

Energy Decay of Solutions to the Wave Equations with Linear Dissipation Localized Near Infinity

By

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§1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be an unbounded domain with smooth boundary. We consider the mixed initial-boundary value problem

$$(1.1) \quad \begin{cases} w_{tt} - \Delta w + b(x, t)w_t = 0, & (x, t) \in \Omega \times (0, \infty), \\ w(x, 0) = w_1(x), w_t(x, 0) = w_2(x), & x \in \Omega, \\ w(x, t) = 0, & (x, t) \in \partial\Omega \times (0, \infty), \end{cases}$$

where $w_t = \partial w / \partial t$, $w_{tt} = \partial^2 w / \partial t^2$, Δ is the N -dimensional Laplacian and $b(x, t)$ is a nonnegative, bounded C^1 -function. Our aim is to study the energy decay of solutions when the dissipative term $b(x, t)w_t$ is effective only near infinity.

Let $H^k(\Omega)$, $k = 0, 1, 2$, be the usual Sobolev space with norm

$$\|f\|_{H^k} = \left\{ \sum_{|\alpha| \leq k} \int_{\Omega} |\nabla^{\alpha} f(x)|^2 dx \right\}^{1/2}$$

(α being multi-indices), and $H_0^1(\Omega)$ be the completion in $H^1(\Omega)$ of the set of all smooth functions with compact support in Ω . We write $H^0(\Omega) = L^2(\Omega)$ and

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$\|f\|_{L^2} = \|f\|$. Let E be the space of all pairs $f = \{f_1, f_2\}$ of functions such that

$$\|f\|_E^2 = \|\{f_1, f_2\}\|_E^2 = \frac{1}{2} (\|f_2\|^2 + \|\nabla f_1\|^2) < \infty.$$

For solution $w(t)$ of (1.1), we simply write

$$\|w(t)\|_E^2 = \|\{w(t), w_t(t)\}\|_E^2$$

and call it the energy of $w(t)$ at time t .

Now assume

$$(1.2) \quad \{w_1, w_2\} \in [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega).$$

Then as is well known, the initial-boundary value problem (1.1) has a global solution in the class

$$(1.3) \quad w(\cdot, t) \in C^0([0, \infty); H^2(\Omega)) \cap C^1([0, \infty); H_0^1(\Omega)) \cap C^2([0, \infty); L^2(\Omega)).$$

Moreover, the energy equation

$$(1.4) \quad \|w(t)\|_E^2 + \int_0^t \int_{\Omega} b(x, \tau) w_t(x, \tau)^2 dx d\tau = \|w(0)\|_E^2$$

holds for any $t > 0$.

Since $b(x, t) \geq 0$, we see from (1.4) that the energy $\|w(t)\|_E^2$ of solution is decreasing in $t > 0$. Thus, a question naturally rises whether it decays or not as t goes to infinity.

The decay and nondecay problems have been studied in works of Matsumura [1], Rauch–Taylor [6], Mochizuki [2], [3] and Mochizuki–Nakazawa [4]. It is proved e.g., in [4] that the energy decays if

$$b_0 \left\{ (e_n + r + t) \log(e_n + r + t) \cdots \log^{[n]}(e_n + r + t) \right\}^{-1} \leq b(x, t) \leq b_1$$

in $\Omega \times [0, \infty)$ for some $b_0, b_1 > 0$, and it does not in general decay if

$$0 \leq b(x) \leq b_2 \left\{ (e_n + r) \log(e_n + r) \cdots [\log^{[n]}(e_n + r)]^\gamma \right\}^{-1}$$

in $\Omega \times [0, \infty)$ for some $b_2 > 0$ and $\gamma > 1$. Here $r = |x|$, and the positive number e_n and the function $\log^{[n]}$ ($n = 0, 1, 2, \dots$) are defined by

$$e_0 = 1, e_1 = e, \dots, e_n = e^{e^{n-1}},$$

$$\log^{[0]} a = a, \log^{[1]} a = \log a, \dots, \log^{[n]} a = \log \log^{[n-1]} a.$$

In this paper we restrict ourselves to the energy decay problem, and obtain a similar result as above when the dissipative term $b(x,t)w_t$ is effective only near infinity.

We require

(A1) $N \geq 3$ and $\mathbb{R}^N \setminus \Omega$ is starshaped with respect to the origin $x = 0$.

(A2) $b(x,t)$ is a bounded, nonnegative C^1 -function, and is not increasing in t ;

$$b_t(x,t) \leq 0 \quad \text{in } \Omega \times (0, \infty).$$

As for the function $b(x,t)$, we consider the following three conditions:

(A3) There exist $R_0 > 0$ and $n \geq 0$ such that

$$\begin{aligned} b_0 \left\{ (e_n + t) \log(e_n + t) \cdots \log^{[n]}(e_n + t) \right\}^{-1} &\leq b(x,t) \\ &\leq b_1 \left\{ (e_n + t) \log(e_n + t) \cdots \log^{[n-1]}(e_n + t) [\log^{[n]}(e_n + t)]^\delta \right\}^{-1} \end{aligned}$$

in $E_\Omega(R_0) \times (0, \infty)$, where $E_\Omega(R_0) = \{x \in \Omega; |x| > R_0\}$, for some $b_0, b_1 > 0$ and $\delta \geq 0$ satisfying $1 - 4b_0/3 < \delta < 1$.

(A3)' There exist $R_0 > 0$ and $n \geq 0$ such that

$$\begin{aligned} b_0 \left\{ (e_n + r + t) \log(e_n + r + t) \cdots \log^{[n]}(e_n + r + t) \right\}^{-1} &\leq b(x,t) \\ &\leq b_1 \left\{ (e_n + r + t) \log(e_n + r + t) \cdots \log^{[n-1]}(e_n + r + t) \right. \\ &\quad \left. \times [\log^{[n]}(e_n + r + t)]^\delta \right\}^{-1} \end{aligned}$$

in $E_\Omega(R_0) \times (0, \infty)$ for some $b_0, b_1 > 0$ and $\delta \geq 0$ satisfying $1 - b_0 < \delta < 1$.

