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The Asymptotic Behavior of Eisenstein Series and a Comparison of the Weil-Petersson and the Zograf-Takhtajan Metrics

By

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Abstract

The asymptotic behavior of Eisenstein series for degenerating hyperbolic surfaces with cusps is of interest to us. In order to investigate it we use integral representations of eigenfunctions for the Laplacian, the collar lemma, the interior Schauder estimates, the maximum principles for subharmonic functions and the Harnack Inequalities. As an application, we will compare the Weil-Petersson and the Zograf-Takhtajan metrics near the boundary of moduli spaces.

§0. Introduction

The Quillen metric defined for the determinant line bundle of the Laplacian over the Teichmüller space T_g of compact hyperbolic surfaces with genus g has played an important role in the moduli theory ([14]). The metric is described as the product of a special value of the Selberg zeta function and the usual L^2-fibre metric with respect to the Poincaré metric. The first Chern form of the metric is represented by the Weil-Petersson two-form for T_g, whose formula has been shown by various methods.

Zograf and Takhtajan proved the formula by quasiconformal deformation theory. Moreover they defined the regularized metric for the determinant bundle of the Laplacian for the Teichmüller space $T_{g,n}$ of hyperbolic surfaces with cusps of type (g, n) and calculated its first Chern form, which is described in

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terms of the Weil-Petersson metric and a new Kähler metric as called Zograf-Takhtajan metric. They showed in [16] that the Zograf-Takhtajan metric is Kählerian and invariant under the action of the mapping class group as the Weil-Petersson metric is. It has been recently shown that the Zograf-Takhtajan metric is incomplete for $T_{g,n}$ as the Weil-Petersson metric is ([13]). The proof has been accomplished by showing that the length of a curve approaching the boundary of $T_{g,n}$ with respect to the Zograf-Takhtajan metric is finite. The construction of the curve is due to Wolpert ([25] II).

On the other hand, recently Fujiki and Weng shed new light on the geometry of the moduli spaces of punctured Riemann surfaces and the Zograf-Takhtajan metric ([4], [19]). From Arakelov geometric points of view, Weng found an arithmetic Riemann-Roch theorem for singular metrics, and established a generalization of Mumford type isometries. Fujiki and Weng have observed that the Zograf-Takhtajan metric is algebraic, and Weng proposed a general arithmetic factorization in terms of the Weil-Petersson metric and the Zograf-Takhtajan metric and Selberg zeta functions.

Therefore the asymptotic behavior of the Zograf-Takhtajan metric near the boundary of moduli space is of importance and of interest for studying compactifications of moduli spaces of punctured Riemann surfaces. In the previous paper ([13]), we observed that the metric is incomplete. In that proof we obtained an estimate of Eisenstein series of index 2 just around pinching geodesics, which is regrettably very rough and far from the precise asymptotic behavior, and a rough estimate of the Zograf-Takhtajan metric ([13]).

In this paper we find the asymptotic behavior of Eisenstein series of index s with Re $s > 1$ in Theorem 1 and 2. As a simple application, we improve an estimate of the Zograf-Takhtajan metric near the boundary of the moduli space (Theorem 3). As a result it turns out that the magnitude of the Zograf-Takhtajan norm is less than or equal to that of the Weil-Petersson norm.

We outline the content of this paper. In Section 1, we review elementary properties of Eisenstein series and definitions of the Weil-Petersson metric and the Zograf-Takhtajan metric and Wolf's degenerating family of Riemann surfaces with cusps by constructing infinite-energy harmonic maps which are defined to be maps from a hyperbolic surface with nodes into smooth hyperbolic surfaces (Though he has constructed the harmonic map for the degenerating family of compact Riemann surfaces in [21], we can use it for punctured Riemann surfaces. cf. [26]). One of the important features of Wolf's construction is that the term of the Eisenstein series of index 2 associated to the nodes appears in the asymptotic behavior of the pull-back of hyperbolic metrics of target surfaces by the infinite-energy harmonic maps (1.7). The infinite-energy harmonic map $w^l : S_0 \to S_l$ converges uniformly to id in the C^k-norm on compact subsets of S_0 ([21]). And $(w^l)^*\Delta_l$, the pull-back of the negative hyperbolic Laplacian Δ_l on S_l by w^l converges uniformly to Δ_0, the Laplacian on S_0 also in the same sense as above ([24], [25] II, [26], cf. [9]). We will apply the nice parametrization of the degenerating family to a search of the asymptotic behavior of Eisenstein series (Theorems 1 and 2).

In Section 2, we shall prove two fundamental lemmas. In Lemma 1, we give an estimate of the Eisenstein series for punctured hyperbolic surfaces on any horocycle around the associated cusp. Amazingly it is independent of the complex structure and the topological type of the surfaces. To prove this, we use integral representations of eigenfunctions of the Laplacian that turn out to be very powerful ([10]). By a similar way we give an estimate of the Eisenstein series on a collar neighborhood of a separating pinching geodesic, which shows the order of convergence of the Eisenstein series to 0 on the component of S_0 not containing the associated cusp (cf. Theorems 1 (2)(ii), and 2).

In Section 3, we shall prove one of the main theorems. Now for simplicity, we assume here that there exists just one pinching geodesic l, separating S^l into two parts. Let $E^l_p(z,s)$ be the Eisenstein series associated to a cusp p for S^l. We consider $(w^l)^*E^l_p(z,s)$, the pull-back of $E^l_p(z,s)$ by $w^l : S_0 \to S_l \setminus \{l\}$. S_0 is now divided into two parts; $S_{0,1}$ containing p, and the other component $S_{0,2}$. Then Theorem 1 claims that $(w^l)^*E^l_p(z,s)$ converges on $S_{0,1}$ to E^0_p, the Eisenstein series associated to p for S_0, and $(w^l)^*E^l_p(z,s)$ converges to 0 on $S_{0,2}$.

The outline of the proof is as follows. Denote by $C(a)$ the cusp region around p with length $1/a$ horocycle. First of all we can obtain the uniform boundedness of the sup-norm of $(w^l)^*E^l_p(z,s)$ over a region $S_0 - C(a)$ by using the maximum principles for subharmonic functions. Then using the interior Schauder estimates, taking an exhausting family of compact sets for S_0 and by the standard diagonal argument, we have obtained a convergent subsequence to a limit function $F_0(z,s)$. Noticing the zeroth Fourier coefficient of Eisenstein series and $F_0(z,s)$ around all cusps of S_0, we can see that $F_0(z,s)$ is independent of how to choose subsequences. In Theorem 2, we shall show that a subsequence of $(w^l)^*E^l_p(z,s)$ multiplied with some constants K_l, which could go to ∞, converges on $S_{0,2}$ to the Eisenstein series for $S_{0,2}$ associated to the new cusp arising from the pinching geodesic l.

