On a Discrete Analogue of the Two-Dimensional Toda Lattice Hierarchy

By

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Abstract

A discrete analogue of the two-dimensional Toda lattice hierarchy, which includes the discrete analogue of the two-dimensional Toda lattice equation, is obtained by extending the theory of the Toda lattice hierarchy. Special solutions and bilinear equations for the discrete analogue of the two-dimensional Toda lattice hierarchy are presented. It is shown that the subjects originated from the Sato theory are useful for the discrete-time integrable systems.

§1. Introduction

In early 80's, Mikio Sato constructed a remarkable theory of the soliton equations [11], [13], [14], [15]. He discovered that the τ-function of the Kadomtsev-Petviashvili (KP) equation is closely related to certain algebraic identities, and found that the totality of solutions for the KP equation and its higher order equations forms an infinite dimensional Grassmann manifold. Subsequently, Ueno-Takasaki developed the Sato theory and presented the 2-dimensional Toda lattice (2DTL) equation and its higher order equations [19]. Here the 2DTL equation can be regarded as a spatial discretization of the KP equation. In these theories, continuous-time soliton equations such as the KP

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equation, the 2DTL equation, have been discussed in detail, however discrete-time soliton equations have not yet been fully understood.

In the last decade, discrete integrable system has been getting a lot of attention from the viewpoints of difference scheme and algorithm. For example, the discrete-time Toda lattice [7], [16] is related to the LR factorization algorithm for calculating eigenvalues of a square matrix [9], [12], [17]. Already there have been many discussions about discretizations of integrable nonlinear differential equations preserving their integrability from various points of view. Miwa revealed the link between the continuous variables x and the discrete variables m [10]

$$x_j = m_1 \frac{(c_1)^j}{j} + m_2 \frac{(c_2)^j}{j} + m_3 \frac{(c_3)^j}{j} + \cdots .$$

It is well-known that the following discrete bilinear equation

$$
\begin{aligned}
(1) \qquad & c_1(c_2 - c_3)\tau(m_1 - c_1, m_2, m_3)\tau(m_1, m_2 - c_2, m_3 - c_3) \\
& + c_2(c_3 - c_1)\tau(m_1, m_2 - c_2, m_3)\tau(m_1 - c_1, m_2, m_3 - c_3) \\
& + c_3(c_1 - c_2)\tau(m_1, m_2, m_3 - c_3)\tau(m_1 - c_1, m_2 - c_2, m_3) \\
& = 0,
\end{aligned}
$$

where c_1, c_2, c_3 are constants related to the difference step size of the discrete independent variables [8], [10], can be derived from the bilinear form of the KP equation

$$(2) \qquad\qquad (D_{x_1}^4 - 4D_{x_1}D_{x_3} + 3D_{x_2}^2)\tau \cdot \tau = 0,$$

and its higher order equations by introducing infinitely many discrete-time variables $m = (m_1, m_2, m_3, \dots)$[10]. The operator D is defined as

$$(3) \qquad\qquad D_x^m a(x) \cdot b(x) = \frac{\partial}{\partial x'}a(x + x')b(x - x')\Big|_{x'=0}.$$

A special solution of Casorati determinant form to Eqs. (1) and (2) is expressed as

$$\tau = \det |f_i(s + j - 1, x, m)|_{1 \le i,j \le N} ,$$

where all f_i, which are functions of $s, x = (x_1, x_2, x_3, \dots), m = (m_1, m_2, m_3, \dots)$, satisfy the following linear equations

$$(4) \qquad\qquad \frac{\partial}{\partial x_j}f_i(s, x, m) = f_i(s + j, x, m),$$

$$(5) \qquad \frac{1}{c_j}\left(1 - \exp\left(-\frac{\partial}{\partial m_j}\right)\right)f_i(s, x, m) = f_i(s + 1, x, m),$$

for $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots$. Hereafter the unshifted independent variables are often omitted and only the shifted variables are written down explicitly. For example, Eq. (5) is rewritten as $(f_i(s; m_j) - f_i(s; m_j - 1))/c_j = f_i(s + 1; m_j)$. Furthermore, Date, Jimbo and Miwa developed Sato's idea as a method for generating discrete soliton equations from the bilinear differential equations [1], [2], [3], [4], [5], [6]. Their method is based on the "transformation groups" theory developed by themselves and their collaborators. In this paper, we will introduce sets of infinite number of discrete variables suitable for the discrete analogues of the 2DTL hierarchy.

The purpose of this paper is to present a discrete analogue of the 2DTL hierarchy which includes the discrete-time 2DTL equation [9] by extending the theory of the 2DTL hierarchy, and to investigate special solutions expressed by Casorati determinants and a bilinear expression of the discrete analogue of the 2DTL hierarchy. A part of this paper's result was announced in [18]. The key is a discrete analogue of the Sato equation introduced in Section 3.

This paper is planned as follows. We will build up a theory of the discrete-analogue of the 2DTL hierarchy. Our theory is much indebted to the results of Ueno-Takasaki [19]. In Section 2, we give a brief review of the continuous-time 2DTL hierarchy. In Section 3, we present suitable Sato type equations and obtain the Lax type equations and the Zakharov-Shabat type equations of the discrete analogue of the 2DTL hierarchy. In Section 4, we discuss special solutions of Casorati determinant form to the discrete Sato equations by considering a simultaneous equation. In Section 5, we derive a part of infinitely many bilinear equations explicitly. In Section 6, an infinite matrix expression of the hierarchy of the discrete-time 2DTL equations is given in a discrete Lax form. The last section is devoted to concluding remarks.

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§2. Toda Lattice Hierarchy

In this section, we give a brief review of the 2DTL hierarchy proposed by Ueno-Takasaki [19]. First of all, we start from the following difference operators

$$(6a) \qquad W^{(\infty)} = 1 + w_1^{(\infty)} e^{-\partial_s} + w_2^{(\infty)} e^{-2\partial_s} + w_3^{(\infty)} e^{-3\partial_s} + \cdots,$$

$$(6b) \qquad W^{(0)} = w_0^{(0)} + w_1^{(0)} e^{\partial_s} + w_2^{(0)} e^{2\partial_s} + w_3^{(0)} e^{3\partial_s} + \cdots,$$

where ∂_s denotes $\partial/\partial s$, e^{∂_s} is a shift operator, and the coefficients of $e^{j\partial_s}$ of L and M are functions of $s, x = (x_1, x_2, \ldots), y = (y_1, y_2, \ldots)$, namely

$w_1^{(\infty)} = w_1^{(\infty)}(s, x, y)$, and so on. Here x and y indicate continuous independent variables, and s denotes a discrete independent variable where s is sometimes called a discrete spatial variable. These operators $W^{(\infty)}$ and $W^{(0)}$ play an important role not only in the 2DTL hierarchy, but also in the discrete analogue of the 2DTL hierarchy.

Let L and M be difference operators defined by

$$
(7a) \qquad L = W^{(\infty)} e^{\partial_s} W^{(\infty)-1}
$$
$$
= e^{\partial_s} + u_1 + u_2 e^{-\partial_s} + u_3 e^{-2\partial_s} + \cdots ,
$$
$$
(7b) \qquad M = W^{(0)} e^{-\partial_s} W^{(0)-1}
$$
$$
= v_0 e^{-\partial_s} + v_1 + v_2 e^{\partial_s} + v_3 e^{2\partial_s} + \cdots .
$$

From parts of the operators L and M, the operators B_n, C_n are defined by

$$
B_n = (L^n)_+,
$$
$$
C_n = (M^n)_-,
$$

where the suffices $_+$ and $_-$ mean parts which do not include $e^{-\partial_s}, e^{-2\partial_s}, \ldots$, and $e^{0\partial_s}, e^{\partial_s}, e^{2\partial_s}, \ldots$, respectively. We introduce time evolution equations of $W^{(\infty)}$ and $W^{(0)}$, the Sato type equations of the 2DTL hierarchy, by

$$
(8a) \qquad \frac{\partial W^{(\infty)}}{\partial x_n} = B_n W^{(\infty)} - W^{(\infty)} e^{n\partial_s},
$$

$$
(8b) \qquad \frac{\partial W^{(\infty)}}{\partial y_n} = C_n W^{(\infty)} - W^{(\infty)} e^{-n\partial_s},
$$

$$
(8c) \qquad \frac{\partial W^{(0)}}{\partial x_n} = B_n W^{(0)} - W^{(0)} e^{n\partial_s},
$$

$$
(8d) \qquad \frac{\partial W^{(0)}}{\partial y_n} = C_n W^{(0)} - W^{(0)} e^{-n\partial_s},
$$

for $n = 1, 2, 3, \ldots$.