(A3)'' There exist $R_0 > 0$ and $n \geq 0$ such that

$$\begin{aligned} b_0 \left\{ (e_n + r + t) \log(e_n + r + t) \cdots \log^{[n]}(e_n + r + t) \right\}^{-1} &\leq b(x,t) \\ &\leq b_1 \left\{ (e_n + r) \log(e_n + r) \cdots \log^{[n-1]}(e_n + r) [\log^{[n]}(e_n + r)]^\delta \right\}^{-1} \end{aligned}$$

in $E_\Omega(R_0) \times (0, \infty)$ for some $b_0, b_1 > 0$ and $\delta > 0$ satisfying $1 - b_0/2 < \delta < 1$.

With these conditions we shall prove the following theorem.

Theorem 1. *Assume (A1) and (A2).*

(i) *If $b(x,t)$ satisfies (A3) and $\{w_1, w_2\}$ satisfy (1.2), then the energy of the solution to (1.1) decays as t goes to infinity like*

$$(1.5) \quad \|w(t)\|_E^2 \leq K \left[\log^{[n]}(e_n + t) \right]^{-\mu}$$

for some $K = K(w_0, w_1, n) > 0$, where μ is a positive constant satisfying

$$(1.6) \quad \frac{1 - \delta}{2} < \mu \leq 1 \quad \text{and} \quad \mu < \frac{2b_0}{3}.$$

(ii) If $b(x, t)$ satisfies (A3)', and $\{w_1, w_2\}$ satisfy (1.2) and

$$(1.7) \quad \int_{\Omega} \log^{[n]}(e_n + r) \{w_2^2 + |\nabla w_1|^2\} dx < \infty,$$

then the energy of the solution to (1.1) decays like (1.5) as t goes to infinity, where μ is a positive constant satisfying

$$(1.8) \quad \frac{1 - \delta}{2} < \mu \leq 1 \quad \text{and} \quad \mu < \frac{b_0}{2}.$$

(iii) If $b(x, t)$ satisfies (A3)'', and $\{w_1, w_2\}$ satisfy the same conditions in (ii), then the energy of the solution to (1.1) decays like (1.5) as t goes to infinity, where μ is a positive constant satisfying

$$(1.9) \quad 1 - \delta < \mu \leq 1 \quad \text{and} \quad \mu < \frac{b_0}{2}.$$

Remark 1. In (A2) the condition $b_t(x, t) \leq 0$ is not essential for our decay estimate above. The same conclusions of Theorem 1 hold if we require that there exists a nonnegative function $\beta(t) \in L^1((0, t))$ such that $b_t(x, t) \leq \beta(t)$ in $\Omega \times (0, \infty)$.

Remark 2. If $b_0 > 3/4$ in (A3), then we can choose $\delta = 0$. So, in this case (A3) with $n = 0$ is reduced to

$$(1.10) \quad b_0(1+t)^{-1} \leq b(x, t) \leq b_1 \quad \text{in} \quad E_{\Omega}(R_0) \times [0, \infty),$$

and our result is expressed as

$$(1.11) \quad \|w(t)\|_E^2 \leq K(1+t)^{-\mu}$$

for μ satisfying $1/2 < \mu \leq 1$ and $\mu < 2b_0/3$. Similarly, if $b_0 > 1$ in (A3)', then we can choose $\delta = 0$. Therefore (A3)' with $n = 0$ is reduced to

$$(1.12) \quad b_0(1+r+t)^{-1} \leq b(x, t) \leq b_1 \quad \text{in} \quad E_{\Omega}(R_0) \times [0, \infty),$$

and (1.11) holds with μ satisfying $1/2 < \mu \leq 1$ and $\mu < b_0/2$. On the other hand, the simplest case of (A3)'' is

$$(1.13) \quad b_0(1+r+t)^{-1} \leq b(x, t) \leq b_1(1+r)^{-\delta} \quad \text{in} \quad E_{\Omega}(R_0) \times [0, \infty),$$

where δ is chosen arbitrarily in $0 < \delta < 1$ if $b_0 > 2$, and (1.11) holds with $\mu = 1$.

Our argument is based on two weighted energy inequalities. The first one is obtained from equations (1.1) multiplied by $\{\eta w\}_t$ with $\eta = \varphi(t)$ or $= \varphi(r + t)$. On the other hand, the second one is obtained from equations (1.1) multiplied by $\psi(w_r + \frac{N-1}{2r}w)$ with $\psi = \psi(r)$. If $b(x, t)$ is effective in the whole Ω , the first weighted energy inequality is enough to obtain the energy decay. See e.g., [4]. In [4] the second inequality is used to obtain some nondecay results. In this paper we use this inequality to estimate the local energy which is not controlled by the dissipative term in which three conditions (A3), (A3)' or (A3)'' are required.

The energy decay for localized dissipation has been studied by Zuazua [7] and Nakao [5] when Ω is bounded. When Ω is unbounded, there are few works on this problem. Note that in Zuazua [8] is treated the Klein-Gordon equations with dissipative term:

$$w_{tt} - \Delta w + \alpha w + f(w) + b(x)w_t = 0, \quad \text{in } \mathbb{R}^N \times (0, \infty),$$

where $\alpha > 0$ and $b(x) \geq b_0 > 0$ in $|x| > R_0$. Under suitable conditions on $f(w)$, he proved the exponential decay of energy based on a weighted energy method. However, the existence of the term αw is essential in his theory, and it seems difficult to apply it directly to our problem with $\alpha = 0$.