In Section 4, we shall apply the asymptotic behavior of the Eisenstein series and compare the magnitudes of two norms near the boundary of the moduli space along the degenerating family constructed by S. Wolpert. It follows that the magnitude of the Zograf-Takhtajan metric is less than or equal to that of the Weil-Petersson metric. The author believes that the two metrics are comparable near the boundary of the moduli space, which has not been proved. The most difficult point is to analyze the asymptotic behavior of $(w^l)^* E_p^l(z,s)$ near the pinching geodesic l because $E_p^0(z,s)$ assumes 0 at the node of S_0. This theme is what we should investigate in the future.

We close this chapter with surveying and proposing some approaches to studying the asymptotic behavior of the Eisenstein series. Wolpert has investigated Eisenstein series $E(z,s)$ with $\{s \in \mathbb{C} | \mathrm{Re}\ s = 1/2,\ s \neq 1/2\}$, while we shall investigate that for Re $s > 1$ ([25] I). He has shown that a subsequence of $\hat{E}(z,s)$, the normalized Eisenstein series, the L^2-norm of which on a thick part of S_l could be constant, converges to a non-trivial sum of the Eisenstein series for S_0. The beautiful proof has been accomplished by investigating Legendre functions and showing his original Schauder-type inequalities. It seems hard to apply our method directly to the case where Re $s \leq 1$. The reason for difficulty is that we can not use the maximum principles because $E(z,s)$ with Re $s \leq 1$ is not subharmonic, and we can not extend Lemma 1 to the case of Re $s \leq 1$, and $E(z,s)$ has poles on $\{s \in [0,1]\}$ (For example, we have observed that the constants $M_1(\mathrm{Re}\ s, a)$ assumes the infinity at Re $s = 1$). What we have to study seems the behavior of the scattering matrix $\Phi(s)$ in (1.4).

The reader is also referred to the recent manuscript of T. Falliero ([3]). She has investigated the asymptotic behavior of hyperbolic Eisenstein series, which is introduced by S. Kudla and J. Millson, for pinching the associated simple closed geodesic.

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§1. Preliminaries

§1.1. Eisenstein Series

Let S be a punctured hyperbolic surface of type (g,n) $(n > 0)$. It can be represented as a quotient H/Γ of the upper half plane $H = \{z \in \mathbb{C} | \mathrm{Im} z > 0\}$ by the action of a torsion free finitely generated Fuchsian group $\Gamma \in \mathrm{PSL}_2(\mathbb{R})$.

The group is generated by $2g$ hyperbolic transformations $A_1, B_1, \dots, A_g, B_g$ and parabolic transformations $P_1, \dots, P_n$ satisfying the relation

$$A_1 B_1 A_1^{-1} B_1^{-1} \dots A_g B_g A_g^{-1} B_g^{-1} P_1 \dots P_n = 1.$$

The fixed points of the parabolic elements $P_1, \dots, P_n$ will be denoted by $z_1, z_2, \dots, z_n \in \mathbb{R} \cup \{\infty\}$ respectively and called inequivalent cusps. The projection of the cusps $z_1, z_2, \dots, z_n$ are the punctures $p_1, p_2, \dots, p_n$ of S. For each $i = 1, \dots, n$, denote by Γ_i the stabilizer of z_i in Γ that is the cyclic subgroup of Γ generated by P_i. Pick $\sigma_i \in \mathrm{PSL}_2(\mathbb{R})$ such that $\sigma_i \infty = z_i$ and $\langle \sigma_i^{-1} P_i \sigma_i \rangle = \langle z \mapsto z + 1 \rangle$. Then, for $a > 1$, the a-cusp region $C_i(a)$ associated to p_i is represented as a quotient $\langle \sigma_i^{-1} P_i \sigma_i \rangle \diagdown \{z \in H | \mathrm{Im} z > a\} \simeq \Gamma \diagdown \{z \in H | \mathrm{Im} z > a\}$, equipped with the metric $ds^2 = (dy^2 + dx^2)/y^2$;

$$C_i(a) \simeq [a, \infty) \times S^1.$$

Let $\Delta : C^\infty(S) \to C^\infty(S)$ be the negative hyperbolic Laplacian of S. Regarded as an operator in $L^2(S)$ with domain $C_0^\infty(S)$, Δ is essentially self-adjoint. Denote by $\overline{\Delta}$ the unique self-adjoint extension (that is, the Friedrichs extension). Then the continuous spectrum of $\overline{\Delta}$ can be described in terms of the Eisenstein series ([6] Chapter VII, [10] Chapter V, [17] Section 3.2).

The Eisenstein series attached to z_i is defined by

$$E_i(z, s) = \sum_{\gamma \in \langle P_i \rangle \diagdown \Gamma} \mathrm{Im}(\sigma_i^{-1} \gamma z)^s, \qquad \mathrm{Re}\ s > 1.$$

We remark that the sum is independent of how we choose each representative in the coset decomposition above. The series is absolutely convergent in the upper half-plane and in the half-plane Re $s > 1$ and it satisfies

$$\Delta E_i(z, s) = s(s-1) E_i(z, s). \tag{1.1}$$

A. Selberg originally showed that the series admits meromorphic continuation to the whole complex s-plane, is continuous on $\{\mathrm{Re}\ s = 1/2\}$, and satisfies a system of functional equations ([15] Section 7). Several mathematicians also verified this by the various methods ([2], [6] Theorem 11.6, [10] pp.23–46, [12]). $E_i(z, s)$ has Fourier expansions at the punctures p_j, ([6] Proposition 8.6, [10] Section 2.2, [11] Section 8, [17] Section 3.1)

(1.2)
$$E_i(\sigma_j z, s) = \delta_{ij} y^s + \phi_{ij}(s) y^{1-s} + \sum_{m \neq 0} c_m(s) y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi |m| y) e^{2\pi \sqrt{-1} m x},$$

$K_{s-1/2}(z)$ the MacDonald-Bessel function ([18], p.78) which has the following asymptotics ([18], p.202)

$$y^{\frac{1}{2}} K_{s-\frac{1}{2}}(y) \sim \sqrt{\frac{\pi}{2}} e^{-y}, \text{ as } y(\in \mathbb{R}) \to \infty, \qquad \text{for any complex } s. \tag{1.3}$$

The scattering matrix $\Phi(s) = \big(\phi_{ij}(s) \big)$ enters in the functional equations ([6] Theorem 11.8, [10] Theorem 4.4.2, [15] (7.36), [17] Theorem 3.5.1),

$$\mathbb{E}(z,s) = \Phi(s)\mathbb{E}(z,1-s), \qquad \Phi(s)\Phi(1-s) = 1, \tag{1.4}$$

where $\mathbb{E}(z,s)$ is the vector of the Eisenstein series.

Thanks to Y. C. de Verdiére's keen observation, the Eisenstein series turns out to have the following characterization ([2], [12]).

