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# Errata to "On Defining Relations of Affine Lie Superalgebras and Affine Quantized Universal Enveloping Superalgebras"

By

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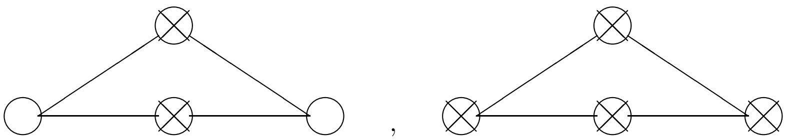
The paper [1] contains mistakes; Theorems 4.1.1, 4.5.1 and 8.4.3 for the data corresponding to the Dynkin diagrams below are incorrect.

(1) 

In Section 2, the statements and proofs of the theorems shall be corrected.

**1** *Preliminary.* Keep the notation and terminology of [1]. We say that the datum $(\mathcal{E}, \Pi, p)$ is of $(\mathrm{A}(1,1)^{(1)})^{\mathcal{H}}$ type if $(\mathcal{E}, \Pi, p)$ is of affine ABCD type (see Definition 1.4.1), $\Pi = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$ and if the Dynkin diagram of $(\mathcal{E}, \Pi, p)$ is either of the two Dynkin diagrams of (1). Until the end of this section, we assume that $(\mathcal{E}, \Pi, p)$ is of $(\mathrm{A}(1,1)^{(1)})^{\mathcal{H}}$ type. Then $\mathcal{G}(\mathcal{E}, \Pi, p) \cong (\mathrm{A}(1,1)^{(1)})^{\mathcal{H}}$, $\mathcal{G}^{\natural}(\mathcal{E}, \Pi, p) \cong (sl(2,2)^{(1)})^{\mathcal{H}}$, and $\Phi(\mathcal{E}, \Pi, p) = \{\pm(m+1)\delta, \pm(m\delta + \alpha_i), \pm(m\delta + \alpha_i + \alpha_{i-1}), \pm(m\delta + \alpha_i + \alpha_{i-1} + \alpha_{i-2}) | i = 0,1,2,3 \in \mathbf{Z}/4\mathbf{Z}, m \geq 0\}$. (See Subsections 1.5 and 3.5, and notice that $(\mathrm{A}(1,1)^{(1)})^{\mathcal{H}}$ and $\mathrm{A}(1,1)^{(1)}$ (resp. $(sl(2,2)^{(1)})^{\mathcal{H}}$ and $sl(2,2)^{(1)}$) are not the same; however they are closely related.) Define $E_i^{(m)}, E_{ii-1}^{(m)}, E_{ii-1i-2}^{(m)} \in \widetilde{\mathcal{N}}_+ (\subset \widetilde{\mathcal{G}}(\mathcal{E}, \Pi, p))$ $(i \in \mathbf{Z}/4\mathbf{Z}, m \geq 0)$ inductively by $E_i^{(0)} = E_i$, $E_i^{(m)} = [E_i, [E_{i-1}, E_{i-2i-3i}^{(m-1)}]]$, $E_{ii-1}^{(m)} = [E_i, E_{i-1}^{(m)}]$, $E_{ii-1i-2}^{(m)} = [E_i, E_{i-1i-2}^{(m)}]$. Let $E_{ii-1} = E_{ii-1}^{(0)}$ and $E_{ii-1i-2} = E_{ii-1i-2}^{(0)}$.

Let $\mathbf{x} : \widetilde{\mathcal{N}}_+ \to \widetilde{\mathcal{N}}_-$ be the isomorphism such that $\mathbf{x}(E_i) = F_i$ $(0 \leq i \leq 3)$. Denote $(\mathcal{E}, \Pi, p)$ by $(\mathrm{AA})^{\star}$ if its Dynkin diagram is the left one of (1) and if

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$(\alpha_1, \alpha_1) = 2$. Define $\mathcal{G}^\star$ to be the admissible Lie superalgebra $\widetilde{\mathcal{G}}(\mathrm{AA})^\star/(r_+^\star + \mathbf{x}(r_+^\star))$, where $r_+^\star$ is the ideal of $\widetilde{\mathcal{N}}_+$ generated by the elements:

$$\begin{cases} [E_0, E_0],\ [E_2, E_2],\ [E_0, E_2],\ [E_1, E_3], \\ [E_1, E_{10}],\ [E_1, E_{21}],\ [E_3, E_{03}],\ [E_3, E_{32}], \\ [E_2, E_{321}],\ [E_{210}, E_{032}],\ [E_2, [E_0, E_{103}]]\,. \end{cases}$$

