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Scattering Theory for a Stratified Acoustic Strip with Short- or Long-Range Perturbations

By

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Abstract

We consider the acoustic propagator $H = -\nabla \cdot \rho\nabla$ acting in $L^2(\Omega)$ with $\Omega := \Omega' \times \mathbb{R}$ and Ω' a bounded open set in $\mathbb{R}^{n-1}$, $n \geq 2$. The real-valued function ρ belongs to $L^\infty(\Omega)$, and is bounded from below by $c > 0$. We assume there exist two strictly positive constants c_1 and c_2 and two perturbations, δ^S of short-range type and δ^L of long-range type, such that $\rho = c_j + \delta^S + \delta^L$ on $\Omega_j := \{(x', x_n) \in \Omega | (-1)^j x_n > 0\}$, $j = 1, 2$. We build two modified free evolutions $U_j(t)$, $j = 1, 2$, such that the wave operators $\Omega_j^\pm := s - \lim_{t\to\pm\infty} e^{itH} U_j(t)$, $j = 1, 2$, exist and are asymptotically complete.

Contents

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§1. Introduction and Results

Let $\Omega' \subset \mathbb{R}^{n-1}, n \geq 2$ be a Lipschitz bounded open set (or according to Stein [13] a domain with a "boundary with minimal regularity"). Then the cylinder $\Omega := \Omega' \times \mathbb{R}$ is also a Lipschitz domain and the Sobolev spaces $\mathcal{H}^s(\Omega)$ and $\mathcal{H}^s_0(\Omega)$, $s \in \mathbb{R}$, have the usual properties.

We consider a function $\rho : \Omega \mapsto \mathbb{R}^*_+$ satisfying the following assumptions:

(i) ρ and $1/\rho$ belong to $L^\infty(\Omega)$.

(ii) There exist two strictly positive constants c_1, c_2 and two real-valued functions δ^S, $\delta^L : \Omega \mapsto \mathbb{R}$, such that:

a) $\rho = c_j + \delta^S + \delta^L$ on $\Omega_j := \{x = (x', x_n) \in \Omega/(-1)^j x_n > 0\}, j = 1, 2$.

b) $\delta^S \in L^\infty(\Omega)$ is a short-range perturbation, i.e. there exist constants $C > 0$ and $\theta \in (0, 1]$ such that

$$|\delta^S(x)| \leq C\langle x\rangle^{-1-\theta} \quad \text{a.e. (almost everywhere) on } \Omega, \tag{1.1}$$

where $\langle x\rangle := (1 + |x|^2)^{1/2}$ and $|x|$ is the Euclidean norm of $x \in \mathbb{R}^n$.

c) $\delta^L \in C^\infty(\overline{\Omega})$ is a long range perturbation, i.e. there exists a constant $\theta \in (0, 1]$ such that

$$\forall \alpha \in \mathbb{N}^n, \exists C_\alpha > 0, |\partial^\alpha \delta^L(x)| \leq C_\alpha \langle x\rangle^{-\theta-|\alpha|} \quad \text{on} \quad \Omega. \tag{1.2}$$

Remark 1.1. a) We can assume that δ^L depends only on x_n. Indeed, according to [11], each connect component of Ω' is a finte union of open sets starred with respect to a ball. Therefore by applying conveniently Taylor's formula with respect to x', we get $\delta^L(x) = \delta^L_0(x_n) + r^S(x)$ where r^S is short-range perturbation and δ^L_0 is a long-range perturbation.

b) We can also assume with a modification of δ^S that

$$|\delta^L(x)| \leq 1/2 \min\{c_1, c_2\}, \ x \in \Omega. \tag{1.3}$$

The quadratic form h, with domain $D(h) := \mathcal{H}^1_0(\Omega)$, defined by

$$h(u, v) := \int_\Omega \rho \nabla u \cdot \nabla \bar{v} dx, \quad u, v \in \mathcal{H}^1_0(\Omega), \tag{1.4}$$

is symmetric, non negative and closed. Kato's representation theorem (Chapter VI of [10]) gives a unique self-adjoint operator in $L^2(\Omega)$ with domain

$$\begin{aligned} D(H) &= \{u \in \mathcal{H}^1_0(\Omega)/ - \nabla \cdot \rho \nabla u \in L^2(\Omega)\}, \\ Hu &= -\nabla \cdot \rho \nabla u \quad \text{if } u \in D(H). \end{aligned} \tag{1.5}$$

The spectral theory of the operator H, and a limiting absorption principle (under more general assumptions), have been studied in [7] (refer also to [4] at the origin of these works). When the perturbation δ^L is zero (and $c_j, j = 1, 2$, not necessarily constants), the scattering theory for H has been made in [5] by comparing H to the "free" operators $H_j, j = 1, 2$, defined analogously to H by substituting c_j for ρ and by using some aspects in relation with a 3-body Hamiltonian of quantum mechanics.

In our case where the functions $c_j, j = 1, 2$, are constant, the "free" operators $H_j, j = 1, 2$, are the self-adjoint operators in $L^2(\Omega)$ defined by

$$(1.6) \qquad \begin{aligned} D(H_j) &= \{u \in \mathcal{H}_0^1(\Omega) / -\Delta u \in L^2(\Omega)\}, \\ H_j u &= -c_j \Delta u \quad \text{if} \quad u \in D(H_j). \end{aligned}$$

Note that $D(H_j) = \mathcal{H}_0^1(\Omega) \cap \mathcal{H}^2(\Omega)$ if Ω' would have the exterior ball property (see [2]).

In the case of a long-range perturbation, it is no longer possible to compare the evolution e^{-itH} with the free evolutions $e^{-itH_j}, j = 1, 2$, and we have to find modified free evolutions $U_j(t), j = 1, 2$. There exist a lot of results concerning the scattering theory for stratified media with short-range perturbations (see for instance [3], [6], [14]). As far as we know, there is no result for such media with long-range perturbations.

The main result of this paper is the following theorem:

Theorem 1.1 (*Existence and completeness of the modified wave operators*). *There exist modified free evolutions $\{U_j(t)\}_{t\in\mathbb{R}}, j = 1, 2$, where $U_j(t)$ are bounded operators in $L^2(\Omega)$ determined by c_j, Ω' and δ^L, such that the modified wave operators*

$$(1.7) \qquad \Omega_j^{\pm} := s - \lim_{t\mapsto\pm\infty} e^{itH} U_j(t)$$

exist and are asymptotically complete, i.e.

$$(1.8) \qquad \mathcal{H}_{ac}(H) := \bigoplus_{j=1,2} \operatorname{Ran} \Omega_j^{\pm}$$

where $\mathcal{H}_{ac}(H)$ is the subspace of absolute continuity of the operator H.

Let $\mathcal{F}_n$ be the partial Fourier transform on $L^2(\Omega)$ defined for $u \in C_0^\infty(\Omega)$ and $(x', \xi) \in \Omega' \times \mathbb{R}$ by

$$(1.9) \qquad \mathcal{F}_n u(x', \xi) := (2\pi)^{-1/2} \int_{\mathbb{R}} u(x', x_n) e^{-ix_n\xi} dx_n.$$

We set $\mathcal{H} = L^2(\Omega), \mathcal{H}^\pm := \{u \in \mathcal{H}/(\mathcal{F}_{x_n} f)u(x', \xi_n) = 0 \text{ if } x' \in \Omega', \mp\xi_n > 0\}$ and $\mathcal{H}^\pm_{ac,j} := \operatorname{Ran}\Omega_j^\pm$. Then we have the following corollary:

Corollary 1.1. 1) *One has* $\operatorname{Ker}\Omega_1^\pm = \operatorname{Ker}\Omega_2^\mp = \mathcal{H}^\pm$.

2) *The operators* $\Omega_1^\pm : \mathcal{H}^\mp \to \mathcal{H}^\pm_{ac,1}$ *and* $\Omega_2^\pm : \mathcal{H}^\pm \to \mathcal{H}^\pm_{ac,2}$ *are unitary.*

3) *Setting* $\Omega^\pm := \Omega_1^\pm \oplus \Omega_2^\pm$, *the scattering operator* $S := (\Omega^+)^*\Omega^-$ *is unitary from* $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ *onto* $\mathcal{H}^- \oplus \mathcal{H}^+ = \mathcal{H}$.

