Contractions of Angles in Symmetric Cones

To Jacques Faraut on his sixtieth birthday

By

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Abstract

For a symmetric cone Ω we compute its Riemannian distance in terms of the singular values of a generalized cross-ratio and prove that the semigroup of the compressions of Ω decreases the compounds distance.

§1.   Introduction

Let $Sp(m, \mathbb{R})$ be the real symplectic group acting on the Siegel upper half plane $T_{\Omega_{Sym}}$ of complex symmetric matrices with positive definite imaginary part via the rational transformations

$$g \cdot z = (Az + B)(Cz + D)^{-1}, \quad g = \begin{pmatrix} A\, B \\ C\, D \end{pmatrix}$$

Let Ω_{Sym} be the set of real positive definite symmetric matrices. We can view it as the Riemannian symmetric space $GL(m, \mathbb{R})/O(m)$. C. L. Siegel's study of the symplectic geometry [16] allows one to get an explicit formula for the geodesic distance in Ω_{Sym} associated with the $GL(m, \mathbb{R})$ invariant Riemannian metric, which coincides with the Euclidean arc length on the logarithms of

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the diagonal matrices in Ω_{Sym}. Namely, if $x, y \in \Omega_{Sym}$, then the Riemannian distance of x and y is given by

$$(1.1) \qquad \delta(x,y) = \left(\sum_{k=1}^{m} \log^2 \lambda_k(x,y) \right)^{1/2},$$

where $\lambda_1(x,y) \geq \cdots \geq \lambda_m(x,y) > 0$ are the *characteristic value* of (x,y), i.e. the eigenvalues of the matrix $y^{-1}x$. The eigenvalues $\lambda_k(x,y)$ are the unique solutions of the equation

$$(1.2) \qquad \mathrm{Det}(x - \lambda y) = 0.$$

Furthermore, if we put

$$(1.3) \qquad q(x,y) := (x-y)(x+y)^{-1}(x-y)(x+y)^{-1}$$

then, the real numbers

$$(1.4) \qquad \xi_j(x,y) := \left(\frac{1 - \lambda_j(x,y)}{1 + \lambda_j(x,y)} \right)^2$$

are the unique eigenvalues of the cross-ratio $q(x,y)$ and the formula (1.1) can be written as follows:

$$(1.5) \qquad \delta(x,y) = \left(\sum_{k=1}^{m} \log^2 \frac{1 + \sqrt{\xi_k(x,y)}}{1 - \sqrt{\xi_k(x,y)}} \right)^{1/2}.$$

Let us consider now the set $\mathcal{H}$ of the matrices in $Sp(m, \mathbb{R})$ that compress Ω_{Sym}:

$$\mathcal{H} := \{ g \in Sp(m, \mathbb{R}) \mid g \cdot \Omega_{Sym} \subset \Omega_{Sym} \}.$$

It is a semigroup of $Sp(m, \mathbb{R})$ and it contains $GL(m, \mathbb{R})$ and the inversion $x \mapsto x^{-1}$. One can show that it is the semigroup of the Hamiltonian matrices:

$$(1.6) \qquad \mathcal{H} = \left\{ \begin{pmatrix} A\ B \\ C\ D \end{pmatrix} \mid A \text{ invertible}, \ BA^* \geq 0, A^*C \geq 0 \right\}$$

where $M \geq 0$ means that M is a positive symmetric matrix, i.e. $M \in \overline{\Omega}_{Sym}$. A natural question arises now : what is the behaviour of the *angles* $\mu_k(x,y) := \log^2 \lambda_k(x,y)$ and therefore of the Riemannian distance upon the action of the semigroup $\mathcal{H}$?

This problem was solved by Ph. Bougerol in [1] where he proved that the elements of $\mathcal{H}$ are contractions (and sometimes uniform contractions) of the $\mu_k(x, y)$ and then for the Riemannian distance.

This problem has a good interpretation in Euclidean Jordan algebras and causal symmetric spaces theories. Indeed Ω_{Sym} is the symmetric cone associated with the Euclidean Jordan algebra $Sym(m, \mathbb{R})$ and $\mathcal{H}$ is the causal semigroup associated with the ordered symmetric space $Sym(m, \mathbb{R})/GL(m, \mathbb{R}) \otimes \mathbb{R}$.

Let V be a Euclidean Jordan algebra and Ω the symmetric cone associated with V. In this paper we shall deal with an explicit distance formula for Ω. We call it *Siegel's distance formula*, since it has close analogies with formulas (1.1) and (1.5) introduced by C. L. Siegel for symmetric matrices. The Siegel distance formula involves a generalized cross-ratio for a quadruple

$$D(w, x, y, z) := P(w - x)P(x - y)^{-1}P(z - y)P(z - w)^{-1}$$

introduced for special cases by L. K. Hua and C. L. Siegel and generalized to any Jordan algebra by H. Braun. The fundamental property of this cross-ratio is, as Siegel observed, that if z and w are two points of the tube $V + i\Omega$, then $D(z, w, \bar{z}, \bar{w})$ always has real eigenvalues.

