = {}^t(-\omega,0,\omega,1), \\ v_{34} = {}^t(0,1,0,0), & v_{35} = {}^t(0,-\omega,\omega,1), & v_{36} = {}^t(0,-\omega,-1,1), \\ v_{45} = {}^t(0,1,\omega,1), & v_{46} = {}^t(0,1,-1,1), & v_{56} = {}^t(0,0,1,0). \end{array}$$

The reflections correspond to the following movements of λ_i's. When λ_i goes near to λ_j in the upper half space and turns around λ_j and returns, x becomes $R^{\omega}_{ij}x$. When λ_i and λ_j are exchanged in the upper half space, x becomes $R^{\zeta}_{ij}x$. Since $R^{\zeta}_{i,i+1}$'s are representations of braids, they satisfy

$$R^{\zeta}_{i-1,i}R^{\zeta}_{i,i+1}R^{\zeta}_{i-1,i} = R^{\zeta}_{i,i+1}R^{\zeta}_{i-1,i}R^{\zeta}_{i,i+1} \quad (2 \leq i \leq 5).$$

The embedding $\jmath$ induces the following homomorphism from $U(3,1;\mathbb{C})$ to

$$Sp(8,\mathbb{R}) = \left\{ g \in GL_8(\mathbb{R}) \mid {}^t gJg = J = \begin{pmatrix} O & -I_4 \\ I_4 & O \end{pmatrix} \right\} :$$

$$\tilde{\jmath} : U(3,1;\mathbb{C}) \ni P + \omega Q \mapsto \begin{pmatrix} P & QH \\ -HQ & H(P-Q)H \end{pmatrix} \in Sp(8,\mathbb{R}),$$

where P and Q are real 4×4 matrices. Note that

$$\begin{aligned} \tilde{\jmath}^{-1} : Sp(8,\mathbb{R}) \supset \tilde{\jmath}(U(3,1;\mathbb{C})) \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\ \mapsto A + \omega BH = (-HC + HDH) - \omega HC \in U(3,1;\mathbb{C}). \end{aligned}$$

Let us express the images of $R^{\omega}(v)$ and $R^{\zeta}(v)$ under the map $\tilde{\jmath}$. The image of ωI_4 under $\tilde{\jmath}$ is given by

$$W = \begin{pmatrix} O & H \\ -H & -I_4 \end{pmatrix} \in Sp(8,\mathbb{Z}).$$

For a column vector $v = a + \omega b$ $(a, b \in \mathbb{R}^4)$, define column vectors $v_1 = \left(\begin{smallmatrix} a \\ -Hb \end{smallmatrix}\right)$ and $v_2 = Wv_1$ and form a (8×2) matrix $V = (v_1, v_2)$. Straightforward calculation shows the following.

Proposition 4.1. *If* ${}^t\bar{v}Hv \neq 0$, *then* $\tilde{\jmath}(R^{\omega}(v)) = \tilde{R}^{\omega}(v)$ *and* $\tilde{\jmath}(R^{\zeta}(v)) = \tilde{R}^{\zeta}(v)$ *are given by*

$$I_8 - (I_8 - W)V({}^tVJV)^{-1}\,{}^tVJ, \quad I_8 - (I_8 + W^2)V({}^tVJV)^{-1}\,{}^tVJ,$$

respectively.

Systems of generators of $\tilde{\Gamma}(1-\omega) = \tilde{\jmath}(\Gamma(1-\omega))$ and $\tilde{\Gamma} = \tilde{\jmath}(\Gamma)$ are given by $\tilde{R}^{\omega}_{ij}$'s and $\tilde{R}^{\zeta}_{i,i+1}$'s.

§5. Riemann Theta Constants

The Riemann theta function

$$\vartheta(z,\tau)=\sum_{n=(n_1,\dots,n_r)\in\mathbb{Z}^r}\exp[\pi\sqrt{-1}(n\tau\,{}^t n+2n\,{}^t z)]$$

is holomorphic on $\mathbb{C}^r\times\mathbb{S}^r$ and satisfies

$$\vartheta(z+p,\tau)=\vartheta(z,\tau),\quad \vartheta(z+p\tau,\tau)=\exp[-\pi\sqrt{-1}(p\tau\,{}^t p+2z\,{}^t p)]\vartheta(z,\tau),$$

where $\mathbb{S}^r$ is the Siegel upper half space of degree r and $p\in\mathbb{Z}^r$. It is well known that for $(z,\tau)\in\mathbb{C}\times\mathbb{H}$, $\vartheta(z,\tau)=0$ if and only if $z=(1+\tau)/2+p+q\tau$ $(p,q\in\mathbb{Z})$.

The theta function $\vartheta_{a,b}(z,\tau)$ with characteristics a,b is defined by

$$\begin{aligned}(6)\quad \vartheta_{a,b}(z,\tau)&=\exp[\pi\sqrt{-1}(a\tau\,{}^t a+2a\,{}^t(z+b))]\vartheta(z+a\tau+b,\tau)\\ &=\sum_{n\in\mathbb{Z}^n}\exp[\pi\sqrt{-1}((n+a)\tau\,{}^t(n+a)+2(n+a)\,{}^t(z+b))],\end{aligned}$$

where $a,b\in\mathbb{Q}^r$. Note that

$$(7)\quad \vartheta_{-a,-b}(z,\tau)=\vartheta_{a,b}(-z,\tau),\quad \vartheta_{a+p,b+q}(z,\tau)=\exp(2\pi\sqrt{-1}a\,{}^t q)\vartheta_{a,b}(z,\tau).$$

The function $\vartheta_{a,b}(\tau)=\vartheta_{a,b}(0,\tau)$ of τ is called the theta constant with characteristics a,b. If τ is diagonal, then this function becomes the product of Jacobi's theta constants:

$$\vartheta_{a,b}(\tau)=\prod_{i=1}^{r}\vartheta_{a_i,b_i}(\tau_i),$$

where

$$a=(a_1,\dots,a_r),\ b=(b_1,\dots,b_r),\ \tau=\mathrm{diag}(\tau_1,\dots,\tau_r).$$

The following transformation formula can be found in [7] p.176.