Claim. *For* $\operatorname{Re} s > 1/2, s \notin (1/2, 1]$ $E_i(z,s)$ *is a unique solution of the equation* $\Delta E_i(z,s) = s(s-1)E_i(z,s)$ *such that* $E_i(z,s) - \operatorname{Im}(\sigma_i^{-1} z)^s \chi|_{C_i(1)}$ *is square integrable on the whole surface, where* $\chi|_{C_i(1)}$ *is the characteristic function of* $C_i(1)$.

Remark. We will use the above Claim just for the case where $\operatorname{Re} s > 1$ (Theorems 1 and 2).

§1.2. The Weil-Petersson and the Zograf-Takhtajan Metrics

Denote by $T_{g,n}$ Teichmüller space of hyperbolic surfaces of type (g,n). Now we consider the tangent and cotangent spaces at S of $T_{g,n}$. The cotangent space is $Q(S)$, the integrable holomorphic quadratic differentials on S. Let $B(S)$ be the L^∞-closure of Γ-invariant, bounded, $(-1,1)$-forms, i.e. the Beltrami differentials for S. For $\mu \in B(S), \varphi \in Q(S)$ the integral $\int_S \mu\varphi$ defines a paring, let $Q(S)^\perp$ be the annihilator of $Q(S)$. The tangent space at S to $T_{g,n}$ is $B(S)/Q(S)^\perp \simeq HB(S)$, the Serre dual space of $Q(S)$, i.e. the harmonic Beltrami differentials on S. Then for $\mu, \nu \in HB(S)$, the Weil-Petersson and the Zograf-Takhtajan metrics are defined as follows ([16]),

$$\langle \mu, \nu \rangle_{\mathrm{WP}} = \iint_S \mu(z)\overline{\nu(z)}\rho(z)\frac{\sqrt{-1}}{2} dz \wedge d\bar{z} \tag{1.5}$$

$$\langle \mu, \nu \rangle_{(i)} = \iint_S E_i(z,2)\mu(z)\overline{\nu(z)}\rho(z)\frac{\sqrt{-1}}{2} dz \wedge d\bar{z} \tag{1.6}$$

$$\langle \mu, \nu \rangle_{\mathrm{ZT}} = \sum_{i=1}^{n} \langle \mu, \nu \rangle_{(i)},$$

where $\rho(z)|dz|^2$ denotes the Poincaré metric on S.

Both the Weil-Petersson and the Zograf-Takhtajan metric are Kählerian and incomplete ([13], [16]).

§1.3. Degenerating Parameters and Infinite-Energy Harmonic Maps

In this part, we consider degeneration of hyperbolic surfaces. Denote by $(S_l(l > 0), \rho_l(w)|dw|^2)$ a degenerating family of hyperbolic surfaces of type (g, n). We assume that several disjoint simple closed geodesics $l_1, l_2, \dots, l_m$ on S_l will be pinched (We denote their hyperbolic lengths by the same notations). Let Δ_l be the negative Laplacian of S_l. To compare functions on the limit surface $(S_0, \rho(z)|dz|^2)$ and $(S_l, \rho_l(w)|dw|^2)$, we use the infinite-energy harmonic maps $w^l : S_0 \to S_l \diagdown \{l_1, l_2, \dots, l_m\}$ constructed by M. Wolf ([9], [21], [26]). A node on S_0 is a pair of cusps and distinct nodes involve distinct cusps.

We can select simple closed geodesics $\{l_{m+1}, \dots, l_{3g-3+n}\}$ so that the set $\{l_1, \dots, l_m\} \cup \{l_{m+1}, \dots, l_{3g-3+n}\}$ is a pair of pants decomposition for S_l. Let $\{\theta_1, \dots, \theta_m, \theta_{m+1}, \dots, \theta_{3g-3+n}\}$ be twist angles around the corresponding geodesics. Then $\{l_1, \dots, l_{3g-3+n}, \theta_1, \dots, \theta_{3g-3+n}\}$ represent the Fenchel-Nielsen coordinates that are global real analytic for $T_{g,n}$. Here we set $\overrightarrow{l} = (l_1, l_2, \dots, l_{3g-3+n})$, and $\overrightarrow{\theta} = (\theta_1, \dots, \theta_{3g-3+n})$.

Then precise real-analytic parameterization is obtained by Wolf ([21], [26])

$$(1.7)\quad (w^l)^* \rho_l |dw|^2 = \Psi(\overrightarrow{l}, \overrightarrow{\theta}) dz^2 + (H(\overrightarrow{l}, \overrightarrow{\theta}) + L(\overrightarrow{l}, \overrightarrow{\theta}))\rho |dz|^2 + \overline{\Psi(\overrightarrow{l}, \overrightarrow{\theta}) dz^2},$$

where $\Psi(\overrightarrow{l}, \overrightarrow{\theta}) = \rho_l w^l_z \overline{w^l_{\bar{z}}}$, $H(\overrightarrow{l}, \overrightarrow{\theta}) = [\rho_l(w^l(z)/\rho(z)]|w^l_z|^2$, $L(\overrightarrow{l}, \overrightarrow{\theta}) = [\rho_l(w^l(z)/\rho(z)]|w^l_{\bar{z}}|^2$.

Choose a real basis $\{\Psi_1, \dots, \Psi_m, \Psi_{m+1}, \dots, \Psi_{6g-6+n-m}\}$ of quadratic differentials on S_0 with the property that, for $1 \le i \le m, \Psi_i$ has a second order pole at the i-th node with leading coefficient equal to one and is otherwise regular or has at worst first order poles at the other nodes and punctures, and is the only basis that has a second pole at the i-th node; and furthermore, for $i > m$, $\{\Psi_{m+1}, \dots, \Psi_{6g-6+n-m}\}$ forms a real basis of integrable holomorphic quadratic differentials on S_0.

Wolf finds also that $\Psi(\overrightarrow{l}, \overrightarrow{\theta})$ in (1.7) can be described in terms of the above basis as

$$(1.8)\qquad \Psi(\overrightarrow{l}, \overrightarrow{\theta}) = \sum_{i=1}^{m} \frac{l_i^2}{4} \Psi_i + \sum_{j=m+1}^{6g-6+n-m} t_j(\overrightarrow{l}, \overrightarrow{\theta}) \Psi_j,$$

where $t_j(\overrightarrow{l}, \overrightarrow{\theta})$ is real analytic in $(\overrightarrow{l}, \overrightarrow{\theta})$ ([21], p.535).

The Beltrami differential of w^l is $\mu_l = w^l_{\bar{z}}/w^l_z = \overline{\Psi}/\rho H$. Wolf shows that when S_l degenerates to S_0, the Beltrami differential μ_l converges uniformly to zero on compact subsets of S_0, and the harmonic map w^l converges uniformly to id on compact subsets of S_0.