The second aim of this paper is to study the monotone behaviour of the angles upon the action of the compression semigroup

$$S = \{\gamma \in G \mid \gamma \cdot \Omega \subset \Omega\},$$

where G is the conformal group of V.

In Section 2 we collect basic facts about Euclidean Jordan algebras needed in this paper. In Section 3 we connect the characteristic values with the eigenvalues of a generalized cross-ratio. In Section 4 we give a generalization of a U. Hirzebruch theorem which is the Fischer min-max theorem on the eigenvalues of elements in V. In Section 5 we prove the Siegel distance formula for Ω. In Section 6 we give some properties of the compression semigroup S and finally in Section 7 we prove that the elements of the semigroup S are contractions for the angles and then for the Riemannian distance, thus generalizing the result of Ph. Bougerol.

§2.   Preliminaries

Let Ω be an open convex cone in a Euclidean vector space V of dimension n. Let $G(\Omega)$ be the group of linear automorphisms of Ω

$$G(\Omega) = \{g \in GL(V) \mid g\Omega = \Omega\}.$$

Then Ω is said to be *homogeneous* if $G(\Omega)$ acts on it transitively. If $\overline{\Omega}$ is pointed, then Ω is said to be *symmetric* if it is homogeneous and self-dual.

A *Euclidean Jordan algebra* is a Euclidean vector space V equipped with a bilinear product such that

$$xy = yx,$$
$$x(x^2 y) = x^2(xy),$$
$$(xy|z) = (y|xz).$$

It is shown in [11] and [17] that the interior Ω of the set of squares in V is a symmetric cone, and every symmetric cone is given in this way.

We define the (left) multiplication L by $L(x)y = xy$ and the so-called *quadratic representation* P by $P(x) = 2L^2(x) - L(x^2)$. For any $x \in V$, the endomorphisms $L(x)$ and $P(x)$ are self-adjoint.

For example, $V = Sym(m, \mathbb{R})$ is a Jordan algebra for the product $x \circ y = (1/2)(xy + yx)$ and it is Euclidean for the scalar product $(x|y) = \mathrm{Tr}(xy)$. The symmetric cone is the cone Ω_{Sym} of positive definite symmetric matrices and $G(\Omega_{Sym})$ is the linear group $GL(m, \mathbb{R})$. In this case, $P(x)y = xyx$.

For any $x \in \Omega$, $P(x)$ is positive definite and then the bilinear forms

$$\mathrm{g}_x(u, v) = (P(x)^{-1}u|v), \quad x \in \Omega, \quad u, v \in V$$

define on Ω a structure of Riemannian symmetric space isomorphic to $G(\Omega)_\circ / K(\Omega)_\circ$ where $G(\Omega)_\circ$ is the identity component of $G(\Omega)$, $K(\Omega)_\circ = \{g \in G(\Omega)_\circ \mid ge = e\}$ and e the identity element of V.

Let r be the rank of V. A *Jordan frame* $\{c_1, \dots, c_r\}$ of V is a complete system of non-zero orthogonal primitive idempotents:

$$c_i^2 = c_i, \quad c_i \text{ indecomposable,}$$
$$c_i c_j = 0 \quad \text{if } i \neq j,$$
$$c_1 + \cdots + c_r = e.$$

Suppose V is simple. In other words, there is no non-trivial ideal in V. Then each element x in V can be written as

$$(2.1) \qquad x = k \sum_{j=1}^{r} \lambda_j c_j, \quad k \in K(\Omega)_\circ, \quad \lambda_j \in \mathbb{R}.$$

The *determinant* is defined by $\det(x) = \prod_{j=1}^{r} \lambda_j$ and the *trace form* by $\mathrm{tr}(x) = \sum_{j=1}^{r} \lambda_j$. The real numbers $\lambda_1, \dots, \lambda_r$ are the *eigenvalues* of x.

In the case of $V = Sym(m, \mathbb{R})$, (2.1) corresponds to the polar decomposition (diagonalization) of symmetric matrices. det and tr are the usual determinant Det and trace Tr of matrices.

If $\lambda_1, \ldots, \lambda_r$ are the eigenvalues of x, then the eigenvalues of $P(x)$ are $\eta_{jk} := \lambda_j \lambda_k$, where $j, k \in \{1, \ldots, r\}$. We will call the square roots of $\eta_{11}, \ldots, \eta_{rr}$ the *singular values* of $P(x)$. They are the absolute values of the eigenvalues of x.

Let T_Ω be the tube domain

$$T_\Omega = V + i\Omega = \{z = x + iy \mid x \in V, \ y \in \Omega\}.$$

It is a Hermitian symmetric space isomorphic to G/K where G is the group of holomorphic automorphisms of T_Ω and K is the stabilizer of ie in G.

In the case of $V = Sym(m, \mathbb{R})$, the tube domain is the Siegel upper half plane $T_{\Omega_{Sym}}$, the group G is the symplectic group $Sp(m, \mathbb{R})$ and K is the unitary group $U(m)$.