The rest of the paper is organized as follows: In Section 2 we develop a semi-abstract theory for the energy decay. Sufficient conditions are given on the dissipative term (Proposition 2.1). The results are applied in Section 3 to prove Theorem 1. Finally, in Section 4 we discuss a more general equation of the form

$$w_{tt} - \nabla \cdot \{a(x)\nabla w\} + b(x, t)w_t = 0, \quad \text{in } \Omega \times (0, \infty).$$

§2. Weighted Energy Estimates and the Energy Decay

Let $\eta(r, t)$ be a smooth function of $r = |x|$ and $t \geq 0$. We multiply by $\{\eta(r, t)w\}_t$ the both sides of (1.1). It then follows that

$$(2.1) \quad X_t + \nabla \cdot Y + Z = 0,$$

where

$$\begin{aligned} X &= \frac{1}{2}\eta\{w_t^2 + |\nabla w|^2\} + \eta_t w_t w + \frac{1}{2}(\eta_t b - \eta_{tt})w^2, \\ Y &= -(\eta w_t + \eta_t w)\nabla w, \\ Z &= \left(\eta b - \frac{3}{2}\eta_t\right)w_t^2 + \frac{1}{2}\eta_t|\nabla w|^2 + \eta_r w_r w_t + \eta_{tr} w_r w \\ &\quad + \frac{1}{2}(\eta_{ttt} - \eta_{tt}b - \eta_t b_t)w^2. \end{aligned}$$

We use this equation choosing $\eta = \varphi(t)$ or $\eta = \varphi(r + t)$ with $\varphi(s)$, $s \geq 0$, satisfying

$$(2.2) \quad \varphi(s) \geq 1, \varphi'(s) > 0, \varphi''(s) \leq 0, \varphi'''(s) \geq 0 \quad \text{and} \quad \lim_{s \rightarrow \infty} \varphi(s) = \infty,$$

$$(2.3) \quad \varphi'(s)\varphi'''(s) - \varphi''(s)^2 \geq 0.$$

In the case $\eta(r, t) = \varphi(r + t)$, we define a weighted energy of solutions as follows:

$$(2.4) \quad \|w(t)\|_{E_\varphi}^2 = \frac{1}{2} \int_{\Omega} \varphi(r + t)(w_t^2 + |\nabla w|^2) dx,$$

and require that the initial data satisfy other than (1.2),

$$(2.5) \quad \|w(0)\|_{E_\varphi}^2 < \infty \quad (\text{cf., (1.5)}).$$

In the following we denote by C_i , $i = 0, 1, 2, \dots$, positive constants independent of $w(t)$.

Lemma 2.1. *Let $\varphi(t)$ satisfies (2.2), (2.3) and*

$$(2.6) \quad (2 - \epsilon)\varphi(t)b(x, t) \geq 3\varphi'(t) \quad \text{in} \quad E_\Omega(R_0) \times [0, \infty)$$

for some $0 < \epsilon < 1$. Then the solution $w(t)$ of (1.1) admits an inequality

$$(2.7) \quad \frac{1}{3}\varphi(t)\|w(t)\|_E^2 - \int_T^t \int_{B_\Omega(R_0)} \frac{3}{2}\varphi'w_t^2 dx d\tau \\ + \int_T^t \int_\Omega \left\{ \frac{\epsilon}{2}\varphi bw_t^2 + \frac{1}{2}\varphi'|\nabla w|^2 \right\} dx d\tau \leq C_0(1 + T)\|w(0)\|_E^2 + C_1\|w_1\|^2$$

for any $0 \leq T < t$, where $B_\Omega(R_0) = \{x \in \Omega; |x| < R_0\}$.

Proof. We choose $\eta = \varphi(t)$ in (2.1). Let R be enough large as $R > R_0 > 0$. We integrate (2.1) over $B_\Omega(R) \times (0, t)$. Then integrations by parts give

$$(2.8) \quad \int_{B_\Omega(R)} X(x, t) dx + \int_0^t \int_{S_\Omega(R)} Y(x, \tau) \cdot \frac{x}{r} dS d\tau \\ + \int_0^t \int_{B_\Omega(R)} Z(x, \tau) dx d\tau = \int_{B_\Omega(R)} X(x, 0) dx,$$

where $S_\Omega(R) = \{x \in \Omega; |x| = R\}$ and $B_\Omega(R)$ as above. By (2.2) and the

Schwarz inequality, we obtain

$$\begin{aligned} X(x, 0) &\leq \varphi\{w_2^2 + |\nabla w_1|^2\} + \frac{1}{2}(\varphi'^2 + \varphi'b - \varphi'')w_1^2 \\ &\leq \varphi\{w_2^2 + |\nabla w_1|^2\} + C_2w_1^2, \\ X(x, t) &\geq \frac{1}{6}\varphi\{w_t^2 + |\nabla w|^2\} - \frac{3}{4}\varphi^{-1}\varphi'^2w^2 + \frac{1}{2}(\varphi'b - \varphi'')w^2 \\ &\geq \frac{1}{6}\varphi\{w_t^2 + |\nabla w|^2\} + \frac{1}{4}\varphi^{-1}\varphi'(2\varphi b - 3\varphi')w^2 \\ &\geq \frac{1}{6}\varphi\{w_t^2 + |\nabla w|^2\} - \frac{3}{4}\varphi^{-1}\varphi'^2\chi_{R_0}w^2, \end{aligned}$$

where χ_{R_0} is the characteristic function on $B_\Omega(R_0)$, and we have used (2.6) to obtain the last inequality. Similarly, since $\varphi''' - \varphi''b - \varphi'b_t \geq 0$ by (A.2) and (2.2), we have

$$\begin{aligned} Z(x, t) &\geq \frac{1}{2}\{(2 - \epsilon)\varphi b - 3\varphi'\}w_t^2 + \frac{1}{2}\{\epsilon\varphi bw_t^2 + \varphi'|\nabla w|^2\} \\ &\geq -\frac{3}{2}\varphi'\chi_{R_0}w_t^2 + \frac{1}{2}\{\epsilon\varphi bw_t^2 + \varphi'|\nabla w|^2\}. \end{aligned}$$

Moreover,

$$(2.9) \quad \left|Y(x, \tau) \cdot \frac{x}{r}\right| \leq \varphi(w_t^2 + w_r^2) + \frac{1}{2}\varphi^{-1}\varphi'^2w^2.$$

Since $w_t, w_r, w \in H^1(\Omega)$ by (1.3), we obtain

$$(2.10) \quad \liminf_{R \rightarrow \infty} \left\{R \int_{S_\Omega(R)} (w_t^2 + w_r^2 + w^2) dS\right\} = 0.$$

Therefore it follows from (2.9) and (2.10) that

$$\liminf_{R \rightarrow \infty} \int_0^t \int_{S_\Omega(R)} \left|Y(x, \tau) \cdot \frac{x}{r}\right| dS d\tau = 0.$$