Of interest to us now is the behavior of Δ_l. We review the discussion of Wolpert ([24], [25], [26]). There exists a basic map $\sigma_l : L^2(S_l) \to L^2(S_0)$, for $f \in L^2(S_l)$, $\sigma_l(f) = f(w^l)$. Define $(w^l)^*\Delta_l = \sigma_l \Delta_l \sigma_l^{-1}$. Then $(w^l)^*\Delta_l$ is real-analytic family ([24], p.450, [25], p.98, [26], pp.254–258). From [24] Lemma 5.3, we easily see that for any $k \in \mathbb{N}$ in the C^k-norm on compact subsets of components of S_0, $(w^l)^*\Delta_l$ converges uniformly to Δ_0 that is defined to be the formal sum of the hyperbolic Laplacians for components of S_0.

Many mathematicians investigated, by various parametrizations, degeneration of hyperbolic surfaces and the asymptotic behavior of several functions; for example, Green's functions ([7], [8], [9], [20]), eigenfunctions of the Laplacian and its eigenvalues ([7], [8], [9], [25], [26]), the Riemann matrix and Faltings invariant ([20]).

Remark. Basically those various parameterizations turn out to be almost the same powerful tools. But what we should pay attention to is that the parameters by the infinite-energy harmonic maps are independent of twist-angles around the pinching geodesics. Nevertheless, the family S_l with pairs of opened collars glued by adequate twist-angles agrees with R_l constructed by Wolpert [25], pp.103–104 ([23], Appendix, [26], pp.251–252).

§2. Some Elementary Estimates of Eisenstein Series

We give key lemmas which play important roles in the proof of the main theorem (cf. [13], Lemma 4).

Lemma 1. *We use the same notations as in Section* 1. *Let the index of Eisenstein series* $\mathrm{Re}\, s > 1$. *For any* $i = 1, 2, \dots, n$ *and any* $a > 1$,

$$|E_i(z,s)| < M_1(\mathrm{Re}\ s, a), \qquad \text{on } \partial C_i(a).$$

Here $M_1(\mathrm{Re}\ s, a)$ *is a constant depending only on* $\mathrm{Re}\ s, a$, *independent of complex structure and topological type of the surface.*

Proof. We recall a fundamental fact. For $\epsilon > 0$, set a $\mathrm{PSL}_2(\mathbb{R})$-invariant kernel function

$$k_\epsilon(z,z') = \begin{cases} 1, & \text{if } d(z,z') < \epsilon \\ 0, & \text{otherwise.} \end{cases}$$

Then there exists a constant $\Lambda_\epsilon(s)$ depending only on ϵ and the index s such that for any $\sigma \in \mathrm{PSL}_2(\mathbb{R})$,

$$\Lambda_\epsilon(s)\mathrm{Im}(\sigma z)^s = \iint_H k_\epsilon(z,z')\mathrm{Im}(\sigma z')^s \frac{dx'dy'}{y'^2}, \qquad (z' = x' + y'). \tag{2.1}$$

([10], Theorem 1.3.2).

Now without losing generality, we may assume that $\langle P_i \rangle = \langle z \mapsto z+1 \rangle$.

For any $z_0 \in \partial C_i(a)$, we get

$$\begin{aligned}
\Lambda_\epsilon(s)E_i(z_0,s) &= \Lambda_\epsilon(s) \sum_{\delta \in \langle P_i \rangle \diagdown \Gamma} (\mathrm{Im}\delta z_0)^s \\
&= \sum_{\delta \in \langle P_i \rangle \diagdown \Gamma} \iint_H k_\epsilon(z_0,z)\,(\mathrm{Im}\delta z)^s\,\frac{dxdy}{y^2} \\
&= \sum_{\delta \in \langle P_i \rangle \diagdown \Gamma} \iint_H k_\epsilon(\delta z_0,\delta z)\,(\mathrm{Im}\delta z)^s\,\frac{dxdy}{y^2} \\
&= \sum_{\delta \in \langle P_i \rangle \diagdown \Gamma} \iint_H k_\epsilon(\delta z_0,z)\,(\mathrm{Im} z)^s\,\frac{dxdy}{y^2} \\
&= \sum_{\delta \in \langle P_i \rangle \diagdown \Gamma} \iint_{B(\delta z_0,\epsilon)} y^{s-2}dxdy\ .
\end{aligned}$$

Here we set $B(\delta z_0,\epsilon) = \{z \in H \mid d(\delta z_0,z) < \epsilon\}$.

Due to the Shimizu-Leutbecher lemma, we select $\epsilon_0 = \epsilon_0(a) > 0$ depending only on $a > 1$ so that the injectivity radius at any point on $\partial C_i(a)$ could be larger than ϵ_0.

Because $B(\delta z_0,\epsilon_0)$ $(\delta \in \langle P_i \rangle \diagdown \Gamma)$ are mutually disjoint and $\sum_{\delta \in \langle P_i \rangle \diagdown \Gamma} B(\delta z_0,\ \epsilon_0) \subset \{z \in H | -1 < \mathrm{Re}\ z < 2,\ 0 < \mathrm{Im}\ z < 2a\}$ (replacing ϵ_0 by a smaller positive number if necessary, and taking appropriate representatives δ if necessary), we see

$$\begin{aligned}
\Lambda_{\epsilon_0}(\mathrm{Re}\ s)|E_i(z_0,s)| &\le \Lambda_{\epsilon_0}(\mathrm{Re}\ s)E_i(z_0,\mathrm{Re}\ s) \\
&= \iint_{\sum_{\delta \in \langle P_i \rangle \diagdown \Gamma} B(\delta z_0,\epsilon_0)} y^{\mathrm{Re}s-2}dxdy \\
&\le \iint_{\substack{-1<x<2 \\ 0<y<2a}} y^{\mathrm{Re}s-2}dxdy \\
&= \frac{3\cdot 2^{\mathrm{Re}s-1}}{\mathrm{Re}\ s-1}\ a^{\mathrm{Re}s-1}.
\end{aligned}$$

Finally we conclude the proof of this lemma with setting

$$M_1(\text{Re } s, a) = \frac{3 \cdot 2^{\text{Re}s-1} a^{\text{Re}s-1}}{(\text{Re } s - 1)\Lambda_{\epsilon_0}(\text{Re } s)}. \qquad \square$$

Let $l_1, \dots, l_m$ be pinching geodesics on S_l. For $0 < k \le 1$ and $j = 1, \dots, m$, set

$$N_{l_j}(k) = \left\{ p \in S_l \middle| d(p, l_j) < k \sinh^{-1}\left(1/\sinh\frac{l_j}{2}\right)\right\},$$

the collar neighborhood around l_j in S_l ([1], 4.1). Here we quote an important claim due to S. Wolpert ([25] II, Lemma 2.1).