Let c be the Cayley transform $c : z \mapsto (e - z)(e + z)^{-1}$. Then it is shown in [9], that $G/cG(\Omega)_\circ c$ is an ordered symmetric space of Cayley type and any symmetric space of Cayley type is obtained in this way (see [6] for the definition). One proves (see [10]) that the corresponding causal semigroup is the semigroup of compressions of Ω:

$$S = \{\gamma \in G \mid g \cdot \Omega \subset \Omega\}$$

and it has the following decomposition where $S = S^+ G(\Omega)_\circ S^-$, S^+ and S^- are Abelian semigroups which are isomorphic to the cone $\overline{\Omega}$. Using this decomposition and infinitesimal arguments we also proved in [10] that the elements of S are contractions of the $G(\Omega)$-invariant Riemannian metric of Ω.

§3. The Characteristic Values and Cross-ratio

Let $x, y \in V$. A root of the *characteristic equation*

(3.1) $$\det(x - \lambda y) = 0$$

is called a *characteristic value* of the pair (x, y).

Let $w, x, y, z \in V$, and let $D(w, x, y, z)$ be the generalized *cross-ratio* studied by H. Braun [2, p. 26] given by

(3.2) $$D(w, x, y, z) := P(w - x)P(x - y)^{-1}P(y - z)P(z - w)^{-1}$$

when the appropriate inverses exist. This was first introduced by L. K. Hua [8, p. 452] for the symmetric, skew-symmetric and Hermitian matrices. See also Siegel [16, p. 3] and Maaß [13, p. 39] for symmetric matrices.

For $x, y \in \Omega$ we put

(3.3)
$$\rho(x, y) = D(x, y, -x, -y),$$
$$= P(x - y)P(x + y)^{-1}P(x - y)P(x + y)^{-1}.$$

Observe that $\rho(x, y)$ is well defined, since $x + y$ is invertible because $x + y \in \Omega$.

Proposition 3.1.   1. *For $x \in V$ and $y \in \Omega$, the characteristic equation $\det(x - \lambda y) = 0$ always has r real characteristic values $\lambda_1(x, y) \geq \cdots \geq \lambda_r(x, y)$.*

2. For $x, y \in \Omega$, the cross-ratio $\rho(x, y)$ always has real eigenvalues and its r singular values $\eta_1(x, y), \ldots, \eta_r(x, y)$ may be related to the characteristic values by

(3.4)
$$\eta_k(x, y) = \Big(\frac{1 - \lambda_k(x, y)}{1 + \lambda_k(x, y)}\Big)^2.$$

Proof.

1. By the Spectral Theorem ([4, Theorem III.1.2]) there exist $g \in G(\Omega)$, a Jordan frame $\{c_1, \ldots, c_r\}$ and r real numbers $\lambda_1(x, y) \geq, \ldots, \geq \lambda_r(x, y)$ such that $y = g \cdot e$ and $x = g \cdot \sum_{j=1}^r \lambda_j(x, y)c_j$. Then

$$\det(x - \lambda y) = \det\left(g \cdot \left(\sum_{j=1}^r \lambda_j(x, y)c_j - \lambda e \right) \right),$$
$$= \mathrm{Det}(g)^{\frac{r}{n}} \prod_{j=1}^r (\lambda_j(x, y) - \lambda).$$

   This proves that $\lambda_1(x, y), \ldots, \lambda_r(x, y)$ are the unique solutions of the characteristic equation since $\mathrm{Det}(g) \neq 0$. Moreover, these solutions are exactly the eigenvalues of $P(y^{-1/2})x$.

2. Using the fundamental formula

$$P(P(y)x) = P(y)P(x)P(y)$$

   ([4, Proposition II.3.3]) we obtain

$$\rho(x, y) = P(x - y)P(x + y)^{-1}P(x - y)P(x + y)^{-1}$$
$$= P(y^{\frac{1}{2}})P(z - e)P(z + e)^{-1}P(z - e)P(z + e)^{-1}P(y^{-\frac{1}{2}})$$

where $z = P(y^{-1/2})x$. But if u and v are in the associative sub-algebra $\mathbb{R}[z]$, then $P(u)P(v) = P(uv)$ ([4, the proof of Proposition II.2.2]). Thus

$$\rho(x, y) = P(y^{\frac{1}{2}})P((z - e)^2(z + e)^{-2})P(y^{-\frac{1}{2}}).$$

Since $\lambda_1(x, y), \dots, \lambda_r(x, y)$ are the unique eigenvalues of z, it follows from the last formula, that $((1 - \lambda_j(x, y))/(1 + \lambda_j(x, y)))^2((1 - \lambda_k(x, y))/(1 + \lambda_k(x, y)))^2$, $(j, k) \in \{1, \dots, r\}^2$, are the unique eigenvalues of the cross-ratio $\rho(x, y)$. Hence $((1 - \lambda_j(x, y))/(1 + \lambda_j(x, y)))^2$, $j \in \{1, \dots, r\}$ are the singular values of $\rho(x, y)$. $\qquad\square$

Example 3.1.   If $V = Sym(m, \mathbb{R})$, we saw that the quadratic representation is given by $P(x)y = xyx$. Then

$$\begin{aligned}
\rho(x, y)w &= P(x - y)P(x + y)^{-1}P(x - y)P(x + y)^{-1}w \\
&= q(x, y)wq(x, y)^t
\end{aligned}$$

where $q(x, y)$ is the cross-ratio

$$q(x, y) = (x - y)(x + y)^{-1}(x - y)(x + y)^{-1}$$

introduced in (1.3). Therefore the eigenvalues of $q(x, y)$ are the singular values $\xi_1(x, y), \dots, \xi_m(x, y)$ of $\rho(x, y)$ (see (1.4)).