From Eqs. (7) and (8), the 2DTL hierarchy is formulated as a system of infinite number of the Lax type equations

$$
(9a) \qquad \frac{\partial L}{\partial x_n} = [B_n, L],
$$

$$
(9b) \qquad \frac{\partial L}{\partial y_n} = [C_n, L],
$$

$$
(9c) \qquad \frac{\partial M}{\partial x_n} = [B_n, M],
$$

$$
(9d) \qquad \frac{\partial M}{\partial y_n} = [C_n, M],
$$

or equivalently, the Zakharov-Shabat type equations

(10a) $$\frac{\partial B_m}{\partial x_n} - \frac{\partial B_n}{\partial x_m} + [B_m, B_n] = 0,$$

(10b) $$\frac{\partial C_m}{\partial y_n} - \frac{\partial C_n}{\partial y_m} + [C_m, C_n] = 0,$$

(10c) $$\frac{\partial B_m}{\partial y_n} - \frac{\partial C_n}{\partial x_m} + [B_m, C_n] = 0,$$

which gives an infinite number of nonlinear differential-difference equations. The bracket $[X, Y]$ means $XY - YX$. From (9) or (10) by setting $m = n = 1$, we obtain the 2DTL equation

$$\frac{\partial}{\partial x_1} v_0(s) = v_0(s) \left(u_1(s) - u_1(s-1) \right),$$

$$\frac{\partial}{\partial y_1} u_1(s) = -v_0(s+1) + v_0(s),$$

as a simple example.

§3. Discrete Analogue of Toda Lattice Hierarchy

A natural extension of the 2DTL theory will be made in this section. It is shown that discrete analogues of the Sato equations, the Lax type equations and the Zakharov-Shabat type equations for the 2DTL hierarchy can be derived from suitable difference operators in which we introduce infinite number of discrete variables. We give some explicit examples including the discrete 2DTL equation [9] as the simplest nontrivial one.

Using the difference operators $W^{(\infty)}, W^{(0)}$ which are introduced in the preceding section, we define the following difference operators $\mathcal{L}_n$ and $\mathcal{M}_n$ by

(11a) $$\mathcal{L}_n = e^{\partial_{k_n}} W^{(\infty)} e^{-\partial_{k_n}} e^{n\partial_s} W^{(\infty)-1}$$
$$= e^{n\partial_s} + u_1^{(n)} e^{(n-1)\partial_s} + u_2^{(n)} e^{(n-2)\partial_s} + u_3^{(n)} e^{(n-3)\partial_s} + \cdots,$$

(11b) $$\mathcal{M}_n = e^{\partial_{l_n}} W^{(0)} e^{-\partial_{l_n}} e^{-n\partial_s} W^{(0)-1}$$
$$= v_0^{(n)} e^{-n\partial_s} + v_1^{(n)} e^{(1-n)\partial_s} + v_2^{(n)} e^{(2-n)\partial_s} + v_3^{(n)} e^{(3-n)\partial_s} + \cdots,$$

where the coefficients of $e^{j\partial_s}$ in $\mathcal{L}_n$ and $\mathcal{M}_n$ are functions of s, $x = (x_1, x_2, \dots)$, $y = (y_1, y_2, \dots)$, and also $k = (k_1, k_2, \dots), l = (l_1, l_2, \dots)$, namely $u_j^{(n)} = u_j^{(n)}(s, x, y, k, l)$, $v_j^{(n)} = v_j^{(n)}(s, x, y, k, l)$. Here s, k, l denote discrete independent variables.

Let us suppose that $W^{(\infty)}$ and $W^{(0)}$ satisfy discrete analogues of the Sato equations

$$(12a) \qquad \Delta_{k_n}\left(W^{(\infty)}\right) = \mathcal{B}_n W^{(\infty)} - e^{\partial_{k_n}} W^{(\infty)} e^{-\partial_{k_n}} e^{n\partial_s},$$

$$(12b) \qquad \Delta_{l_n}\left(W^{(\infty)}\right) = \mathcal{C}_n W^{(\infty)} - e^{\partial_{l_n}} W^{(\infty)} e^{-\partial_{l_n}} e^{-n\partial_s},$$

$$(12c) \qquad \Delta_{k_n}\left(W^{(0)}\right) = \mathcal{B}_n W^{(0)} - e^{\partial_{k_n}} W^{(0)} e^{-\partial_{k_n}} e^{n\partial_s},$$

$$(12d) \qquad \Delta_{l_n}\left(W^{(0)}\right) = \mathcal{C}_n W^{(0)} - e^{\partial_{l_n}} W^{(0)} e^{-\partial_{l_n}} e^{-n\partial_s},$$

where the difference operators $\mathcal{B}_n, \mathcal{C}_n$ are defined by

$$\mathcal{B}_n = (\mathcal{L}_n)_+,$$
$$\mathcal{C}_n = (\mathcal{M}_n)_-.$$

Here we also define the difference operators Δ_{k_n} and Δ_{l_n} by

$$\Delta_{k_n} = \frac{1}{a_n}(e^{\partial_{k_n}} - 1),$$
$$\Delta_{l_n} = \frac{1}{b_n}(e^{\partial_{l_n}} - 1),$$

where $a_1, a_2, \dots, b_1, b_2, \dots$ are difference step sizes for the discrete independent variables $k_1, k_2, \dots, l_1, l_2, \dots$, respectively.

Now let us derive a discrete analogue of the Lax equations (9) from Eqs. (11) and (12). If we notice that

$$\Delta_{k_n}(W^{(\infty)}W^{(\infty)-1}) = \Delta_{k_n}(W^{(\infty)})W^{(\infty)-1} + e^{\partial_{k_n}}W^{(\infty)}e^{-\partial_{k_n}}\Delta_{k_n}(W^{(\infty)-1})$$
$$= 0,$$

we get

$$\Delta_{k_n}(W^{(\infty)-1}) = -e^{\partial_{k_n}}W^{(\infty)-1}e^{-\partial_{k_n}}\Delta_{k_n}\left(W^{(\infty)}\right)W^{(\infty)-1}$$
$$= -e^{\partial_{k_n}}W^{(\infty)-1}e^{-\partial_{k_n}}\mathcal{B}_n + e^{n\partial_s}W^{(\infty)-1}.$$

By employing the equation above, a discrete Lax equation corresponding to the

Lax equation (9a) is given as follows

$$
\begin{aligned}
\Delta_{k_n}(\mathcal{L}_m) &= e^{\partial_{k_m}}\Delta_{k_n}(W^{(\infty)})e^{-\partial_{k_m}}e^{m\partial_s}W^{(\infty)-1} \\
&\quad + e^{\partial_{k_m}+\partial_{k_n}}W^{(\infty)}e^{-\partial_{k_m}-\partial_{k_n}}e^{m\partial_s}\Delta_{k_n}(W^{(\infty)-1}) \\
&= e^{\partial_{k_m}}(\mathcal{B}_nW^{(\infty)} - e^{\partial_{k_n}}W^{(\infty)}e^{-\partial_{k_n}}e^{n\partial_s})e^{-\partial_{k_m}}e^{m\partial_s}W^{(\infty)-1} \\
&\quad + e^{\partial_{k_m}+\partial_{k_n}}W^{(\infty)}e^{-\partial_{k_m}-\partial_{k_n}}+e^{m\partial_s}(e^{n\partial_s}W^{(\infty)-1} - e^{\partial_{k_n}}W^{(\infty)-1}e^{-\partial_{k_n}}\mathcal{B}_n) \\
&= e^{\partial_{k_m}}\mathcal{B}_nW^{(\infty)}e^{-\partial_{k_m}}e^{m\partial_s}W^{(\infty)-1} \\
&\quad - e^{\partial_{k_m}+\partial_{k_n}}W^{(\infty)}e^{-\partial_{k_m}}e^{m\partial_s}W^{(\infty)-1}e^{-\partial_{k_n}}\mathcal{B}_n \\
&= e^{\partial_{k_m}}\mathcal{B}_ne^{-\partial_{k_m}}\mathcal{L}_m - e^{\partial_{k_n}}\mathcal{L}_me^{-\partial_{k_n}}\mathcal{B}_n.
\end{aligned}
$$