Thus, using (2.6) and above estimates of $X(x, t)$, $X(x, 0)$ and $Z(x, t)$, and letting $R \rightarrow \infty$ in (2.8), we obtain

$$(2.11) \quad \begin{aligned} &\frac{1}{3}\|w(t)\|_E^2 - \frac{3}{4}\varphi(0)^{-1}\varphi'(0)^2 \int_{B_\Omega(R_0)} w^2 dx - \int_0^t \int_{B_\Omega(R_0)} \frac{3}{2}\varphi'w_t^2 dx d\tau \\ &+ \int_T^t \int_\Omega \left\{\frac{\epsilon}{2}\varphi bw_t^2 + \frac{1}{2}\varphi'|\nabla w|^2\right\} dx d\tau \leq 2\varphi(0)\|w(0)\|_E^2 + C_3\|w_1\|^2, \end{aligned}$$

where we have used the fact that $\frac{\epsilon}{2}\varphi bw_t^2 + \frac{1}{2}\varphi'|\nabla w|^2 \geq 0$ and have neglected the term

$$\int_0^T \int_\Omega \left\{\frac{\epsilon}{2}\varphi bw_t^2 + \frac{1}{2}\varphi'|\nabla w|^2\right\} dx d\tau.$$

In the left-hand side, the Hardy inequality gives

$$(2.12) \quad \int_{B_\Omega(R_0)} w^2 dx \leq \int_\Omega \frac{4C_4}{(N-2)r^2} w^2 dx \\ \leq C_4 \|\nabla w(t)\|^2 \leq C_4 \|w(0)\|_E^2.$$

Moreover noting (1.4) and (2.2), we obtain

$$(2.13) \quad \int_0^T \int_{B_\Omega(R_0)} \frac{3}{2} \varphi' w_t^2 dx d\tau \leq 3T \max_{0 \leq t \leq T} [\varphi'(t) \left\{ \frac{1}{2} \int_\Omega w_t^2 dx \right\}] \leq 3\varphi'(0)T \|w(0)\|_E^2.$$

Applying (2.12) and (2.13) in (2.11), we conclude the assertion of the lemma. \square

Lemma 2.2. *Let $\varphi(r+t)$ satisfies (2.2), (2.3) and*

$$(2.14) \quad (2-\epsilon)\varphi(r+t)b(x,t) \geq (3+k)\varphi'(r+t) \quad \text{in } E_\Omega(R_0) \times [0, \infty)$$

for some $0 < \epsilon < 1$ and $k \geq 1$. Then the solution $w(t)$ of (1.1) admits an inequality

$$(2.15) \quad \frac{1}{2} \|w(t)\|_{E_\varphi}^2 - \int_T^t \int_{B_\Omega(R_0)} \frac{3+k}{2} \varphi' w_t^2 dx d\tau \\ + \int_T^t \int_\Omega \left\{ \frac{\epsilon}{2} \varphi b w_t^2 + \frac{k-1}{2k} \varphi' |\nabla w|^2 \right\} dx d\tau \\ \leq 2 \|w(0)\|_{E_\varphi}^2 + C_5(1+T) \|w(0)\|_E^2 + C_6 \|w_1\|^2$$

for any $0 \leq T < t$.

Proof. We choose $\eta = \varphi(r+t)$ in (2.1). Note that

$$\frac{k}{2} \varphi' w_t^2 + \frac{1}{2k} \varphi' |\nabla w|^2 + \varphi' w_r w_t + \varphi'' w_r w \\ = \frac{k}{2} \varphi' \left| \frac{x}{r} w_t + \frac{1}{k} \nabla w + \frac{x}{r} \varphi'^{-1} \varphi'' w \right|^2 - \frac{k}{2} \varphi'^{-1} \varphi''^2 w^2 - k \varphi'' w_t w,$$

where

$$-k \varphi'' w_t w = -\frac{k}{2} \partial_t [\varphi'' w^2] + \frac{k}{2} \varphi''' w^2.$$

Then by means of (A2), (2.2) and (2.3), we have

$$Z \geq \left(\varphi b - \frac{k+3}{2} \varphi' \right) w_t^2 + \frac{k-1}{2k} \varphi' |\nabla w|^2 \\ + \frac{1}{2} \{ (k+1) \varphi''' - k \varphi'^{-1} \varphi''^2 \} w^2 - \frac{k}{2} \partial_t [\varphi'' w^2] \\ \geq \left(\varphi b - \frac{k+3}{2} \varphi' \right) w_t^2 + \frac{k-1}{2k} \varphi' |\nabla w|^2 - \frac{k}{2} \partial_t [\varphi'' w^2].$$

Moreover, as is easily verified,

$$\begin{aligned} X(x, t) &= \frac{k}{2} \varphi''(r+t)w(x, t)^2 \\ &\geq \frac{1}{4} \varphi \{w_t^2 + |\nabla w|^2\} + \frac{1}{2} \varphi^{-1} \varphi' \{\varphi b - 2\varphi'\} w^2, \\ X(x, 0) &= \frac{k}{2} \varphi''(r)w(x, 0)^2 \leq \varphi \{w_2^2 + |\nabla w_1|^2\} + C_7 w_1^2. \end{aligned}$$

We integrate (2.1) over $B_\Omega(R) \times (0, t)$, where $R > R_0$. Then the above inequalities and (2.14) give

$$\begin{aligned} (2.16) \quad & \frac{1}{4} \int_{B_\Omega(R)} \varphi \{w_t^2 + |\nabla w|^2\} dx \\ & - \varphi(0)^{-1} \varphi'(0)^2 \int_{B_\Omega(R_0)} w(t)^2 dx - \int_0^t \int_{B_\Omega(R_0)} \frac{k+3}{2} \varphi' w_t^2 dx d\tau \\ & + \int_T^t \int_{B_\Omega(R)} \left\{ \frac{\epsilon}{2} \varphi b w_t^2 + \frac{k-1}{2k} \varphi' |\nabla w|^2 \right\} dx d\tau \\ & \leq \int_{B_\Omega(R)} \{ \varphi (w_2^2 + |\nabla w_1|^2) + C_7 w_1^2 \} dx - \int_0^t \int_{S_\Omega(R)} Y(x, \tau) \cdot \frac{x}{r} dS d\tau. \end{aligned}$$

Note that

$$\left| Y(x, \tau) \cdot \frac{x}{r} \right| \leq \varphi(w_t^2 + w_r^2) + \frac{1}{2} \varphi^{-1} \varphi'^2 w^2.$$