§4.   A Generalized Min-Max Theorem

Let V be a simple Euclidean Jordan algebra of rank $r \geq 2$. We consider the scalar product on V

$$(x|y) = \operatorname{tr}(xy),$$

and denote by $\mathcal{J}(V)$ the set of primitive idempotents of V. An element c of V is said to be *almost primitive idempotent*, if $c \neq 0$ and

$$c^2 = (e|c)c.$$

Let $\mathcal{J}^a(V)$ be the set of the almost primitive idempotents of V. Observe that

$$\begin{aligned}
\mathcal{J}(V) &= \{c \in \mathcal{J}^a(V) \mid \operatorname{tr}(c) = 1\} \\
&= \left\{ \frac{c}{(e|c)} \mid c \in \mathcal{J}^a(V) \right\}
\end{aligned}$$

and that for a fixed $x \in V$, the maximum (resp. minimum) of the function $(x|c)$ over $\mathcal{J}(V)$ coincides with the maximum (resp. minimum) of the Rayleigh quotient $(x|c)/(e|c)$ over $\mathcal{J}^a(V)$.

We state, in a different way, the following theorem due to U. Hirzerbruch, see [7].

Theorem 4.1.   *For any x in V, the eigenvalues $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_r(x)$ of x, may be obtained as follows*:

$$\lambda_1(x) = \max_{c \in \mathcal{J}^a(V)} \frac{(x|c)}{(e|c)}, \quad \lambda_r(x) = \min_{c \in \mathcal{J}^a(V)} \frac{(x|c)}{(e|c)},$$

and for $2 \leq k \leq r-1$,

$$\lambda_k(x) = \min_{d_1,\dots,d_{k-1} \in \mathcal{J}^a(V)} \; \max_{c \in V^a(d_1,\dots,d_{k-1};0)} \frac{(x|c)}{(e|c)},$$

where $V^a(d_1,\dots,d_{k-1};0) = \{c \in \mathcal{J}^a(V) \mid cd_1 = \cdots = cd_{k-1} = 0\}$.

To prove a generalization of this theorem, we need the following

Lemma 4.1.   1. *$\mathcal{J}^a(V) = \{c \in \overline{\Omega} \cup -\overline{\Omega} \mid \mathrm{rank}(c) = 1\}$.*
2. *$G(\Omega)$ acts transitively on $\mathcal{J}^a(V)$.*
3. *Let $c, d \in \mathcal{J}^a(V)$. Then $(c|d) = 0$ if, and only if, $cd = 0$.*

Proof.   The assertions 1 and 2 follow from [4, Proposition IV.3.1].
The assertion 3 is a direct consequence of the following fact: If $x, y \in \overline{\Omega}$, then

$$(x|y) = 0 \iff xy = 0$$

see [5, Satz 4.1]. $\square$

Theorem 4.2.   *If $x \in V$ and $y \in \Omega$, then the characteristic values $\lambda_1(x,y) \geq \cdots \geq \lambda_r(x,y)$ of (x,y) are given by*:

$$\lambda_1(x,y) = \max_{c \in \mathcal{J}^a(V)} \frac{(x|c)}{(y|c)}, \quad \lambda_r(x,y) = \min_{c \in \mathcal{J}^a(V)} \frac{(x|c)}{(y|c)},$$

and for $2 \leq k \leq r-1$,

$$\lambda_k(x,y) = \min_{d_1,\dots,d_{k-1} \in \mathcal{J}^a(V)} \; \max_{c \in V^a(d_1,\dots,d_{k-1};0)} \frac{(x|c)}{(y|c)}.$$

Proof. There exists $g \in G(\Omega)$ such that $y = g \cdot e$ and $x = g \cdot \sum_{j=1}^{r} \lambda_j(x, y) c_j$ for some Jordan frame $\{c_1, \ldots, c_r\}$. Then

$$\max_{c \in \mathcal{J}^a(V)} \frac{(x|c)}{(y|c)} = \max_{c \in \mathcal{J}^a(V)} \frac{(g \cdot \sum_{i=1}^{r} \lambda_i(x, y) c_i | c)}{(g \cdot e | c)},$$

$$= \max_{c \in \mathcal{J}^a(V)} \frac{(\sum_{i=1}^{r} \lambda_i(x, y) c_i | g^t \cdot c)}{(e | g^t \cdot c)}.$$

Now, from Lemma 4.1 it follows that $g^t \cdot \mathcal{J}^a(V) = \mathcal{J}^a(V)$. Hence

$$\max_{c \in \mathcal{J}^a(V)} \frac{(x|c)}{(y|c)} = \max_{d \in \mathcal{J}^a(V)} \frac{(\sum_{i=1}^{r} \lambda_i(x, y) c_i | d)}{(e | d)},$$

$$= \lambda_1(x, y), \quad \text{by Theorem 4.1.}$$

We use the same proof to get the extremal property of $\lambda_r(x, y)$.