For the proofs, we use some results of [7], in particular the limiting absorption principle, and an idea of Isozaki and Kitada [9] improved by Yafaev [17]. These authors have built wave operators for the Schrödinger operator with an identification $\mathcal{J}$, defined as a Fourier integral operator and allowing to prove the asymptotic completeness. The modified free evolutions follow by the stationary phase method. Nevertheless, the existence of thresholds gives some problems.

The paper is organized as follows. In Section 2, we collect from [4] and [7] the needed results concerning the operators H and H_j. In Section 3, we study the eikonal equation. It defines the phase function of a Fourier integral operator on $\mathbb{R}$ considered in Section 4. The operators of identification are built in Section 5. The existence of the generalized wave operators is proved in Section 6, and their completeness in Section 7. Finally, the construction of the free evolutions $U_j(t)$ and the proofs of Theorem 1.1 and Corollary 1.1 are given in Section 8.

Let us give some notations. The norm (respectively the scalar product) in a normed (respectively Hilbert) space E is denoted by $\|\cdot\|_E$ (respectively $(\cdot,\cdot)_E$). The space of bounded (respectively compact) linear operators from a Banach space E to a Banach space F equipped with the uniform operator topology is denoted by $\mathcal{B}(E, F)$ and by $\mathcal{B}(E)$ if $E = F$ (respectively $\mathcal{K}(E, F)$ and by $\mathcal{K}(E)$ if $E = F$). If H is a self-adjoint operator in a Hilbert complex space, the spectrum of H, (respectively the essential spectrum, the singular continuous spectrum, the absolutely continuous spectrum, the set of eigenvalues, the orthogonal projection on the subspace of absolute continuity $\mathcal{H}_{ac}(H)$) are denoted by $\sigma(H)$ (respectively $\sigma_{ess}(H), \sigma_{sc}(H), \sigma_{ac}(H), \sigma_p(H), P_{ac}(H)$).

§2. Spectral Preliminaries

We identify the space $L^2(\Omega)$ with the direct integral $\int_{\mathbb{R}}^{\oplus} L^2(\Omega')d\xi$, so that $\mathcal{F}_n$, the partial Fourier transform (1.9), is unitary. We set $\widehat{H}_j := \mathcal{F}_n H_j \mathcal{F}_n^* =$

$\int_{\mathbb{R}}^{\oplus} \widehat{H}_j(\xi)$, with $\widehat{H}_j(\xi)$, $\xi \in \mathbb{R}$, the self-adjoint operator on $L^2(\Omega')$ defined by

$$D(\widehat{H}_j(\xi)) = \{v \in \mathcal{H}_0^1(\Omega')/\Delta' v \in L^2(\Omega')\}, \quad \widehat{H}_j(\xi)v = -c_j\Delta' v + c_j\xi^2 v \quad \text{if } v \in D(\widehat{H}_j(\xi)), \tag{2.1}$$

where Δ' is the Laplace operator on Ω'. The operator $\widehat{H}_j(\xi)$ is with a compact resolvent, and if $(\lambda_k)_{k\geq 1}$, $0 < \lambda_1 \leq \lambda_2 \leq \dots$, is the sequence of eigenvalues of the Dirichlet problem for $-\Delta'$ on Ω' and $(V_k)_{k\geq 1}$ is an orthonormal basis of $L^2(\Omega')$, elements of which are real-valued eigenfunctions of the same problem associated to the above eigenvalues $(\lambda_k)_{k\geq 1}$, then the $V_k, k \geq 1$ are also eigenfunctions of $\widehat{H}_j(\xi)$ associated to the eigenvalues

$$\lambda_{jk} := c_j(\lambda_k + \xi^2), \quad j = 1, 2, k \geq 1. \tag{2.2}$$

Note that $V_k \in C^\infty(\Omega)$.

It is known (see [7]) that H_j is purely absolutely continuous and that $\sigma(H_j) = [c_j\lambda_1, +\infty), j = 1, 2$. The generalized eigenfunctions of H_j are independent of $j = 1, 2$, and defined by

$$\Phi_k(x, \xi) := (2\pi)^{-1/2} e^{ix_n\xi} V_k(x'), \quad x = (x', x_n) \in \Omega,\ \xi \in \mathbb{R},\ k \geq 1. \tag{2.3}$$

For $f \in L^2(\Omega)$ and $k \geq 1$, we set

$$f_k(x_n) := \int_{\Omega'} f(x', x_n)V_k(x')dx',\ x_n \in \mathbb{R}, \quad \widehat{f}_k(\xi) := (2\pi)^{-1/2}\int_{\mathbb{R}} e^{-ix_n\xi} f_k(x_n)dx_n = \int_\Omega f(x)\overline{\Phi_k(x,\xi)}dx,\ \xi \in \mathbb{R}, \tag{2.4}$$

where both last integrals converge in $L^2(\mathbb{R})$.

The following properties can be found in [15], [4] and [7].

Proposition 2.1. 1) *For $x \in \Omega, j = 1, 2, k \geq 1, \xi \in \mathbb{R}$,*

$$-c_j\Delta_x\Phi_k(x, \xi) = \lambda_{j,k}(\xi)\Phi_k(x, \xi).$$

2) *For $f \in L^2(\Omega)$,*

$$f(x) = \sum_{k\geq 1}\int_{\mathbb{R}} \widehat{f}_k(\xi)\Phi_k(x, \xi)d\xi \tag{2.5}$$

where the series converges in $L^2(\Omega)$.

3) *For $f, g \in L^2(\Omega)$,*

$$(f, g)_{L^2(\Omega)} = \sum_{k\geq 1}\int_{\mathbb{R}} \widehat{f}_k(\xi)\overline{\widehat{g}_k(\xi)}d\xi. \tag{2.6}$$

4) *For* $f \in D(H_j)$,

$$\widehat{(H_j f)}_k(\xi) = \lambda_{jk}(\xi)\widehat{f}_k(\xi). \tag{2.7}$$

5) *For* $f \in L^2(\Omega)$ *and* φ *a bounded Borel function on* $\mathbb{R}$,

$$(\widehat{\varphi(H_j) f})_k(\xi) = \varphi(\lambda_{j,k}(\xi))\widehat{f}_k(\xi). \tag{2.8}$$

6) *The operator* $\mathcal{F} : L^2(\Omega) \to \int_{\mathbb{R}}^{\oplus} L^2(\Omega')$ *defined for* $f \in L^2(\Omega)$ *by*

$$(\mathcal{F}f)(\xi) := \sum_{k \geq 1} \widehat{f}_k(\xi) V_k \tag{2.9}$$

is unitary.

Proposition 2.2 (*see Theorem* 0.2 *of* [7])**.**
1) $\inf \sigma(H) > 0$.
2) $\sigma_{ess}(H) = [\mu, +\infty)$, *with* $\mu := \min\{c_1\lambda_1, c_2\lambda_2\}$.
3) $\sigma_{sc}(H) = \emptyset$.
4) *Let* $\tau(H) := \{c_j\lambda_k \,/\, j = 1, 2, k \geq 1\}$ *be the set of thresholds of* H, *then* $\sigma_p(H) \cup \tau(H)$ *is closed and countable. The elements of* $\sigma_p(H)\backslash\tau(H)$ *are eigenvalues of finite multiplicity that can accumulate at the thresholds* (*or at infinity*) *only.*

To formulate the limiting absorption principle for H (see Theorem 0.3 of [7]), we need the weighted Sobolev spaces $\mathcal{H}^s_{(t)}(\Omega) := \{u \in \mathcal{D}'(\Omega)/\langle\cdot\rangle^t u \in \mathcal{H}^s(\Omega)\}$, t and s real numbers. These spaces are equipped with the natural norms $\|\langle\cdot\rangle^t u\|_{\mathcal{H}^s(\Omega)}$. When $t \geq 0$, the injections $\mathcal{H}^{-1}_{(t)}(\Omega) \hookrightarrow \mathcal{H}^{-1}(\Omega)$ and $\mathcal{H}^1_0(\Omega) \hookrightarrow \mathcal{H}^1_{(-t)}(\Omega)$ are continuous.