As a consequence of the same procedure, we obtain discrete analogues of the Lax equations (9)

(13a) $$\Delta_{k_n}(\mathcal{L}_m) = [\mathcal{B}_n, \mathcal{L}_m]_{k_m,k_n}, \qquad m \neq n,$$

(13b) $$\Delta_{l_n}(\mathcal{L}_m) = [\mathcal{C}_n, \mathcal{L}_m]_{k_m,l_n},$$

(13c) $$\Delta_{k_n}(\mathcal{M}_m) = [\mathcal{B}_n, \mathcal{M}_m]_{l_m,k_n},$$

(13d) $$\Delta_{l_n}(\mathcal{M}_m) = [\mathcal{C}_n, \mathcal{M}_m]_{l_m,l_n}, \qquad m \neq n,$$

(13e) $$0 = [\mathcal{L}_m, \mathcal{L}_n]_{k_n,k_m},$$

(13f) $$0 = [\mathcal{M}_m, \mathcal{M}_n]_{l_n,l_m},$$

where $[X,Y]_{x,y}$ indicates $e^{\partial_x}Xe^{-\partial_x}Y - e^{\partial_y}Ye^{-\partial_y}X$. Eqs. (13e) and (13f) are readily verified from the definition of $\mathcal{L}_m, \mathcal{M}_m$.

The equations (13) can be expressed equivalently as

(14a) $$\Delta_{k_m}(\mathcal{B}_n) - \Delta_{k_n}(\mathcal{B}_m) + [\mathcal{B}_n, \mathcal{B}_m]_{k_m,k_n} = 0,$$

(14b) $$\Delta_{l_m}(\mathcal{C}_n) - \Delta_{l_n}(\mathcal{C}_m) + [\mathcal{C}_n, \mathcal{C}_m]_{l_m,l_n} = 0,$$

(14c) $$\Delta_{l_m}(\mathcal{B}_n) - \Delta_{k_n}(\mathcal{C}_m) + [\mathcal{B}_n, \mathcal{C}_m]_{l_m,k_n} = 0,$$

which are discrete analogues of the Zakharov-Shabat equations (10).

Let us give a proof of (14). We can derive Eq. (14a) from Eqs. (13a) and (13e) as the following way

$$
\begin{aligned}
\Delta_{k_m}(\mathcal{L}_n) - \Delta_{k_n}(\mathcal{L}_m) &= [\mathcal{B}_m, \mathcal{L}_n]_{k_n,k_m} - [\mathcal{B}_n, \mathcal{L}_m]_{k_m,k_n} \\
&= [\mathcal{L}_m - \underline{\mathcal{L}}_m, \mathcal{L}_n]_{k_n,k_m} - [\mathcal{B}_n, \mathcal{L}_m]_{k_m,k_n} \\
&= [\mathcal{L}_n, \underline{\mathcal{L}}_m]_{k_m,k_n} - [\mathcal{B}_n, \mathcal{L}_m]_{k_m,k_n} \\
&= [\mathcal{B}_n + \underline{\mathcal{L}}_n, \underline{\mathcal{L}}_m]_{k_m,k_n} - [\mathcal{B}_n, \mathcal{B}_m + \underline{\mathcal{L}}_m]_{k_m,k_n} \\
&= [\underline{\mathcal{L}}_n, \underline{\mathcal{L}}_m]_{k_m,k_n} - [\mathcal{B}_n, \mathcal{B}_m]_{k_m,k_n},
\end{aligned}
$$

where $\underline{\mathcal{L}}_m$ denotes $(\mathcal{L}_m)_-$. Then we take the plus part of the equation above, and obtain (14a).

Next we can obtain Eq. (14c) from Eqs. (13d) and (13f) as follows. Note that

$$
\begin{aligned}
\Delta_{l_m}(\mathcal{M}_n) - \Delta_{l_n}(\mathcal{M}_m) &= [\mathcal{C}_m, \mathcal{M}_n]_{l_n, l_m} - [\mathcal{C}_n, \mathcal{M}_m]_{l_m, l_n} \\
&= [\mathcal{M}_m - \underline{\mathcal{M}}_m, \mathcal{M}_n]_{l_n, l_m} - [\mathcal{C}_n, \mathcal{M}_m]_{l_m, l_n} \\
&= [\mathcal{M}_n, \underline{\mathcal{M}}_m]_{l_m, l_n} - [\mathcal{C}_n, \mathcal{M}_m]_{l_m, l_n} \\
&= [\mathcal{C}_n + \underline{\mathcal{M}}_n, \underline{\mathcal{M}}_m]_{l_m, l_n} - [\mathcal{C}_n, \mathcal{C}_m + \underline{\mathcal{M}}_m]_{l_m, l_n} \\
&= [\underline{\mathcal{M}}_n, \underline{\mathcal{M}}_m]_{l_m, l_n} - [\mathcal{C}_n, \mathcal{C}_m]_{l_m, l_n}.
\end{aligned}
$$

Then we take the plus part of the equation above, and obtain (14c).

Finally we show that (14b) is given from (13). Eqs. (13b) and (13c)

$$
\begin{aligned}
\Delta_{l_n} \mathcal{L}_m &= [\mathcal{C}_n, \mathcal{L}_m]_{k_m, l_n}, \\
\Delta_{k_n} \mathcal{M}_m &= [\mathcal{B}_n, \mathcal{M}_m]_{l_m, k_n},
\end{aligned}
$$

are rewritten as

$$
\text{(15a)} \qquad \Delta_{l_n}(\mathcal{B}_m) - [\mathcal{C}_n, \mathcal{B}_m^{(k)}]_{k_m, l_n} = -\Delta_{l_n}(\underline{\mathcal{L}}_m) + [\mathcal{C}_n, \underline{\mathcal{L}}_m]_{k_m, l_n},
$$

$$
\text{(15b)} \qquad \Delta_{k_n}(\overline{\mathcal{M}}_m) - [\mathcal{B}_n, \overline{\mathcal{M}}_m]_{l_m, k_n} = -\Delta_{k_n}(\mathcal{C}_m) + [\mathcal{B}_n, \mathcal{C}_m]_{l_m, k_n},
$$

where $\overline{\mathcal{M}}_m$ denotes $(\mathcal{M}_m)_+$, respectively. By summing Eqs. (15), we have

$$
\begin{aligned}
\Delta_{l_m}(\mathcal{B}_n) + \Delta_{k_n}(\overline{\mathcal{M}}_m) &- [\mathcal{B}_n, \overline{\mathcal{M}}_m]_{l_m, k_n} \\
&= -\Delta_{k_n}(\mathcal{C}_m) - \Delta_{l_m}(\underline{\mathcal{L}}_n) + [\mathcal{C}_m, \underline{\mathcal{L}}_n]_{k_n, l_m}.
\end{aligned}
$$

Taking the plus part of the equation above, we find that

$$
\text{(16)} \qquad \Delta_{l_m}(\mathcal{B}_n) + \Delta_{k_n}(\overline{\mathcal{M}}_m) - [\mathcal{B}_n, \overline{\mathcal{M}}_m]_{l_m, k_n} = 0.
$$

Substituting (15a) into (16), we obtain (14b).