Let $2 \le k \le r - 1$ and define the quantity

$$\alpha_k(x, y) = \min_{d_1, \ldots, d_{k-1} \in \mathcal{J}^a(V)} \max_{cd_1 = \cdots = cd_{k-1} = 0} \frac{(x|c)}{(y|c)}.$$

Then

$$\alpha_k(x, y) = \min_{d_1, \ldots, d_{k-1} \in \mathcal{J}^a(V)} \max_{cd_1 = \cdots = cd_{k-1} = 0} \frac{(\sum_{j=1}^{n} \lambda_j(x, y) c_j | g^t \cdot c)}{(e | g^t \cdot c)}.$$

Setting $c^1 = g^t c$ and $d_j^1 = g^{-1} d_j$ this becomes using Lemma 4.1

$$\alpha_k(x, y) = \min_{d_1^1, \ldots, d_{k-1}^1 \in \mathcal{J}^a(V)} \max_{c^1 d_1^1 = \cdots = c^1 d_{k-1}^1 = 0} \frac{(\sum_{j=1}^{n} \lambda_j(x, y) c_j | c^1)}{(e | c^1)},$$

$$= \lambda_k(x, y), \quad \text{by Theorem 4.1.}$$

$\square$

The following corollary generalizes the monotone property by U. Hirzebruch [7].

Corollary 4.1. *Let $x, y \in \Omega$ and $v \in \overline{\Omega}$. For any $k \in \{1, \ldots, r\}$ we have*

(4.1) $$\lambda_k(x + v, y) \ge \lambda_k(x, y),$$

(4.2) $$\lambda_k(x, y + v) \le \lambda_k(x, y).$$

The inequalities (4.1) and (4.2) become strict when $v \in \Omega$.

§5.   The Riemannian Distance of Ω

The family of bilinear forms g_x given by,

$$(5.1) \qquad g_x(u,v) = (P(x)^{-1}u|v), \quad x \in \Omega, \quad u,v \in V,$$

defines a $G(\Omega)$-invariant Riemannian metric on Ω, see [4, Theorem III.5.3]. Therefore, Ω is a Riemannian symmetric space isomorphic to $G(\Omega)_\circ/K(\Omega)_\circ$.

Theorem 5.1.   *Let $x, y \in \Omega$. Then there exists a unique curve of shortest length joining x and y. The length of this curve is given by*

$$(5.2) \qquad \delta(x,y) = \left(\sum_{k=1}^{r} \log^2 \lambda_k(x,y) \right)^{1/2},$$

where $\lambda_1(x,y), \ldots, \lambda_r(x,y)$ are the characteristic values of (x,y).
    $\delta(x,y)$ is the Riemannian distance of x and y.

Proof.   Following the proof of Theorem 4.2 we may assume without loss of generality that $x = \sum_{i=1}^{r} \lambda_i(x,y)c_i$ and $y = e$.

The unique curve of shortest length joining e and x is $\gamma(t) = \exp(t \log x)$ where $\log x = \sum_{i=1}^{r} \log \lambda_i(x,y)c_i$. Thus the Riemannian distance between x and e is

$$\delta(x,e) = \int_0^1 \left\{ g_{\gamma(t)}\big(\dot{\gamma}(t), \dot{\gamma}(t)\big) \right\}^{\frac{1}{2}} dt,$$

$$= \int_0^1 (P(\gamma(t)^{-\frac{1}{2}})\dot{\gamma}(t) | P(\gamma(t)^{-\frac{1}{2}})\dot{\gamma}(t))^{\frac{1}{2}} dt,$$

$$= \int_0^1 (P(\gamma(t)^{-\frac{1}{2}})\{\gamma(t)\log x\} | P(\gamma(t)^{-\frac{1}{2}})\{\gamma(t)\log x\})^{\frac{1}{2}} dt.$$

Since $\log x, \gamma(t)$ and $\gamma(t)^{-1/2}$ are in the associative sub-algebra $\oplus_{k=1}^{r}\mathbb{R}c_k$, this implies that

$$\delta(x,e) = \int_0^1 (\{P(\gamma(t)^{-\frac{1}{2}})\gamma(t)\}\log x | \{P(\gamma(t)^{-\frac{1}{2}})\gamma(t)\}\log x)^{\frac{1}{2}} dt,$$

$$= \int_0^1 (\log x | \log x)^{\frac{1}{2}} dt,$$

$$= \int_0^1 \left(\sum_{k=1}^{r} \log \lambda_k(x,y)c_k | \sum_{k=1}^{r} \log \lambda_k(x,y)c_k \right)^{\frac{1}{2}} dt,$$

$$= \left(\sum_{k=1}^{r} \log^2 \lambda_k(x,y) \right)^{\frac{1}{2}}.$$

$\square$

Remark. δ is the Riemannian distance on Ω associated with the involution $x \mapsto x^{-1}$. The main property of this distance is its invariance by $G(\Omega)$.