Then since $\varphi(s) = O(s)$ as $s \rightarrow \infty$, it follows that

$$\liminf_{R \rightarrow \infty} \int_0^t \int_{S_\Omega(R)} \left| Y(x, \tau) \cdot \frac{x}{r} \right| dS d\tau = 0.$$

Thus, letting $R \rightarrow \infty$ in (2.16) and taking account of (2.12) and (2.13), we conclude (2.15). □

Next, let $\psi(s)$, $s \geq 0$, be a smooth function satisfying

$$(2.17) \quad 0 \leq \psi(s) \leq \psi_0, \quad 0 \leq \psi'(s) \leq s^{-1} \psi(s) \quad \text{and} \quad \psi''(s) \leq 0.$$

We multiply by $\psi(r)(w_r + \frac{N-1}{2r}w)$ the both sides of equation (1.1). Then since

$$\begin{aligned} w_{tt}\psi\left(w_r + \frac{N-1}{2r}w\right) &= \left[\psi w_t(w_r + \frac{N-1}{2r}w)\right]_t - \frac{1}{2}\nabla \cdot \left[\frac{x}{r}\psi w_t^2\right] + \frac{1}{2}\psi' w_t^2, \\ \Delta w\psi\left(w_r + \frac{N-1}{2r}w\right) &= -\nabla \cdot \left(\psi \nabla w w_r - \frac{x}{2r}\psi |\nabla w|^2\right) + \psi' w_r^2 \\ &\quad + \frac{1}{r}\psi (|\nabla w|^2 - w_r^2) - \frac{1}{2}\left(\frac{N-1}{r}\psi + \psi'\right) |\nabla w|^2 \\ &\quad - \nabla \cdot \left\{\frac{N-1}{2r}\psi \nabla w w + \frac{x}{2r}\left(\frac{N-1}{2r^2}\psi - \frac{N-1}{2r}\psi'\right) w^2\right\}, \\ b w_t \psi\left(w_r + \frac{N-1}{2r}w\right) &= \psi b w_t w_r + \left(\psi b \frac{N-1}{4r} w^2\right)_t - \psi b_t \frac{N-1}{4r} w^2, \end{aligned}$$

it follows that

$$(2.18) \quad \tilde{X}_t + \nabla \cdot \tilde{Y} + \tilde{Z} = 0,$$

where

$$\begin{aligned} \tilde{X} &= \psi w_t \left(w_r + \frac{N-1}{2r}w\right) + \psi b \frac{N-1}{4r} w^2, \\ \tilde{Y} &= -\frac{x}{2r}\psi \{w_t^2 - |\nabla w|^2\} - \psi \nabla w \left(w_r + \frac{N-1}{2r}w\right) \\ &\quad - \frac{x}{r}(r^{-1}\psi - \psi') \frac{N-1}{4r} w^2, \\ \tilde{Z} &= \psi b w_t w_r + \frac{1}{2}\psi' \{w_t^2 + |\nabla w|^2\} - (\psi'' + \psi b_t) \frac{N-1}{4r} w^2 \\ &\quad + (r^{-1}\psi - \psi') \left\{|\nabla w|^2 - w_r^2 + \frac{(N-1)(N-3)}{4r^2} w^2\right\}. \end{aligned}$$

Lemma 2.3. *The solution $w(t)$ of (1.1) admits an inequality*

$$(2.19) \quad \int_T^t \int_{\Omega} \left\{ \psi b w_t w_r + \frac{1}{2}\psi' (w_t^2 + |\nabla w|^2) \right\} dx d\tau \\ \leq C_8 \left\{ (1+T) \|w(0)\|_E^2 + \|w_1\|^2 \right\}$$

for any $0 \leq T < t$.

Proof. Since $N \geq 3$, it follows from (A2) and (2.17) that

$$(2.20) \quad \tilde{Z} \geq \psi b w_t w_r + \frac{1}{2}\psi' \{w_t^2 + |\nabla w|^2\}.$$

Integrate (2.18) over $\Omega \times (0, t)$. Then (2.20) gives

$$\int_{\Omega} \tilde{X} dx \Big|_{\tau=0}^{\tau=t} + \int_0^t \int_{\partial\Omega} \nu \cdot \tilde{Y} dS d\tau + \int_0^t \int_{\Omega} \left\{ \psi b w_t w_r + \frac{1}{2} \psi' (w_t^2 + |\nabla w|^2) \right\} dx d\tau \leq 0,$$

where ν is the outer unit normal to the boundary $\partial\Omega$. As is easily seen,

$$\int_{\Omega} \tilde{X}(x, t) dx \geq \int_{\Omega} \psi w_t \left(w_r + \frac{N-1}{2r} w \right) dx \geq -C_9 \|w(0)\|_E^2$$

for any $t \geq 0$, and

$$\int_{\Omega} \tilde{X}(x, 0) dx \leq C_{10} \left(\|w(0)\|_E^2 + \|w_1\|^2 \right),$$

$$\int_0^T \int_{\Omega} \left\{ \psi b w_t w_r + \frac{1}{2} \psi' (w_t^2 + |\nabla w|^2) \right\} dx d\tau \geq -C_{11} T \|w(0)\|_E^2.$$

On the other hand, by means of the boundary condition $w|_{\partial\Omega} = 0$,

$$\int_0^t \int_{\partial\Omega} \nu \cdot \tilde{Y} dS d\tau = \frac{1}{2} \int_0^t \int_{\partial\Omega} \psi \left\{ \left(\nu \cdot \frac{x}{r} \right) |\nabla w|^2 - 2(\nu \cdot \nabla w) \left(\frac{x}{r} \cdot \nabla w \right) \right\} dS d\tau$$

$$= -\frac{1}{2} \int_0^t \int_{\partial\Omega} \psi \left(\nu \cdot \frac{x}{r} \right) |\nu \cdot \nabla w|^2 dS d\tau \geq 0,$$

where in the last inequality we have used the fact that $\partial\Omega$ is starshaped, i.e., $(\nu \cdot x/r) \leq 0$ on $\partial\Omega$.

Summarizing these inequalities, we conclude (2.19). □

We are now ready to prove the following proposition which gives sufficient conditions on the energy decay.

