Publl. RIMS, Kyoto Univ.
38 (2002), 135–146

On Hypoellipticity of the Operator $\exp[-|x_1|^{-\sigma}]D_1^2 + x_1^4 D_2^2 + 1$

Dedicated to Professor Mutsuhide Matsumura on his seventieth birthday

By

Nobuo NAKAZAWA* and Seiichiro WAKABAYASHI*

§1. Introduction

We shall consider the operator

$$P(x, D) = f_\sigma(x_1)D_1^2 + x_1^4 D_2^2 + 1$$

in $\mathbb{R}^2$, where

$$f_\sigma(t) = \begin{cases} \exp[-|t|^{-\sigma}] & (t \neq 0), \\ 0 & (t = 0) \end{cases}$$

for $\sigma > 0$, $f_0(t) = 1/e$, $x = (x_1, x_2) \in \mathbb{R}^2$ and $D = (D_1, D_2) = -i(\partial_1, \partial_2) = -i(\partial/\partial x_1, \partial/\partial x_2)$. In [3] the first author proved that $P(x, D)$ is hypoelliptic if $0 < \sigma < 2$ (see Example 4.5 in [3]). It is obvious that $P(x, D)$ is hypoelliptic if $\sigma = 0$ (see [2]). On the other hand, $L(x, D) = x_1^4 D_1^2 + f_\sigma(x_1)D_2^2 + 1$ is hypoelliptic in $\mathbb{R}^2$ for any $\sigma > 0$ (see Example 4.4 in [3]). Moreover, $L(x, D)$ is not hypoelliptic if $\sigma = 0$. Indeed, $u(x) = x_1 \exp[ix_1^{-1} + \sqrt{2e}x_2]$ $(x_1 \neq 0)$ is a non-smooth null solution of $L(x, D)$ if $\sigma = 0$ (see, also, [1] and [4]). In this paper we shall prove that $P(x, D)$ is not hypoelliptic if $\sigma \geq 2$. In doing so, we shall construct asymptotic solutions using the Airy function. Although our operator has a very special form, we believe that our method here can be applicable to a wide class of operators.

Now we shall give the precise definition of hypoellipticity and our main result.

Communicated by T. Kawai. Received April 2, 2001.
2000 Mathematics Subject Classification(s): 35H10.
*Institute of Mathematics, University of Tsukuba, Ibaraki 305-8571, Japan.
e-mail: `nakazawa@math.tsukuba.ac.jp`, `wkbysh@math.tsukuba.ac.jp`

Definition 1.1. Let $x^0 \in \mathbb{R}^2$. We say that P is hypoelliptic at x^0 if there is a neighborhood ω of x^0 such that

$$\omega \cap \text{sing supp } Pu = \omega \cap \text{sing supp } u \quad \text{for } u \in \mathcal{E}', \tag{1.1}$$

where sing supp u denotes the singular support of u and $\mathcal{E}' = \{u \in \mathcal{D}'; \text{supp } u$ is compact$\}$.

Theorem 1.2. *Let $\sigma \geq 0$. Then $P(x, D)$ is hypoelliptic at $x = (0, 0)$ if and only if $(0 \leq)\ \sigma < 2$.*

Remark 1.3. In the above theorem $x = (0, 0)$ can be replaced by $x = (0, a)$ with $a \in \mathbb{R}$. Moreover, $P(x, D)$ is elliptic at $x = (x_1, x_2)$ with $x_1 \neq 0$ and, therefore, $P(x, D)$ is hypoelliptic at $x = (x_1, x_2)$ with $x_1 \neq 0$.

In the rest of the paper we shall prove the above theorem.

§2. Preliminaries

If P is hypoelliptic, then the Banach closed graph theorem implies that some *a priori* estimates hold for P.

Lemma 2.1. *Assume that P is hypoelliptic at x^0. Then there is a neighborhood ω of x^0 such that for any non-void open subsets ω_i $(i = 1, 2)$ of ω with $\omega_1 \subset\subset \omega_2 \subset \omega$ and any $p \in \mathbb{Z}_+$ there exist $q \in \mathbb{Z}_+$ and $C > 0$ satisfying*

$$\sup_{\substack{x \in \omega_1 \\ |\alpha| \leq p}} |D^\alpha u(x)| \leq C\{\sup_{\substack{x \in \omega_2 \\ |\alpha| \leq q}} |D^\alpha Pu(x)| + \sup_{x \in \omega_2} |u(x)|\} \tag{2.1}$$

for any $u \in C^\infty(\overline{\omega_2})$. Here $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $|\alpha| = \alpha_1 + \alpha_2$ and $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2}$ for $\alpha = (\alpha_1, \alpha_2) \in (\mathbb{Z}_+)^2$, $\omega_1 \subset\subset \omega_2$ means that $\overline{\omega_1}$ is a compact subset of the interior $\overset{\circ}{\omega}_2$ of ω_2, and $C^\infty(\overline{\omega_2}) = \{u \in C^0(\overline{\omega_2})$; there is $U(x) \in C^\infty(\mathbb{R}^2)$ such that $U|_{\overline{\omega_2}} = u\}$.

Remark 2.2. If P is hypoelliptic, then the transposed operator ${}^tP$ of P is locally solvable in $\mathcal{D}'$ (see [6], [7]). The estimates (2.1) hold for $u \in C_0^\infty(\omega_1)$ if ${}^tP$ is only locally solvable at x^0.