(See Definition 1.2.1 for the terminology.) Denote $(\mathcal{E}, \Pi, p)$ by $(\mathrm{AA})^\bullet$ if its Dynkin diagram is the right one of (1) and if $(\alpha_1, \alpha_2) = 1$. Define $\mathcal{G}^\bullet$ to be the admissible Lie superalgebra $\widetilde{\mathcal{G}}(\mathrm{AA})^\bullet/(r_+^\bullet + \mathbf{x}(r_+^\bullet))$, where $r_+^\bullet$ is the ideal of $\widetilde{\mathcal{N}}_+$ generated by the elements:

$$\begin{cases} [E_0, E_0],\ [E_1, E_1],\ [E_2, E_2],\ [E_3, E_3],\ [E_0, E_2],\ [E_1, E_3], \\ [E_0, E_{103}],\ [E_1, E_{210}],\ [E_2, E_{321}],\ [E_3, E_{032}]\,. \end{cases}$$

**Lemma 1.** *Let $(\mathcal{E}, \Pi, p) = (\mathrm{AA})^\bullet$, and $C^\bullet := \{\pm(m\delta + \alpha_i + \alpha_{i-2}) | i \in \mathbf{Z}/4\mathbf{Z},\ m \geq 1\}$. Then:*

(1) $\dim \mathcal{G}^\bullet_\beta = 1$ *if $\beta \in \Phi(\mathcal{E}, \Pi, p) \setminus \mathbf{Z}\delta$.*

(2) $\dim \mathcal{G}^\bullet_\beta = 0$ *if $\beta \in (P_+ \cup P_-) \setminus (\Phi(\mathcal{E}, \Pi, p) \cup C^\bullet \cup \{0\})$.*

(3) $\mathcal{G}^\bullet_\beta = [E_i, \mathcal{G}^\bullet_{\beta-\alpha_i}]$, $\mathcal{G}^\bullet_{-\beta} = [F_i, \mathcal{G}^\bullet_{-\beta+\alpha_i}]$ *if $\beta = m\delta + \alpha_i + \alpha_{i-2}$ with $m \geq 1$.*

*Proof.* Let $\mathbf{a}$ be the automorphism of $\mathcal{G}^\bullet$ such that $\mathbf{a}(E_i) = E_{i-2}$, $\mathbf{a}(F_i) = F_{i-2}$. Let $\mathbf{b}_i := \exp(\mathrm{ad}[E_i, E_{i-1}]) \exp(\mathrm{ad}[F_i, F_{i-1}]) \exp(\mathrm{ad}[E_i, E_{i-1}])$, and $\mathbf{c}_i := \mathbf{a} \circ \mathbf{b}_i \circ \mathbf{b}_{i+1}$. (Notice that $\mathrm{ad}[E_i, E_{i-1}]$ and $\mathrm{ad}[F_i, F_{i-1}]$ are locally nilpotent.) If $\beta = a_0\alpha_0 + a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$, then $\mathbf{c}_i(\mathcal{G}^\bullet_\beta) = \mathcal{G}^\bullet_{\beta-(a_i-a_{i-2})\delta}$.