Fact 5.1. *For any* $g=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in Sp(2r,\mathbb{Z})$ *and* $(a,b)\in\mathbb{Q}^{2r}$, *we put*

$$\begin{aligned}g\cdot(a,b)&=(a,b)g^{-1}+\frac{1}{2}(\mathrm{dv}(C\,{}^tD),\mathrm{dv}(A\,{}^tB))\\ \phi_{(a,b)}(g)&=-\frac{1}{2}(a\,{}^tDB\,{}^ta-2a\,{}^tBC\,{}^tb+b\,{}^tCA\,{}^tb)\\ &\quad+\frac{1}{2}(a\,{}^tD-b\,{}^tC)\,{}^t(\mathrm{dv}(A\,{}^tB)),\end{aligned}$$

where $\mathrm{dv}(A)$ *is the row vector consisting of the diagonal components of* A*. Then for every* $g \in Sp(2r,\mathbb{Z})$*, we have*

$$\begin{aligned}&\vartheta_{g\cdot(a,b)}((A\tau+B)(C\tau+D)^{-1})\\&\qquad=\kappa(g)\exp(2\pi\sqrt{-1}\phi_{(a,b)}(g))\det(C\tau+D)^{\frac{1}{2}}\vartheta_{(a,b)}(\tau),\end{aligned}$$

in which $\kappa(g)^2$ *is a* 4*-th root of* 1 *depending only on* g*.*

Proposition 5.1. *There are* $81 = 3^4$ *theta characteristics*

$$(a,b) = (a_1,\dots,a_4,b_1,\dots,b_4)$$

such that

$$g\cdot(a,b) \equiv (a,b) \mod \mathbb{Z}^8$$

for any $g \in \tilde{\Gamma}(1-\omega) \subset Sp(8,\mathbb{Z})$*; they are given by*

$$b = -aH,\ a_i \in \left\{\frac{1}{6},\frac{3}{6},\frac{5}{6}\right\} \quad (i=1,\dots,4). \tag{8}$$

Proof. Since

$$W\cdot(a,b) = (-a+bH,-aH)+\frac{1}{2}(1,1,1,-1,0,0,0,0),$$

we have

$$-aH \equiv b,\ -2a+\frac{1}{2}(1,1,1,-1) \equiv a \mod \mathbb{Z}^4.$$

Thus we have the condition (8). It is easy to check such theta characteristics are invariant under the action on 15 reflections $\tilde{R}^{\omega}_{ij}$. □

We label the 81 characteristics a's by combinatorics of six letters; they are classified to 4 classes. The list of the correspondence between the label of a and $6a$ is as follows:

(12;34;56) $\leftrightarrow \pm(3,3,3,-1)$	(12;35;46) $\leftrightarrow \pm(3,1,1,-3)$	(12;36;45) $\leftrightarrow \pm(3,1,-1,-3)$
(13;24;56) $\leftrightarrow \pm(1,1,3,-3)$	(13;25;46) $\leftrightarrow \pm(1,-1,1,-1)$	(13;26;45) $\leftrightarrow \pm(-1,1,1,1)$
(14;23;56) $\leftrightarrow \pm(1,-1,3,-3)$	(14;25;36) $\leftrightarrow \pm(1,1,1,-1)$	(14;26;35) $\leftrightarrow \pm(1,1,-1,-1)$
(15;23;46) $\leftrightarrow \pm(1,1,-1,1)$	(15;24;36) $\leftrightarrow \pm(1,-1,-1,1)$	(15;26;34) $\leftrightarrow \pm(1,3,1,-3)$
(16;23;45) $\leftrightarrow \pm(1,1,1,1)$	(16;24;35) $\leftrightarrow \pm(1,-1,1,1)$	(16;25;34) $\leftrightarrow \pm(1,3,-1,-3)$

$$(1^2 2) \leftrightarrow (1,3,3,3) \quad (1^2 3) \leftrightarrow (5,1,3,5) \quad (1^2 4) \leftrightarrow (5,5,3,5)$$

$$(1^2 5) \leftrightarrow (5,3,1,1) \quad (1^2 6) \leftrightarrow (5,3,5,1) \quad (2^2 3) \leftrightarrow (1,1,3,5)$$

$$(2^2 4) \leftrightarrow (1,5,3,5) \quad (2^2 5) \leftrightarrow (1,3,1,1) \quad (2^2 6) \leftrightarrow (1,3,5,1)$$

$$(3^2 4) \leftrightarrow (3,1,3,3) \quad (3^2 5) \leftrightarrow (3,5,1,5) \quad (3^2 6) \leftrightarrow (3,5,5,5)$$