Proof. The lemma is well-known. For completeness we shall give the proof. Choose a neighborhood ω of x^0 so that (1.1) holds. Let ω_i $(i = 1, 2)$ be non-void open subsets of ω satisfying $\omega_1 \subset\subset \omega_2 \subset \omega$. We define

$$X = \{u \in C^\infty(\omega_2) \cap \mathcal{B}^0(\omega_2);\ Pu \in \mathcal{B}(\omega_2)\},$$

where $\mathcal{B}^k(\omega_2) = \{u \in C^k(\omega_2); \sup_{x\in\omega_2,|\alpha|\le k} |D^\alpha u(x)| < \infty\}$ ($k \in \mathbb{Z}_+$) and $\mathcal{B}(\omega_2) = \bigcap_{k=0}^{\infty} \mathcal{B}^k(\omega_2)$. We introduce a topology into X which is defined by the seminorms $|\cdot|_{X,p}$ ($p \in \mathbb{Z}_+$), where

$$|u|_{X,p} \equiv \sup_{x\in\omega_2,\, |\alpha|\le p} |D^\alpha P u(x)| + \sup_{x\in\omega_2} |u(x)| \quad \text{for } u \in X.$$

Then X becomes a Fréchet space. Indeed, let $\{u_j\}$ be a Cauchy sequence of X. This implies that there are $u \in \mathcal{B}^0(\omega_2)$ and $f \in \mathcal{B}(\omega_2)$ such that $u_j \to u$ in $\mathcal{B}^0(\omega_2)$ and $Pu_j \to f$ in $\mathcal{B}(\omega_2)$, *i.e.*, $\sup_{x\in\omega_2} |u_j(x) - u(x)| \to 0$ and $\sup_{x\in\omega_2,\ |\alpha|\le k} |D^\alpha P u_j(x) - D^\alpha f(x)| \to 0$ for every $k \in \mathbb{Z}_+$ as $j \to \infty$. Note that $f = Pu$ in $\mathcal{D}'(\omega_2)$. By assumption we have $u \in C^\infty(\omega_2)$, which implies that X is complete. It follows from the closed graph theorem that $X \ni u \longmapsto u \in C^\infty(\omega_2)$ is continuous. This proves the lemma since $C^\infty(\overline{\omega_2}) \subset X$. □

We shall construct asymptotic solutions $u_\rho(x)$, which violate (2.1), in the form

$$u_\rho(x) = U_\rho(x_1) \exp[(4\log\rho)^{2/\sigma} x_2]$$

when $\sigma \ge 2$. Write

$$P_\rho(x_1, \partial_1) U_\rho(x_1) = -\exp[-(4\log\rho)^{2/\sigma} x_2] P(x, D) u_\rho(x),$$

where $\rho \ge 4$. Then we have

$$P_\rho(x_1, \partial_1) = f_\sigma(x_1)\partial_1^2 + (4\log\rho)^{4/\sigma} x_1^4 - 1.$$

Asymptotic solutions will be constructed in two intervals $[t_\rho^-, t_\rho^+]$ and $[t_\rho, 1]$, respectively, where

$$t_\rho^\pm = (4\log\rho)^{-1/\sigma}(1 \pm 2\rho^{-1}) \text{ and } t_\rho = (4\log\rho)^{-1/\sigma}(1 + \rho^{-1}).$$

In order to estimate and connect these asymptotic solutions we need the following

Lemma 2.3. *Let $\rho \ge 4$ and let $R(t;\rho)$ be a real-valued function defined for $\rho \ge 4$ and $t \in [t_\rho, 1]$ such that, with some $M \in \mathbb{R}$,*

$$|\partial_t^k R(t;\rho)| \le C_k \rho^{-M+3k/2}$$

for $\rho \ge 4$, $t \in [t_\rho, 1]$ and $k \in \mathbb{Z}_+$. Moreover, let $u(t;\rho)$ be a solution of the initial-value problem

$$\begin{cases} (\partial_t^2 + p(t;\rho))u(t;\rho) = R(t;\rho) \quad (t \in [t_\rho, 1]), \\ u(t_\rho;\rho) = \alpha(\rho), \quad (\partial_t u)(t_\rho;\rho) = \beta(\rho), \end{cases} \tag{2.2}$$

where $p(t;\rho) = f_\sigma(t)^{-1}((4\log\rho)^{4/\sigma}t^4 - 1)$ *and* $\alpha(\rho)$ *and* $\beta(\rho)$ *are real-valued functions of* $\rho(\geq 4)$.

(i) *Assume that* $R(t;\rho) \equiv 0$. *Then we have*

$$|u(t;\rho)| \leq C(\rho^4(\log\rho)^{2/\sigma}|\alpha(\rho)| + \rho^{5/2}(\log\rho)^{2/\sigma}|\beta(\rho)|) \tag{2.3}$$

for $t \in [t_\rho, 1]$.

(ii) *Assume that* $\alpha(\rho) \equiv \beta(\rho) \equiv 0$. *Then we have*

$$|\partial_t^k u(t;\rho)| \leq C_k \rho^{-M-2+3k/2}(\log\rho)^{-1/(2\sigma)} \tag{2.4}$$

for $t \in [t_\rho, t_\rho^+]$ *and* $k \in \mathbb{Z}_+$.