Proposition 2.3. *Let* $t > 1/2$ *be a real number. Then for each compact* K *included in* $\mathbb{R}\backslash(\sigma_p(H) \cup \tau(H))$, *one has*

$$\sup_{\lambda \in K, 0<\varepsilon<1} \|(H - \lambda - i\varepsilon)^{-1}\|_{\mathcal{B}(\mathcal{H}^{-1}_{(t)}(\Omega), \mathcal{H}^1_{(-t)}(\Omega))} < +\infty. \tag{2.10}$$

Finally, we need the following lemma.

Lemma 2.1 (*see Proposition* 5.7 *of* [7])**.** *Let* $\alpha_1, \alpha_2 \in C^\infty(\Omega)$ *be two functions depending only on* x_n, *such that* $\alpha_1 = 0$ *when* x_n *is large enough,* $\alpha_2 = 0$ *when* $(-x_n)$ *is large enough, and* $\alpha_1 + \alpha_2 = 1$ *when* $|x_n|$ *is large enough. Then for every* $\beta \in C_0^\infty(\mathbb{R})$, *the operator* $\beta(H) - \sum_{j=1,2} \alpha_j \beta(H_j)$ *belongs to* $\mathcal{K}(L^2(\Omega))$.

§3. Eikonal Equation

We fix $\Lambda := [a, b]$ a compact interval in $\mathbb{R}\backslash(\tau(H) \cup \sigma_p(H))$ and $\beta \in C_0^\infty(\mathbb{R})$ a real-valued function equal to 1 on Λ such that supp β (the support of β) is an interval $\Lambda_0 := [a_0, b_0]$ also in $\mathbb{R}\backslash(\tau(H) \cup \sigma_p(H))$.

For $j = 1, 2$, there exists a unique $k_j(\Lambda) \in \mathbb{N}^*$ such that $c_j\lambda_{k_j(\Lambda)} < a < b < c_j\lambda_{k_j(\Lambda)+1}$. We define the real-valued function $\beta_{jk} \in C_0^\infty(\mathbb{R})$ by

$$\beta_{jk}(\xi) := \beta(\lambda_{jk}(\xi)), \quad \xi \in \mathbb{R}, j = 1, 2, k \geq 1. \tag{3.1}$$

We have $\beta_{jk} = 0$ if $k \geq k_j(\Lambda)+1$ and there exists $\varepsilon_j(\Lambda) > 0$ such that $\beta_{jk}(\xi) = 0$ if $|\xi| \leq \varepsilon_j(\Lambda)$ and $k \leq k_j(\Lambda)$.

For $j = 1, 2, k \geq 1$, let $\alpha_{jk} \in C^\infty(\mathbb{R})$ be a real-valued function such that

$$\alpha_{jk}(x_n) = 1 \text{ if } (-1)^j x_n \geq 2R_k,\ \alpha_{jk}(x_n) = 0 \text{ if } (-1)^j x_n \leq R_k,\ x_n \in \mathbb{R}, \tag{3.2}$$

where R_k will be chosen large enough.

Finally, we define the real-valued function $\gamma_{jk} \in C^\infty(\mathbb{R}^2)$ by

$$\gamma_{jk}(x_n, \xi) =: \alpha_{jk}(x_n)\beta_{jk}(\xi), \quad x_n, \xi \in \mathbb{R},\ j = 1, 2, k \geq 1. \tag{3.3}$$

We can assume $\theta < 1$ in (1.2). Then we have the following proposition:

Proposition 3.1. *For every $j = 1, 2$ and $k \geq 1$, there exist functions ψ_{jk} and r_{jk} in $C^\infty(\mathbb{R} \times \mathbb{R}^*)$, with the following properties*:

1) *ψ_{jk} is real-valued and if $m \in \mathbb{N}^*$ and $m\theta > 1$, for all $p, q \in \mathbb{N}$, one has, with constants C_{pq} depending also on j and k, the inequalities*:

$$|\partial_{x_n}^p \partial_\xi^q [\xi^{2m-1}\psi_{jk}(x_n, \xi)]| \leq C_{pq}\langle\xi\rangle^{2m-q}\langle x_n\rangle^{1-\theta-p}, \quad x_n \in \mathbb{R},\ \xi \in \mathbb{R}^*. \tag{3.4}$$

2) *One has* $\operatorname{supp} r_{jk} \subset \operatorname{supp}\gamma_{jk}$, *and for every $p, q \in \mathbb{N}$, there exist constants C'_{pq} (depending also on j and k) such that*

$$|\partial_{x_n}^p \partial_\xi^q r_{jk}(x_n, \xi)| \leq C'_{pq}\langle x_n\rangle^{-1-\theta-p}, \quad x_n \in \mathbb{R},\ \xi \in \mathbb{R}^*. \tag{3.5}$$

3) *One has the equality*

$$-\nabla \cdot (c_j + \delta^L)\nabla(e^{i\psi_{jk}}\Phi_k\gamma_{jk}) = e^{i\psi_{jk}}\Phi_k[\lambda_{jk}(\xi)\gamma_{jk} + r_{jk}], \quad x \in \Omega,\ \xi \in \mathbb{R}^*. \tag{3.6}$$

Proof. Suppose ψ_{jk} satisfying 1): a direct calculation gives easily that

$$\begin{aligned} &-\nabla \cdot (c_j + \sigma^L)\nabla(e^{i\psi_{jk}}\Phi_k\gamma_{jk}) \\ &\quad = e^{i\psi_{jk}}\Phi_k[\lambda_{jk}(\xi)\gamma_{jk} + \gamma_{jk}E_{jk}(\partial_{x_n}\psi_{jk}) + r'_{jk}] \quad \text{for } x \in \Omega,\ \xi \in \mathbb{R}^*, \end{aligned} \tag{3.7}$$

where r'_{jk} satisfies 2) and

$$E_{jk}(t) := (c_j + \delta^L)t^2 + 2\xi(c_j + \delta^L)t + (\lambda_k + \xi^2)\delta^L, \quad t \in \mathbb{R}.$$

We determine ψ_{jk} as an approximate solution of the eikonal equation $E_{jk}(\partial_{x_n}\psi_{jk}) = 0$. We set

$$A_{jk}(x_n, \xi) := \frac{\lambda_k + \xi^2}{\xi^2(c_j + \delta^L(x_n))}\delta^L(x_n), \quad x_n \in \mathbb{R},\ \xi \in \mathbb{R}^*, \tag{3.8}$$

and choose R_k in (3.2) such that $|A_{jk}| \le 1/2$ on $\operatorname{supp}\gamma_{jk}$. Then on this support of γ_{jk}, we have $E_{jk}(B'_{jk}) = 0$ where $B'_{jk} := \xi(\sqrt{1 - A_{jk}} - 1)$. Now, we define $B_{jk} : \mathbb{R} \times \mathbb{R}^* \to \mathbb{R}$ by

$$B_{jk} := \xi \sum_{s=1}^{m} \frac{1}{2}\left(\frac{1}{2} - 1\right)\cdots\left(\frac{1}{2} - s + 1\right)\frac{(-A_{jk})^s}{s!}, \tag{3.9}$$

and $\psi_{jk} : \mathbb{R} \times \mathbb{R}^* \to \mathbb{R}$ by

$$\psi_{jk}(x_n, \xi) := \int_0^{x_n} B_{jk}(t, \xi)dt. \tag{3.10}$$

Using (1.2), (3.8), (3.9) and (3.10), we see that ψ_{jk} satisfies 1). Now with Taylor's formula, using (1.2), (3.8), (3.9) and with the choice of m, we see that, if $D_{jk} := B_{jk} - B'_{jk}$ on $\operatorname{supp}\gamma_{jk}$, then $\gamma_{jk}D_{jk}$ satisfies inequalities of type (3.5). Finally we have $\partial_{x_n}\psi_{jk} = B_{jk} = B'_{jk} + D_{jk}$ and then

$$\gamma_{jk}E_{jk}(\partial_{x_n}\psi_{jk}) = \gamma_{jk}E_{jk}(B'_{jk}) + (c_j + \delta^L)\gamma_{jk}D_{jk}(D_{jk} + 2B'_{jk} + 2\xi).$$

To get (3.6), we use (3.7) and choose $r_{jk} := r'_{jk} + \gamma_{jk}D_{jk}(c_j + \delta^L)(D_{jk} + 2B'_{jk} + 2\xi)$. □

§4. A Class of Fourier Integral Operators

With the notations of Section 3 and for $j = 1, 2, k \ge 1$, we define the real-valued function $\varphi_{jk} \in C^\infty(\mathbb{R} \times \mathbb{R}^*)$ by $\varphi_{jk}(x_n, \xi) := x_n\xi + \varphi_{jk}(x_n, \xi)$ and the operator F_{jk} by

(4.1)
$$F_{jk}f(x_n) := (2\pi)^{-1/2}\int_{\mathbb{R}} e^{i\varphi_{jk}(x_n,\xi)}\gamma_{jk}(x_n, \xi)\widehat{f}(\xi)d\xi, \quad f \in \mathcal{S}(\mathbb{R}),\ x_n \in \mathbb{R},$$

with $\widehat{f}(\xi) := (2\pi)^{-1/2}\int_{\mathbb{R}} e^{-it\xi}f(t)dt$.