$$(4^2 5) \leftrightarrow (3,1,1,5) \quad (4^2 6) \leftrightarrow (3,1,5,5) \quad (5^2 6) \leftrightarrow (3,3,1,3)$$

$$(ij^2) \leftrightarrow -a \text{ for } (i^2 j) \quad 1 \leq i < j \leq 6,$$

$$(123) \leftrightarrow (3,1,3,5) \quad (124) \leftrightarrow (3,5,3,5) \quad (125) \leftrightarrow (3,3,1,1)$$

$$(126) \leftrightarrow (3,3,5,1) \quad (134) \leftrightarrow (1,3,3,1) \quad (135) \leftrightarrow (1,1,1,3)$$

$$(136) \leftrightarrow (1,1,5,3) \quad (145) \leftrightarrow (1,5,1,3) \quad (146) \leftrightarrow (1,5,5,3)$$

$$(156) \leftrightarrow (1,3,3,5) \quad (lmn) \leftrightarrow -a \text{ for } (ijk) \quad \{i,j,k,l,m,n\} = \{1,\dots,6\}$$

$$(123456) \leftrightarrow (3,3,3,3).$$

The first class is characterized by $(6a)H\ {}^t(6a) \equiv 2 \bmod 24$ and the characteristics $(a, -aH)$ with label $(ij;kl;mn)$ is invariant under the actions $\tilde{R}^{\zeta}_{ij}$, $\tilde{R}^{\zeta}_{kl}$ and $\tilde{R}^{\zeta}_{mn}$; the second class is characterized by $(6a)H\ {}^t(6a) \equiv 10 \bmod 24$ and the characteristics $(a, -aH)$ with label $(i^2 j)$ is invariant under the actions $\tilde{R}^{\zeta}_{kl}$ ($\{i,j\} \cap \{k,l\} = \emptyset$) and $\tilde{R}^{\zeta}_{ij} \cdot (a, -aH)$ is $(-a, aH)$ with label (ij^2); the third class is characterized by $(6a)H\ {}^t(6a) \equiv 18 \bmod 24$ and the characteristics $(a, -aH)$ with label (ijk) is invariant under the actions $\tilde{R}^{\zeta}_{lm}$ ($\{i,j,k\} \cap \{l,m\} = \emptyset$ or $\{l,m\}$).

We denote $\vartheta_{a,-aH}(\Omega)$ by $\vartheta_{[6a]}(\Omega)$ or $\vartheta(ij;kl;mn)$, $\vartheta(i^2 j)$, $\vartheta(ijk)$ and $\vartheta(123456)$ for corresponding characteristics a. Note that for $p, q \in \mathbb{Z}^4$,

$$\begin{aligned}
&\vartheta(a(\Omega - H) + p\Omega + q, \Omega) \\
&\quad = \exp[-\pi\sqrt{-1}(p\Omega\ {}^t p + 2p(\Omega - H)\ {}^t a)]\vartheta(a(\Omega - H), \Omega) \\
&\quad = \exp[-\pi\sqrt{-1}(p\Omega\ {}^t p + 2p(\Omega - H)\ {}^t a + a\Omega\ {}^t a - 2aH\ {}^t a)]\vartheta_{a,-aH}(\Omega) \\
&\quad = \exp[2\pi\sqrt{-1}(a+p)H\ {}^t(a+p)]\exp[-\pi\sqrt{-1}(a+p)\Omega\ {}^t(a+p)]\vartheta_{[a]}(\Omega).
\end{aligned}$$

Proposition 5.2. *The theta constants $\vartheta(i^2j)$, $\vartheta(ijk)$ and $\vartheta(123456)$ are identically zero on $\jmath(\mathbb{B}^3)$. The theta constants $\vartheta(ij;kl;mn)$ are not identically zero on $\jmath(\mathbb{B}^3)$.*

Proof. We apply Fact 5.1 for

$$\tau = \Omega = \jmath(x), \quad g = W = \begin{pmatrix} 0 & H \\ -H & -I_4 \end{pmatrix}, \quad (a,b) = (a, -aH).$$

Note that

$$W \cdot \Omega = \Omega, \quad W \cdot (a, -aH) = \left(a - (3a - \frac{1}{2}\mathrm{diag}(H)), -aH\right)$$

and that

$$\phi_{(a,-aH)}(W) = \frac{3}{2}aH\,{}^ta = \frac{1}{24}(6a)H\,{}^t(6a), \quad \det(C\Omega + D) = \omega.$$

Since $\kappa(W)$ is an 8-th root of 1, the sufficient condition for

$$\kappa(W)\exp(2\pi\sqrt{-1}\phi_{(a,b)}(W))\det(C\Omega+D)^{\frac{1}{2}} = 1 \tag{9}$$

is $(6a)H\,{}^t(6a) \equiv 2 \mod 24$. If $(6a)H\,{}^t(6a) \not\equiv 2 \mod 24$, then $\vartheta_{a,-aH}(\Omega)$ vanishes. Thus the theta constants $\vartheta(i^2j)$, $\vartheta(ijk)$ and $\vartheta(123456)$ are identically zero on $\jmath(\mathbb{B}^3)$.