Proof. Put

$$U(t;\rho) = p(t;\rho)u(t;\rho)^2 + (\partial_t u(t;\rho))^2$$

for $t \in [t_\rho, 1]$. From (2.2) we have

$$2^{-1}\partial_t U(t;\rho) = 2^{-1}(\partial_t p(t;\rho))u(t;\rho)^2 + R(t;\rho)\partial_t u(t;\rho)$$

and, therefore,

$$\begin{aligned} &2^{-1}U(t;\rho) - 2^{-1}U(t_\rho;\rho) \\ &\quad = 2^{-1}\int_{t_\rho}^t (\partial_s p(s;\rho))u(s;\rho)^2 ds + \int_{t_\rho}^t R(s;\rho)\partial_s u(s;\rho)ds. \end{aligned}$$

Since

$$\begin{aligned} \partial_t p(t;\rho) &= -\sigma t^{-\sigma-1}p(t;\rho) + 4(4\log\rho)^{4/\sigma}t^3\exp[t^{-\sigma}] \\ &\leq 4(4\log\rho)^{1/\sigma}((4\log\rho)^{1/\sigma}t - 1)^{-1}p(t;\rho), \\ 4(4\log\rho)^{1/\sigma}&((4\log\rho)^{1/\sigma}t-1)^{-1} \geq 4 \end{aligned}$$

for $t \in [t_\rho, 1]$, we have

$$\begin{aligned} U(t;\rho) \leq\, &U(t_\rho;\rho) + \int_{t_\rho}^t R(s;\rho)^2 ds \\ &+4\int_{t_\rho}^t (4\log\rho)^{1/\sigma}((4\log\rho)^{1/\sigma}s - 1)^{-1}U(s;\rho)ds \end{aligned}$$

for $t \in [t_\rho, 1]$. Putting $\tau = (4\log\rho)^{1/\sigma}t - 1$, $V(\tau) = U(t;\rho)$ and $S(\tau) = R(t;\rho)$, we have

$$V(\tau) \leq V(1/\rho) + \int_{1/\rho}^\tau (4\log\rho)^{-1/\sigma}S(s)^2 ds + 4\int_{1/\rho}^\tau \frac{V(s)}{s}ds$$

for $\tau \in [1/\rho, (4\log\rho)^{1/\sigma} - 1]$. Therefore, $F(\tau) \equiv \tau^{-4}\int_{1/\rho}^{\tau}(V(s)/s)ds$ satisfies

$$\tau^5 F'(\tau) \le V(1/\rho) + \int_{1/\rho}^{\tau}(4\log\rho)^{-1/\sigma}S(s)^2 ds.$$

This gives

$$\begin{aligned} F(\tau) &\le (\rho^4/4 - 1/(4\tau^4))V(1/\rho) \\ &\quad + \int_{1/\rho}^{\tau}(1/(4s^4) - 1/(4\tau^4))(4\log\rho)^{-1/\sigma}S(s)^2 ds, \\ V(\tau) &\le \rho^4\tau^4 V(1/\rho) + \int_{1/\rho}^{\tau}(\tau/s)^4(4\log\rho)^{-1/\sigma}S(s)^2 ds, \end{aligned}$$

$$\begin{aligned} U(t;\rho) &\le \rho^4((4\log\rho)^{1/\sigma}t - 1)^4 U(t_\rho;\rho) \\ &\quad + \int_{t_\rho}^{t}((4\log\rho)^{1/\sigma}t - 1)^4((4\log\rho)^{1/\sigma}s - 1)^{-4}R(s;\rho)^2 ds \\ &\qquad\qquad \text{for } t \in [t_\rho, 1]. \end{aligned} \tag{2.5}$$

(i) We first assume that $R(t;\rho) \equiv 0$. Since $p(t_\rho;\rho) \le C\rho^3$ and $p(t;\rho)^{-1} \le \rho/(4e)$ for $t \in [t_\rho, 1]$, (2.5) yields (2.3).

(ii) Assume that $\alpha(\rho) \equiv \beta(\rho) \equiv 0$. From (2.5) we have

$$U(t;\rho) \le C\rho^{-2M-1}(\log\rho)^{-1/\sigma} \quad \text{for } t \in [t_\rho, t_\rho^+].$$

Since $p(t;\rho)^{-1} \le C\rho^{-3}$ for $t \in [t_\rho, t_\rho^+]$, this proves that (2.4) is valid for $k = 0, 1$. Note that

$$|\partial_t^k p(t;\rho)| \le \begin{cases} C\rho^3 & (\ k = 0), \\ C_k\,\rho^4(\log\rho)^{k/\sigma + k - 1} & (\ k \ge 1) \end{cases} \tag{2.6}$$

for $t \in [t_\rho, t_\rho^+]$. Now suppose that (2.4) is valid for $k \le l$, where $l \ge 1$. Let $k = l + 1$. Then, from (2.2) and (2.6) we have

$$\begin{aligned} |\partial_t^k u(t;\rho)| &\le \sum_{j=0}^{k-2}\binom{k-2}{j}|\partial_t^j u(t;\rho)|\,|\partial_t^{k-2-j}p(t;\rho)| + |\partial_t^{k-2}R(t;\rho)| \\ &\le C_k\rho^{-M-2+3k/2}(\log\rho)^{-1/(2\sigma)} \qquad \text{for } t \in [t_\rho, t_\rho^+], \end{aligned}$$

which proves the assertion (ii). □

§3. Proof of Theorem 1.2

In order to prove Theorem 1.2 it suffices to show that $P(x, D)$ is not hypoelliptic at $x = 0$ when $\sigma \geq 2$. Assume that $\sigma \geq 2$. As we stated in Section 2, we shall construct asymptotic solutions $u_\rho(x)$ in the form $u_\rho(x) = U_\rho(x_1)\exp[(4\log\rho)^{2/\sigma}x_2]$. Note that

$$\sup_{|x_2|\leq 1} \exp[(4\log\rho)^{2/\sigma}x_2] \leq \rho^4.$$

First we shall construct asymptotic solutions $U_\rho(x_1)$ satisfying $P_\rho(x_1, \partial_1)U_\rho(x_1) \sim 0$ in $[t_\rho^-, t_\rho^+]$. Putting $t = \rho\{(4\log\rho)^{1/\sigma}x_1 - 1\}$ and $V_\rho(t) = U_\rho(x_1)$, we can write