From Eqs. (13) or (14), we obtain an infinite number of nonlinear difference equations that we name the discrete analogue of the Toda lattice hierarchy. For example, from (14c) by setting $m = n = 1$, we obtain

$$
\text{(17)} \qquad
\begin{cases}
\quad \Delta_{l_1} u_1^{(1)}(s; k_1, l_1) + v_0^{(1)}(s+1; k_1, l_1) \\
\qquad\qquad - v_0^{(1)}(s; k_1+1, l_1) = 0 \\
-\Delta_{k_1} v_0^{(1)}(s; k_1, l_1) + u_1^{(1)}(s; k_1, l_1+1) v_0^{(1)}(s; k_1, l_1) \\
\qquad\qquad - v_0^{(1)}(s; k_1+1, l_1) u_1^{(1)}(s-1; k_1, l_1) = 0
\end{cases},
$$

which is just the discrete 2DTL equation [9]. Moreover, we derive from (14c) by setting $m = 1, n = 2$

$$\begin{cases} \Delta_{l_1} u_1^{(2)}(s; k_2, l_1) + v_0^{(1)}(s+2; k_2, l_1) - v_0^{(1)}(s; k_2+1, l_1) = 0 \\ \Delta_{l_1} v_2^{(2)}(s; k_2, l_1) + u_1^{(2)}(s; k_2, l_1+1)v_0^{(1)}(s+1; k_2, l_1) \\ \qquad - v_0^{(1)}(s; k_2+1, l_1)u_1^{(2)}(s-1; k_2, l_1) = 0 \\ -\Delta_{k_2} v_0^{(1)}(s; k_2, l_1) + u_2^{(2)}(s; k_2, l_1+1)v_0^{(1)}(s; k_2, l_1) \\ \qquad - v_0^{(1)}(s; k_2+1, l_1)u_2^{(2)}(s-1; k_2, l_1) = 0 \end{cases} .$$

An infinite number of differential-difference equations with respect to the variables x, y, k, l is also obtained from Eqs. (8) and (12) as

$$\frac{\partial \mathcal{B}_n}{\partial x_m} - (\Delta_{k_n} B_m) = \left(e^{\partial_{k_n}} B_m\right) \mathcal{B}_n - \mathcal{B}_n B_m,$$

$$\frac{\partial \mathcal{B}_n}{\partial y_m} - (\Delta_{k_n} C_m) = \left(e^{\partial_{k_n}} C_m\right) \mathcal{B}_n - \mathcal{B}_n C_m,$$

$$\frac{\partial \mathcal{C}_n}{\partial x_m} - (\Delta_{l_n} B_m) = \left(e^{\partial_{l_n}} B_m\right) \mathcal{C}_n - \mathcal{C}_n B_m,$$

$$\frac{\partial \mathcal{C}_n}{\partial y_m} - (\Delta_{l_n} C_m) = \left(e^{\partial_{l_n}} C_m\right) \mathcal{C}_n - \mathcal{C}_n C_m.$$

Some examples of integrable differential-difference equations are then

$$\begin{cases} u_1^{(1)}(s; k_1) + u_1(s+1; k_1) = u_1^{(1)}(s+1; k_1) + u_1(s; k_1+1) \\ \frac{\partial}{\partial x_1} u_1^{(1)}(s; k_1) - \frac{u_1(s; k_1+1) - u_1(s; k_1)}{a_1} \\ \qquad = u_1^{(1)}(s; k_1) \left(u_1(s; k_1+1) - u_1(s; k_1)\right) \end{cases} ,$$

$$\begin{cases} \frac{\partial}{\partial y_1} u_1^{(1)}(s; k_1) = v_0(s; k_1+1) - v_0(s+1; k_1) \\ \frac{v_0(s; k_1+1) - v_0(s; k_1)}{a_1} \\ \qquad = v_0(s; k_1)u_1^{(1)}(s; k_1) - v_0(s; k_1+1)u_1^{(1)}(s-1; k_1) \end{cases} ,$$

$$\begin{cases} \frac{\partial}{\partial x_1} \log v_0^{(1)}(s; l_1) = u_1(s; l_1+1) - u_1(s-1; l_1) \\ \frac{u_1(s; l_1+1) - u_1(s; l_1)}{b_1} = v_0^{(1)}(s; l_1) - v_0^{(1)}(s-1; l_1) \end{cases} ,$$

$$\begin{cases} \frac{\partial}{\partial y_1} v_0^{(1)}(s; l_1) = \frac{v_0(s; l_1+1) - v_0(s; l_1)}{b_1} \\ v_0(s; l_1+1) - v_0^{(1)}(s-1; l_1) = v_0^{(1)}(s; l_1) - v_0(s-1; l_1) \end{cases} .$$

§4.   Solutions of Casorati Determinant Form

In this section, we give special solutions of Casorati determinant form to the discrete analogue of the 2DTL hierarchy, originally proposed by Ueno-Takasaki [19].

We first consider a formal difference operator W_N, which depends on $s, x = (x_1, x_2, \dots), y = (y_1, y_2, \dots), k = (k_1, k_2, \dots)$, and $l = (l_1, l_2, \dots)$, defined by

$$(18) \qquad W_N(s, x, y, k, l) = e^{N\partial_s} + w_1 e^{(N-1)\partial_s} + w_2 e^{(N-2)\partial_s} + \cdots + w_N.$$

The operator W_N is determined by the simultaneous difference equation

$$(19) \qquad\qquad W_N(s, x, y, k, l) f_j(s, x, y, k, l) = 0$$

for $j = 1, \dots, N$, where f_j are the linearly independent functions in s. Namely, if we give a set of functions $\{f_j\}_{j=1,2,\dots,N}$, the dependent variables w_j is expressed as

(20)
$$w_j = -\frac{|f(s), \dots, f(s+N-j-1), f(s+N), f(s+N-j+1), \dots, f(s+N-1)|}{|f(s), \dots, f(s+N-1)|}$$

for $j = 1, \dots, N$, where we have introduced a new notation of determinant for simplicity,

$$|f(s), f(s+1), \cdots, f(s+N-1)|$$

$$\equiv \det \begin{vmatrix} f_1(s) & f_1(s+1) & \cdots & f_1(s+N-1) \\ f_2(s) & f_2(s+1) & \cdots & f_2(s+N-1) \\ \vdots & \vdots & \ddots & \vdots \\ f_N(s) & f_N(s+1) & \cdots & f_N(s+N-1) \end{vmatrix}.$$

The determinant $|f(s), \cdots, f(s+N-1)|$ is called a Casorati determinant which is a discrete analogue of Wronskian.

Let us define (x, y, k, l)-dependence of $f_j(s, x, y, k, l)$ as

$$(21a) \qquad\qquad\qquad \frac{\partial f_j(s)}{\partial x_n} = f_j(s+n),$$

$$(21b) \qquad\qquad\qquad \frac{\partial f_j(s)}{\partial y_n} = f_j(s-n),$$

$$(21c) \qquad\qquad\qquad \Delta_{k_n} f_j(s) = f_j(s+n),$$

$$(21d) \qquad\qquad\qquad \Delta_{l_n} f_j(s) = f_j(s-n),$$

for $j = 1, 2, \dots, N$ and $n = 1, 2, \dots$.

Under the above conditions, one can show that the difference operator W_N satisfies the following equations,

$$(22a) \qquad \frac{\partial W_N}{\partial x_n} = B_n W_N - W_N e^{n\partial_s},$$

$$(22b) \qquad \frac{\partial W_N}{\partial y_n} = C_n W_N - W_N e^{-n\partial_s},$$

$$(22c) \qquad \Delta_{k_n}(W_N) = \mathcal{B}_n W_N - e^{\partial_{k_n}} W_N e^{-\partial_{k_n}} e^{n\partial_s},$$

$$(22d) \qquad \Delta_{l_n}(W_N) = \mathcal{C}_n W_N - e^{\partial_{l_n}} W_N e^{-\partial_{l_n}} e^{-n\partial_s},$$

where the operators $B_n, C_n, \mathcal{B}_n, \mathcal{C}_n$ satisfy

$$B_n = (B_n)_+, \qquad C_n = (C_n)_-,$$
$$\mathcal{B}_n = (\mathcal{B}_n)_+, \qquad \mathcal{C}_n = (\mathcal{C}_n)_-.$$

To prove Eqs. (22), we use the following lemma. (A proof of Eqs. (22a) and (22b) can be found in [19].)