Claim. *Let $\rho(z)$ be the injectivity radius of S_l at z. There is an absolute positive constant C_0 such that for $l < 2sinh^{-1}1$, and any $z \in N_l(1)$, then $\rho(z)e^{d(z,\partial N_l(1))} \ge C_0$.*

We improve the estimate of the Eisenstein series in [13], Lemma 4 just for the case where there exist separating pinching geodesics on S_l. Let $E_i^l(z, s)$ be the Eisenstein series attached to p_i for S_l.

Lemma 2. *Let the index* $\text{Re}\, s > 1$. *Assume that there is only one pinching geodesic $l = l_1$ on S_l, separating S_l into two parts*; *$S_{l,1}$ containing the puncture p_i and the other component $S_{l,2}$. Then for $l < 2sinh^{-1}1$,*

$$|E_i^l(z, s)| < M_2(\text{Re } s)\ l^{\text{Re}s(1+k)-2}, \qquad \text{on } \partial N_l(k) \cap S_{l,2}.$$

Here $M_2(\text{Re } s)$ is an absolute constant depending only on Re s.

Proof. Without losing generality, we may assume that $\langle P_i \rangle = \langle z \mapsto z+1 \rangle$. First we fix a constant $\epsilon = \epsilon_1 < \sinh^{-1}1$. As in the proof of Lemma 1, for any $z_1 \in \partial N_l(k) \cap S_{l,2}$, we apply (2.1) for ϵ_1. We have

$$\Lambda_{\epsilon_1}(s)\ E_i^l(z_1, s) = \sum_{\delta \in \langle P_i \rangle \diagdown \Gamma} \iint_{B(\delta z_1, \epsilon_1)} y^{s-2} dx dy.$$

Next we repeat the same discussion in the proof of [13], Lemma 4. Notice that if $B(z_1, \epsilon_1) \cap B(\delta z_1, \epsilon_1) \ne \phi$ then $d(z_1, \delta z_1) < 2\epsilon_1$. And we easily see that the multiplicity of $B(z_1, 2\epsilon_1) \to B(z_1, 2\epsilon_1)/\Gamma$ is at most $c\rho^{-1}(z_1)$ (c is an absolute constant). Here we set $\mathcal{B} = \bigcup_{\delta \in \langle P_i \rangle \diagdown \Gamma} B(\delta z_1, \epsilon_1)$.

Thus from simple consideration, the multiplicity of the points of $\mathcal{B}$ are at most $c\rho^{-1}(z_1)$. Another important fact is that $\mathcal{B}$ is included in the set

$$\mathcal{R} = \{\ z \in H \mid d(z, \{\text{Im } z = 1\}) \ge -(1+k)\log l - \epsilon_1\ \} \quad \text{(See [25] II, p.102).}$$

Combining the above arguments, we have obtained

$$\begin{aligned}
&\Lambda_{\epsilon_1}(\text{Re } s)|E_i^l(z_1,s)| \le \Lambda_{\epsilon_1}(\text{Re } s)E_i^l(z_1,\text{Re } s)\\
&= \sum_{\delta\in\langle P_i\rangle\setminus\Gamma} \iint_{B(\delta z_1,\epsilon_1)} y^{\text{Re}s-2}dxdy\\
&\le c\rho(z_1)^{-1}\iint_{\mathcal{B}} y^{\text{Re}s-2}dxdy\\
&\le c\rho(z_1)^{-1}\iint_{\mathcal{R}\cap\{-1<\text{Re}z<2\}} y^{\text{Re}s-2}dxdy\\
&\le c\rho(z_1)^{-1}\iint_{\substack{-1<x<2\\ 0<y<e^{\epsilon_1}\ l^{1+k}}} y^{\text{Re}s-2}dxdy\\
&\le c' e^{d(z_1,\partial N_l(1))}\ \frac{3\cdot e^{(\text{Re}s-1)\epsilon_1}}{\text{Re } s-1}\ l^{(\text{Re}s-1)(1+k)} \qquad \text{(cf. Claim in §2)}\\
&\le c' l^{-(1-k)}\ \frac{3\cdot e^{(\text{Re}s-1)\epsilon_1}}{\text{Re } s-1}\ l^{(\text{Re}s-1)(1+k)}\ \ (\text{ By } d(z_1,\partial N_l(1))\approx -(1-k)\log\ l\ )\\
&= c'\ \frac{3\cdot e^{(\text{Re}s-1)\epsilon_1}}{\text{Re } s-1}\ l^{\text{Re}s(1+k)-2}.
\end{aligned}$$

We may set

$$M_2(\text{Re } s) = c'\ \frac{3\cdot e^{(\text{Re}s-1)\epsilon_1}}{(\text{Re } s-1)\ \Lambda_{\epsilon_1}(\text{Re } s)}. \qquad \square$$

Remark. For any s with Re $s > 1$, there is $0 < k \le 1$ such that Re $s\ (1+k) - 2 > 0$.

§3. The Asymptotic Behavior of Eisenstein Series

Our aim is to prove one of the main theorems. From now on, the cusps of S_0 that arise from the cusps of S_l are called the *old cusps* and the cusps of S_0 that arise from the pinching geodesics of S_l are called the *new cusps*.

Theorem 1. *We set the same notations as in Section* 1. *Let the index* $\text{Re}\, s > 1$.

(1) *If* $\{l_1,\dots,l_m\}$ *do not separate* S_l, *then for any* $i = 1,\dots,n$, *as* $l_1,\dots,l_m \to 0$,

$$(w^l)^* E_i^l(z,s) \quad \longrightarrow \quad E_i^0(z,s) \tag{3.1}$$

uniformly on any compact subset of S_0. *Here* $E_i^0(z,s)$ *is the Eisenstein series attached to the old puncture* p_i *for* S_0.

(2) *Assume that* $\{l_1,\dots,l_m\}$ *separate* S_l. *Denote by* $S_{0,1}^i$ *and* $S_{0,2}^i$ *respectively the component of* S_0 *containing* p_i *and the union of the components of* S_0 *not containing* p_i. *Let* q_j $(j=1,\dots,m)$ *be the new cusp arising from* l_j. *Denote by* $C_j(b)$ $(b>1)$ *be the cusp region around* q_j *in* S_0, *each composed of usual two b-cusp regions. Then*

(i) *For any* $i=1,\dots,n$, *as* $l_1,\dots,l_m\to 0$,

$$(w^l)^*E_i^l(z,s) \quad\longrightarrow\quad E_i^0(z,s) \tag{3.2}$$

uniformly on any compact subset of $S_{0,1}^i$. *Here* $E_i^0(z,s)$ *is the Eisenstein series attached to* p_i *for* $S_{0,1}^i$.

(ii) *For any* $i=1,\dots,n$ *and any* $b>1$, *as* $l_1,\dots,l_m\to 0$,

$$(w^l)^*E_i^l(z,s) \quad\longrightarrow\quad 0 \tag{3.3}$$

uniformly on $S_{0,2}^i-\bigcup_{j=1}^m C_j(b)$.