$$\rho^{-2}(4\log\rho)^{-2/\sigma} f_\sigma(x_1)^{-1} P_\rho(x_1, \partial_1)U_\rho(x_1) = \widetilde{P}_\rho(t, \partial_t)V_\rho(t)$$

for $t \in [-2, 2]$, where

$$\begin{aligned}\widetilde{P}_\rho(t, \partial_t) = {}& \partial_t^2 + 4\rho(4\log\rho)^{-2/\sigma}t + (4\log\rho)^{-2/\sigma}(6t^2 + 4\rho^{-1}t^3 + \rho^{-2}t^4) \\ & + \sum_{j\geq k\geq 1} c_{j,k}\rho^{1-j}(\log\rho)^{-2/\sigma+k}t^{j+1}.\end{aligned}$$

Indeed, we have

$$\begin{aligned}(1 + \rho^{-1}t)^{-\sigma} &= 1 + \sum_{k=1}^{\infty} \binom{-\sigma}{k} \rho^{-k}t^k, \\ f_\sigma(x_1)^{-1} &= \exp[(4\log\rho)(1 + \rho^{-1}t)^{-\sigma}] \\ &= \rho^4 \left(1 + \sum_{j\geq k\geq 1} c'_{j,k}\rho^{-j}(\log\rho)^k t^j\right),\end{aligned}$$

if $\rho \geq 4$ and $|t| \leq 2$. Write

$$V_\rho(t) = \mathrm{Ai}(-c_\rho t)V_\rho^0(t) + \rho^{-1/6}(\log\rho)^{1/(3\sigma)}\,\mathrm{Ai}'(-c_\rho t)V_\rho^1(t), \tag{3.1}$$

where $c_\rho = 4^{1/3}\rho^{1/3}(4\log\rho)^{-2/(3\sigma)}$ and $\mathrm{Ai}(t)$ denotes the Airy function. The Airy function $\mathrm{Ai}(t)$ is defined, for example, by

$$\mathrm{Ai}(t) = \pi^{-1}\int_0^\infty \cos(s^3/3 + ts)ds$$

and satisfies $\mathrm{Ai}''(t) = t\,\mathrm{Ai}(t)$. A simple calculation gives

$$\begin{aligned}
&\widetilde{P}_\rho(t,\partial_t)V_\rho(t)\\
&\quad=\rho^{1/3}(\log\rho)^{-2/(3\sigma)}\operatorname{Ai}'(-c_\rho t)\Big\{-2\cdot 4^{(1-2/\sigma)/3}\partial_t V^0_\rho(t)\\
&\qquad+\rho^{-1/2}(\log\rho)^{1/\sigma}\partial_t^2 V^1_\rho(t)+\sum_{\substack{j\ge k\ge 0\\ j\ge 1}} c_{j,k}\,\rho^{1/2-j}(\log\rho)^{-1/\sigma+k}\,t^{j+1}V^1_\rho(t)\Big\}\\
&\qquad+\rho^{1/2}(\log\rho)^{-1/\sigma}\operatorname{Ai}(-c_\rho t)\Big\{2\cdot 4^{2(1-2/\sigma)/3}\,t\partial_t V^1_\rho(t)+4^{2(1-2/\sigma)/3}V^1_\rho(t)\\
&\qquad+\rho^{-1/2}(\log\rho)^{1/\sigma}\partial_t^2 V^0_\rho(t)+\sum_{\substack{j\ge k\ge 0\\ j\ge 1}} c_{j,k}\,\rho^{1/2-j}(\log\rho)^{-1/\sigma+k}\,t^{j+1}V^0_\rho(t)\Big\},
\end{aligned}$$

where $c_{1,0}=6\cdot 4^{-2/\sigma}$, $c_{2,0}=4^{1-2/\sigma}$, $c_{3,0}=4^{-2/\sigma}$ and $c_{j,0}=0$ for $j\ge 4$. Put

$$J(j)=\{-\mu/\sigma+l;\ \mu,l\in\mathbb{Z},\ |\mu|\le j \text{ and } 0\le l\le j\}$$

for $j\in\mathbb{Z}_+$, and write $J(j)=\{\nu_{j,1},\nu_{j,2},\dots,\nu_{j,r(j)}\}$, where $\nu_{j,1}>\nu_{j,2}>\cdots>\nu_{j,r(j)}$. Note that $J(0)=\{0\}$, $r(0)=1$ and $\nu_{0,1}=0$. We define $I(\cdot\,;j):\mathbb{R}\ni\delta\longmapsto I(\delta;j)\in\{0,1,\dots,r(j)\}$ $(j\in\mathbb{Z}_+)$ by

$$I(\delta;j)=\begin{cases}k & \text{if } \delta\in J(j) \text{ and } \delta=\nu_{j,k},\\ 0 & \text{if } \delta\notin J(j).\end{cases}$$