Lemma 4.1. *1) $F_{jk} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is continuous.*
2) F_{jk} has a unique extension as an operator of $\mathcal{B}(L^2(\mathbb{R}))$.

Proof. For 1), we use the inequalities (3.4).
For 2), we remark firstly that the formal adjoint of F_{jk} satisfies the equality

(4.2)
$$\widehat{(F_{jk}^* g)}(\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-i\varphi_{jk}(y_n,\xi)} \gamma_{jk}(y_n,\xi) g(y_n) dy_n, \quad g \in \mathcal{S}(\mathbb{R}),\ \xi \in \mathbb{R}.$$

Then

$$(F_{jk}F_{jk}^* g)(x_n) = (2\pi)^{-1} \int_{\mathbb{R}} K_{jk}(x_n,y_n) g(y_n) dy_n, \quad g \in \mathcal{S}(\mathbb{R}), \tag{4.3}$$

where

(4.4)
$$K_{jk}(x_n,y_n) = \int_{\mathbb{R}} e^{i[\varphi_{jk}(x_n,\xi)-\varphi_{jk}(y_n,\xi)]} \gamma_{jk}(x_n,\xi)\gamma_{jk}(y_n,\xi) d\xi, \quad x_n, y_n \in \mathbb{R}.$$

We have

$$\varphi_{jk}(x_n,\xi) - \varphi_{jk}(y_n,\xi) = (x_n - y_n)(\xi + \zeta_{jk}(x_n,y_n,\xi)) \tag{4.5}$$

with

$$\zeta_{jk}(x_n,y_n,\xi) := \int_0^1 (\partial_{x_n}\psi_{jk})(tx_n + (1-t)y_n) dt. \tag{4.6}$$

If (x_n,ξ) and (y_n,ξ) are in $\operatorname{supp}\gamma_{jk}$, we have $|tx_n + (1-t)y_n| = t|x_n| + (1-t)|y_n| \geq R_k$ for every $t \in (0,1)$. Then using (3.4), we find a constant $C > 0$ such that $|\partial_\xi \zeta_{jk}(x_n,y_n,\xi)| \leq CR_k^{-\theta} \leq 1/2$ if R_k is large enough. In the integral (4.4), we can do the change of variables $\eta = \xi + \zeta_{jk}(x_n,y_n,\xi)$, i.e. $\xi = \xi_{jk}(x_n,y_n,\eta)$, since the derivatives with respect to η of ξ_{jk} are bounded on the support of the integrand. After an integration by part, we get for every $p \in \mathbb{N}$

(4.7)
$$\begin{aligned} K_{jk}(x_n,y_n) = \langle x_n - y_n\rangle^{-2p} \int_{\mathbb{R}} & e^{i(x_n-y_n)\eta} (1-\partial_\eta^2)^p [\gamma_{jk}(x_n,\xi_{jk}(x_n,y_n,\eta)) \\ & \times \gamma_{jk}(y_n,\xi_{jk}(x_n,y_n,\eta))(1 + (\partial_\xi\zeta_{jk})(x_n,y_n,\xi_{jk}(x_n,y_n,\eta)))^{-1}] d\eta. \end{aligned}$$

Thus there exists a constant $C > 0$ such that $|K_{jk}(x_n,y_n)| \leq C\langle x_n - y_n\rangle^{-2p}$ for every $x_n, y_n \in \mathbb{R}$ and $p \in \mathbb{N}$. Now using the Schur lemma, we deduce that $F_{jk}F_{jk}^*$ has a unique extension as an operator of $\mathcal{B}(L^2(\mathbb{R}))$, so that F_{jk}^* and F_{jk} have the same property. $\square$

Remark 4.1. In fact $F_{jk} \in \mathcal{B}(L^2(\mathbb{R}), \mathcal{H}^s(\mathbb{R}))$ for all $s \in \mathbb{R}$.

Lemma 4.2. *The operator* $L_{jk} := \langle \cdot \rangle^{-(1+\theta)/2}[\langle \cdot \rangle^{(i+\theta)/2}, F_{jk}]$*, well defined on* $\mathcal{S}(\mathbb{R})$*, has a unique extension as an operator of* $\mathcal{B}(L^2(\mathbb{R}))$*.*

Proof. If $a(x_n) := \langle x_n \rangle^{(1+\theta)/2}$, $b(x_n, y_n) := a(x_n)^{-1}[a(x_n) - a(y_n)]$ and $\rho_{jk}(x_n, y_n, \xi) := (x_n - y_n)\xi + \psi_{jk}(x_n, \xi)$, we have

$$(L_{jk}f)(x_n) = (2\pi)^{-1/2} \int_{\mathbb{R}} \widetilde{K_{jk}}(x_n, y_n) f(y_n) dy_n, \ f \in \mathcal{S}(\mathbb{R}), \ x_n \in \mathbb{R}, \tag{4.8}$$

where

$$\widetilde{K_{jk}}(x_n, y_n) := \int_{\mathbb{R}} e^{i\rho_{jk}(x_n, y_n, \xi)} b(x_n, y_n) \gamma_{jk}(x_n, \xi) d\xi, \ x_n, y_n \in \mathbb{R}. \tag{4.9}$$

If $|x_n - y_n| > (1/2)|x_n|$ and $q \in \mathbb{N}$, $q \geq 3/(2\theta)$, we have $|\widetilde{K_{jk}}(x_n, y_n)| = \langle x_n - y_n \rangle^{-2q} b(x_n, y_n) \int_{\mathbb{R}} e^{i(x_n - y_n)\xi}(1 - \partial_\xi^2)^q [e^{i\psi_{jk}(x_n, \xi)} \gamma_{jk}(x_n, \xi)] d\xi$ and there exist constants C'_{jk} and C''_{jk} such that

$$|\widetilde{K_{jk}}(x_n, y_n)| \leq C'_{jk} \langle x_n - y_n \rangle^{-2q} |x_n - y_n| \langle x_n \rangle^{2q(1-\theta)} \leq C''_{jk} \langle x_n - y_n \rangle^{-2}. \tag{4.10}$$

If $|x_n - y_n| \leq (1/2)|x_n|$, then $(2/3)|y_n| \leq |x_n| \leq 2|y_n|$. For $p \in \mathbb{N}$, $p \geq (2-\theta)/\theta$, we write

$$b(x_n, y_n) = s_p(x_n, y_n) + r_p(x_n, y_n), \tag{4.11}$$

where

$$s_p(x_n, y_n) := -a(x_n)^{-1} \sum_{t=1}^{p} \frac{1}{t!} (y_n - x_n)^t a^{(t)}(x_n) \tag{4.12}$$

and

$$r_p(x_n, y_n) := -a(x_n)^{-1} \frac{1}{p!} (y_n - x_n)^{p+1} \int_0^1 (1-\tau)^p a^{(p+1)}(x_n + \tau(y_n - x_n)) d\tau. \tag{4.13}$$

According to the decomposition (4.11), we write

$$\widetilde{K_{jk}}(x_n, y_n) := \widetilde{K_{jk}}'(x_n, y_n) + \widetilde{K_{jk}}''(x_n, y_n). \tag{4.14}$$