Proposition 2.1. *Let $w(t)$ be the solution to (1.1).*

(a) *Assume other than (2.6),*

$$(2.21) \quad \psi'(r) - 3\varphi'(t) \geq 0 \quad \text{in } B_{\Omega}(R_0) \times [T_0, \infty),$$

$$(2.22) \quad \epsilon\varphi(t)\varphi'(t) - \psi_0^2 b(x, t) \geq 0 \quad \text{in } \Omega \times [T_0, \infty)$$

for some $0 < \epsilon < 1$ and $T_0 \geq 0$. Then we have

$$(2.23) \quad \varphi(t) \|w(t)\|_E^2 \leq C_{12} \left\{ (1 + T_0) \|w(0)\|_E^2 + \|w_1\|^2 \right\} < \infty.$$

(b) Assume other than (2.14),

$$(2.24) \quad \psi'(r) - (k + 3)\varphi'(r + t) \geq 0 \quad \text{in } B_\Omega(R_0) \times [T_0, \infty),$$

$$(2.25) \quad \epsilon\varphi(r + t)\varphi'(r + t) - \frac{k}{k - 1}\psi_0^2 b(x, t) \geq 0 \quad \text{in } \Omega \times [T_0, \infty)$$

for some $0 < \epsilon < 1$, $k > 1$ and $T_0 \geq 0$. Then we have

$$(2.26) \quad \|w(t)\|_{E_\varphi}^2 \leq C_{13} \left\{ \|w(0)\|_{E_\varphi}^2 + (1 + T_0)\|w(0)\|_E^2 + \|w_1\|^2 \right\} < \infty.$$

(c) Assume other than (2.14) and (2.24) with $k = 1$,

$$(2.27) \quad \epsilon\varphi(r + t)\psi'(r) - \psi_0^2 b(x, t) \geq 0 \quad \text{in } \Omega \times [T_0, \infty)$$

for some $0 < \epsilon < 1$ and $T_0 \geq 0$. Then we have (2.26) with different positive constant C_{13} .

Proof. (a) We put together inequalities (2.7) and (2.19). Then

$$\begin{aligned} & \frac{1}{3}\varphi(t)\|w(t)\|_E^2 + \frac{1}{2} \int_{T_0}^t \int_{B_\Omega(R_0)} (\psi' - 3\varphi')w_t^2 dx d\tau \\ & \quad + \frac{1}{2} \int_{T_0}^t \int_\Omega \{ \epsilon\varphi bw_t^2 + \varphi'|\nabla w|^2 + 2\psi bw_t w_r \} dx d\tau \\ & \leq C_{14} \left\{ (1 + T_0)\|w(0)\|_E^2 + \|w_1\|^2 \right\}. \end{aligned}$$

By (2.21) and (2.22) we see that the second and third terms of the left-hand side is nonnegative. Thus, (2.23) is concluded.

(b) We put together inequalities (2.15) and (2.19). Then

(2.28)

$$\begin{aligned} & \frac{1}{2}\|w(t)\|_{E_\varphi}^2 + \frac{1}{2} \int_{T_0}^t \int_{B_\Omega(R_0)} \{ \psi' - (k + 3)\varphi' \} w_t^2 dx d\tau \\ & \quad + \frac{1}{2} \int_{T_0}^t \int_\Omega \left\{ \epsilon\varphi bw_t^2 + \left(\frac{k - 1}{k} \varphi' + \psi' \right) |\nabla w|^2 + 2\psi bw_t w_r \right\} dx d\tau \\ & \leq C_{15} \left\{ \|w(0)\|_{E_\varphi}^2 + (1 + T_0)\|w(0)\|_E^2 + \|w_1\|^2 \right\}. \end{aligned}$$

By (2.24) and (2.25) we see that the second and third terms of the left-hand side is nonnegative. Thus, (2.26) is concluded.

(c) We choose $k = 1$ in (2.28). By assumptions (2.24) with $k = 1$ and (2.27) we also have the nonnegativity of the two terms of the left. Thus, (2.26) is concluded with some different constant C_{13} . \square

§3. Proof of Theorem 1

In this section we prove Theorem 1 applying Proposition 2.1. The results (a), (b) and (c) correspond respectively to the conditions (A3), (A3)' and (A3)''.

We put

$$(3.1) \quad \varphi(s) = \left[\log^{[n]}(e_n + s) \right]^\mu,$$

where n and μ are as given in Theorem 1, and put

$$(3.2) \quad \psi(s) = \psi_0 \{ 1 - \alpha(1 + s)^{-\gamma+1} \}$$

when (A3) or (A3)' is satisfied, and

$$(3.3) \quad \psi(s) = \psi_0 \left\{ 1 - \alpha [\log^{[n]}(e_n + s)]^{-\gamma+1} \right\}$$

when (A3)'' is satisfied. Here $0 < \alpha < \gamma^{-1} < 1$ and $\psi_0 > 0$ is determined later. In case of (A3)'', we further require $\gamma < \mu + \delta$ in (3.3).

First, we shall show that the above φ satisfies (2.2) and (2.3). Differentiating (3.1), we have

$$\begin{aligned} \varphi' &= \mu [n]^{\mu-1} [n-1]^{-1} \dots [2]^{-1} [1]^{-1} [0]^{-1}, \\ \varphi'' &= -\mu [n]^{\mu-1} [n-1]^{-1} \dots [2]^{-1} [1]^{-1} [0]^{-2} \\ &\quad -\mu [n]^{\mu-1} [n-1]^{-1} \dots [2]^{-1} [1]^{-2} [0]^{-2} \\ &\quad \dots \\ &\quad -\mu [n]^{\mu-1} [n-1]^{-2} \dots [2]^{-2} [1]^{-2} [0]^{-2} \\ &\quad -\mu(1-\mu) [n]^{\mu-2} [n-1]^{-2} \dots [2]^{-2} [1]^{-2} [0]^{-2}, \\ \varphi''' &= \left\{ -2 \sum_{i=0}^{n-1} [i]^{-1} \dots [0]^{-1} - (2-\mu) [n]^{-1} \dots [0]^{-1} \right\} \varphi'' \\ &\quad -\mu \sum_{k=1}^n [k]^{-1} \dots [0]^{-1} \sum_{i=1}^k [n]^{\mu-1} \dots [i]^{-1} [i-1]^{-2} \dots [0]^{-2}, \end{aligned}$$

where $[k] = \log^{[k]}(e_n + s)$ ($k = 0, 1, \dots, n$). These show (2.2). Note that

$$\begin{aligned} \varphi''' &\geq \left\{ -2 \sum_{i=0}^{n-1} [i]^{-1} \dots [0]^{-1} - (2-\mu) [n]^{-1} \dots [0]^{-1} \right\} \varphi'' \\ &\quad + \sum_{k=1}^n [k]^{-1} \dots [0]^{-1} \varphi'' \\ &= \left\{ -2 [0]^{-1} - \sum_{k=1}^{n-1} [k]^{-1} \dots [0]^{-1} - (1-\mu) [n]^{-1} \dots [0]^{-1} \right\} \varphi''. \end{aligned}$$