We also define $I(\delta;j)=0$ if $j<0$. Let us determine $V^i_\rho(t)$ $(i=0,1)$ in the form

$$V^i_\rho(t)\sim\sum_{j=0}^{\infty}\sum_{k=1}^{r(j)}\rho^{-j/2}(\log\rho)^{\nu_{j,k}}V^i_{j,k}(t) \tag{3.2}$$

so that $\widetilde{P}_\rho(t,\partial_t)V_\rho(t)\sim 0$, *i.e.*,

$$\begin{aligned}
&\Big|\partial_t^l\widetilde{P}_\rho(t,\partial_t)\sum_{j=0}^{N}\sum_{k=1}^{r(j)}\rho^{-j/2}(\log\rho)^{\nu_{j,k}}\{\operatorname{Ai}(-c_\rho t)V^0_{j,k}(t)\\
&\qquad+\rho^{-1/6}(\log\rho)^{1/(3\sigma)}\operatorname{Ai}'(-c_\rho t)V^1_{j,k}(t)\}\Big|\le C_{N,l}\rho^{-a_{N,l}}
\end{aligned}$$

for $\rho\ge 4$, $t\in[-2,2]$ and $N,l\in\mathbb{Z}_+$, where $a_{N,l}\to\infty$ as $N\to\infty$. Then we have the transport equations

$$\begin{cases}-2\cdot 4^{(1-2/\sigma)/3}\partial_t V^0_{j,k}(t)+\partial_t^2 V^1_{j-1,I(\nu_{j,k}-1/\sigma;j-1)}(t)\\ \quad+\sum_{\mu\ge\nu\ge 0,\,\mu\ge 1}c_{\mu,\nu}t^{\mu+1}V^1_{j-2\mu+1,I(\nu_{j,k}+1/\sigma-\nu;j-2\mu+1)}(t)=0,\\ 4^{2(1-2/\sigma)/3}(2t\partial_t+1)V^1_{j,k}(t)+\partial_t^2 V^0_{j-1,I(\nu_{j,k}-1/\sigma;j-1)}(t)\\ \quad+\sum_{\mu\ge\nu\ge 0,\,\mu\ge 1}c_{\mu,\nu}t^{\mu+1}V^0_{j-2\mu+1,I(\nu_{j,k}+1/\sigma-\nu;j-2\mu+1)}(t)=0\end{cases} \tag{3.3}$$

for $j \in \mathbb{Z}_+$ and $1 \le k \le r(j)$, where $V^i_{j,0} \equiv 0$ for $j \in \mathbb{Z}_+$ and $i = 0, 1$. Let $V^i_{j,k}(t)$ ($i = 0, 1$, $j \in \mathbb{Z}_+$ and $1 \le k \le r(j)$) be solutions of (3.3) with the initial conditions

$$\begin{cases} V^0_{0,1}(0) = 1, \\ V^i_{j,k}(0) = 0 \quad \text{if } i = 0, 1,\ i + j \ge 1 \text{ and } 1 \le k \le r(j). \end{cases}$$

Then the $V^i_{j,k}(t)$ are determined inductively by

$$(3.4)\quad \begin{cases} V^0_{j,k}(t) = \delta_{j,0} + 2^{-1} \cdot 4^{-(1-2/\sigma)/3} \Big\{ \partial_t V^1_{j-1,I(\nu_{j,k}-1/\sigma;j-1)}(t) \\ \quad -(\partial_t V^1_{j-1,I(\nu_{j,k}-1/\sigma;j-1)})(0) \\ \quad + \sum_{\mu \ge \nu \ge 0,\, \mu \ge 1} c_{\mu,\nu} \int_0^t s^{\mu+1} V^1_{j-2\mu+1,I(\nu_{j,k}+1/\sigma-\nu;j-2\mu+1)}(s)ds \Big\}, \\ V^1_{j,k}(t) = -2^{-1} \cdot 4^{-2(1-2/\sigma)/3} \int_0^t t^{-1/2} s^{-1/2} \Big\{ \partial_s^2 V^0_{j-1,I(\nu_{j,k}-1/\sigma;j-1)}(s) \\ \quad + \sum_{\mu \ge \nu \ge 0,\, \mu \ge 1} c_{\mu,\nu} s^{\mu+1} V^0_{j-2\mu+1,I(\nu_{j,k}+1/\sigma-\nu;j-2\mu+1)}(s) \Big\} ds \end{cases}$$

($j \in \mathbb{Z}_+$ and $1 \le k \le r(j)$). Since $t^{-1/2}(t\theta)^{-1/2} = t^{-1}\theta^{-1/2}$ for $t \ne 0$ and $0 \le \theta \le 1$ and

$$\int_0^t t^{-1/2} s^{-1/2} f(s) ds = \int_0^1 \theta^{-1/2} f(t\theta) d\theta \in C^\infty([-2, 2])$$

for $f(t) \in C^\infty([-2, 2])$, we have $V^i_{j,k}(t) \in C^\infty([-2, 2])$ ($i = 0, 1$, $j \in \mathbb{Z}_+$ and $1 \le k \le r(j)$). Substituting (3.2) and (3.4) in (3.1), we have $\widetilde{P}_\rho(t, \partial_t)V_\rho(t) \sim 0$. Indeed, we see that

$$\{1, 2, \ldots, r(j)\} \subset \{I(\nu_{j+1,k} - 1/\sigma; j)\,;\, 1 \le k \le r(j + 1)\},$$
$$\{1, 2, \ldots, r(j)\} \subset \{I(\nu_{j+2\mu-1,k} + 1/\sigma - \nu; j)\,;\, 1 \le k \le r(j + 2\mu - 1)\}$$

if $j \in \mathbb{Z}_+$, $\mu \ge 1$ and $\mu \ge \nu \ge 0$. We have also used the estimates

$$(3.5)\quad \begin{cases} |\operatorname{Ai}^{(k)}(t)| \le C_k(1 + t)^{-1/4+k/2} \exp[-2t^{3/2}/3], \\ |\operatorname{Ai}^{(k)}(-t)| \le C_k(1 + t)^{-1/4+k/2} \end{cases}$$