On the one hand, the identity

$$(y_n - x_n)^{p+1} e^{i\rho_{jk}(x_n, y_n, \xi)} = [(-D_\xi)^{p+1} e^{i(x_n - y_n)\xi}] e^{i\psi_{jk}(x_n, \xi)}$$

and integrations by part give a constant $C_{p,j,k}$ such that, for $|x_n - y_n| \leq (1/2)|x_n|$,

$$|\widetilde{K_{jk}}''(x_n, y_n)| \leq C_{p,j,k}\langle x_n\rangle^{-(p+1)\theta} \leq C_{p,j,k}\langle x_n - y_n\rangle^{-2}. \tag{4.15}$$

On the other hand, with similar integrations by parts, we see that the kernel $\widetilde{K_{jk}}'(x_n, y_n)$ verifies an inequality similar to (4.10) for $|x_n - y_n| \geq (1/2)|x_n|$. Thus for every $x_n, y_n \in \mathbb{R}$, there exists a constant C_{jk} such that $|\widetilde{K_{jk}}''(x_n, y_n)| \leq C_{j,k}\langle x_n - y_n\rangle^{-2}$, so that the operator T''_{jk} with the integral kernel $(2\pi)^{-1/2}\widetilde{K_{jk}}''$ is bounded on $L^2(\mathbb{R})$.

Finally the operator T'_{jk} with the integral kernel $(2\pi)^{-1/2}\widetilde{K_{jk}}'$ is written as

$$(T'_{jk}f)(x_n) = \int_{\mathbb{R}} e^{i\varphi_{jk}(x_n,\xi)} e_{jk}(x_n, \xi)\widehat{f}(\xi)d\xi, \quad f \in \mathcal{S}(\mathbb{R}),\ x_n \in \mathbb{R},$$

where $e_{jk} \in C^\infty(\mathbb{R} \times \mathbb{R})$, $\operatorname{supp} e_{jk} \subset \operatorname{supp} \gamma_{jk}$ and all the derivatives of e_{jk} are bounded. Using the proof of Lemma 4.1, we get $T'_{jk} \in \mathcal{B}(L^2(\mathbb{R}))$. The proof is complete since $L_{jk} = T'_{jk} + T''_{jk}$ according to (4.8) and (4.14). □

§5. Identification Operators

With the notations of the previous sections, we consider the operators $G_j = G_j(\Lambda), j = 1, 2$, defined for $f \in C_0^\infty(\Omega)$ and $x \in \Omega$ by

$$(G_j f)(x) := \sum_{k\geq 1} \int_{\mathbb{R}} e^{i\psi_{jk}(x_n,\xi)} \Phi_k(x, \xi)\gamma_{jk}(x_n, \xi)\widehat{f_k}(\xi)d\xi. \tag{5.1}$$

Remark 5.1. The sum in (5.1) is finite since $\gamma_{jk} = 0$ if $k \geq k_j(\Lambda) + 1$.

Lemma 5.1. 1) *G_j has a unique extension as an operator of $\mathcal{B}(L^2(\Omega), \mathcal{H}_0^1(\Omega))$.*

2) *For all $f \in L^2(\Omega), G_j f$ is in $C^\infty(\Omega)$.*

Proof. It is sufficient to take into account the following expression of G_j:

$$(G_j f)(x) = \sum_{k\geq 1} V_k(x')(F_{jk} f_k)(x_n), \quad f \in C_0^\infty(\Omega), \tag{5.2}$$

where f_k is defined by (2.4) and to use Remark 4.1 and the fact that $V_k \in \mathcal{H}_0^1(\Omega') \cap C^\infty(\Omega')$. □

Remark now that for a fixed interval Λ and for $1 \leq k \leq k_j(\Lambda), j = 1, 2$, we can choose R_k in (3.2) independent of k, and consequently $\alpha_{jk} =: \alpha_j$.

Proposition 5.1. *The following relation holds*:

$$\sum_{j=1,2}[\alpha_j^2\beta^2(H_j)-G_jG_j^*]\in\mathcal{K}(L^2(\Omega)). \tag{5.3}$$

Proof. Using (2.5) and (2.8), we get

$$\begin{aligned}\sum_{j=1,2}&[\alpha_j^2\beta^2(H_j)f](x) \\ &=(2\pi)^{-1/2}\sum_{j=1,2}\sum_{k\geq1}V_k(x')\int_{\mathbb{R}}e^{ix_n\xi}\alpha_j^2(x_n)\beta^2(\lambda_{jk}(\xi))\widehat{f}_k(\xi)d\xi\end{aligned} \tag{5.4}$$

for $f\in C_0^\infty(\Omega), x\in\Omega$. On the other hand, the formal adjoint of G_j satisfies

$$(G_j^*g)_k=F_{jk}^*g_k,\ g\in C_0^\infty(\Omega),\quad j=1,2,\ k\geq1. \tag{5.5}$$

Then we have

$$(G_jG_j^*f)(x)=(2\pi)^{-1}\sum_{k\geq1}V_k(x')\int_{\mathbb{R}}K_{jk}(x_n,y_n)f_k(y_n)dy_n,\quad f\in C_0^\infty(\Omega), \tag{5.6}$$

where K_{jk} is defined by (4.4) or(4.7). Using the identities

$$\begin{aligned}\alpha_j(y_n)-\alpha_j(x_n)&=(y_n-x_n)\int_0^1\alpha_j'(x_n+t(y_n-x_n))dt,\\ 1-(1+\partial_\xi\zeta_{jk})^{-1}&=(\partial_\xi\zeta_{jk})(1+\partial_\xi\zeta_{jk})^{-1},\quad \eta-\xi_{jk}=\zeta_{jk}(x_n,y_n,\xi_{jk}),\end{aligned}$$

(4.6) and the inequalities (3.4), it is easily seen that $K_{jk}=K_{jk}'+K_{jk}''$ with

$$K_{jk}'(x_n,y_n)=\int_{\mathbb{R}}e^{i(x_n-y_n)\eta}\gamma_{jk}^2(x_n,\eta)d\eta,\quad x_n,y_n\in\mathbb{R}, \tag{5.7}$$

while, for every $p,q\in\mathbb{N}$, there exists a constant C_{pq} (depending also on j and k) such that

$$|\partial_{x_n}^qK_{jk}''(x_n,y_n)|\leq C_{pq}\langle x_n\rangle^{-\theta}\langle x_n-y_n\rangle^{-2p},\quad x_n,y_n\in\mathbb{R}. \tag{5.8}$$

Then the operator T_j defined by

$$(T_jf)(x):=(2\pi)^{-1}\sum_{k\geq1}V_k(x')\int_{\mathbb{R}}K_{jk}''(x_n,y_n)f_k(y_n)dy_n,\quad f\in C_0^\infty(\Omega), \tag{5.9}$$

is such that $\langle\cdot\rangle^\theta T_j\in\mathcal{B}(L^2(\Omega),\mathcal{H}_0^1(\Omega))$, so that $T_j\in\mathcal{K}(L^2(\Omega)), j=1,2$.

Finally, comparing the relations (5.4), (5.6) and (5.7), the operator on the left-hand side of (5.3) is equal to T_1+T_2, and the proof is ended. □

§6. Existence of the Generalized Wave Operators

We use the following result of existence for the wave operators.