Then

$$\begin{aligned} \frac{\varphi''}{\varphi'} &= -\sum_{k=0}^{n-1} [k]^{-1} \cdots [0]^{-1} - (1 - \mu)[n]^{-1} \cdots [0]^{-1}, \\ \frac{\varphi'''}{\varphi''} &\leq -2[0]^{-1} - \sum_{k=1}^{n-1} [k]^{-1} \cdots [0]^{-1} - (1 - \mu)[n]^{-1} \cdots [0]^{-1}, \end{aligned}$$

and it follows that

$$\frac{\varphi'''}{\varphi''} - \frac{\varphi''}{\varphi'} \leq -[0]^{-1} \leq 0.$$

Thus (2.3) is proved.

Next, we shall show that the above ψ satisfies (2.17). Differentiating (3.3) ((3.2) is the special case of (3.3) with $n = 0$), we have

$$\begin{aligned} \psi' &= \psi_0 \alpha (\gamma - 1) [n]^{-\gamma} [n - 1]^{-1} \cdots [2]^{-1} [1]^{-1} [0]^{-1} \\ &\leq \psi_0 (1 - \alpha) [0]^{-1} \leq \psi_0 \{1 - \alpha [n]^{-\gamma+1}\} s^{-1} = s^{-1} \psi(s), \\ \psi'' &= -\psi_0 \alpha (\gamma - 1) [n]^{-\gamma} [n - 1]^{-1} \cdots [2]^{-1} [1]^{-1} [0]^{-2} \\ &\quad - \psi_0 \alpha (\gamma - 1) [n]^{-\gamma} [n - 1]^{-1} \cdots [2]^{-1} [1]^{-2} [0]^{-2} \\ &\quad \cdots \\ &\quad - \psi_0 \alpha (\gamma - 1) [n]^{-\gamma} [n - 1]^{-2} \cdots [2]^{-2} [1]^{-2} [0]^{-2} \\ &\quad - \psi_0 \alpha (\gamma - 1) \gamma [n]^{-\gamma-1} [n - 1]^{-2} \cdots [2]^{-2} [1]^{-2} [0]^{-2} \leq 0. \end{aligned}$$

These show (2.17).

Now, to show Theorem 1, we have to verify that the conditions of (a), (b) or (c) in Proposition 2.1 are satisfied by the above φ and ψ .

(i) First, we consider the case (A3). Note that

$$(2 - \epsilon)\varphi(t)b(x, t) \geq (2 - \epsilon)b_0\mu^{-1}\varphi'(t)$$

in $E_\Omega(R_0) \times [0, \infty)$. Since $\mu < 2b_0/3$, we can choose $\epsilon > 0$ very small to satisfy $(2 - \epsilon)b_0\mu^{-1} > 3$. Then (2.6) holds. Note that

$$\begin{aligned} \psi'(r) &\geq \psi_0 \alpha (\gamma - 1) (1 + R_0)^{-\gamma}, \\ 3\varphi'(t) &\leq 3\mu \left\{ (e_n + t) \cdots \log^{[n-1]}(e_n + t) [\log^{[n]}(e_n + t)]^{1-\mu} \right\}^{-1}. \end{aligned}$$

Then, if $n \geq 1$ or $\mu < 1$, we can choose $\psi_0 = 1$ and T_0 sufficiently large to obtain (2.21). If $n = 0$ and $\mu = 1$, we also obtain (2.21) choosing ψ_0 sufficiently large. Further, note that

$$\epsilon\varphi(t)\varphi'(t) = \epsilon\mu \left\{ (e_n + t) \cdots \log^{[n-1]}(e_n + t) [\log^{[n]}(e_n + t)]^{1-2\mu} \right\}^{-1}$$

and $1 - 2\mu < \delta$. Then for any $\psi_0 > 0$, we can choose T_0 sufficiently large to obtain (2.22). Thus, the conditions of (a) are all verified to hold.

(ii) Next we consider the case (A3)'. Note that

$$(2 - \epsilon)\varphi(r + t)b(x, t) \geq (2 - \epsilon)b_0\mu^{-1}\varphi'(r + t)$$

in $E_\Omega(R_0) \times [0, \infty)$. Since $\mu < b_0/2$ by assumption, We can choose $\epsilon > 0$ and $k > 1$ to satisfy

$$\frac{\mu}{b_0} \leq \frac{2 - \epsilon}{k + 3} < \frac{1}{2},$$

which shows (2.14). (2.24) and (2.25) are easily shown if we follow the above argument. Thus, the conditions of (b) are all verified to hold.

(iii) Finally, we consider the case (A3)'' . (2.14) and (2.24) with $k = 1$ are already established in the above proof. Note that

$$\begin{aligned} \epsilon\varphi(r + t)\psi'(r) &\geq \epsilon\psi_0(1 - \gamma^{-1}) \left\{ (e_n + r) \cdots \log^{[n-1]}(e_n + r) \right. \\ &\quad \left. \times [\log^{[n]}(e_n + r)]^\gamma [\log^{[n]}(e_n + r + t)]^{-\mu} \right\}^{-1}. \end{aligned}$$

Then, since $\gamma - \delta < \mu$ by assumption, for any ψ_0 , we can choose T_0 sufficiently large to obtain (2.27). Thus, the conditions of (c) are all verified to hold.