for $t \ge 0$ (see, e.g., [5]). We note that $V^0_{2j-1,k}(t) \equiv V^1_{2j,l}(t) \equiv 0$ if $j \in \mathbb{N}$, $1 \le k \le r(2j - 1)$ and $1 \le l \le r(2j)$. Put

$$(3.6)\quad U^N_\rho(x_1) = \sum_{j=0}^{N} \sum_{k=1}^{r(j)} \rho^{-j/2} (\log \rho)^{\nu_{j,k}} [\operatorname{Ai}(-c'_\rho(x_1 - s_\rho)) V^0_{j,k}(c''_\rho(x_1 - s_\rho)) + \rho^{-1/6} (\log \rho)^{1/(3\sigma)} \operatorname{Ai}'(-c'_\rho(x_1 - s_\rho)) V^1_{j,k}(c''_\rho(x_1 - s_\rho))],$$
$$R^N_\rho(x_1) = \rho^{-2}(4 \log \rho)^{-2/\sigma} f_\sigma(x_1)^{-1} P_\rho(x_1, \partial_1) U^N_\rho(x_1)$$

for $x_1 \in [t_\rho^-, t_\rho^+]$ and $N \in \mathbb{Z}_+$, where $c'_\rho = 4^{1/3}\rho^{4/3}(4\log\rho)^{1/(3\sigma)}$, $c''_\rho = \rho(4\log\rho)^{1/\sigma}$ and $s_\rho = (4\log\rho)^{-1/\sigma}$. Then we have

$$
\begin{aligned}
(3.7)\quad & R_\rho^N(x_1)\\
&= \mathrm{Ai}'(-c'_\rho(x_1-s_\rho))\Big[\sum_{k=1}^{r(N)} \rho^{-N/2-1/6}(\log\rho)^{1/(3\sigma)+\nu_{N,k}}\partial_t^2 V_{N,k}^1(t)\\
&\quad + \sum_{j=0}^{N} \sum_{\substack{2\mu\ge N+2-j\\ \mu\ge\nu\ge 0}} \sum_{k=1}^{r(j)} c_{\mu,\nu}\,\rho^{5/6-\mu-j/2}(\log\rho)^{-5/(3\sigma)+\nu+\nu_{j,k}}\, t^{\mu+1}V_{j,k}^1(t)\Big]\\
&\quad + \mathrm{Ai}(-c'_\rho(x_1-s_\rho))\Big[\sum_{k=1}^{r(N)} \rho^{-N/2}(\log\rho)^{\nu_{N,k}}\partial_t^2 V_{N,k}^0(t)\\
&\quad + \sum_{j=0}^{N} \sum_{\substack{2\mu\ge N+2-j\\ \mu\ge\nu\ge 0}} \sum_{k=1}^{r(j)} c_{\mu,\nu}\,\rho^{1-\mu-j/2}(\log\rho)^{-2/\sigma+\nu+\nu_{j,k}}\, t^{\mu+1}V_{j,k}^0(t)\Big],
\end{aligned}
$$

where $t = \rho\{(4\log\rho)^{1/\sigma}x_1 - 1\}$. From (3.5) we have

$$
\begin{aligned}
(3.8)\qquad & |\partial_1^k R_\rho^N(x_1)|\\
&\le \begin{cases} C_{N,0}\rho^{-N/2}(\log\rho)^{1-2/\sigma+(1+1/\sigma)N} & (k=0),\\ C_{N,k}\rho^{-1/12+3k/2-N/2}(\log\rho)^{1-11/(6\sigma)+(1+1/\sigma)N} & (k\ge 1)\end{cases}
\end{aligned}
$$

for $x_1 \in [s_\rho, t_\rho^+]$, and

$$
(3.9)\qquad |\partial_1^k R_\rho^N(x_1)| \le \begin{cases} C_{N,0}\rho^{-N/2}(\log\rho)^{1-2/\sigma+(1+1/\sigma)N}\\ \qquad\times\exp[-2(c'_\rho)^{3/2}(s_\rho-x_1)^{3/2}/3] \quad (k=0),\\ C_{N,k}\rho^{-1/12+3k/2-N/2}(\log\rho)^{1-11/(6\sigma)+(1+1/\sigma)N}\\ \qquad\times\exp[-2(c'_\rho)^{3/2}(s_\rho-x_1)^{3/2}/3] \quad (k\ge 1)\end{cases}
$$

for $x_1 \in [t_\rho^-, s_\rho]$. Indeed, for example, we have

$$
\begin{aligned}
|\partial_1^k\,\mathrm{Ai}(-c'_\rho(x_1-s_\rho))| &\le \begin{cases} C_0 & (k=0),\\ C_k\rho^{-1/12+3k/2}(\log\rho)^{1/(6\sigma)} & (k\ge 1),\end{cases}\\
|\partial_1^k\,\mathrm{Ai}'(-c'_\rho(x_1-s_\rho))| &\le C_k\rho^{1/12+3k/2}(\log\rho)^{-1/(6\sigma)}
\end{aligned}
$$

for $x_1 \in [s_\rho, t_\rho^+]$. Moreover, we have

$$
\rho^{1-\mu-j/2}(\log\rho)^{\nu+\nu_{j,k}} \le \rho^{-N/2}(\log\rho)^{1+(1+1/\sigma)N}
$$

if $0 \le j \le N$, $1 \le k \le r(j)$, $2\mu \ge N+2-j$ and $\mu \ge \nu \ge 0$, and

$$(\log \rho)^{1/(3\sigma)+\nu_{N,k}} \le (\log \rho)^{1-5/(3\sigma)+(1+1/\sigma)N}$$

if $1 \le k \le r(N)$. Similarly, we have

$$\begin{cases} |U_\rho^N(t_\rho)| \le C_N\, \rho^{-1/12}(\log \rho)^{1/(6\sigma)}, \\ |(\partial_1 U_\rho^N)(t_\rho)| \le C_N\, \rho^{17/12}(\log \rho)^{1/(6\sigma)}. \end{cases} \tag{3.10}$$