Proposition 6.1. *Let $\mathcal{J}$ be an open set in $\mathbb{R}$, $\mathcal{H}$ be a separable complex Hilbert space, T_1 and T_2 be two self-adjoint operators in $\mathcal{H}$, S be an operator in $\mathcal{B}(\mathcal{H})$, A_k and $B_k, k = 1, \dots, N$, be operators with dense domains in $\mathcal{H}$. Suppose that*

1) *$\mathcal{J} = \bigcup_{i\in\mathbb{N}^*} \mathcal{J}_i$ where each $\mathcal{J}_i$ is a bounded open interval, and $\mathcal{J}_i \cap \mathcal{J}_k = \emptyset$ if $i \neq k$.*

2) *A_k is T_1-bounded and locally T_1-smooth on $\mathcal{J}_i$, for $1 \leq k \leq N, i \geq 1$.*

3) *B_k is T_2-bounded and locally T_2-smooth on $\mathcal{J}_i$, for $1 \leq k \leq N, i \geq 1$.*

4) *$T_2S - ST_1 = \sum_{k=1}^{N} B_k^* A_k$ holds in the sense of forms, that is*

$$(T_2u, Sv)_{\mathcal{H}} - (u, ST_1v)_{\mathcal{H}} = \sum_{1\leq k\leq N} (B_ku, A_kv)_{\mathcal{H}}, \quad u \in D(T_2),\ v \in D(T_1).$$

5) *Both sets $\sigma(T_1)\backslash\mathcal{J}$ and $\sigma(T_2)\backslash\mathcal{J}$ have Lebesgue measure 0.*

Then the generalized wave operators $s - \lim_{t\to\pm\infty} e^{itT_2} S e^{-itT_1} P_{ac}(T_1)$ and $s - \lim_{t\to\pm\infty} e^{itT_1} S^* e^{-itT_2} P_{ac}(T_2)$ exist.

The proof of this proposition is in [12] for $S = Id_{\mathcal{H}}$ and in [16] for the general case S in $\mathcal{B}(\mathcal{H})$.

Remark 6.1 (see [12] and [1]). If T is a self-adjoint operator on $\mathcal{H}$ and A is a T-bounded operator, in order to verify that A is locally T-smooth on an open interval $I \subset \mathbb{R}$, it is suffcient to verify, for every compact set $K \subset I$,

$$\sup_{\lambda\in K, 0<\epsilon<1} \|A(T - \lambda - i\epsilon)^{-1} A^*\|_{\mathcal{B}(\mathcal{H})} < +\infty. \tag{6.1}$$

Theorem 6.1. *With the assumptions* (i) *and* (ii) *of the introduction and with G_j defined by* (5.1)*, there exist the following generalized wave operators*:

$$\Omega_j^{\pm}(\Lambda) := s - \lim_{t\to\pm\infty} e^{itH} G_j e^{-itH_j}, \quad j = 1, 2, \tag{6.2}$$

$$W_j^{\pm}(\Lambda) := s - \lim_{t\to\pm\infty} e^{itH_j} G_j^* e^{-itH} P_{ac}(H), \quad j = 1, 2. \tag{6.3}$$

Proof. We apply Proposition 6.1 with $T_1 = H_j, T_2 = H, \mathcal{J} = \mathbb{R}\backslash(\tau(H) \cup \sigma_p(H)), S = G_j(\Lambda), \mathcal{H} = L^2(\Omega)$. The assumptions 1) and 5) are fullfilled. Using point 4) of Proposition 2.1 and (5.1), we have, for $v \in D(H_j), j = 1, 2, x \in \Omega$,

$$G_jH_jv(x) = \sum_{k\geq 1} \int_{\mathbb{R}} e^{i\psi_{jk}(x_n,\xi)} \Phi_k(x, \xi)\gamma_{jk}(x_n, \xi)\lambda_{jk}(\xi)\widehat{v_k}(\xi)d\xi. \tag{6.4}$$

On the other hand, if $u \in D(H), v \in D(H_j)$ and if we take into account the support of γ_{jk}, we have

$$(6.5) \qquad (Hu, G_j v)_{L^2(\Omega)} = \int_\Omega (c_j + \delta^L)\nabla u \cdot \nabla G_j v\, dx + \int_\Omega \delta^S \nabla u \cdot \nabla G_j v\, dx.$$

With (3.6) and (6.4), we see that the first integral on the right-hand side of (6.5) with $u \in C_0^\infty(\Omega)$ is equal to

$$(6.6) \qquad -\int_\Omega u\nabla \cdot (c_j + \delta^L)\nabla G_j v\, dx = (u, G_j H_j v)_{L^2(\Omega)} + (u, \widetilde{G_j} v)_{L^2(\Omega)},$$

where

$$(6.7) \quad \widetilde{G_j} v(x) := \sum_{k\geq 1} \int_{\mathbb{R}} e^{i\psi_{jk}(x_n,\xi)} \Phi_k(x,\xi) r_{jk}(x_n,\xi)\widehat{v_k}(\xi) d\xi, \quad j = 1, 2,\ x \in \Omega.$$

Then the assumption 4) will be satisfied with $N = n + 1$,

$$B_k u := \langle x_n\rangle^{1+\theta}\delta^S\langle x_n\rangle^{-(1+\theta)/2}\partial_{x_k} u,\ A_k v := \langle x_n\rangle^{-(1+\theta)/2}\partial_{x_k} G_j v \text{ if } 1 \leq k \leq n,$$
$$B_{n+1} u := \langle x_n\rangle^{-(1+\theta)/2} u \quad \text{and} \qquad A_{n+1} v := \langle x_n\rangle^{(1+\theta)/2}\widetilde{G_j} v.$$

Using (2.10) with $t = (1+\theta)/2$ and (6.1), we see that the operators $B_k, 1 \leq k \leq N$, satisfy the assumption 3). We write now the identity $\langle x_n\rangle^{-t} G_j v = \langle x_n\rangle^{-t}[G_j, \langle x_n\rangle^t]\langle x_n\rangle^{-t} v + G_j\langle x_n\rangle^{-t} v$. Lemma 4.2 and the relation (5.2) imply that $A_k, 1 \leq k \leq n$, satisfy the assumption 2). Finally, with the inequalities (3.5), we see that the operator $\langle x_n\rangle^{1+\theta}\widetilde{G_j}$ has the properties of G_j. In order to verify the assumption 2) for A_{n+1}, it is now sufficient to remark the identity

$$\langle x_n\rangle^t \widetilde{G_j} v = (\langle x_n\rangle^{-t}[\langle x_n\rangle^{2t}\widetilde{G_j}, \langle x_n\rangle^t]\langle x_n\rangle^{-t} + \langle x_n\rangle^{2t}\widetilde{G_j}\langle x_n\rangle^{-t})v. \qquad \square$$

§7. Asymptotic Completeness

Remark 7.1. The following statements follow directly from the general properties of the generalized wave operators.

a) $(\Omega_j^\pm(\Lambda))^* = W_j^\pm(\Lambda), \quad j = 1, 2.$

b) $\operatorname{Ran}\Omega_1^\pm(\Lambda) \perp \operatorname{Ran}\Omega_2^\pm(\Lambda).$

c) If $E(\cdot)$ (respectively $E_j(\cdot)$) is the spectral measure of H (respectively H_j), one has, for every Borel set $\mathcal{J} \subset \mathbb{R}$,

$$(7.1) \qquad E(\mathcal{J})\Omega_j^\pm(\Lambda) = \Omega_j^\pm(\Lambda)E_j(\mathcal{J}), \quad j = 1, 2.$$

d) $\operatorname{Ran}\Omega_j^\pm(\Lambda) \subset \mathcal{H}_{ac}(H), \quad j = 1, 2.$

Theorem 7.1. *Under the assumptions* (i) *and* (ii) *of the introduction, one has the following property of asymptotic completeness*:

$$P_{ac}(H) = \sum_{j=1,2} \Omega_j^{\pm}(\Lambda) W_j^{\pm}(\Lambda) \text{ on } E(\Lambda)\mathcal{H}. \tag{7.2}$$

Proof. It is sufficient to prove that if $f \in E(\Lambda)\mathcal{H}$ (in particular $f \in \mathcal{H}_{ac}(H)$), and $f_j^{\pm} := W_j^{\pm}(\Lambda)f, j = 1, 2$, then the following equality holds:

$$\lim_{t\to\pm\infty} \|e^{-itH}f - \sum_{j=1,2} G_j e^{-itH_j} f_j^{\pm}\|_{L^2(\Omega)} = 0. \tag{7.3}$$

From $\lim_{t\to\pm\infty} \|G_j^* e^{-itH}f - e^{-itH_j} f_j^{\pm}\|_{L^2(\Omega)} = 0,\ j = 1, 2$, we get

$$\lim_{t\to\pm\infty} \|G_j G_j^* e^{-itH}f - G_j e^{-itH_j} f_j^{\pm}\|_{L^2(\Omega)} = 0, \quad j = 1, 2. \tag{7.4}$$

Using Lemma 2.1, we see that the operator $K_1 := \beta^2(H) - \sum_{j=1,2} \alpha_j^2 \beta^2(H_j)$ is in $\mathcal{K}(L^2(\Omega))$. From Proposition 5.1, the operator $K_2 := \sum_{j=1,2}[\alpha_j^2\beta^2(H_j) - G_j G_j^*]$ is also in $\mathcal{K}(L^2(\Omega))$. Since $\lim_{t\to\pm\infty} e^{-itH}f = 0$ for the weak topology of $L^2(\Omega)$, we get $\lim_{t\to\pm\infty} \|(K_1 + K_2)e^{-itH}f\| = 0$. Finally, the relation (7.4) and the equality $\beta^2(H)e^{-itH}f = e^{-itH}f$ imply (7.3). □

§8. Free Modified Evolutions

We use the notations of the Sections 3 to 5, and denote by $\chi_j : \Omega \to \mathbb{R}$ the characteristic function of $\{x_n \in \mathbb{R}/(-1)^j x_n > 0\},\ j = 1, 2$.