Notice that $\mathcal{G}^\bullet$ has a triangular decomposition $\mathcal{G}^\bullet = \mathcal{N}^\bullet_+ \oplus \mathcal{H} \oplus \mathcal{N}^\bullet_-$, where $\mathcal{N}^\bullet_+ = \widetilde{\mathcal{N}}_+/r^\bullet_+$ and $\mathcal{N}^\bullet_- = \widetilde{\mathcal{N}}_-/\mathbf{x}(r^\bullet_+)$. Assume that $\beta \in P_+ \setminus \{0\}$ and that we have already proved the lemma for $\beta' \in P_+ \setminus \{0\}$ with $\beta - \beta' \in P_+ \setminus \{0\}$. Then the lemma can be proved as follows; here we shall treat two cases as examples. First example is the case where $\beta = m\delta + \alpha_{i+1} + 2\alpha_i + \alpha_{i-1}$ with $m \geq 0$. If $m$ is even, it follows that $\dim \mathcal{G}^\bullet_\beta = \dim \mathbf{c}_i^{m/2}(\mathcal{G}^\bullet_\beta) = \dim \mathcal{G}^\bullet_{\alpha_{i+1}+2\alpha_i+\alpha_{i-1}} = \dim \mathcal{N}^\bullet_+ \cap \mathcal{G}^\bullet_{\alpha_{i+1}+2\alpha_i+\alpha_{i-1}} = 0$, where the last equality can be proved directly. If $m$ is odd, it follows that $\dim \mathcal{G}^\bullet_\beta = \dim \mathcal{G}^\bullet_{\alpha_i - \alpha_{i-2}} = 0$. Second example is the case where $\beta = m\delta + \alpha_i + \alpha_{i-2}$ with $m \geq 0$. By the assumption, we have $\mathcal{G}^\bullet_\beta = \sum_{j \in \mathbf{Z}/4\mathbf{Z}} [E_j, \mathcal{G}^\bullet_{\beta-\alpha_j}] = [E_i, \mathcal{G}^\bullet_{m\delta+\alpha_{i-2}}] + [E_{i-2}, \mathcal{G}^\bullet_{m\delta+\alpha_i}] = [E_i, [E_{i-2}, \mathcal{G}^\bullet_{m\delta}]] + [E_{i-2}, [E_i, \mathcal{G}^\bullet_{m\delta}]] = [E_i, [E_{i-2}, \mathcal{G}^\bullet_{m\delta}]] = [E_i, \mathcal{G}^\bullet_{m\delta+\alpha_{i-2}}]$. Other cases can be treated similarly.

The case where $\beta \in P_- \setminus \{0\}$ can also be treated similarly. □

**Lemma 2.** *Let* $(\mathcal{E},\Pi,p)=(\mathrm{AA})^\star$*, and* $C^\star := \{\pm((m+1)\delta+\alpha_3+2\alpha_2+\alpha_1),\, \pm(m\delta+\alpha_1+2\alpha_0+\alpha_3)\,|\, m\geq 0\}$*. Then:*

(1) $\dim \mathcal{G}^\star_\beta = 1$ *if* $\beta\in\Phi(\mathcal{E},\Pi,p)\setminus \mathbf{Z}\delta$.

(2) $\dim \mathcal{G}^\star_\beta = 0$ *if* $\beta\in (P_+\cup P_-)\setminus(\Phi(\mathcal{E},\Pi,p)\cup C^\star\cup\{0\})$.

(3) $\mathcal{G}^\star_\beta=[E_i,\mathcal{G}^\star_{\beta-\alpha_i}]$, $\mathcal{G}^\star_{-\beta}=[F_i,\mathcal{G}^\star_{-\beta+\alpha_i}]$ *if* $\beta=m\delta+\alpha_{i+1}+2\alpha_i+\alpha_{i-1}$ *with* $m\geq 0$.

*Proof.* Recall the definition of $s_2$ from Proposition 2.4.3. We can easily show that there exists an isomorphism $\mathbf{e}:\mathcal{G}^\bullet\to\mathcal{G}^\star$ such that $\mathbf{e}(H_\gamma)=H_{s_2(\gamma)}$, $\mathbf{e}(E_j)=-(\alpha_j,\alpha_2)[E_j,E_2]$, $\mathbf{e}(F_j)=-[F_j,F_2]$ $(j=1,\,3)$, $\mathbf{e}(E_2)=F_2$, $\mathbf{e}(F_2)=-E_2$, $\mathbf{e}(E_0)=E_0$, $\mathbf{e}(F_0)=F_0$. Then the statements (1) and (2) follow from Lemma 1 and from the fact that $\mathbf{e}(\mathcal{G}^\bullet_\alpha)=\mathcal{G}^\star_{s_2(\alpha)}$. The statement (3) can be proved by an exactly similar argument to that for Lemma 1 (3). □