(2.22) implies (1.5) with

$$K = C_{10} \left\{ (1 + T_0)\|w(0)\|_E^2 + \|w_1\|^2 \right\}$$

and (2.25) or (2.27) implies (1.5) with

$$K = C_{11} \left\{ \|w(0)\|_{E_\varphi}^2 + (1 + T_0)\|w(0)\|_E^2 + \|w_1\|^2 \right\}$$

Thus, the proof of Theorem 1 is complete.

§4. A Generalization of the Problem

In this final section, we shall remark that our results are generalized to the initial-boundary value problem

$$(4.1) \quad \begin{cases} w_{tt} - \nabla \cdot \{a(x)\nabla w\} + b(x, t)w_t = 0, & (x, t) \in \Omega \times (0, \infty), \\ w(x, 0) = w_1(x), \quad w_t(x, 0) = w_2(x), & x \in \Omega, \\ w(x, t) = 0, & (x, t) \in \partial\Omega \times (0, \infty), \end{cases}$$

with a small varying coefficient $a(x)$. For the sake of simplicity, we only consider the case where Ω and $b(x, t)$ satisfy (A1), (A2) and (A3)'. On the other hand, $a(x)$ is uniformly positive and small varying in the following sense.

(A4) There exist $a_0, a_1 > 0$ such that $a_0 \leq a(x) \leq a_1$ in Ω . Moreover, $a(x) \in C^1(\Omega)$ and

$$(4.2) \quad \frac{\beta(N-2)^2 \psi'(r)}{2(N-1) \psi(r)} \leq \frac{a_r(x)}{a(x)} \leq (1-\beta) \frac{\psi'(r)}{\psi(r)}$$

in Ω for some $0 \leq \beta \leq 1$, where $\psi(r)$ is given in (3.2).

Multiply by $\psi(r)(w_r + \frac{N-1}{2r}w)$ the both sides of (4.1). Then we have in (2.18)

$$\begin{aligned} \tilde{Z} &= \psi b w_t w_r + \frac{1}{2} \psi' w_t^2 + \frac{1}{2} \{ (a\psi' - a_r \psi') |\nabla w|^2 \\ &\quad - a\psi'' \frac{N-1}{2r} w^2 + a_r \left(\frac{\psi}{r} - \psi' \right) \frac{N-1}{2r} w^2 \} - b_t \psi \frac{N-1}{4r} w^2 \\ &\quad + a \left(\frac{\psi}{r} - \psi' \right) \left\{ |\nabla w| - w_r^2 + \frac{(N-1)(N-3)}{2r^2} w^2 \right\} \\ &\geq \psi b w_t w_r + \frac{1}{2} \psi' w_t^2 + \frac{1}{2} \{ (a\psi' - a_r \psi) |\nabla w|^2 \\ &\quad - a\psi'' \frac{N-1}{2r} w^2 + a_r \left(\frac{\psi}{r} - \psi' \right) \frac{N-1}{2r} w^2 \}. \end{aligned}$$

Integrate \tilde{Z} over $\Omega \times (0, t)$. Then since we have

$$\int_{\Omega} a\psi' |\nabla w|^2 dx \geq \int_{\Omega} \left\{ a\psi' \frac{(N-2)^2}{4r^2} + (a\psi')_r \frac{N-2}{2r} \right\} w^2 dx,$$

it follows that

$$\begin{aligned} \int_0^t \int_{\Omega} \tilde{Z} dx d\tau &\geq \int_0^t \int_{\Omega} \left\{ \psi b w_t w_r + \frac{1}{2} \psi' w_t^2 \right\} dx d\tau \\ &\quad + \frac{1}{2} \int_0^t \int_{\Omega} \{ ((1-\beta)a\psi' - a_r \psi) |\nabla w|^2 \\ &\quad + \left(a\psi' \frac{\beta(N-2)^2}{4r^2} + a_r \psi \frac{N-1}{2r^2} \right) w^2 \\ &\quad + (-a\psi'' - a_r \psi') \frac{N-1-\beta(N-2)}{2r} w^2 \} dx d\tau. \end{aligned}$$

Note that $-\psi'/\psi'' \geq (\gamma-1)(1+r)^{-1} \geq \psi'/\psi$. Then using inequalities (4.2), we obtain

$$(4.3) \quad \int_0^t \int_{\Omega} \tilde{Z} dx d\tau \geq \int_0^t \int_{\Omega} \left\{ \psi b w_t w_r + \frac{1}{2} \psi' w_t^2 \right\} dx d\tau.$$

With this inequality we can follow the argument of Lemma 2.3 to obtain the

Lemma 4.1. *The solution $w(t)$ of (4.1) admits an inequality*

$$\int_T^t \int_\Omega \left\{ \psi b w_t w_r + \frac{1}{2} \psi' w_t^2 \right\} dx d\tau \leq C_{16} \{ (1+T) \|w(0)\|_E^2 + \|w_1\|^2 \}$$

for any $0 \leq T < 1$, where

$$\|w(t)\|_E^2 = \frac{1}{2} \int_\Omega \{ w_t^2 + a(x) |\nabla w|^2 \} dx.$$

Lemma 4.1 and a modified version of Lemma 2.2 show the following proposition which corresponds to Proposition 2.1 (b).

Proposition 4.1. *Assume other than (2.14),*

$$(4.4) \quad a(x)\psi'(r) - (k+3)\varphi'(r+t) \geq 0 \quad \text{in } B_\Omega(R_0) \times [T_0, \infty),$$

$$(4.5) \quad \epsilon\varphi(r+t)\varphi'(r+t) - \frac{k}{k-a_1}\psi_0^2 b(x,t) \geq 0 \quad \text{in } \Omega \times [T_0, \infty)$$

for some $0 < \epsilon < 1, k > \max\{1, a_1\}$ and $T_0 \geq 0$. Then we have

$$\|w(t)\|_{E_\varphi}^2 \leq C_{17} \left\{ \|w(0)\|_{E_\varphi}^2 + (1+T_0) \|w(0)\|_E^2 + \|w_1\|^2 \right\} < \infty,$$

where

$$\|w(t)\|_{E_\varphi}^2 = \frac{1}{2} \int_\Omega \varphi(r+t) \{ w_t^2 + a(x) |\nabla w|^2 \} dx.$$

With this proposition we can follow the same line of proof of Theorem 1 to obtain the same conclusion for problem (4.1).

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