For $a = (\frac{1}{6}, \dots, \frac{1}{6})$ and $x = (0,0,0,1)$, $\vartheta_{a,-aH}(\Omega)$ reduces to

$$\vartheta_{(\frac{1}{6},\frac{-1}{6})}(\omega)^3\vartheta_{(\frac{1}{6},\frac{1}{6})}(-\omega^2),$$

which does not vanish. Hence $\vartheta(ij;kl;mn)$'s survive. Note that $\kappa(W)^2 = -1$ by (9). □

Proposition 5.3. *We have*

$$\vartheta(i,i+1;kl;mn)(\tilde{R}^{\zeta}_{i,i+1}\cdot\jmath(x))^3 = -\chi(\tilde{R}^{\zeta}_{i,i+1})\vartheta(i,i+1;kl;mn)(\jmath(x))^3,$$
$$\vartheta(ik;i+1,l;mn)(\tilde{R}^{\zeta}_{i,i+1}\cdot\jmath(x))^3 = \chi(\tilde{R}^{\zeta}_{i,i+1})\vartheta(il;i+1,k;mn)(\jmath(x))^3,$$

where

$$\chi(\tilde{R}^{\zeta}_{i,i+1}) = \left(\frac{{}^t(R^{\zeta}_{i,i+1}x)H(R^{\zeta}_{i,i+1}x)}{{}^txHx}\right)^{3/2},$$

which takes 1 *on the mirror of* $R^{\zeta}_{i,i+1}$.

Proof. For $\tilde{R}^{\zeta}_{i,i+1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, straightforward calculation shows

$$\det(C\jmath(x) + D) = \frac{{}^t(R^{\zeta}_{i,i+1}x)H(R^{\zeta}_{i,i+1}x)}{\det(R^{\zeta}_{i,i+1})\ {}^t xHx} = \frac{{}^t(R^{\zeta}_{i,i+1}x)H(R^{\zeta}_{i,i+1}x)}{-\omega^2\ {}^t xHx}.$$

By computing $\phi_{a,b}(\tilde{R}^{\zeta}_{i,i+1})$ in Fact 5.1 and using (7), we have

$$\vartheta(i,i+1;kl;mn)(\tilde{R}^{\zeta}_{i,i+1}\cdot \jmath(x))^3 = -c\chi(\tilde{R}^{\zeta}_{i,i+1})\vartheta(i,i+1;kl;mn)(\jmath(x))^3,$$
$$\vartheta(ik;i+1,l;mn)(\tilde{R}^{\omega}_{i,i+1}\cdot \jmath(x))^3 = c\chi(\tilde{R}^{\zeta}_{i,i+1})\vartheta(il;i+1,k;mn)(\jmath(x))^3,$$

where c is a certain constant depending only on $\tilde{R}^{\zeta}_{i,i+1}$. If we restrict $\jmath(x)$ on the mirror of $R^{\zeta}_{i,i+1}$, we have

$$\tilde{R}^{\zeta}_{i,i+1}\cdot \jmath(x) = \jmath(x), \quad \chi(\tilde{R}^{\zeta}_{i,i+1}) = \left(\frac{{}^t(R^{\zeta}_{i,i+1}x)H(R^{\zeta}_{i,i+1}x)}{{}^t xHx}\right)^{3/2} = 1.$$

Since $\vartheta(i,k;i+1,l;mn) = \vartheta(i,l;i+1,k;mn)$ on the mirror of $R^{\zeta}_{i,i+1}$ and it does not vanish, the constant c must be 1. □

Since $\tilde{R}^{\zeta}_{pq}$ can be expressed in terms of $\tilde{R}^{\zeta}_{i,i+1}$ and $\tilde{R}^{\omega}_{pq} = (\tilde{R}^{\zeta}_{pq})^2$, we have the following two propositions.

Proposition 5.4. *We have*

$$\vartheta(ij;kl;mn)(\tilde{R}^{\omega}_{pq}\cdot \jmath(x))^3 = \chi(\tilde{R}^{\omega}_{pq})\vartheta(ij;kl;mn)(\jmath(x))^3,$$

where

$$\chi(\tilde{R}^{\omega}_{pq}) = \left(\frac{{}^t(R^{\omega}_{pq}x)H(R^{\omega}_{pq}x)}{{}^t xHx}\right)^{3/2},$$

which takes 1 *on the mirror of* R^{ω}_{pq}.

Proposition 5.5. *The function $\vartheta(ij;kl;mn)(\jmath(x))$ vanishes on the $\Gamma(1-\omega)$ orbits of the mirrors of R^{ω}_{ij}, R^{ω}_{kl} and R^{ω}_{mn}.*

Proof. By Proposition 5.3, when we restrict $\jmath(x)$ on the mirrors of R^{ω}_{12}, R^{ω}_{34} and R^{ω}_{56}, we have

$$\vartheta(12;34;56)(\jmath(x))^3 = -\vartheta(12;34;56)(\jmath(x))^3 = 0.$$

For the $\Gamma(1-\omega)$ orbits, use the previous proposition. In oder to show for general $\vartheta(ij;kl;mn)(\jmath(x))$'s, use Proposition 5.3. □

§6. The Inverse of the Period Map

Proposition 6.1. *Let Ω be the period matrix of*

$$C(\lambda): w^3 = z(z-1)(z-\ell_1)(z-\ell_2)(z-\ell_3)$$

given in Proposition 3.1. *We have*

$$\ell_1 = \frac{\vartheta^3(13;24;56)(\Omega)}{\vartheta^3(14;23;56)(\Omega)}, \tag{10}$$

$$\ell_2 = \frac{\vartheta^3(13;25;46)(\Omega)}{\vartheta^3(15;23;46)(\Omega)}, \tag{11}$$

$$\ell_3 = \frac{\vartheta^3(13;26;45)(\Omega)}{\vartheta^3(16;23;45)(\Omega)}. \tag{12}$$