Lemma 4.1 ([19]).   *For any difference operator* $U = u_0 + u_1 e^{\partial_s} + u_2 e^{2\partial_s} + \cdots$, *there exist two operators* Q *and* R *uniquely such that*

$$\begin{cases} U = QW_N + R, \\ Q = \sum_{j \geq 0} q_j(s) e^{j\partial_s}, \qquad R = \sum_{j=0}^{N-1} r_j(s) e^{j\partial_s}. \end{cases}$$

Similarly, for any difference operator $U' = u'_0 + u'_1 e^{-\partial_s} + u'_2 e^{-2\partial_s} + \cdots$, *there exist two operators* Q' *and* R' *uniquely such that*

$$\begin{cases} U' = Q'W_N e^{-N\partial_s} + R', \\ Q' = \sum_{j \leq 0} q'_j(s) e^{j\partial_s}, \qquad R' = \sum_{j=1-N}^{0} r'_j(s) e^{j\partial_s}. \end{cases}$$

We first prove Eq. (22c). By using a difference analogue of the Leibniz rule

$$\Delta_n(f(n)g(n)) = \Delta_n(f(n))g(n) + f(n+1)\Delta_n(g(n)),$$

we can find

$$(\Delta_{k_n}(W_N) + e^{\partial_{k_n}} W_N e^{\partial_{-k_n}} e^{n\partial_s}) f_n = 0 \qquad (n = 1, \dots, N)$$

from Eq. (19). Lemma 4.1 implies that there exist certain difference operators $\mathcal{B}_n$, and $\mathcal{R}_n$ of the form

$$\mathcal{B}_n = \sum_{j \geq 0} \beta_j(s, x, y, k, l) e^{j \partial_s},$$

$$\mathcal{R}_n = \sum_{j=0}^{N-1} \gamma_j(s, x, y, k, l) e^{j \partial_s}$$

such that

$$\Delta_{k_n}(W_N) + e^{\partial_{k_n}} W_N e^{-\partial_{k_n}} e^{n \partial_s} = \mathcal{B}_n W_N + \mathcal{R}_n.$$

Then we get

$$(\Delta_{k_n}(W_N) + e^{\partial_{k_n}} W_N e^{-\partial_{k_n}} e^{n \partial_s}) f_j = (\mathcal{B}_n W_N + \mathcal{R}_n) f_j = 0$$

and

$$\mathcal{R}_n f_j = 0$$

for $j = 1, \dots, N$. We conclude that $\mathcal{R}_n = 0$, since f_j are the linearly independent functions in s. Hence Eq. (22c) holds. The proof of Eqs. (22a), (22b) and (22d) are given in the same procedure.

From Eqs. (22), we find that the difference operators $B_n, C_n, \mathcal{B}_n, \mathcal{C}_n$ are written in term of W_N as

$$B_n = (W_N e^{-N \partial_s} e^{n \partial_s} (W_N e^{-N \partial_s})^{-1})_+,$$

$$C_n = (W_N e^{-n \partial_s} (W_N)^{-1})_-,$$

$$\mathcal{B}_n = (e^{\partial_{k_n}} W_N e^{\partial_{-k_n}} e^{-N \partial_s} e^{n \partial_s} (W_N e^{-N \partial_s})^{-1})_+,$$

$$\mathcal{C}_n = (e^{\partial_{l_n}} W_N e^{-\partial_{l_n}} e^{n \partial_s} (W_N)^{-1})_-,$$

and the coefficients of these operators are represented by using the Casorati-type determinants (20). If we set

$$W^{(\infty)} = W_N e^{-N \partial_s},$$

$$W^{(0)} = W_N,$$

we can show that these operators $W^{(\infty)}$ and $W^{(0)}$ solve the Sato equations (8) and (12), as a consequence of Eqs. (22).

§5. τ Function and Hirota's Bilinear Equations

Here we introduce a τ function which is a dependent variable in Hirota's bilinear equations. In this section, we reveal the τ function of the discrete analogue of the 2DTL hierarchy which is proposed in Section 3, and derive a part of infinitely many bilinear equations explicitly.

In the theory of the 2DTL hierarchy [19], it is known that the τ functions for the 2DTL hierarchy are defined through the functions $w_j^{(\infty)}$ and $w_j^{(0)}$. As we have shown in the preceding Section 3, the discrete analogue of the 2DTL hierarchy shares the difference operators $W^{(\infty)}, W^{(0)}$ with the continuous-time 2DTL hierarchy. The τ function for the discrete analogue of the 2DTL hierarchy depends on $s, x = (x_1, x_2, \dots), y = (y_1, y_2, \dots), k = (k_1, k_2, \dots)$, and $l = (l_1, l_2, \dots)$, and is also defined through the function

$$(24a) \qquad w_j^{(\infty)} = \frac{p_j(-\widetilde{\partial_x})\tau(s, x, y, k, l)}{\tau(s, x, y, k, l)},$$

$$(24b) \qquad w_j^{(0)} = \frac{p_j(-\widetilde{\partial_y})\tau(s+1, x, y, k, l)}{\tau(s, x, y, k, l)},$$

where $p_j, j = 1, 2, \dots$, are polynomials defined by

$$(25) \qquad \exp\left(\sum_{n=1}^{\infty} t_n \lambda^n\right) = \sum_{j=0}^{\infty} p_j(t_1, t_2, \dots)\lambda^j,$$

and $\widetilde{\partial_x}, \widetilde{\partial_y}$ are differential operators given by

$$(26a) \qquad \widetilde{\partial_x} = \left(\frac{\partial}{\partial x_1}, \frac{1}{2}\frac{\partial}{\partial x_2}, \frac{1}{3}\frac{\partial}{\partial x_3}, \dots\right),$$

$$(26b) \qquad \widetilde{\partial_y} = \left(\frac{\partial}{\partial y_1}, \frac{1}{2}\frac{\partial}{\partial y_2}, \frac{1}{3}\frac{\partial}{\partial y_3}, \dots\right).$$

Moreover, it is known that the difference operator $W^{(\infty)-1}$ and $W^{(0)-1}$ are written in terms of the τ functions as

$$(27a) \quad W^{(\infty)-1} = 1 + e^{-\partial_s}w_1^{(\infty)*}(s+1) + e^{-2\partial_s}w_2^{(\infty)*}(s+1) + \cdots,$$

$$(27b) \quad W^{(0)-1} = w_0^{(0)*}(s+1) + e^{\partial_s}w_1^{(0)*}(s+1) + e^{2\partial_s}w_2^{(0)*}(s+1) + \cdots,$$

where

$$(28a) \qquad w_j^{(\infty)*} = \frac{p_j(\widetilde{\partial_x})\tau(s, x, y, k, l)}{\tau(s, x, y, k, l)},$$

$$(28b) \qquad w_j^{(0)*} = \frac{p_j(\widetilde{\partial_y})\tau(s-1, x, y, k, l)}{\tau(s, x, y, k, l)}.$$

From Eqs. (12) we have

$$
\begin{aligned}
\mathcal{B}_n &= \left(\Delta_{k_n}(W^{(\infty)}) + e^{\partial_{k_n}} W^{(\infty)} e^{-\partial_{k_n}} e^{n\partial_s} \right) W^{(\infty)-1} \\
&= \left(\Delta_{k_n}(W^{(0)}) + e^{\partial_{k_n}} W^{(0)} e^{-\partial_{k_n}} e^{n\partial_s} \right) W^{(0)-1}, \\
\mathcal{C}_n &= \left(\Delta_{l_n}(W^{(\infty)}) + e^{\partial_{l_n}} W^{(\infty)} e^{-\partial_{l_n}} e^{-n\partial_s} \right) W^{(\infty)-1} \\
&= \left(\Delta_{l_n}(W^{(0)}) + e^{\partial_{l_n}} W^{(0)} e^{-\partial_{l_n}} e^{-n\partial_s} \right) W^{(0)-1}.
\end{aligned}
$$