Proposition 8.1. *There exist real-valued functions $a_{jk}, b_{jk} \in C^\infty(\mathbb{R} \times \mathbb{R}^*), j = 1, 2, k \geq 1$, uniquely determined by the functions ψ_{jk} of Proposition* 3.1, *such that*

1) *The following identity holds for $x_n \in \mathbb{R}$ and $\pm t > 0$:*

$$a_{jk}(x_n, t) = \mp\frac{\pi}{4} + \frac{x_n^2}{4c_j^2 t}(2c_j - 1) - c_j\lambda_k t + b_{jk}(x_n, t). \tag{8.1}$$

2) *b_{jk} is zero if $\psi_{jk} = 0$.*

3) *The following relations hold for $j = 1, 2$ and $f \in L^2(\Omega)$:*

$$\lim_{t\to\pm\infty} \|G_j(\Lambda)e^{-itH_j}f - U_j(t)\beta(H_j)f\|_{L^2(\Omega)} = 0, \tag{8.2}$$

where $U_j(t), j = 1, 2, t \in \mathbb{R}^*$, are the operators of $\mathcal{B}(L^2(\Omega))$ defined by

(8.3)

$$[U_j(t)f](t) := \chi_j(x_n)|2c_j t|^{-1/2} \sum_{k\geq 1} V_k(x')e^{ia_{jk}(x_n,t)} \widehat{f}_k\left(\frac{x_n}{2c_j t}\right), \quad f \in L^2(\Omega).$$

Proof. It is clear that $U_j(t) \in \mathcal{B}(L^2(\Omega))$ since, for $j = 1, 2, \pm t > 0$ and $f \in L^2(\Omega)$,

$$\|U_j(t)f\|^2 = \sum_{k\geq 1} \int_{\mathbb{R}} \chi_j(\pm x_n)|\widehat{f_k}(x_n)|^2 dx_n \leq \|f\|^2_{L^2(\Omega)}. \tag{8.4}$$

We fix $f \in C_0^\infty(\Omega)$. Pointing out that $\lim_{t\to\pm\infty} e^{-itH_j} f = 0$ for the weak topology of $L^2(\Omega)$ and that $\chi_j - \alpha_{jk}, j = 1, 2, k \geq 1$, is with a compact support in $\mathbb{R}$, it is sufficient to prove (8.2) with $\widetilde{G_j}(\Lambda)$ instead of $G_j(\Lambda)$ (where $\widetilde{G_j}(\Lambda)$ is obtained from $G_j(\Lambda)$ by substituting χ_j for α_{jk}) since $G_j - \widetilde{G_j} \in \mathcal{K}(L^2(\Omega))$. Using (2.8) and (5.2), we write

$$\widetilde{G_j}(\Lambda)e^{-itH_j} f(x) = \sum_{k\geq 1} V_k(x')(\widetilde{F_{jk}} f_k)(x_n), \quad x \in \Omega, j = 1, 2, \tag{8.5}$$

where

$$(\widetilde{F_{jk}} g)(x_n) := (2\pi)^{-1/2} \int_{\mathbb{R}} e^{it\Phi_{jk}(x_n,\xi,t)} \widetilde{\gamma_{jk}}(x_n, \xi)\widehat{g}(\xi)d\xi, \quad g \in C_0^\infty(\mathbb{R}), \tag{8.6}$$

with

(8.7)
$$\begin{aligned} \widetilde{\gamma_{jk}}(x_n, \xi) &:= \chi_j(x_n)\beta_{jk}(\xi) \quad \text{and} \\ \Phi_{jk}(x_n, \xi, t) &:= x_n t^{-1}\xi - c_j(\lambda_k + \xi^2) + t^{-1}\psi_{jk}(x_n, \xi), \quad x_n \in \mathbb{R}, \xi, t \in \mathbb{R}^*. \end{aligned}$$

We fix $\varepsilon > 0$ small enough and $\rho \in C_0^\infty(\mathbb{R})$ an even real-valued function, such that $\rho(s) = 1$ if $|s| \leq \varepsilon/2$, $\rho(s) = 0$ if $|s| \geq \varepsilon$. We set $\rho'_{jk}(x_n, \xi, t) := \rho((\partial_\xi \Phi_{jk})(x_n, \xi, t))$ and $\rho''_{jk} := 1 - \rho'_{jk}$. According to $1 = \rho'_{jk} + \rho''_{jk}$, we write $\widetilde{F_{jk}} = \widetilde{F_{jk}}' + \widetilde{F_{jk}}''$. We set also

$$A_{jk} := \{(x_n, \xi, t) \in \operatorname{supp} \rho'_{jk} / \operatorname{dist}(\xi, \operatorname{supp} \beta_{jk}) = d/2, |t| \geq T\}$$

where $d := \operatorname{dist}(0, \operatorname{supp} \beta_{jk})$ and $T \geq 1$ chosen large enough. We have

$$(\partial_\xi \Phi_{jk})(x_n, \xi, t) = \frac{x_n}{t} - 2c_j\xi + \frac{1}{t}(\partial_\xi \psi_{jk})(x_n, \xi).$$

Taking (3.4) into account, we can find two constants m and $M, 0 < m < M$, such that if T is large enough, the following estimates hold on A_{jk}:

$$m \leq \left|\frac{x_n}{2c_j t}\right| \leq M \text{ and } \left|\frac{x_n}{2c_j t} - \xi\right| \leq \frac{\varepsilon}{4c_j}. \tag{8.8}$$

We can assume $\varepsilon/(4c_j) \leq d/4$. We have also

$$(\partial_\xi^2 \Phi_{jk})(x_n, \xi, t) = -2c_j + t^{-1}(\partial_\xi^2 \psi_{jk})(x_n, \xi)$$

so that $|\partial_\xi^2 \Phi_{jk}| \geq c_j$ on A_{jk} if T is large enough. We see that for T large enough, for $x_n, t \in \mathbb{R}, |t| \geq T$, such that dist $(x_n/(2c_j t), \mathrm{supp}\beta_{jk}) \leq (d/4)$, there exists a unique solution $\xi_j = \xi_{jk}(x_n, t)$ of the equation $(\partial_\xi \Phi_{jk})(x_n, \xi_j, t) = 0$ such that $|x_n/(2c_j t) - \xi_j| \leq \varepsilon/(4c_j)$ and thus $(x_n, \xi_j, t) \in A_{jk}$. Using (8.8) and (3.4), we get that the derivatives of Φ_{jk} are bounded on A_{jk} and then, with Theorem 7.7.6 of [8], we have

$$\begin{aligned}(8.9)\quad (\widetilde{F_{jk}}' g)(x_n) \sim\ & e^{it\Phi_{jk}(x_n, \xi_{jk}(x_n,t),t)} |(\partial_\xi^2 \psi_{jk})(x_n, \xi_{jk}(x_n, t)) - 2c_j t|^{-1/2} \\ & e^{\frac{\pi i}{4}\mathrm{sgn}[(\partial_\xi^2 \psi_{jk})(x_n, \xi_{jk}(x_n,t)) - 2c_j t]} \widetilde{\gamma_{jk}}(x_n, \xi_{jk}(x_n, t)) \widehat{g}(\xi_{jk}(x_n, t))\end{aligned}$$