Keep the notation of Definition 1.2.1. By the fact that $\mathcal{G}^\natural(\mathrm{AA})^\star\cong(sl(2,2)^{(1)})^{\mathcal{H}}$, we can easily see that $\mathcal{G}^\star\succ\mathcal{G}^\natural(\mathrm{AA})^\star$. Hence $\Psi[\mathcal{G}^\star,\mathcal{G}^\natural(\mathrm{AA})^\star]$ can be defined. Similarly $\Psi[\mathcal{G}^\bullet,\mathcal{G}^\natural(\mathrm{AA})^\bullet]$ can be defined. Let $\Psi^{\star\natural}=\Psi[\mathcal{G}^\star,\mathcal{G}^\natural(\mathrm{AA})^\star]$, $\Psi^{M\star}=\Psi[\widetilde{\mathcal{G}}(\mathrm{AA})^\star,\mathcal{G}^\star]$, $\Psi^{\bullet\natural}=\Psi[\mathcal{G}^\bullet,\mathcal{G}^\natural(\mathrm{AA})^\bullet]$ and $\Psi^{M\bullet}=\Psi[\widetilde{\mathcal{G}}(\mathrm{AA})^\bullet,\mathcal{G}^\bullet]$.

**Theorem 3.** (1) *The ideal* $\ker\Psi^{\star\natural}$ *is generated by the elements* $\Psi^{M\star}([E_0,E^{(m)}_{103}])$, $\Psi^{M\star}([E_2,E^{(m+1)}_{321}])$, $\Psi^{M\star}(\mathbf{x}([E_0,E^{(m)}_{103}]))$, $\Psi^{M\star}(\mathbf{x}([E_2,E^{(m+1)}_{321}]))$ $(m\geq 0)$.

(2) *The ideal* $\ker\Psi^{\bullet\natural}$ *is generated by the elements* $\Psi^{M\bullet}([E_i,E^{(m)}_{i+2}])$, $\Psi^{M\bullet}(\mathbf{x}([E_i,E^{(m)}_{i+2}]))$ $(i=0,\,1,\,m\geq 1)$.

*Proof.* Let $m\geq 0$, and let $Y\in\widetilde{\mathcal{G}}(\mathrm{AA})^\star$ be $E^{(m+1)}_{321}$, $E^{(m)}_{103}$, $\mathbf{x}(E^{(m+1)}_{321})$ or $\mathbf{x}(E^{(m)}_{103})$. Let $\gamma\in P$ be such that $Y\in\widetilde{\mathcal{G}}_\gamma$. By the fact that $\mathcal{G}^\natural(\mathrm{AA})^\star\cong(sl(2,2)^{(1)})^{\mathcal{H}}$, we see that $\Psi^{\star\natural}\circ\Psi^{M\star}(Y)\neq 0$. It follows from Lemma 2 (1) that $\mathcal{G}^\star_\gamma$ is the one dimensional vector space spanned by $\Psi^{M\star}(Y)$. Then the statement (1) follows immediately from Lemma 2. (We see that since $\mathcal{G}^\natural(\mathrm{AA})^\star$ is the maximal affine-admissible Lie superalgebra, $\ker\Psi^{\star\natural}\cap\mathcal{G}^\star_{m\delta}=\{0\}$ for $m\neq 0$.)

(2) We can show this by a similar argument to that for (1), and by Lemma 1. □

We say that $(\mathcal{E},\Pi,p)$ is a *usual* datum of $(\mathrm{A}(1,1)^{(1)})^{\mathcal{H}}$ type, if $(\alpha_i,\alpha_i)\neq 0$ for some $0\leq i\leq 3$. Otherwise we say that $(\mathcal{E},\Pi,p)$ is an *unusual* datum of

$(\mathrm{A}(1,1)^{(1)})^{\mathcal{H}}$ type. If $(\mathcal{E},\Pi,p)$ is a usual datum of $(\mathrm{A}(1,1)^{(1)})^{\mathcal{H}}$ type and if $(\alpha_1,\alpha_1)\neq 0$ (resp. $(\alpha_1,\alpha_1)=0$), let $\kappa(j)=j$ (resp. $\kappa(j)=j+1$) ($j\in\mathbf{Z}/4\mathbf{Z}$).

**2** *Corrections.* We correct Theorems 4.1.1, 4.5.1 and 8.4.3 of [1] as follows:

(2.1) *Correction of Theorem* 4.1.1. We add the following relations to the statement of Theorem 4.1.1.