For the symmetric cone of positive definite symmetric matrices and for the Siegel upper half plane this distance formula was obtained by C. L. Siegel [16, Theorem 3], see also Maaß [13, pp. 27, 39] and Neretin [15, Thoerem 6.3.5].

Corollary 5.1.   *Let $x, y \in \Omega$. Then the Riemannian distance of x and y may be given by*

$$(5.3) \qquad \delta(x,y) = \left(\sum_{k=1}^{r} \log^2 \frac{1 + \sqrt{\eta_k(x,y)}}{1 - \sqrt{\eta_k(x,y)}} \right)^{1/2} ,$$

where $\eta_1(x,y), \dots, \eta_r(x,y)$ are the singular values of the cross-ratio $\rho(x,y)$.

Proof.   This is a direct consequence of Theorem 5.1 and Proposition 3.1. $\qquad\square$

§6.   Compression Semigroup

Let G be the conformal group of the Jordan algebra V. It is the group of all holomorphic automorphisms of the tube domain $T_\Omega = V + i\Omega$. The group G is generated by the affine group P of transformations

$$z \mapsto gz + v \quad (g \in G(\Omega),\ v \in V),$$

and by the symmetry,

$$z \mapsto -z^{-1}.$$

The homogeneous space G/P is compact, and V can be embedded into G/P

$$V \to G/P$$
$$v \mapsto gP$$

where $g(z) = -z^{-1} + v$. G/P is the conformal compactification of V.

In the action of G on G/P, we introduced (see [10]) the compression semigroup S of Ω:

$$(6.1) \qquad S = \{\gamma \in G \mid \gamma \cdot \Omega \subset \Omega\}.$$

For $v \in \overline{\Omega}$, the translation

$$\gamma_v^+(z) = z + v$$

is a holomorphic automorphism of T_Ω and the semigroup of all real translations γ_v^+ is an Abelian sub-semigroup S^+ of S isomorphic to $\overline{\Omega}$.

Put $j : z \mapsto z^{-1}$. This is an isometry of Ω. Let $S^- = j \circ S^+ \circ j$, be the semigroup of the maps

$$\gamma_v^-(z) = (z^{-1} + v)^{-1}, \quad v \in \overline{\Omega}.$$

It is an Abelian sub-semigroup of S isomorphic to $\overline{\Omega}$.

Theorem 6.1 ([10]).   *The sub-semigroups S^+ and S^-, together with the subgroup $G(\Omega)_\circ$, generate S. More precisely, one has the following decomposition*

$$(6.2) \qquad\qquad\qquad S = S^+ G(\Omega)_\circ S^-.$$

If $\gamma = \gamma_u^+ g \gamma_v^- \in S$, then we write

$$(6.3) \qquad\qquad n^+(\gamma) := u, \quad A(\gamma) := g \quad \text{and} \quad n^-(\gamma) := v.$$

Remark.   We have also the other side decomposition

$$S = S^- G(\Omega)_\circ S^+.$$

§7.   Contraction Semigroup

It is shown in [10], by using infinitesimal arguments, that the elements of S are contractions of the $G(\Omega)$-invariant Riemannian metric (5.1) on Ω.

The aim of this section is to study the monotone behaviour of the *angles*:

$$(7.1) \qquad\qquad\qquad \mu_k(x, y) := \log^2 \lambda_k(x, y),$$

and prove refined contraction properties of the Riemannian distance (5.2) of Ω.

Recall that for $x \in V$, the number $\sup\{(x|c) \mid c \in \mathcal{J}(V)\}$, which is the largest eigenvalue of x (see Theorem 4.1), defines the (*operator*) *norm* $|x|$ of x.

Lemma 7.1.   *Let $x, y \in V$.*

1. *If $x - y \in \Omega$, then $|x| \geq |y|$.*

2. *If $x, y \in \Omega$, then $x - y \in \Omega$ if, and only if, $y^{-1} - x^{-1} \in \Omega$.*

Proof.   Observe that every element in Ω is invertible.

1. If $x - y \in \Omega$, then for any $c \in \mathcal{J}(V)$, $(x - y|c) > 0$ and thus $|x| \geq |y|$.

2. Let $x, y \in \Omega$, then there exist $g \in G(\Omega)$, a Jordan frame $\{c_1, \ldots, c_r\}$ and r real numbers $\lambda_j > 0$ such that $x = g^t \cdot e$ and $y = g^t \cdot \sum_{j=1}^{r} \lambda_j c_j$. Since $x - y \in \Omega$ we have $1 > \lambda_j$. Furthermore,

$$y^{-1} - x^{-1} = g^{-1} \cdot \sum_{j=1}^{r} \frac{1}{\lambda_j} c_j - g^{-1} \cdot e,$$

$$= g^{-1} \cdot \left(\sum_{j=1}^{r} \left(\frac{1}{\lambda_j} - 1 \right) c_j \right) \in \Omega.$$