Proposition 6.2. *For the period matrix Ω of $C(\lambda)$, linear and cubic relations among $\vartheta^3(ij;kl;mn)(\Omega)$ coincide with the defining equations of $Y \subset \mathbb{P}^{14}$:*

$$\vartheta^3(ij;kl;mn)(\Omega) - \vartheta^3(ik;jl;mn)(\Omega) + \vartheta^3(il;jk;mn)(\Omega) = 0, \tag{13}$$

$$\begin{aligned}\vartheta^3(ij;kl;mn)(\Omega)\vartheta^3(ik;jn;lm)(\Omega)\vartheta^3(im;jl;kn)(\Omega)\\ = \vartheta^3(ij;kn;lm)(\Omega)\vartheta^3(ik;jl;mn)(\Omega)\vartheta^3(im;jn;kl)(\Omega).\end{aligned} \tag{14}$$

Propositions 6.1 and 6.2 imply the following.

Theorem 6.1. *Let Θ be the map from $\mathbb{B}^3/\Gamma(1-\omega)$ to Y defined by*

$$x \mapsto [\ldots, y_{\langle ij;kl;mn\rangle}, \ldots] = [\ldots, \vartheta^3(ij;kl;mn)(\jmath(x)), \ldots].$$

We have the following S_6-equivariant commutative diagram:

$$\begin{array}{ccc} \Lambda & \xrightarrow{\psi} & \mathbb{B}^3/\Gamma(1-\omega) \\ \iota\downarrow & \Theta\swarrow & \\ Y\subset\mathbb{P}^{14}. & & \end{array}$$

In order to prove Propositions 6.1 and 6.2, we state two facts in [8]; the one is Riemann's theorem and the other is Abel's theorem.

Fact 6.1. *We suppose z is a fix point on the Jacobi variety $Jac(R)$ of a Riemann surface R of genus r. The multi-valued function $\vartheta(z + \int_{P_0}^{P} \varphi, \tau)$ of*

P on X has r zeros $P_1,\dots,P_r$ provided not to be constantly zero, where $\varphi = (\varphi_1,\dots,\varphi_r)$ is the normalized basis of the vector space of holomorphic 1-forms on R such that $(\int_{B_i}\varphi_j)_{ij} = I_r$ for a symplectic basis $\{A_1,\dots,A_r,B_1,\dots,B_r\}$ of $H_1(R,\mathbb{Z})$, and $\tau = (\int_{A_i}\varphi_j)_{ij}$. Moreover, there exists a point Δ on $Jac(R)$ called Riemann's constant such that

$$z = \Delta - \sum_{i=1}^{r}\int_{P_0}^{P_i}\varphi.$$

Fact 6.2. *Let R be a Riemann surface of genus r with an initial point P_0. Suppose $\sum_{i=1}^d P_i$ and $\sum_{i=1}^d Q_i$ be effective divisors of degree d satisfying*

$$\sum_{i=1}^{d}\int_{P_0}^{P_i}\varphi = \sum_{i=1}^{d}\int_{P_0}^{Q_i}\varphi, \tag{15}$$

where φ is the normalized basis of vector space of holomorphic 1-forms on R. Then there exists a meromorphic function f on R such that

$$(f) = \sum_{i=1}^{d} Q_i - \sum_{i=1}^{d} P_i;$$

f can be expressed as

$$f(P) = c\frac{\prod_{i=1}^d \vartheta(e+\int_{Q_i}^P\varphi,\tau)}{\prod_{i=1}^d \vartheta(e+\int_{P_i}^P\varphi,\tau)},$$

where c is a constant, τ is the period matrix of R, e satisfies $\vartheta(e)=0$,

$$\vartheta\left(e+\int_{P_i}^{P}\varphi,\tau\right)\not\equiv 0,\quad \vartheta\left(e+\int_{Q_i}^{P}\varphi,\tau\right)\not\equiv 0,$$

as multi-valued functions of P on R, and paths from P_i and Q_i to P are the inverse of the paths in (15) followed by a common path from P_0 to P.

Proof of Proposition 6.1. We take R as

$$C : w^3 = z(z-1)(z-\ell_1)(z-\ell_2)(z-\ell_3)$$

with the initial point $P_0 = (0,0)$ and put

$$P_\infty = (\infty,\infty),\quad P_1 = (1,0),\quad P_{\ell_i} = (\ell_i, 0)\ (i=1,2,3).$$

Let us define a meromorphic function f on C by $C \ni (z,w) \mapsto z$, then

$$(f) = 3P_0 - 3P_\infty.$$

We construct a meromorphic function on C with poles $3P_\infty$ and zeros $3P_0$ by following the recipe given in Fact 6.2. Let $\gamma_i(z_1,z_2)$ $(i=1,2,3)$ be a path in C from (z_1,w_1) to (z_2,w_2) in the i-th sheet. Since $\omega^2+\omega+1=0$, we have

$$\sum_{i=1}^{3}\int_{\gamma_i(0,\infty)}\varphi = (0,0,0,0)$$

for three paths $\gamma_i(0,\infty)$ from P_0 to P_∞. We give the following table:

$$\int_{\gamma_1(\infty,0)}\varphi = \frac{1}{3}\int_{A_1-B_1}\varphi, \quad \int_{\gamma_2(\infty,0)}\varphi = \frac{1}{3}\int_{-2A_1-B_1}\varphi,$$
$$\int_{\gamma_3(\infty,0)}\varphi = \frac{1}{3}\int_{A_1+2B_1}\varphi, \quad \int_{\gamma_1(0,1)}\varphi = \frac{1}{3}\int_{-2A_1+A_2-A_4-B_1+2B_2+2B_4}\varphi,$$
$$\int_{\gamma_2(0,1)}\varphi = \frac{1}{3}\int_{A_1+A_2-A_4+2B_1-B_2-B_4}\varphi,$$
$$\int_{\gamma_3(0,1)}\varphi = \frac{1}{3}\int_{A_1-2A_2+2A_4-B_1-B_2-B_4}\varphi, \quad \int_{\gamma_1(1,\ell_1)}\varphi = \frac{1}{3}\int_{A_2-B_2}\varphi,$$
$$\int_{\gamma_1(\ell_1,\ell_2)}\varphi = \frac{1}{3}\int_{-2A_2+A_3+2A_4-B_2+2B_3-B_4}\varphi, \quad \int_{\gamma_1(\ell_2,\ell_3)}\varphi = \frac{1}{3}\int_{A_3-B_3}\varphi.$$

Put

$$e = \frac{1}{6}\int_{3A_1+A_2+3A_3+5A_4-3B_1-B_2-3B_3+5B_4}\varphi,$$

corresponding to the characteristic $(3,1,3,5)/6$ with label (123), and define a meromorphic function F of $P=(z,w)$ on C as

$$F(P) = \frac{\vartheta\left(e+\int_{\gamma_1(0,z)}\varphi, \Omega\right)^3}{\prod_{i=1}^{3}\vartheta\left(e+\int_{\gamma_i(\infty,0)+\gamma_1(0,z)}\varphi,\Omega\right)}, \tag{16}$$

where Ω is the period matrix of C. Since $\vartheta(123)$ vanishes, we have $\vartheta(e)=0$. We check that neither the denominator nor the numerator of F identically vanishes. We put $P = P_{\ell_1}, P_{\ell_2}, P_{\ell_3}$ and use (6) and (7), then we have

$$F(P_{\ell_1}) = cf(P_{\ell_1}) = c\ell_1 = \exp\left[\frac{\pi\sqrt{-1}}{3}(2\Omega_{11}+1)\right]\frac{\vartheta^3_{[-1,-1,3,-3]}(\Omega)}{\vartheta^3_{[1,-1,3,-3]}(\Omega)},$$
$$F(P_{\ell_2}) = cf(P_{\ell_2}) = c\ell_2 = \exp\left[\frac{\pi\sqrt{-1}}{3}(2\Omega_{11}+1)\right]\frac{\vartheta^3_{[-1,1,-1,1]}(\Omega)}{\vartheta^3_{[1,1,-1,1]}(\Omega)},$$

$$F(P_{\ell_3}) = cf(P_{\ell_3}) = c\ell_3 = \exp\left[\frac{\pi\sqrt{-1}}{3}(2\Omega_{11}+1)\right]\frac{\vartheta^3_{[-1,1,1,1]}(\Omega)}{\vartheta^3_{[1,1,1,1]}(\Omega)},$$

where c is a constant depending on Ω. By Proposition 5.2, neither the denominator nor the numerator of F identically vanishes.

We put $P = P_\infty, P_0, P_1$; the denominator and the numerator of F vanish at these points by Proposition 5.2. Since $(F) = 3P_0 - 3P_\infty$, P_∞ and P_0 are zeros of higher order of the denominator and numerator of F, respectively. The number of zeros of the denominator and numerator of F are 4 by Fact 6.1, thus P_1 is a simple zero. We consider $\lim_{P\to P_1} F(P)$. Let t be a local coordinate for P around P_1 and $z(t)$ be $\int_{P_1}^{P}\varphi$. We have

$$F(P) = \exp\left[\frac{\pi\sqrt{-1}}{3}(2\Omega_{11}-2)\right]\frac{\vartheta^3_{[-1,-3,-3,-3]}(z(t),\Omega)}{\vartheta^3_{[1,3,3,3]}(z(t),\Omega)}.$$

When $P \to P_1$, we have $t \to 0$ and $z(t) \to (0,0,0,0)$. Since $t = 0$ is simple zero, we have

$$\lim_{t\to 0}\frac{\vartheta^3_{[-1,-3,-3,-3]}(z(t),\Omega)}{\vartheta^3_{[1,3,3,3]}(z(t),\Omega)} = \lim_{t\to 0}\frac{\vartheta^3_{[1,3,3,3]}(-z(t),\Omega)}{\vartheta^3_{[1,3,3,3]}(z(t),\Omega)} = -1,$$

which implies $c = \exp[(\pi\sqrt{-1}/3)(2\Omega_{11}+1)]$. Hence we have the expressions (10), (11) and (12). □