Hence we obtain the following relations

$$
\text{(30a)} \qquad \frac{1}{a_n} e^{\partial_{k_n}} W^{(\infty)} e^{-\partial_{k_n}} W^{(\infty)-1} + e^{\partial_{k_n}} W^{(\infty)} e^{-\partial_{k_n}} e^{n\partial_s} W^{(\infty)-1}
$$
$$
= \frac{1}{a_n} e^{\partial_{k_n}} W^{(0)} e^{-\partial_{k_n}} W^{(0)-1} + e^{\partial_{k_n}} W^{(0)} e^{-\partial_{k_n}} e^{n\partial_s} W^{(0)-1},
$$

$$
\text{(30b)} \qquad \frac{1}{b_n} e^{\partial_{l_n}} W^{(\infty)} e^{-\partial_{l_n}} W^{(\infty)-1} + e^{\partial_{l_n}} W^{(\infty)} e^{-\partial_{l_n}} e^{-n\partial_s} W^{(\infty)-1}
$$
$$
= \frac{1}{b_n} e^{\partial_{l_n}} W^{(0)} e^{-\partial_{l_n}} W^{(0)-1} + e^{\partial_{l_n}} W^{(0)} e^{-\partial_{l_n}} e^{-n\partial_s} W^{(0)-1},
$$

which are generating relations of infinite number of bilinear equations. From the definition of the D operator (3), we have the following formula

$$
\text{(31)} \qquad \sum_{i+j=k} p_i(-\widetilde{\partial}_x) u(x) \cdot p_j(\widetilde{\partial}_x) v(x) = p_k(-\widetilde{D}_x) u(x) \cdot v(x),
$$

where $\widetilde{D}_x, \widetilde{D}_y$ are defined by

$$
\widetilde{D}_x = \left(D_{x_1}, \frac{1}{2} D_{x_2}, \frac{1}{3} D_{x_3}, \dots \right), \quad \widetilde{D}_y = \left(D_{y_1}, \frac{1}{2} D_{y_2}, \frac{1}{3} D_{y_3}, \dots \right).
$$

Hereafter the unshifted independent variables of the τ function are often omitted for simplicity and only the shifted variables are written down explicitly. For example, $e^{\partial_{k_n}} \tau(s, x, y, k, l)$ and $e^{\partial_{l_n}} \tau(s, x, y, k, l)$ are written as $\tau(s; k_n + 1)$

and $\tau(s; l_n + 1)$, respectively. Substituting Eqs. (6) into (30a), we have

l.h.s. of (30a)

$$= \left(\sum_{i=0}^{\infty} \frac{p_i(-\widetilde{\partial}_x)\tau(s; k_n + 1)}{\tau(s; k_n + 1)} e^{-i\partial_s} \right) \left(\frac{1}{a_n} \sum_{j=0}^{\infty} e^{-j\partial_s} \frac{p_j(\widetilde{\partial}_x)\tau(s + 1; k_n)}{\tau(s + 1; k_n)} \right)$$

$$+ \left(\sum_{i=0}^{\infty} \frac{p_i(-\widetilde{\partial}_x)\tau(s; k_n + 1)}{\tau(s; k_n + 1)} e^{-i\partial_s} \right) e^{n\partial_s} \left(\sum_{j=0}^{\infty} e^{-j\partial_s} \frac{p_j(\widetilde{\partial}_x)\tau(s + 1; k_n)}{\tau(s + 1; k_n)} \right)$$

$$= \frac{1}{a_n} \sum_{i,j=0}^{\infty} \frac{p_i(-\widetilde{\partial}_x)\tau(s; k_n + 1) \cdot p_j(\widetilde{\partial}_x)\tau(s - i - j + 1; k_n)}{\tau(s; k_n + 1)\tau(s - i - j + 1; k_n)} e^{-(i+j)\partial_s}$$

$$+ \sum_{i,j=0}^{\infty} \frac{p_i(-\widetilde{\partial}_x)\tau(s; k_n + 1) \cdot p_j(\widetilde{\partial}_x)\tau(s - i - j + n + 1; k_n)}{\tau(s; k_n + 1)\tau(s - i - j + n + 1; k_n)} e^{(n-i-j)\partial_s},$$

and

r.h.s. of (30a)

$$= \left(\sum_{i=0}^{\infty} \frac{p_i(-\widetilde{\partial}_y)\tau(s + 1; k_n + 1)}{\tau(s; k_n + 1)} e^{i\partial_s} \right) \left(\frac{1}{a_n} \sum_{j=0}^{\infty} e^{j\partial_s} \frac{p_j(\widetilde{\partial}_y)\tau(s; k_n)}{\tau(s + 1; k_n)} \right)$$

$$+ \left(\sum_{i=0}^{\infty} \frac{p_i(-\widetilde{\partial}_y)\tau(s + 1; k_n + 1)}{\tau(s; k_n + 1)} e^{i\partial_s} \right) e^{n\partial_s} \left(\sum_{j=0}^{\infty} e^{j\partial_s} \frac{p_j(\widetilde{\partial}_y)\tau(s; k_n)}{\tau(s + 1; k_n)} \right)$$

$$= \frac{1}{a_n} \sum_{i,j=0}^{\infty} \frac{p_i(-\widetilde{\partial}_y)\tau(s + 1; k_n + 1) \cdot p_j(\widetilde{\partial}_y)\tau(s + i + j; k_n)}{\tau(s; k_n + 1)\tau(s + i + j + 1; k_n)} e^{(i+j)\partial_s}$$

$$+ \sum_{i,j=0}^{\infty} \frac{p_i(-\widetilde{\partial}_y)\tau(s + 1; k_n + 1) \cdot p_j(\widetilde{\partial}_y)\tau(s + i + j + n; k_n)}{\tau(s; k_n + 1)\tau(s + i + j + n + 1; k_n)} e^{(i+j+n)\partial_s}.$$

Thus from the m-th coefficients of (30a), we have

$$\frac{1}{a_n} \sum_{i+j=-m \geq 0} \frac{p_i(-\widetilde{\partial}_x)\tau(s; k_n + 1) \cdot p_j(\widetilde{\partial}_x)\tau(s + m + 1; k_n)}{\tau(s; k_n + 1)\tau(s + m + 1; k_n)}$$

$$+ \sum_{i+j=n-m \geq 0} \frac{p_i(-\widetilde{\partial}_x)\tau(s; k_n + 1) \cdot p_j(\widetilde{\partial}_x)\tau(s + m + 1; k_n)}{\tau(s; k_n + 1)\tau(s + m + 1; k_n)}$$

$$= \frac{1}{a_n} \sum_{i+j=m \geq 0} \frac{p_i(-\widetilde{\partial}_y)\tau(s + 1; k_n + 1) \cdot p_j(\widetilde{\partial}_y)\tau(s + m; k_n)}{\tau(s; k_n + 1)\tau(s + m + 1; k_n)}$$

$$+ \sum_{i+j=m-n \geq 0} \frac{p_i(-\widetilde{\partial}_y)\tau(s + 1; k_n + 1) \cdot p_j(\widetilde{\partial}_y)\tau(s + m; k_n)}{\tau(s; k_n + 1)\tau(s + m + 1; k_n)}$$

or equivalently, a system of bilinear equations

$$(32) \quad \begin{cases} (p_{-m}(-\widetilde{D}_x)+a_n p_{n-m}(-\widetilde{D}_x))\tau(s;k_n+1)\cdot\tau(s+m+1;k_n) \\ \qquad = 0, \hspace{4cm} -\infty < m < 0 \\ (a_n p_n(-\widetilde{D}_x)+1)\tau(s;k_n+1)\cdot\tau(s+1;k_n) \\ \qquad = \tau(s+1;k_n+1)\tau(s;k_n), \hspace{2cm} m = 0 \\ a_n p_{n-m}(-\widetilde{D}_x)\tau(s;k_n+1)\cdot\tau(s+m+1;k_n) \\ \qquad = p_m(-\widetilde{D}_y)\tau(s+1;k_n+1)\cdot\tau(s+m;k_n), \hspace{0.5cm} 0 < m < n \\ a_n\tau(s;k_n+1)\tau(s+n+1;k_n) \\ \qquad = (p_n(-\widetilde{D}_y)+a_n)\tau(s+1;k_n+1)\cdot\tau(s+n;k_n), \hspace{0.2cm} m = n \\ (p_m(-\widetilde{D}_y)+a_n p_{m-n}(-\widetilde{D}_y))\tau(s+1;k_n+1)\cdot\tau(s+m;k_n) \\ \qquad = 0, \hspace{4cm} n < m < \infty \end{cases}$$

for $n = 1, 2, \ldots, m = \ldots, -1, 0, 1, 2, \ldots$, where we have used Eq. (31). Similarly, we obtain the other infinite number of bilinear difference equations from (30b)