Let $\widetilde{U}_\rho^N(t)$ be a solution of (2.2) with $R(t;\rho) \equiv 0$, $\alpha(\rho) = U_\rho^N(t_\rho)$ and $\beta(\rho) = (\partial_1 U_\rho^N)(t_\rho)$. We choose a function $\chi(t) \in C^\infty(\mathbb{R})$ so that $\chi(t) = 1$ for $t \le 1$ and $\chi(t) = 0$ for $t \ge 2$, and put

$$u_\rho^N(x_1) = \{\chi_\rho^0(x_1) U_\rho^N(x_1) + \chi_\rho^1(x_1) \widetilde{U}_\rho^N(x_1)\} \exp[(4\log\rho)^{2/\sigma} x_2],$$

where $N \in \mathbb{Z}_+$, $\rho \ge 4$ and

$$\begin{aligned} \chi_\rho(x_1) &= \chi(c''_\rho(x_1 - s_\rho)), \\ \chi_\rho^0(x_1) &= \chi_\rho(x_1)\chi(c''_\rho(s_\rho - x_1)), \\ \chi_\rho^1(x_1) &= 1 - \chi_\rho(x_1). \end{aligned}$$

Lemma 3.1. (i) *For every $k, N \in \mathbb{Z}_+$*

$$|(\partial_1^k u_\rho^N)(s_\rho, 0)| = 4^{k/3}\rho^{4k/3}(4\log\rho)^{k/(3\sigma)}\left(|\operatorname{Ai}^{(k)}(0)| + \mathrm{o}(1)\right) \tag{3.11}$$

as $\rho \to \infty$. In particular, there are $c_{N,k} > 0$ and $\rho_{N,k} \ge 4$ such that

$$|(\partial_1^{3k} u_\rho^N)(s_\rho, 0)| \ge c_{N,k}\rho^{4k}(\log\rho)^{k/\sigma} \tag{3.12}$$

if $\rho \ge \rho_{N,k}$.

(ii) *For $N \in \mathbb{Z}_+$ and $\rho \ge 4$*

$$\sup_{|x_2| \le 1} |u_\rho^N(x)| \le \begin{cases} C_N \rho^4 \exp[-2(c'_\rho)^{3/2}(s_\rho - x_1)^{3/2}/3] & \text{if } x_1 \le s_\rho, \\ C_N\rho^4 \quad \text{if } \ x_1 \in [s_\rho, t_\rho], \\ C_N \rho^{95/12}(\log\rho)^{13/(6\sigma)} \quad \text{if } \ x_1 \in [t_\rho, 1]. \end{cases}$$

(iii) *For* $N \in \mathbb{Z}_+$, $\alpha = (\alpha_1, \alpha_2) \in (\mathbb{Z}_+)^2$ *and* $\rho \geq 4$

$$\sup_{|x_2| \leq 1} |D^\alpha P(x,D) u_\rho^N(x)| \leq \begin{cases} 0 \quad \textit{if} \ \ x_1 \in (-\infty, t_\rho^-] \cup [t_\rho^+, 1], \\ C_{N,M,\alpha}\rho^{-M} \quad \textit{if} \ \ x_1 \in [t_\rho^-, s_\rho^-] \ \ \textit{and} \ \ M \in \mathbb{Z}_+, \\ C_{N,\alpha}\rho^{2+3\alpha_1/2-N/2}(\log \rho)^{1+1/(6\sigma)+2\alpha_2/\sigma+(1+1/\sigma)N} \\ \qquad \textit{if} \ \ x_1 \in [s_\rho^-, t_\rho], \\ C_{N,\alpha}\rho^{3+3\alpha_1/2-N/2}(\log \rho)^{1-1/(3\sigma)+2\alpha_2/\sigma+(1+1/\sigma)N} \\ \qquad \textit{if} \ \ x_1 \in [t_\rho, t_\rho^+], \end{cases} \tag{3.13}$$

where $s_\rho^- = (4 \log \rho)^{-1/\sigma}(1 - \rho^{-1})$.

Proof. (i) Note that $u_\rho^N(x_1, 0) = U_\rho^N(x_1)$ for $x_1 \in [s_\rho^-, t_\rho]$. This, together with (3.6), yields (3.11). Since $\mathrm{Ai}(0) = 3^{-2/3}\Gamma(2/3)^{-1}$ and $\mathrm{Ai}^{(3k)}(0) = 1 \cdot 4 \cdot \cdots \cdot (3k-2)\,\mathrm{Ai}(0)$ ($\neq 0$) for $k \in \mathbb{N}$, we have (3.12).

(ii) Note that

$$u_\rho^N(x) = \begin{cases} 0 \quad \text{if } x_1 \leq t_\rho^-, \\ \chi_\rho^0(x_1) U_\rho^N(x_1) \exp[(4\log\rho)^{2/\sigma} x_2] \quad \text{if } x_1 \leq t_\rho, \\ U_\rho^N(x_1) \exp[(4\log\rho)^{2/\sigma} x_2] \quad \text{if } x_1 \in [s_\rho^-, t_\rho], \\ \widetilde{U}_\rho^N(x_1) \exp[(4\log\rho)^{2/\sigma} x_2] \quad \text{if } x_1 \in [t_\rho^+, 1]. \end{cases} \tag{3.14}$$

From Lemma 2.3 and (3.10) it follows that

$$|\widetilde{U}_\rho^N(x_1)| \leq C_N\, \rho^{47/12}(\log \rho)^{13/(6\sigma)} \quad \text{for } x_1 \in [t_\rho, 1].$$

This, together with (3.5), (3.6) and (3.14), proves the assertion (ii).