modulo terms the $L^2(\mathbb{R})$-norms of which go to zero if $|t| \to \infty$. Thanks to the inequalities (3.4) and (8.8), for every $s \in \mathbb{N}$, there exists a positive constant C_s such that $|(\partial_\xi^s \psi_{jk})(x_n, \xi_{jk}(x_n, t))| \leq C_s |t|^{1-\theta}, |t| \geq T$. In particular, using the equality

$$(8.10)\qquad \xi_{jk}(x_n, t) = \frac{x_n}{2c_j t} + \frac{1}{2c_j t}(\partial_\xi \psi_{jk})(x_n, \xi_{jk}(x_n, t)),$$

we get the inequality $|\xi_{jk}(x_n, t) - (x_n/2c_j t)| \leq (2c_j)^{-1} C_s |t|^{-\theta}$, $|t| \geq T$. On the other hand, to eliminate some terms (depending explicitely on ξ_{jk}) of Φ_{jk}, we replace ξ_{jk} by its expression coming from (8.10) and we use a Taylor's development of finite order to express $(\partial_\xi^s \psi_{jk})(x_n, \xi_{jk}(x_n, t))$ in function of the derivatives $(\partial_\xi^r \psi_{jk})(x_n, (x_n/2c_j t))$, the powers of $\xi_{jk}(x_n, t) - (x_n/2c_j t)$ and a convenient remainder. Repeating several times these two operations, we find the functions a_{jk} and b_{jk} satisfying (8.1), independent of $\xi_{jk}(x_n, t)$ and such that $|t\Phi_{jk}(x_n, \xi_{jk}(x_n, t)) - a_{jk}(x_n, t)| \leq C|t|^{-\delta}$ holds on A_{jk} with two positive constants C and δ. We deduce

$$(8.11)\qquad (\widetilde{F_{jk}}' g)(x_n) \sim \chi_j(x_n) |2c_j t|^{-1/2} e^{i a_{jk}(x_n, t)} \beta_{jk}\left(\frac{x_n}{2c_j t}\right) \widehat{g}\left(\frac{x_n}{2c_j t}\right).$$

To estimate $\widetilde{F_{jk}}''$, we write it under the form

$$(8.12)\qquad (\widetilde{F_{jk}}'' g)(x_n) = -(2\pi)^{-1/2} t^{-1} \int_{\mathbb{R}} e^{it\Phi_{jk}(x_n, \xi)} D_\xi \left(\frac{\rho_{jk}''}{\partial_\xi \Phi_{jk}} \widetilde{\gamma_{jk}} \widehat{g}\right) d\xi.$$

In (8.8), we can assume M large enough. If $|x_n/(2c_j t)| \leq M, |t| \geq T$, T large enough, we have also $\langle x_n \rangle / 2c_j |t| \leq M + 1$ and the derivatives of Φ_{jk} are bounded. Therefore there exists a constant $C > 0$ such that for $|x_n/(2c_j t)| \leq M, |t| \geq T$, we have

$$|(\widetilde{F_{jk}}'' g)(x_n)| \leq C|t|^{-1} \leq C(2c_j(M+1))^{3/4}|t|^{-1/4}\langle x_n\rangle^{-3/4}. \tag{8.13}$$

If $|x_n/(2c_j t)| > M$, the derivative of ρ''_{jk} is zero for $\xi \in \operatorname{supp}\beta_{jk}$ and the only term with problems is $(\partial_\xi^2\Phi_{jk})(\partial_\xi\Phi_{jk})^{-2}$. But if M and T are large enough, there exist positive constants c and C such that $|\partial_\xi^2\Phi_{jk}(x_n,\xi)| \leq C(1+|t|^{-1}\langle x_n\rangle^{1-\theta})$ and $|\partial_\xi\Phi_{jk}(x_n,\xi)| \geq ct^{-1}\langle x_n\rangle$. On the support of ρ''_{jk}, we have also $|\partial_\xi\Phi_{jk}(x_n,\xi)| \geq \varepsilon/2$ and then we can find another constant C_0 such that for $|x_n/(2c_j t)| > M, |t| \geq T$,

$$|(\widetilde{F_{jk}}'' g)(x_n)| \leq C_0\langle x_n\rangle^{-1} \leq C_0(2c_j M)^{-1/4}|t|^{-1/4}\langle x_n\rangle^{-3/4}. \tag{8.14}$$

From (8.13) and (8.14), we get

$$\lim_{t\to\pm\infty} \|\widetilde{F_{jk}}'' g\| = 0. \tag{8.15}$$

The relation (8.2) follows from (2.8), (8.3), (8.5), (8.11) and (8.15). □

Proof of Theorem 1.1. Using Theorem 6.1 and (8.2), we deduce

$$\Omega_j^\pm(\Lambda) = s - \lim_{t\to\pm\infty} e^{itH}U_j(t)\beta(H_j), \quad j = 1, 2, \tag{8.16}$$

so that there exist the operators $\Omega_j^\pm$ defined by (1.7) and

$$\Omega_j^\pm(\Lambda) = \Omega_j^\pm\beta(H_j), \quad j = 1, 2. \tag{8.17}$$

Property (1.8) of asymptotic completeness follows from (8.17) and Theorem (7.1). □

Proof of Corollary 1.1. It is sufficient to use the relation (8.4) which implies

$$\|\Omega_j^\pm f\|^2_{L^2(\Omega)} = \sum_{k\geq 1}\int_{\mathbb{R}} \chi_j(\pm x_n)|\tilde{f}_k(x_n)|^2 dx_n, \quad f \in L^2(\Omega). \qquad \square$$

References

[1] Amrein, W., Boutet de Monvel, A. and Georgescu, V., *C_0-groups, commutator methods and spectral theory of N-body Hamiltonians*, Birkhäuser, 1996.

[2] Boutet de Monvel, A., Georgescu, V. and Soffer, A., N-body Hamiltonians with hard-core interactions, *Rev. Math. Phys.*, **6**-4 (1994), 515–596.

[3] Boutet de Monvel, A. and Manda, D., Spectral and scattering theory for wave propagation in perturbed stratified media, *J. Math. Anal. Appl.*, **191** (1995), 137–167.

[4] Croc, E. and Dermenjian, Y., Analyse spectrale d'une bande acoustique multistratifiée I, Principe d'absorption limite pour une stratification simple. *SIAM J. Math. Anal.*, **26** (1995), 880–924.

[5] Croc, E. and Iftimie, V., Wave operators in a multistratified strip, *Integr. Equ. Oper. Theory*, **33** (1999), 389–405.

[6] DeBièvre, S. and Pravica, D. W., Spectral analysis for optical fibers and stratified fluids I., *J. Funct. Anal.*, **98** (1991), 404–436.

[7] Dermenjian, Y., Durand, M. and Iftimie, V., Spectral Analysis of an acoustic multistratified perturbed cylinder, *Comm. Partial Diff. Eq.*, **23** (1998), 141–169.

[8] Hörmander, L., *The analysis of linear partial differential operators I*, Springer-Verlag, 1983.

[9] Isozaki, H. and Kitada, H., Modified wave operators with time-independent modifiers, *J. Fac. Sci. Univ. Tokyo*, **32** (1985), 77–104.

[10] Kato, T., *Perturbation theory for linear operators*, Springer-Verlag, 1980.

[11] Mazja, V. G., *Sobolev spaces*, Springer-Verlag, 1985.

[12] Reed, M. and Simon, B., *Methods of modern mathematical physics IV: Analysis of operators*, Academic Press, 1978.

[13] Stein, E., *Singular integrals and differentiability properties of functions*, Princeton University Press, 1970.

[14] Weder, R., *Spectral and scattering theory for wave propagation in perturbed stratified media*, Springer-Verlag, 1991.

[15] Wilcox, C. H., *Sound propagation in stratified fluids*, Springer-Verlag, 1984.

[16] Yafaev, D. R., Mathematical scattering theory, *Amer. Math. Soc.*, 1992.

[17] ———, The scattering amplitude for the Schrödinger equation with a long-range potential, *Comm. Math. Phys.*, **191** (1998), 183–218.