$$(33) \quad \begin{cases} (p_{-m}(-\widetilde{D}_x)+b_n p_{-n-m}(-\widetilde{D}_x))\tau(s;l_n+1)\cdot\tau(s+m+1;l_n) \\ \qquad = 0, \hspace{4cm} -\infty < m < -n \\ (p_n(-\widetilde{D}_x)+b_n)\tau(s;l_n+1)\cdot\tau(s-n+1;l_n) \\ \qquad = b_n\tau(s+1;l_n+1)\tau(s-n;l_n), \hspace{1.5cm} m = -n \\ p_{-m}(-\widetilde{D}_x)\tau(s;l_n+1)\cdot\tau(s+m+1;l_n) \\ \qquad = b_n p_{m+n}(-\widetilde{D}_y)\tau(s+1;l_n+1)\cdot\tau(s+m;l_n), -n < m < 0 \\ \tau(s;l_n+1)\tau(s+1;l_n) \\ \qquad = (b_n p_n(-\widetilde{D}_y)+1)\tau(s+1;l_n+1)\cdot\tau(s;l_n), \hspace{1cm} m = 0 \\ (p_m(-\widetilde{D}_y)+b_n p_{m+n}(-\widetilde{D}_y))\tau(s+1;l_n+1)\cdot\tau(s+m;l_n) \\ \qquad = 0, \hspace{4cm} 0 < m < \infty \end{cases}$$

for $n = 1, 2, \ldots, m = \ldots, -1, 0, 1, 2, \ldots$. Eqs. (32) and (33) are the bilinear equations of the discrete analogue of the 2DTL hierarchy. Starting from these bilinear equations, we can derive the discrete equations. For example, from Eq. (32) by setting $m = 0, n = 1$

$$(-a_1 D_{x_1} + 1)\tau(s;k_1 + 1) \cdot \tau(s + 1;k_1) = \tau(s + 1;k_1 + 1)\tau(s;k_1)$$

and Eq. (33) by setting $m = -1, n = 1$

$$(-D_{x_1} + b_1)\tau(s;l_1 + 1) \cdot \tau(s;l_1) = b_1\tau(s + 1;l_1 + 1)\tau(s - 1;l_1)$$

we obtain the discrete 2DTL equation (17). Here we set the dependent variables

$u_1^{(1)}, v_0^{(1)}$ as

$$(34) \qquad u_1^{(1)} = w_1^{(\infty)}(s; k_1 + 1, l_1) + w_1^{(\infty)*}(s + 1; k_1, l_1),$$

$$(35) \qquad v_0^{(1)} = w_0^{(0)}(s; k_1, l_1 + 1) w_0^{(0)*}(s; k_1, l_1),$$

which are determined by the definition (11).

§6.   Matrix Form of Discrete 2DTL Hierarchy

We have developed a theory of a discrete analogue of the 2DTL hierarchy by operator formalism. In Section 3 we have proposed three types of expressions (12), (13) and (14) for the discrete analogue of the 2DTL hierarchy. Those expressions are slightly complicated, so we give another expression by using matrix forms.

First we rewrite the discrete Zakharov-Shabat equations (14) by using difference operators $\mathcal{B}_n$ and $\mathcal{C}_n$ as

$$(1 + a_n \mathcal{B}_n(k_m + 1, k_n))(1 + a_m \mathcal{B}_m(k_m, k_n))$$
$$= (1 + a_m \mathcal{B}_m(k_m, k_n + 1))(1 + a_n \mathcal{B}_n(k_m, k_n)),$$
$$(1 + b_n \mathcal{C}_n(l_m + 1, l_n))(1 + b_m \mathcal{C}_m(l_m, l_n))$$
$$= (1 + b_m \mathcal{C}_m(l_m, l_n + 1))(1 + b_n \mathcal{C}_n(l_m, l_n)),$$
$$(1 + a_n \mathcal{B}_n(k_n, l_m + 1))(1 + b_m \mathcal{C}_m(k_n, l_m))$$
$$= (1 + b_m \mathcal{C}_m(k_n + 1, l_m))(1 + a_n \mathcal{B}_n(k_n, l_m)).$$

The discrete analogue of the 2DTL hierarchy can be formulated by infinite matrices R_n, L_n which are equivalent to the difference operators $(1 + a_n \mathcal{B}_n)$ and $(1 + b_m \mathcal{C}_m)$, respectively. Let us introduce

$$R_n = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & & \mathbf{0} & \\ & I_{s-1}^{(n,0)} & \cdots & \cdots & \cdots & 1 & & \\ & & I_s^{(n,0)} & \cdots & \cdots & I_s^{(n,n-1)} & 1 & \\ & & & I_{s+1}^{(n,0)} & \cdots & \cdots & I_{s+1}^{(n,n-1)} & \\ & & & & \ddots & \ddots & \ddots & \ddots \\ & & & & & I_{s+n-1}^{(n,1)} & \ddots & \ddots \\ & & & & & & I_{s+n}^{(n,0)} & \ddots \\ & \mathbf{0} & & & & & & \ddots \\ & & & & & & & \ddots \end{pmatrix},$$

$$
L_n =
\begin{array}{c}
\\
\vdots \\
-n \\
1-n \\
\vdots \\
-1 \\
0 \\
1 \\
\vdots
\end{array}
\begin{array}{ccccccc}
\cdots & -n & 1-n & \cdots & -1 & 0 & 1 & \cdots
\end{array}
\left(
\begin{array}{ccccccc}
\ddots & & & & & & \\
\ddots & 1 & & & & 0 & \\
\ddots & \ddots & 1 & & & & \\
\ddots & \ddots & \ddots & \ddots & & & \\
\ddots & \ddots & \ddots & \ddots & 1 & & \\
 & V_s^{(n,n)} & \cdots & \cdots & V_s^{(n,1)} & 1 & \\
 & & V_{s+1}^{(n,n)} & \cdots & \cdots & V_{s+1}^{(n,1)} & 1 \\
 & 0 & & \ddots & \ddots & \ddots & \ddots & \ddots
\end{array}
\right).
$$

Here the dependent variables $I_s^{(n,0)}, I_s^{(n,1)}, I_s^{(n,2)}, \ldots, V_s^{(n,1)}, V_s^{(n,2)}, \ldots$ are given by

$$
\begin{aligned}
I_s^{(n,0)} &= 1 + a_n u_n^{(n)}(s), & I_s^{(n,1)} &= a_n u_{n-1}^{(n)}(s), \\
I_s^{(n,2)} &= a_n u_{n-2}^{(n)}(s), & & \cdots, \\
V_s^{(n,1)} &= b_n v_{n-1}^{(n)}(s), & V_s^{(n,2)} &= b_n v_{n-2}^{(n)}(s), \\
V_s^{(n,3)} &= b_n v_{n-3}^{(n)}(s), & & \cdots.
\end{aligned}
$$

The products infinite matrices $L_m R_n$ and $R_m L_n$ for $m, n = 1, 2, \ldots$ are well-defined. See Ueno-Takasaki [19], p. 4.

Then the discrete analogue of the 2DTL hierarchy (13) or (14) can be written in the following discrete Lax forms

(36a) $R_m(k_m, k_n + 1)R_n(k_m, k_n) = R_n(k_m + 1, k_n)R_m(k_m, k_n),$

(36b) $L_m(l_m, l_n + 1)L_n(l_m, l_n) = L_n(l_m + 1, l_n)L_m(l_m, l_n),$

(36c) $R_n(k_m + 1, l_n)L_m(k_m, l_n) = L_m(k_m, l_n + 1)R_n(k_m, l_n),$

for $m, n = 1, 2, \ldots$, which is a generalization to infinite matrices of the LR factorization algorithm.

§7.   Concluding Remarks

In this paper, we have presented a discrete analogue of the 2DTL hierarchy by using the formulation of the continuous-time 2DTL hierarchy [19]. It has been shown that the subjects originated from the Sato theory are useful for the discrete-time integrable systems. The difference operators $W^{(\infty)}$ and $W^{(0)}$ introduced by Ueno-Takasaki [19] also play an important role in the discrete-time analogue of the 2DTL hierarchy, likewise in the continuous-time 2DTL

hierarchy. This framework might be applicable to other discrete-time soliton equations.