Furthermore, for $b>1$ *fixed,*

$$|(w^l)^*E_i^l(z,s)| = O\,(\max_{j=1,\dots,m} l_j^{(2-\delta)\mathrm{Re}s-2}), \quad \textit{for any small } \delta>0$$

on $S_{0,2}^i-\bigcup_{j=1}^m C_j(b)$.

Proof. (1) First of all, we estimate $|E_i^l(z,s)|$ on $S_l-C_i(a)$. It is easily seen that $|E_i^l(z,s)|\le E_i^l(z,\mathrm{Re}\ s)$. Since the index $\mathrm{Re}\ s>1$, $E_i^l(z,\mathrm{Re}\ s)$ clearly turns out to be a subharmonic function of z (Cf. (1.1)). Thanks to this property, we can apply the maximum principle and Lemma 1. Then observing that $E_i^l(z,s)$ vanishes at all the cusps except for the i-th cusp since $\mathrm{Re}\ s>1$, we obtain

$$\begin{aligned}\|E_i^l(z,s)\|_{C^0(S_l-C_i(a))} &\le \|E_i^l(z,\mathrm{Re}\ s)\|_{C^0(S_l-C_i(a))}\\ &\le \sup_{z\in\partial C_i(a)} E_i^l(z,\mathrm{Re}\ s)\le M_1(\mathrm{Re}\ s,a).\end{aligned} \tag{3.4}$$

For any region $\Omega\subset\subset S_0$, we select $a=a(\Omega)>1$ such that $w^l(\Omega)\subset\subset S_l-C_i(a)$. From (3.4), for sufficiently small $l_1,\dots,l_m$,

$$\|(w^l)^*E_i^l(z,s)\|_{C^0(\Omega)} < M_1(\mathrm{Re}\ s,a). \tag{3.5}$$

For $z\in\Omega$, from the definition of $(w^l)^*\Delta_l$ in Section 1, 1.3,

$$((w^l)^*\Delta_l - s(s-1))\ (w^l)^*E_i^l(z,s)=0.$$

As stated in Section 1.1.3, $(w^l)^*\Delta_l$ converges uniformly to Δ_0 in the C^2-norm on compact subsets of Ω as $l_1, \dots, l_m \to 0$. Then by (3.5) and the interior Schauder estimates ([5], Theorem 6.2), for any $0 < \alpha < 1$ and $\Omega^{'} \subset\subset \Omega$, there exists a positive constant $L' = L'(\Omega^{'}, \Omega, s, \alpha)$ such that

$$\|(w^l)^* E_i^l(z,s)\|_{C^{2,\alpha}(\Omega')} \leq L'.$$

By the standard compactness argument ([5]), and taking an exhausting family of compact subsets of S_0, the diagonal method and (3.4), there is a subsequence $\{l_1^j, \dots, l_m^j\}$ such that

$$\lim_{j\to\infty} (w^{l^j})^* E_i^{l^j}(z,s) = F_0(z,s)$$

in the C^2-norm on compact subsets of S_0 for a function $F_0(z,s) \in C^2(S_0)$, which satisfies on S_0,

$$(3.6) \qquad \begin{aligned} &(\Delta_0 - s(s-1))\ F_0(z,s) = 0. \\ &\|F_0(z,s)\|_{C^0(S_0 - C_i(a))} \leq M_1(\mathrm{Re}\ s, a). \end{aligned}$$

By elliptic regularity, it turns out that $F_0(z,s) \in C^\infty(S_0)$. We have to show just that the limit function $F_0(z,s)$ equals to $E_i^0(z,s)$ and is independent of how we choose subsequences.

From [10], Theorem 3.2.1, we see that the Fourier expansion of $F_0(z,s)$ at any cusp of Γ_0 (we set $S_0 \simeq H/\Gamma_0$) is of the form

$$c_0 y^s + c_0^{'} y^{1-s} + \mathrm{O}\ (e^{-2\pi\sqrt{-1}\ y}), \qquad \text{as } y \to \infty.$$

It should be noted that w^l converges uniformly to id on compact subsets of S_0 (Section 1, 1.3), and the 0-th Fourier coefficients of $E_i^l(\sigma_j z, s)$ at the old cusps (1.2) are of the form

$$\delta_{ij} y^s + \phi_{ij}(s) y^{1-s} = \int_0^1 E_i^l(\sigma_j z, s) dx.$$

Thus by the above remarks, Lemma 1, (1.2) and (3.6), we easily see that the 0-th coefficients of $F_0(z,s)$ at the old cusp p_i and at the other old and the new cusps are respectively of the form $y^s + cy^{1-s}$, $c'y^{1-s}$ with constants c, c'. The conclusion follows from the above observation, (3.6), Claim in Section 1.

(2) (i) We can show (3.2) in much the same way as in the proof of (3.1).

(2) (ii) First we prove the convergence on compact subsets of S_0 as in (1). We can obtain the limit function $f_0(z,s)$ of a subsequence of $\{\ (w^l)^* E_i^l(z,s)\ \}$,

satisfying

$$(3.7) \qquad \begin{aligned}&(\Delta_0 - s(s-1))\ f_0(z,s) = 0.\\ &\|f_0(z,s)\|_{C^0(S_0 - C_i(a))} \le M_1(\text{Re } s, a).\end{aligned}$$

From the inequality in (3.7), we can show that the 0-th Fourier coefficients of $f_0(z,s)$ at all cusps of $S^i_{0,2}$ are of the form $b'' y^{1-s}$, which means that $f_0(z,s)$ is square integrable. Thus $f_0(z,s)$ is clearly included in the domain of the self-adjoint extension of the Laplacian $\overline{\Delta_0}$. Since $-s(s-1)$ is not an eigenvalue of $\overline{\Delta_0}$, $(\overline{\Delta_0} - s(s-1))$ has a densely defined bounded inverse. By the above arguments and the equation in (3.7), we obtain $f_0(z,s) = 0$.

Next we prove the uniform convergence on $S^i_{0,2} - \bigcup_{j=1}^{m} C_j(b)$ for any $b > 1$. We already know

$$\begin{aligned}\sup_{z \in S^i_{0,2} - \bigcup_{j=1}^{m} C_j(b)} |(w^l)^* E^l_i(z,s)| &\le \sup_{z \in S^i_{0,2} - \bigcup_{j=1}^{m} C_j(b)} (w^l)^* E^l_i(z, \text{Re } s)\\ &\le \max_{j=1,\dots,m} \left\{ \sup_{z \in \partial C_j(b)} (w^l)^* E^l_i(z, \text{Re } s) \right\}.\end{aligned}$$

Because we have observed that the last term in the above inequalities converges to 0, the conclusion easily follows.