Proof of Proposition 6.2. In order to obtain a cubic relation among $\vartheta^3(ij;kl;mn)$'s, put

$$e = \frac{1}{6}\int_{3A_1+5A_2+3A_3+5A_4-3B_1-5B_2-3B_3+5B_4}\varphi,$$

corresponding to the characteristic $(3,5,3,5)/6$ with label (124), then $\vartheta(e) = 0$; and define a meromorphic function F by (16). We have

$$F(P_1) = cf(P_1) = c = \exp\left[\frac{\pi\sqrt{-1}}{3}(2\Omega_{11}+1)\right]\frac{\vartheta^3_{[-1,1,3,-3]}(\Omega)}{\vartheta_{[1,1,3,-3]}(\Omega)},$$

$$F(P_{\ell_2}) = cf(P_{\ell_2}) = c\ell_2 = \exp\left[\frac{\pi\sqrt{-1}}{3}(2\Omega_{11}+1)\right]\frac{\vartheta^3_{[-1,-1,-1,1]}(\Omega)}{\vartheta^3_{[1,-1,-1,1]}(\Omega)},$$

$$F(P_{\ell_3}) = cf(P_{\ell_3}) = c\ell_3 = \exp\left[\frac{\pi\sqrt{-1}}{3}(2\Omega_{11}+1)\right]\frac{\vartheta^3_{[-1,-1,1,1]}(\Omega)}{\vartheta^3_{[1,-1,1,1]}(\Omega)},$$

and

$$\begin{aligned} c\ell_1 = cf(P_{\ell_1}) &= \lim_{P\to P_{\ell_1}} F(P) \\ &= \exp\left[\frac{\pi\sqrt{-1}}{3}(2\Omega_{11}-2)\right] \frac{\vartheta^3_{[-1,-3,-3,-3]}(\int_{P_{\ell_1}}^P \varphi, \Omega)}{\vartheta^3_{[1,3,3,3]}(\int_{P_{\ell_1}}^P \varphi, \Omega)} \\ &= \exp\left[\frac{\pi\sqrt{-1}}{3}(2\Omega_{11}+1)\right]. \end{aligned}$$

These imply

$$\begin{aligned} \ell_1 &= \frac{cf(P_{\ell_1})}{cf(P_1)} = \frac{\vartheta^3(13;24;56)(\Omega)}{\vartheta^3(14;23;56)(\Omega)}, \\ \ell_2 &= \frac{cf(P_{\ell_2})}{cf(P_1)} = \frac{\vartheta^3(14;25;36)(\Omega)\vartheta^3(13;24;56)(\Omega)}{\vartheta^3(15;24;36)(\Omega)\vartheta^3(14;23;56)(\Omega)}, \\ \ell_3 &= \frac{cf(P_{\ell_3})}{cf(P_1)} = \frac{\vartheta^3(14;26;35)(\Omega)\vartheta^3(13;24;56)(\Omega)}{\vartheta^3(16;24;35)(\Omega)\vartheta^3(14;23;56)(\Omega)}. \end{aligned}$$

Compare with the above expression of ℓ_2 and (11), we have a cubic relation among the $\vartheta^3(ij;kl;mn)$'s. By letting $S_6 \simeq \tilde{\Gamma}/\langle\tilde{\Gamma}(1-\omega), -I_4\rangle$ act on theta constants, we have more cubic relations among $\vartheta^3(ij;kl;mn)$'s.

Let us lead a linear relation among the $\vartheta^3(ij;kl;mn)$'s. We start with the meromorphic function $f' : (z,w) \mapsto z-1$; note that $(f') = 3P_1 - 3P_\infty$. Put

$$e = \frac{1}{6}\int_{3A_1+A_2+3A_3+5A_4-3B_1-B_2-3B_3+5B_4} \varphi,$$

corresponding to the characteristic $(3,1,3,5)/6$ with label (123), and define a meromorphic function F' of $P=(z,w)$ on C as

$$F'(P) = \frac{\prod_{i=1}^3 \vartheta\left(e + \int_{\gamma_i(1,0)+\gamma_1(0,z)} \varphi, \Omega\right)}{\prod_{i=1}^3 \vartheta\left(e + \int_{\gamma_i(\infty,0)+\gamma_1(0,z)} \varphi, \Omega\right)}.$$

Since $\vartheta(123)$ vanishes, we have $\vartheta(e)=0$. We consider $\lim_{P\to P_0} F'(P)$ and put $P = P_{\ell_1}$ then we have

$$F'(P_0) = cf'(P_0) = -c = \lim_{P\to P_0} K\exp\left[\frac{4\pi\sqrt{-1}}{3}\right] \frac{\vartheta^3_{[-5,-1,-3,-5]}(\int_{P_0}^P \varphi, \Omega)}{\vartheta^3_{[5,1,3,5]}(\int_{P_0}^P \varphi, \Omega)}$$

$$F'(P_{\ell_1}) = cf'(P_{\ell_1}) = c(\ell_1 - 1) = K\exp\left[\frac{4\pi\sqrt{-1}}{3}\right] \frac{\vartheta^3_{[3,3,3,-1]}(\Omega)}{\vartheta^3_{[1,-1,3,-3)}(\Omega)},$$

where

$$K = \exp\left[-\frac{2\pi\sqrt{-1}}{3}e'\Omega\ {}^t(e'-e_1) + \frac{4\pi\sqrt{-1}}{3}e'H\ {}^t(e'-e_1)\right]$$

and $e' = (1,-1,0,1)$. Now we have the expression

$$\ell_1 - 1 = \frac{\vartheta^3(12;34;56)(\Omega)}{\vartheta^3(14;23;56)(\Omega)}.$$

Since we had in (10)

$$\ell_1 = \frac{\vartheta^3(13;24;56)(\Omega)}{\vartheta^3(14;23;56)(\Omega)},$$

we get a relation

$$\frac{\vartheta^3(12;34;56)(\Omega)}{\vartheta^3(14;23;56)(\Omega)} - \frac{\vartheta^3(13;24;56)(\Omega)}{\vartheta^3(14;23;56)(\Omega)} + 1 = 0,$$

which is equivalent to

$$\vartheta^3(12;34;56)(\Omega) - \vartheta^3(13;24;56)(\Omega) + \vartheta^3(14;23;56)(\Omega) = 0.$$

Action of $S_6 \simeq \tilde{\Gamma}/\langle\tilde{\Gamma}(1-\omega), -I_4\rangle$ produces the other linear relations among the $\vartheta^3(ij;kl;mn)$'s. □

§7. Appendix

In this section, we give a geometrical meaning of the label of a's. In order to do this, we determine Riemann's constant Δ.