(S4)(18) $[E_{\kappa(2)\kappa(1)\kappa(0)},E_{\kappa(0)\kappa(3)\kappa(2)}]=0$ *and* $[E_{\kappa(2)},E^{(m)}_{\kappa(3)\kappa(2)\kappa(1)}]=0$, $[E_{\kappa(0)},E^{(m)}_{\kappa(1)\kappa(0)\kappa(3)}]=0$ $(m\geq 1)$ *if* $(\mathcal{E},\Pi,p)$ *is a usual datum of* $(\mathrm{A}(1,1)^{(1)})^{\mathcal{H}}$ *type,*

(S4)(19) $[E_i,E^{(m)}_{i+2}]=0$ $(i=0,\,1,\,m\geq 1)$ *if* $(\mathcal{E},\Pi,p)$ *is an unusual datum of* $(\mathrm{A}(1,1)^{(1)})^{\mathcal{H}}$ *type,*

(S5)($b$) ($b=18,\,19$) *The same relations with* $E_r$*'s in place of* $F_r$*'s in* (S4)($b$)

*Proof of the corrected statement of Theorem* 4.1.1. If $(\mathcal{E},\Pi,p)$ is not of $(\mathrm{A}(1,1)^{(1)})^{\mathcal{H}}$ type, use the original proof. Otherwise use Theorem 3. □

(2.2) *Correction of Theorem* 4.5.1. We add the relations (S4)($b$), (S5)($b$) ($b=18,\,19$) to the statement of Theorem 4.5.1.

The corrected statement follows immediately from Theorem 3.5.1 and the corrected Theorem 4.1.1.

(2.3) *Correction of Theorem* 8.4.3. We add the following relations to the statement of Theorem 8.4.3.

(QS4)(18) $Z_{\kappa(0)}([T_{\omega_3}(K_3^{-1}F_3),T_{\omega_1}(K_1^{-1}F_1)])=0$ *and* $Z_{\kappa(0)}([E_0,T^{m+1}_{\omega_2}(K_2^{-1}F_2)])=0$, $Z_{\kappa(2)}([E_0,T^{m+1}_{\omega_2}(K_2^{-1}F_2)])=0$ $(m\geq 1)$ *if* $(\mathcal{E},\Pi,p)$ *is a usual datum of* $(\mathrm{A}(1,1)^{(1)})^{\mathcal{H}}$ *type,*

(QS4)(19) $[E_i,T^{m+1}_{\omega_{i+2}}(K^{-1}_{i+2}F_{i+2})]=0$ $(i=0,\,1,\,m\geq 1)$ *if* $(\mathcal{E},\Pi,p)$ *is an unusual datum of* $(\mathrm{A}(1,1)^{(1)})^{\mathcal{H}}$ *type,*

(QS5)($b$) ($b=18,\,19$) *The same relations with* $E_r$*'s in place of* $F_r$*'s in* (QS4)($b$).

Using Proposition 8.4.2 and using $T_{\omega_i}$'s, the corrected statement is obtained by exactly the same argument as in the proof of Theorem 6.8.2.

*Remark* 4. Let $\mathcal{D}$ be the finite dimensional Lie superalgebra such that $\dim\mathcal{D}=\dim\mathrm{D}(2,1;x)$ and such that its defining relations are obtained from those of $\mathrm{D}(2,1;x)$ by substituting $-1$ for $x$. Then there exists an epimorphism

$\mathcal{D} \to sl(2,2)$. (See also [2].) Let $L(\mathcal{D},1)$ be the infinite dimensional Lie superalgebra obtained from $\mathcal{D}$ by the same way as in Subsection 1.1. We can utilize $L(\mathcal{D},1)$ for giving concrete bases of $\mathcal{G}^{\star}$ and $\mathcal{G}^{\bullet}$. After doing so, we will see that the elements in the statement of Theorem 3 are indeed non-zero, which implies that the relations of the corrected statements are necessary.

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## References

[1] Yamane, H., On defining relations of affine Lie superalgebras and affine quantized universal enveloping superalgebras, *Publ. RIMS, Kyoto Univ.*, **35** (1999), 321–390.

[2] Iohara, K. and Koga, Y., Central extension of Lie superalgebras, *Comment. Math. Helv.*, **76** (2001), 110–154.