Finally we give another proof of (2) (ii) that gives us the order of convergence. As is stated in the remark following Lemma 2, for any given $s = s_1$ with $1 < \text{Re } s_1$, we can find $0 < k = k_1 < 1$ such that $\text{Re } s_1(1 + k_1) - 2 > 0$. Then for any $b = b_1 > 1$, we easily see that for sufficiently small $l_1, \dots, l_m$,

$$w^l \left(S^i_{0,2} - \bigcup_{j=1}^{m} C_j(b_1) \right) \subset w^l(S^i_{0,2}) - \bigcup_{j=1}^{m} N_{l_j}(k_1).$$

Then

$$(3.8) \qquad \begin{aligned}\sup_{z \in S^i_{0,2} - \bigcup_{j=1}^{m} C_j(b_1)} |(w^l)^* E^l_i(z,s_1)| &\le \sup_{w \in w^l(S^i_{0,2}) - \bigcup_{j=1}^{m} N_{l_j}(k_1)} E^l_i(w, \text{Re } s_1)\\ &\le \max_{j=1,\dots,m} \sup_{\partial N_{l_j}(k_1) \cap w^l(S^i_{0,2})} E^l_i(w, \text{Re } s_1)\\ &\le \mathrm{M}_2(\text{Re } s_1) \max_{j=1,\dots,m} l_j^{\text{Re} s_1(1+k_1)-2}.\end{aligned}$$

From this we clearly obtain that $(w^l)^* E_i^l(z, s_1)$ converges uniformly to 0 on $S_{0,2}^i - \bigcup_{j=1}^{m} C_j(b_1)$ for any $b = b_1 > 1$ with the order of its convergence as in (3.8). $\square$

Theorem 2. *We set the same assumption and notations as in Theorem 1 (2). Pick R one of the components of $S_{0,2}^i$, which has the new cusps $q_1, q_2, \dots, q_t$, arising from the pinching geodesics $l_1, \dots, l_m$ on S_l, and has the old cusps which may be denoted by $p_1, \dots, p_u$, differing from p_i, replacing the enumeration if necessary. Denote by $E_{q_j}(z, s)$ the Eisenstein series attached to q_j for R $(j = 1, \dots, t)$.*

Then there exist some constants $K_l \to \infty$ and a subsequence $l_1^{(h)} = \cdots = l_m^{(h)} = l^{(h)} \to 0$,

$$K_{l^{(h)}} (w^{l^{(h)}})^* E_i^{l^{(h)}}(z, s) \quad \longrightarrow \quad G_0(z, s)$$

on any compact subset of R, where $G_0(z, s)$ is a non-trivial smooth function on R satisfying

$$(\Delta_0 - s(s-1))\ G_0(z, s) = 0 \qquad \text{on } R.$$

And $\lim_{l \to 0} K_l\ l^{2(\mathrm{Re} s - 1) - \delta} = \infty$, for any $\delta > 0$.

Moreover $G_0(z, s)$ is of the form,

$$G_0(z, s) = \sum_{j=1}^{t} B_j E_{q_j}(z, s), \tag{3.9}$$

where B_j $(j = 1, \dots, t)$ are some constants.

Proof. Let $C_v^o(a), C_j^n(a)$ $(a \geq 1)$ be respectively the a-cusp regions around p_v, q_j in R $(v = 1, \dots, u, j = 1, \dots, t)$. For any $a \geq 1$, set $\Omega(a) = R - \bigcup_{v=1}^{u} C_v^o(a) - \bigcup_{j=1}^{t} C_j^n(a)$.

Now we set constants $K_l > 0$ such that

$$K_l \sup_{z \in \Omega(1)} (w^l)^* E_i^l(z, \mathrm{Re}\ s) = K_l \times \sup_{z \in \bigcup_{v=1}^{u} \partial C_v^o(1) \cup \bigcup_{j=1}^{t} \partial C_j^n(1)} (w^l)^* E_i^l(z, \mathrm{Re}\ s) = 1.$$

Then we can show the following.

Lemma 3. *For any $a > 1$ fixed,*

$$\begin{aligned} L_l(a) &= K_l \sup_{z \in \Omega(a)} (w^l)^* E_i^l(z, \mathrm{Re}\ s) \\ &= K_l \times \sup_{z \in \bigcup_{v=1}^{u} \partial C_v^o(a) \cup \bigcup_{j=1}^{t} \partial C_j^n(a)} (w^l)^* E_i^l(z, \mathrm{Re}\ s) \end{aligned}$$

is uniformly bounded as $l_1 = \cdots = l_m = l \to 0$.

Proof of Lemma 3. Let $a > 1$ be fixed. Then taking a region $\Omega' \supset\supset \Omega(a)$ and applying Cor. 8.21 in [5], we have a constant $C = C(\Omega(a), \Omega')$ such that for l small enough, the Harnack inequalities

$$\sup_{\Omega(a)} K_l (w^l)^* E_i^l(z, \mathrm{Re}\ s) \leq C \inf_{\Omega(a)} K_l (w^l)^* E_i^l(z, \mathrm{Re}\ s)$$

hold. Thus we get

$$\begin{aligned} L_l(a) = \sup_{\Omega(a)} K_l(w^l)^* E_i^l(z, \mathrm{Re}\ s) &\leq C \inf_{\Omega(a)} K_l(w^l)^* E_i^l(z, \mathrm{Re}\ s) \\ &\leq C \inf_{\Omega(1)} K_l(w^l)^* E_i^l(z, \mathrm{Re}\ s) \\ &\leq C \sup_{\Omega(1)} K_l(w^l)^* E_i^l(z, \mathrm{Re}\ s) \\ &= C < \infty. \end{aligned}$$

We have concluded the proof of Lemma 3. □

We continue to the proof of Theorem 3. In the same way in the proof of Theorem 1, we take an exhausting family of compact subsets of R and apply the maximum principles, the interior Schauder estimates and the standard diagonal method to $\{K_l\ (w^l)^* E_i^l(z, \mathrm{Re}\ s)\}$. Thus we observe that there is a subsequence $l_1^{(h)} = \cdots = l_m^{(h)} = l^{(h)} \to 0$ and $K_{l^{(h)}}$,

$$\lim_{h\to\infty} K_{l^{(h)}}(w^{l^{(h)}})^* E_i^{l^{(h)}}(z, s) \;=\; G_0(z, s)$$

uniformly on any compact subset of R, for a non-trivial function $G_0(z, s)$ satisfying

$$(\Delta_0 - s(s-1))\ G_0(z, s) \;=\; 0. \tag{3.10}$$

Take a region D in R such that for sufficiently small $l_1, \ldots, l_m$, $w^l(D) \subseteq w^l(R) - \bigcup_{j=1}^{m} N_{l_j}(1)$. By Lemma 2, we see

$$\begin{aligned} \sup_{z\in D} |(w^l)^* E_i^l(z, s)| &\leq \max_{j=1,\ldots,m} \sup_{\partial N_{l_j}(1)} E_i^l(w, \mathrm{Re}\ s) \\ &\leq \mathrm{M}_2(1, \mathrm{Re}\ s) \max_{j=1,\ldots,m} l_j^{2(\mathrm{Re}\, s-1)}. \end{aligned}$$

By the above inequality, it is necessary that $K_l \geq \hat{c}\ l^{-2(\mathrm{Re}\, s-1)}$ for a constant $\hat{c}$.