We have introduced an infinite number of discrete-time variables in the discrete Sato equation (12) which governs the discrete-time evolutions. Miwa [10] also introduced an infinite number of discrete-time variables of the discrete KP equation as we have seen in Introduction. There is an interesting contrast in the linear equations appeared in the special solutions of the Casorati determinant form. Namely, the orders of the linear equations (5) of the discrete KP equation are all same, while the orders of the linear equations (21c), (21d) of the discrete analogue of the 2DTL hierarchy are different from each other. We can introduce a more general case where the linear equations of the discrete analogue of the 2DTL hierarchy are given as

$$(37a) \qquad \Delta_{\widetilde{k}_n} f_j(s) = f_j(s+n) + \sum_{i=1}^{n-1} \alpha_{ni} f_j(s+n-i),$$

$$(37b) \qquad \Delta_{\widetilde{l}_n} f_j(s) = f_j(s-n) + \sum_{i=1}^{n-1} \beta_{ni} f_j(s-n+i),$$

where $\widetilde{k}_n, \widetilde{l}_n, n = 1, 2, \ldots$ are new discrete variables, and $\alpha_{ij}, \beta_{ij}, i, j = 1, 2, \ldots$ are constants. In this case, we can derive a generalized discrete Sato equations

$$(38a) \quad \Delta_{\widetilde{k}_n}\left(W^{(\infty)}\right) = \widetilde{\mathcal{B}}_n W^{(\infty)} - e^{\partial_{\widetilde{k}_n}} W^{(\infty)} e^{-\partial_{\widetilde{k}_n}} \left(e^{n\partial_s} + \sum_{i=1}^{n-1} \alpha_{ni} e^{(n-i)\partial_s} \right),$$

$$(38b) \quad \Delta_{\widetilde{l}_n}\left(W^{(\infty)}\right) = \widetilde{\mathcal{C}}_n W^{(\infty)} - e^{\partial_{\widetilde{l}_n}} W^{(\infty)} e^{-\partial_{\widetilde{l}_n}} \left(e^{-n\partial_s} + \sum_{i=1}^{n-1} \beta_{ni} e^{(i-n)\partial_s} \right),$$

$$(38c) \quad \Delta_{\widetilde{k}_n}\left(W^{(0)}\right) = \widetilde{\mathcal{B}}_n W^{(0)} - e^{\partial_{\widetilde{k}_n}} W^{(0)} e^{-\partial_{\widetilde{k}_n}} \left(e^{n\partial_s} + \sum_{i=1}^{n-1} \alpha_{ni} e^{(n-i)\partial_s} \right),$$

$$(38d) \quad \Delta_{\widetilde{l}_n}\left(W^{(0)}\right) = \widetilde{\mathcal{C}}_n W^{(0)} - e^{\partial_{\widetilde{l}_n}} W^{(0)} e^{-\partial_{\widetilde{l}_n}} \left(e^{-n\partial_s} + \sum_{i=1}^{n-1} \beta_{ni} e^{(i-n)\partial_s} \right),$$

where the difference operators $\widetilde{\mathcal{B}}_n$ and $\widetilde{\mathcal{C}}_n$ are defined as

$$\widetilde{\mathcal{B}}_n = e^{\partial_{\widetilde{k}_n}} W^{(\infty)} e^{-\partial_{\widetilde{k}_n}} \left(e^{n\partial_s} + \sum_{i=1}^{n-1} \alpha_{ni} e^{(n-i)\partial_s} \right) W^{(\infty)-1},$$

$$\widetilde{\mathcal{C}}_n = e^{\partial_{\widetilde{l}_n}} W^{(0)} e^{-\partial_{\widetilde{l}_n}} \left(e^{-n\partial_s} + \sum_{i=1}^{n-1} \beta_{ni} e^{(i-n)\partial_s} \right) W^{(0)-1}.$$

If we choose the constants $\alpha_{ij}, i, j = 1, 2, \ldots$ suitably, one can find the following relationship between the discrete KP hierarchy and the discrete analogue of the 2DTL hierarchy

$$e^{-\partial_{\widetilde{k}_n}} f(s) = e^{\partial_{m_1} + \partial_{m_2} + \cdots + \partial_{m_n}} f(s).$$

Thus each discrete variable $\widetilde{k}_n$ of the discrete analogue of the 2DTL hierarchy correspond to the unit of n discrete variables $m_1, m_2, \ldots, m_n$ of the discrete KP hierarchy.

References

[1] Date, E., Jimbo, M., Kashiwara, M. and Miwa, T., Transformation groups for soliton equations, in *Nonlinear Integrable Systems*, (M. Jimbo and T. Miwa, eds.), World Scientific, Singapore, 1983, pp. 39–119.

[2] Date, E., Jimbo, M. and Miwa, T., Method for generating discrete soliton equations I, *J. Phys. Soc. Japan*, **51** (1982), 4116–4124.

[3] ———, Method for generating discrete soliton equations II, *J. Phys. Soc. Japan*, **51** (1982), 4125–4131.

[4] ———, Method for generating discrete soliton equations III, *J. Phys. Soc. Japan*, **52** (1983), 388–393.

[5] ———, Method for generating discrete soliton equations IV, *J. Phys. Soc. Japan*, **52** (1983), 761–765.

[6] ———, Method for generating discrete soliton equations V, *J. Phys. Soc. Japan*, **52** (1983), 766–771.

[7] Hirota, R., Nonlinear partial difference equations. II. Discrete-time Toda equation, *J. Phys. Soc. Japan*, **43** (1977), 2074–2078.

[8] ———, Discrete analogue of a generalized Toda equation, *J. Phys. Soc. Japan*, **50** (1981), 3785–3791.

[9] Hirota, R., Tsujimoto, S. and Imai, T., Difference Scheme of Soliton Equations, in *Future Directions of Nonlinear Dynamics in Physical and Biological Systems*, (P. L. Christiansen, J. C. Eilbeck and R. D. Parmentier, eds.), **312** in NATO ASI Ser. B, Physics, Plenum Press, New York, 1993, pp.7–15.

[10] Miwa, T., On Hirota's difference equations, *Proc. Japan Acad.*, **58** (1982), 9–12.

[11] Ohta, Y., Satsuma, J., Takahashi, D. and Tokihiro, T., An elementary introduction to Sato theory, *Prog. Theor. Phys. Suppl.*, **94** (1988), 210–241.

[12] Papageorgiou, V., Grammaticos, B. and Ramani, A., Orthogonal polynomial approach to discrete Lax-pairs for initial boundary-value problems of the QD algorithm, *Lett. Math. Phys.*, **34** (1995), 91–101.

[13] Sato, M., Soliton equations as dynamical systems on infinite dimensional Grassmann manifolds, *RIMS, Kyoto Univ. Kokyuroku*, **439** (1981), 30–46.

[14] ———, The KP hierarchy and infinite-dimensional Grassmann manifolds, in Proc. Symp. Pure Math., *Theta functions*, Bowdin 1987, (L. Ehrenpreis and R. C. Gunning, eds.), vol. 49-Part1, AMS, Providence, 1989, pp. 51–66.

[15] Sato, M. and Sato, Y., Soliton equations as dynamical systems on infinite dimensional Grassmann manifold, in *Nonlinear PDE in Applied Science*, Proc. U. S.-Japan Seminar, Tokyo 1982, Kinokuniya, Tokyo, 1983, pp. 259–271.

[16] Suris, Yu. B., Discrete-time generalized Toda lattices: complete integrability and relations with relativistic Toda lattices, *Phys. Lett. A*, **145** (1990), 113–119.

[17] Suris, Yu. B., Bi-Hamiltonian structure of the qd algorithm and new discretizations of the Toda lattice, *Phys. Lett. A*, **206** (1995), 153–161.

[18] Tsujimoto, S., On exact solutions of discrete soliton equations, *RIMS, Kyoto Univ. Kokyuroku*, **1005** (1997), 28–36 (in Japanese).

[19] Ueno, K. and Takasaki, K., The Toda lattice hierarchy, in *Group Representations and Systems of Differential Equations*, Adv. Stud. Pure Math., **4**, Kinokuniya, Tokyo, 1984, pp. 1–139.