Conversely, if $y^{-1} - x^{-1} \in \Omega$, then $x - y \in \Omega$. □

If $V = \mathbb{R}$ then $\Omega = \mathbb{R}_+^*$ and, for any $x, y \in \Omega$, the equation $x - \lambda y = 0$ has only one solution $\lambda = x/y$ (recall $r = 1$ in this case). Consider the distance between x and y given by $\delta(x, y) = |\log(x/y)| = |\log x - \log y|$. Let $r \geq 0$ (i.e. $r \in \overline{\Omega}$) and compare $\lambda(x + r, y + r)$ to $\lambda(x, y)$. By Rolle's theorem, there exists c, between x and y, such that

$$\frac{\log(x + r) - \log(y + r)}{\log x - \log y} = \frac{c}{c + r}.$$

Therefore, if $0 < x, y \leq m$,

(7.2)
$$\log \frac{x + r}{y + r} \leq \frac{m}{m + r} \log^+ \frac{x}{y}$$

where $\log^+ s = \max\{\log s, 0\}$. So that,

$$\log^2 \lambda(x + r, y + r) \leq \left(\frac{m}{m + r} \right)^2 \log^2 \lambda(x, y),$$

and

$$\delta(x + r, y + r) \leq \frac{m}{m + r} \delta(x, y).$$

This example will be used to prove following fundamental proposition:

Proposition 7.1. Let $x, y \in \Omega$ and $v \in \overline{\Omega}$. Then for any $k \in \{1, \ldots, r\}$,

(7.3)
$$|\log \lambda_k(x + v, y + v)| \leq \frac{\alpha}{\alpha + \beta} |\log \lambda_k(x, y)|,$$

where $\alpha = \max\{|x|, |y|\}$ and $\beta = \inf\{(v|c), \ c \in \mathcal{J}(V)\}$.

Proof.   In proving the inequality (7.3) we may assume $\lambda_k(x+v, y+v) > 1$. Indeed, $\lambda_k(x + v, y + v) = 1/\lambda_{r-k+1}(y + v, x + v)$, moreover (7.3) is obvious when $\lambda_k(x + v, y + v) = 1$.

Let $\lambda_k(x + v, y + v) > 1$. Then we have for any $2 \leq k \leq r - 1$

$$
\begin{aligned}
0 &< |\log \lambda_k(x + v, y + v)| \\
&= \log \lambda_k(x + v, y + v) \\
&= \log \min \left\{ \max \left\{ \frac{(x + v|c)}{(y + v|c)} \cdots \right\}, \cdots \right\}, \text{ by Theorem 4.2} \\
&= \min \left\{ \max \left\{ \log \frac{(x|c) + (v|c)}{(y|c) + (v|c)} \cdots \right\}, \cdots \right\} \\
&\leq \frac{\alpha}{\alpha + \beta} \min \left\{ \max \left\{ \log^+ \frac{(x|c)}{(y|c)} \cdots \right\}, \cdots \right\}, \text{ by (7.2)} \\
&= \frac{\alpha}{\alpha + \beta} \log^+ \lambda_k(x, y) \\
&= \frac{\alpha}{\alpha + \beta} |\log \lambda_k(x, y)|.
\end{aligned}
$$

To prove the inequatily (7.3) for λ_1 and λ_r we use the same arguments. $\square$

Remark.   If $v \in \overline{\Omega}$, then $\beta = \inf\{(v|c), \ c \in \mathcal{J}(V)\}$ may be equal to zero, and

$$|\log \lambda_k(x + v, y + v)| \leq |\log \lambda_k(x, y)|.$$

If $v \in \Omega$, then v is invertible and β, the lowest eigenvalue of v, does not vanish. Hence

$$|\log \lambda_k(x + v, y + v)| < |\log \lambda_k(x, y)|.$$

Using the notations (6.3), we set

$$S_1 = \{\gamma \in S \mid n^+(\gamma) \in \Omega\},$$

and

$$S_2 = \{\gamma \in S \mid n^-(\gamma) \in \Omega\}.$$

We state now the main theorem of this section.

Theorem 7.1.   *Let $k \in \{1, \ldots, r\}$. The following holds*:

1. *For any $\gamma \in S$ and for any $x, y \in \Omega : \mu_k(\gamma \cdot x, \gamma \cdot y) \leq \mu_k(x, y)$.*

2. *For any $\gamma \in S_1 \cup S_2$ and for any $x, y \in \Omega : \mu_k(\gamma \cdot x, \gamma \cdot y) < \mu_k(x, y)$.*

3. *For any $\gamma \in S_1 \cap S_2$, there exists $\kappa(\gamma)$, $0 < \kappa(\gamma) < 1$, such that for any $x, y \in \Omega : \mu_k(\gamma \cdot x, \gamma \cdot y) \leq \kappa(\gamma)\mu_k(x, y)$.*

Proof. Let $\gamma \in S$, then by Theorem 6.1,

$$\gamma = \gamma_u^+ g\gamma_v^-,$$
$$= \gamma_u^+ gj\gamma_v^+ j.$$

Here $u, v \in \overline{\Omega}$, $g \in G(\Omega)_\circ$ and j is the involution $j(w) = w^{-1}$ ($w \in \Omega$).