All we have to show is just that $G_0(z,s)$ is of the form (3.9). From [10], Theorem 3.2.1, we see that the Fourier expansion of $G_0(z,s)$ at any cusp of Γ_0 (we set $R \simeq H/\Gamma_0$) is of the form

$$c_0 y^s + c_0^{'} y^{1-s} + \sum_{m\neq 0} c_m(s) y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|m|y) e^{2\pi\sqrt{-1}mx},$$

on $y > 1$. Here $K_{s-\frac{1}{2}}$ is the MacDonald-Bessel function.

Because for any $v = 1, \dots, u$,

$$\sup_{z\in C^o(a)} K_l \; |(w^l)^* E_i^l(z,s)| \leq \sup_{z\in \partial C_v^o(a)} K_l \; (w^l)^* E_i^l(z, \mathrm{Re}\; s) \leq L_l(a)$$

holds, we see that $G_0(z,s)$ is bounded in $C_v^o(a)$.

Therefore by (3.10), Claim in Section 1, the discussion of the proof of Theorem 1 (2) (ii) and the above observations, we have seen that $G_0(z,s)$ is a linear sum of the Eisenstein series attached to the new cusps $q_1, \dots, q_t$. It follows that there are constants $B_1, \dots, B_t$ such that

$$G_o(z,s) \; = \; \sum_{j=1}^{t} B_j E_{q_j}(z,s). \qquad \square$$

§4. A Comparison of the W-P and the Z-T Metrics

We review the construction of the degenerating family of punctured surfaces $\{c(t) \; = \; [S_{l(t)}] \in T_{g,n} \mid \; -\infty < t < 0\}$ in [13] (cf. [25] II). For simplicity assume that there is only one pinching geodesic on $S_{l(t)}$ as denoted by $l(t)$. The feature of the construction is to initiate with defining the vector field of the degenerating family that is represented as harmonic Beltrami differential as follows.

Set $\Gamma_t \subset \mathrm{PSL}_2(\mathbb{R})$ such that $S_{l(t)} \simeq H/\Gamma_t$. Let A_t be the one of primitive hyperbolic elements of Γ_t that cover the pinching geodesic $l(t)$. Define a relative Poincaré series associated to $l(t)$,

$$\Theta_t = \sum_{B\in\langle A_t\rangle\diagdown\Gamma_t} (\omega_{A_t}\circ B)\; B^{'2},$$

where $\omega_{A_t} \; = \; 1/z^2$, if A_t is normalized to fix 0, ∞. We remark that Θ_t is independent of how we choose each representative in the coset decomposition above. Then we define a tangent vector $\tau(t) \; \in HB(\Gamma_t)$,

$$\frac{dc(t)}{dt} \; = \; \tau(t) = p(t)\; (d\rho_t)^{-1}\overline{\Theta_t}, \qquad \iint_{S_{l(t)}} \tau(t)\Theta_t = \frac{\pi}{2} l(t),$$

where $d\rho_t$ is the hyperbolic area element for $S_{l(t)}$. It should be noted here that $p(t)$ is defined by the integration above. Then as is discussed in [13], Section 2, we have obtained the resulting behavior of $p(t)$ and $l(t)$ as follows,

$$\begin{aligned} &p(t) = 1 + \mathrm{O}(l(t)^2) \\ &\frac{dl(t)}{dt} = (dl(t), \tau) \ = \ \frac{2}{\pi} \iint_{S_{l(t)}} \tau \Theta_t = l(t), \qquad l(t) \ = \ le^t. \end{aligned} \tag{4.1}$$

Theorem 3. *The Weil-Petersson and the Zograf-Takhtajan metrics have the following behavior near the boundary of $T_{g,n}$, along the degenerating family of hyperbolic surfaces constructed by Wolpert. That is, let τ be the vector field formed by the degenerating family with only one pinching geodesic for simplicity. Then the norms of τ with respect to the Weil-Petersson and Zograf-Takhtajan metrics satisfy*

$$\|\tau\|_{ZT} \ \leq \ n\tilde{c}\|\tau\|_{WP} \qquad \textit{as } l \to 0,$$

where $\tilde{c}$ is an absolute constant.

Proof. Evaluate the norms of the tangent vectors $\tau(t)$ in terms of the Weil-Petersson and the Zograf-Takhtajan metrics respectively,

$$\begin{aligned} \left\| \frac{dc}{dt} \right\|^2_{\mathrm{WP}} &= \iint_{S_{l(t)}} |\tau(t)|^2 \ d\rho_t \\ \left\| \frac{dc}{dt} \right\|^2_{(\mathrm{i})} &= \iint_{S_{l(t)}} E_i^l(z,2) \ |\tau(t)|^2 \ d\rho_t. \end{aligned} \tag{4.2}$$

From ([25] II Lemma 2.3 and [13] Corollary), we have seen that

$$\left\| \frac{dc}{dt} \right\|^2_{\mathrm{WP}} \approx l(t), \qquad \text{as } l(t) \to 0. \tag{4.3}$$

Therefore we have to just show that the second norm in (4.2) has the same order as (4.3). Divide the Zograf-Takhtajan's norm in (4.2) into three parts as in [13] Theorem,

$$\left\| \frac{dc}{dt} \right\|^2_{(\mathrm{i})} \ = \ \iint_{N_l(\frac{1}{2})} + \iint_{N_l(1) - N_l(\frac{1}{2})} + \iint_{S_{l(t)} - N_l(1)} \ = \ \mathrm{I} \ + \mathrm{II} \ + \mathrm{III}.$$

Since we have already shown in [13] that

$$\mathrm{II} \ = \mathrm{O} \ (l(t)^2), \qquad \mathrm{III} \ = \mathrm{O} \ (l(t)^4), \qquad \text{as } l(t) \to 0$$

(that is, small enough), we may just concentrate on estimating the main term I.

From Lemma 1 and (4.1), it follows that

$$\begin{aligned}\mathrm{I} &= \iint_{N_l(\frac{1}{2})} E_i^l(z,2)\ |\tau(t)|^2\ d\rho_t \\ &= \iint_{N_l(\frac{1}{2})} E_i^l(z,2)\ |p(t)(d\rho_t)^{-1}\overline{\Theta_t}|^2\ d\rho_t \\ &\leq \mathrm{M}_1(2,1) \iint_{S_{l(t)}} 2|(d\rho_t)^{-1}\overline{\Theta_t}|^2 \approx l(t), \qquad \text{as } l(t) \to 0. \qquad \square\end{aligned}$$

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