(iii) It is obvious that

$$|f_\sigma^{(k)}(x_1)| \leq C_k \rho^{-4}(\log\rho)^{(1+1/\sigma)k}$$

for $x_1 \in [t_\rho^-, t_\rho^+]$. For $x_1 \in [t_\rho^-, s_\rho^-]$ we have

$$\begin{aligned} P(x,D)u_\rho^N(x) = &-\exp[(4\log\rho)^{2/\sigma}x_2]\{[P_\rho, \chi_\rho^0(x_1)]U_\rho^N(x_1) \\ &+ \rho^2(4\log\rho)^{2/\sigma} f_\sigma(x_1) R_\rho^N(x_1)\chi_\rho^0(x_1)\}, \\ [P_\rho, \chi_\rho^0(x_1)] =& 2f_\sigma(x_1)(\partial_1\chi_\rho^0(x_1))\partial_1 + f_\sigma(x_1)(\partial_1^2\chi_\rho^0(x_1)). \end{aligned}$$

Therefore, by (3.5), (3.6) and (3.9) we can see that (3.13) is valid if $x_1 \in [t_\rho^-, s_\rho^-]$. For $x_1 \in [s_\rho^-, t_\rho]$ we have

$$P(x,D)u_\rho^N(x) = -\exp[(4\log\rho)^{2/\sigma}x_2]\rho^2(4\log\rho)^{2/\sigma} f_\sigma(x_1) R_\rho^N(x_1).$$

This, together with (3.8) and (3.9), shows that (3.13) is valid if $x_1 \in [s_\rho^-, t_\rho]$. For $x_1 \in [t_\rho, t_\rho^+]$ we have

$$P(x,D)u_\rho^N(x) = -\exp[(4\log\rho)^{2/\sigma}x_2] \\ \times P_\rho(x_1,\partial_1)\{\chi_\rho^0(x_1)(U_\rho^N(x_1) - \widetilde{U}_\rho^N(x_1))\}.$$

Since

$$(\partial_1^2 + p(x_1;\rho))(U_\rho^N(x_1) - \widetilde{U}_\rho^N(x_1)) = \rho^2(4\log\rho)^{2/\sigma}R_\rho^N(x_1),$$
$$U_\rho^N(t_\rho) - \widetilde{U}_\rho^N(t_\rho) = 0,\ (\partial_1 U_\rho^N)(t_\rho) - (\partial_1\widetilde{U}_\rho^N)(t_\rho) = 0,$$

Lemma 2.3 and (3.8) give

$$|\partial_1^k(U_\rho^N(x_1) - \widetilde{U}_\rho^N(x_1))| \le C_k\,\rho^{3k/2-N/2}(\log\rho)^{1-1/(3\sigma)+(1+1/\sigma)N}$$

for $x_1 \in [t_\rho, t_\rho^+]$. This proves that (3.13) is also valid for $x_1 \in [t_\rho, t_\rho^+]$. □

By Lemma 3.1, for any neighborhoods ω_i $(i = 1,2)$ of $x = 0$ with $\omega_1 \subset\subset \omega_2 \subset \{x \in \mathbb{R}^2\ ;\ |x_1| \le 1 \text{ and } |x_2| \le 1\}$ and any $N \in \mathbb{Z}_+$ there are $c > 0$, $\rho_0 \ge 4$, $C > 0$ and $C_j > 0$ $(j \in \mathbb{Z}_+)$ such that

$$\sup_{x\in\omega_1,\,|\alpha|\le 6} |D^\alpha u_\rho^N(x)| \ge c\rho^8,$$
$$\sup_{x\in\omega_2,\,|\alpha|\le q} |D^\alpha P(x,D)u_\rho^N(x)| \le C_q\,\rho^{4+3q/2-N/2},$$
$$\sup_{x\in\omega_2} |u_\rho^N(x)| \le C\rho^{95/12}(\log\rho)^{13/(6\sigma)}$$

if $\rho \ge \rho_0$ and $q \in \mathbb{Z}_+$. This, together with Lemma 2.1, implies that $P(x,D)$ is not hypoelliptic at $x = 0$.

References

[1] Derridj, M., Sur une classe d'opérateurs différentiels hypoelliptiques à coefficients analytiques, Séminaire Goulaouic-Schwartz 1970–1971, Exposé No. 12, Ecole Polytechnique, Paris.

[2] Hörmander, L., Hypoelliptic second order differential equations, *Acta Math.*, **119** (1967), 147–171.

[3] Nakazawa, N., On hypoellipticity for class of pseudo-differential operators, *Tsukuba J. Math.*, **24** (2000), 257–278.

[4] Oleĭnik, O. A. and Radkevič, E. V., *Second Order Equations with Nonnegative Characteristic Form*, Itogi Nauki, Ser. Mat., Mat. Anal., Moscow, 1971. English translation: Plenum Press, New York-London, 1973.

[5] Olver, F. W. J., *Asymptotics and Special Functions*, Academic Press, New York-London, 1974.

[6] Treves, F., *Topological Vector Spaces, Distributions and Kernels*, Academic Press, New York, 1967.

[7] Yoshikawa, A., On the hypoellipticity of differential operators, *J. Fac. Sci. Univ. Tokyo Sect.*, IA, **14** (1967), 81–88.