Recall that the elements of $G(\Omega)_\circ$ are isometries of μ_k. Moreover, the Hua identity [4, Lemma X.4.4]:

$$\det(z^{-1} - w^{-1}) = (\det z)^{-1}(\det w)^{-1}\det(w - z)$$

allows one to get $\lambda_k(x^{-1}, y^{-1}) = 1/\lambda_k(x, y)$. Therefore, j is also an isometry of μ_k. Hence, according to Proposition 7.1 and Remark 7, the first point of the theorem is proved.

Now if u or v is in the cone Ω, then by considering again Remark 7, the second point of the theorem follows.

It remains to prove the last point of the theorem. Let $\gamma = \gamma_u^+ g\gamma_v^- = \gamma_u^+ gj\gamma_v^+ j \in S_1 \cap S_2$, then $u, v \in \Omega$. Since j is an isometry of μ_k, we have

$$\mu_k(\gamma \cdot x, \gamma \cdot y) = \mu_k((\gamma_u^+ gj\gamma_v^+) \cdot x, (\gamma_u^+ gj\gamma_v^+) \cdot y),$$
$$= \mu_k((gj\gamma_v^+) \cdot x + u, (gj\gamma_v^+) \cdot y + u).$$

According to Lemma 7.1, we obtain

$$g \cdot v^{-1} - (gj\gamma_v^+) \cdot x = g \cdot v^{-1} - g \cdot (v + x)^{-1} \in \Omega,$$

and

$$|(gj\gamma_v^+) \cdot x| \leq |g \cdot v^{-1}|, \quad \text{for any } x \in \Omega.$$

Let $\zeta = |g \cdot v^{-1}|$, $\eta = \inf\{(u|c) \mid c \in \mathcal{J}(V)\}$, and put $\kappa(\gamma) = (\zeta/(\zeta + \eta))^2$. Then, it follows from Proposition 7.1 that

$$\mu_k(\gamma \cdot x, \gamma \cdot y) \leq \kappa(\gamma)\mu_k((gj\gamma_v^+) \cdot x, (gj\gamma_v^+) \cdot y),$$
$$\leq \kappa(\gamma)\mu_k(x, y).$$

So that the theorem is proved. □

As an easy consequence, Theorem 7.1 implies that the elements of the semigroup S are contractions of the distance δ. More precisely we have

Corollary 7.1.   *The following holds*:

1. *For any $\gamma \in S$, and $x, y \in \Omega : \delta(\gamma \cdot x, \gamma \cdot y) \leq \delta(x, y)$.*

2. *For any $\gamma \in S_1 \cup S_2$ and $x, y \in \Omega : \delta(\gamma \cdot x, \gamma \cdot y) < \delta(x, y)$.*

3. *For any $\gamma \in S_1 \cap S_2$, there exists $\kappa(\gamma)$, $0 < \kappa(\gamma) < 1$, such that, for all $x, y \in \Omega : \delta(\gamma \cdot x, \gamma \cdot y) \leq \kappa(\gamma)\, \delta(x, y)$.*

§8.   Final Remarks

We refer to [6] for the following definitions. Let $\mathcal{M} = G/H$ be a compactly causal symmetric space. Then $\mathcal{D} = G/K$ is a Hermitian symmetric domain. We can view $H/H \cap K$ as the real points of $\mathcal{D}$, i.e. fixepoint set of a conjugation on $\mathcal{D}$. For example, when $\mathcal{M}$ is a causal symmetric space of Cayley type, then $H/H \cap K$ is isomorphic to a symmetric cone Ω and $\mathcal{D}$ isomorphic to the tube domain $V + i\Omega$.

It is easy to see that the Siegel distance formula (5.3) extends to $V + i\Omega$, namely

$$\delta(z, w) = \left(\sum_{j=1}^{r} \log^2 \frac{1 + \sqrt{\eta_j(z,w)}}{1 - \sqrt{\eta_j(z,w)}} \right)^2$$

where $\eta_1(z, w), \dots, \eta_r(z, w)$ are the singular values of the cross-ratio

$$D(z, w, \bar{z}, \bar{w}) = P(z - w)P(w - \bar{z})^{-1}P(\bar{z} - \bar{w})P(\bar{w} - z)^{-1}.$$

J.-L. Clerc proved that the elements of the Olshanskiĭ holomorphic semigroup $\Gamma = \{\gamma \in G_{\mathbb{C}} \mid \gamma \mathcal{D} \subset \mathcal{D}\}$ are contractions of $\mathcal{D}$ for the Bergman metric, see [3].

We conjecture that the Olshanskiĭ holomorphic semigroup decreases the angles

$$\mu_j(z, w) := \log^2 \frac{1 + \sqrt{\eta_j(z,w)}}{1 - \sqrt{\eta_j(z,w)}}.$$

This was proved, in the particular cases of $G = Sp(n, \mathbb{R})$ and $G = U(p, q)$ by Neretin, see [14] and [15].

We also conjecture that the Siegel distance formula and the contractions properties, with respect to the associated compression semigroup, hold for the real bounded domain $H/H \cap K$.

Many applications of these results could be expected. For example to modular forms, filtering theory, control theory, Riccati equation theory, etc...

After this paper was finished, the author noted that Y. Lim [12] proved independently that the elements of the semigroup S are contractions of the largest angle μ_1.

Acknowledgement

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