Fact 7.1. *Riemann's constant Δ is given by*

$$\Delta = \sum_{i=1}^{m+r-1}\int_{P_0}^{P_i}\varphi - \sum_{j=1}^{m}\int_{P_0}^{Q_j}\varphi \tag{17}$$

for a certain divisor $D_0 = \sum_{i=1}^{m+r-1} P_i - \sum_{j=1}^{m} Q_j$ such that $2D_0$ is linearly equivalent to the canonical divisor of R. It is easy to see that Riemann's constant Δ is a half period on $Jac(R)$ if and only if $(2r-2)P_0$ is a canonical divisor.

For our case, Riemann's constant Δ is a half period on $Jac(C(\lambda))$ since we have $6P_0 = (\varphi_4)$ for any $C(\lambda)$.

Proposition 7.1. *Riemann's constant Δ is invariant under the action of the monodromy group $\tilde{\Gamma}(1-\omega)$. Hence we have*

$$\Delta = \left(\frac{1}{2}, \dots, \frac{1}{2}\right).$$

Proof. Let γ be a closed path in Λ and $g \in \tilde{\Gamma}(1-\omega)$ be its representation. Since Δ is a half period point of $Jac(C(\lambda))$, it is expressed by $c = (c_1, \dots, c_8)$ ($c_i \in \{0, 1/2\}$). When λ moves a little, this vector is invariant and presents Δ. By the continuation along γ, Δ is presented by the vector c with respect to the transformed homology basis by g; i.e., it is presented by $g \cdot c$ with respect to the initial homology basis.

On the other hand, Δ is invariant as a point of $Jac(C(\lambda))$ under the continuation along γ with respect to the initial basis by the expression (17). Thus we have $g \cdot c = c$. There is only one half characteristic $(1/2, \dots, 1/2)$ invariant under $\tilde{\Gamma}(1-\omega)$. □

By straightforward calculation, we have the following proposition giving a geometrical meaning of the label of a's.

Proposition 7.2. *The points $(a, -aH)$ of $Jac(C)$ for a with label (ijk) and (i^2j) are expressed as*

$$\Delta - \int_{P_0}^{P_{\lambda_i}} \varphi - \int_{P_0}^{P_{\lambda_j}} \varphi - \int_{P_0}^{P_{\lambda_k}} \varphi,$$
$$\Delta - 2\int_{P_0}^{P_{\lambda_i}} \varphi - \int_{P_0}^{P_{\lambda_j}} \varphi,$$

respectively.

We have the necessary and sufficient condition for $\vartheta(z, \tau) = 0$.

Fact 7.2. *For a period matrix τ of Riemann's surface R of genus r, $\vartheta(z, \tau) = 0$ if and only if there exists an effective divisor $\sum_{i=1}^{r-1} P_i$ such that*

$$z = \Delta - \sum_{i=1}^{r-1} \int_{P_0}^{P_i} \varphi.$$

Proposition 7.3. *The theta constant $\vartheta(ij; kl; mn)(\jmath(x))$ vanishes only on the $\Gamma(1-\omega)$ orbit of the mirrors of R^{ω}_{ij}, R^{ω}_{kl} and R^{ω}_{mn}.*

Proof. The function $\vartheta(13;24;56)(\jmath(x))$ is a non-zero constant times

$$\vartheta\left(\Delta - \int_{P_0}^{P_\infty}\varphi - \int_{P_0}^{P_0}\varphi - \int_{P_0}^{P_1}\varphi + \int_{P_0}^{P_{\ell_1}}\varphi, \jmath(x)\right).$$

By the previous fact, $\vartheta(13;24;56)(\jmath(x)) = 0$ if and only if there exists an effective divisor $Q_1 + Q_2 + Q_3$ such that

$$Q_1 + Q_2 + Q_3 \equiv P_\infty + 2P_0 + P_1 - P_{\ell_1} = E.$$

By the Riemann-Roch theorem, the dimension of vector space of meromorphic functions f such that $(f) + E \geq 0$ is equal to that of meromorphic 1-forms ϕ such that

$$(\phi) - E \geq 0. \tag{18}$$

Since we have

$$(\varphi_1) = P_\infty + P_0 + P_1 + P_{\ell_1} + P_{\ell_2} + P_{\ell_3}, \quad (\varphi_2) = 6P_\infty,$$

$$(\varphi_3) = 3P_\infty + 3P_0, \quad (\varphi_4) = 6P_0,$$

there does not exist a meromorphic 1-from satisfying (18). Thus if $\lambda \in \Lambda$ then no effective divisor $Q_1 + Q_2 + Q_3$ such that $Q_1 + Q_2 + Q_3 \equiv E$.

The zeros of theta constants on mirrors are studied in [12], which yields this proposition. □

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