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On a Sharp Levi Condition in Gevrey Classes for Some Infinitely Degenerate Hyperbolic Equations and Its Necessity

By

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§1. Introduction

In this article we are concerned with a sharp Levi condition associated with the Cauchy problem on the strip $[0,T]\times\mathbb{R}^n$ $(T>0)$ for linear weakly hyperbolic equations of second order with time dependent coefficients:

$$\text{(CP)}\qquad \begin{cases} L(t,\partial_t,\partial_x)u(t,x)=f(t,x), & (t,x)\in[0,T]\times\mathbb{R}^n,\\ u(t_0,x)=u_0(x),\ \partial_t u(t_0,x)=u_1(x), & x\in\mathbb{R}^n,\end{cases}$$

where $t_0\in[0,T)$,

$$L(t,\partial_t,\partial_x)=\partial_t^2-a_2(t,\partial_x)-a_1(t,\partial_x),$$
$$a_2(t,\partial_x)=\sum_{j,k=1}^{n}a_{jk}(t)\partial_{x_jx_k},$$
$$a_1(t,\partial_x)=\sum_{j=1}^{n}a_j(t)\partial_{x_j}$$

together with $a_{jk}(t)\in C^1([0,T])$ and $a_j(t)\in C^1([0,T])$. Here we prepare some weight functions to describe our assumptions on the coefficients of a_1 and a_2. Let $\lambda(t)\in C^1([0,T])$ be a real-valued function such that $\lambda(0)=\lambda'(0)=0$ and $\lambda'(t)>0$ if $0<t\le T$. Moreover, suppose that for $0<t\le T$

$$c_0\frac{\lambda(t)}{\Lambda(t)}\le\frac{\lambda'(t)}{\lambda(t)}\le c_1\frac{\lambda(t)}{\Lambda(t)}\qquad(1.1)$$

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with some constants $c_0 > s/(2s-2)$ ($s > 2$ fixed) and $c_1 \ge c_0$ when we put $\Lambda(t) = \int_0^t \lambda(\tau)\, d\tau$.

Now we can state our hypotheses on a_1 and a_2 as below:

$$\tag{1.2} \begin{cases} d_0\lambda(t)^2|\xi|^2 \le a_2(t,\xi) \le d_2\lambda(t)^2|\xi|^2 & ((t,\xi) \in [0,T] \times \mathbb{R}^n_\xi), \\ |\partial_t a_2(t,\xi)| \le d_2'\lambda(t)^3\Lambda(t)^{-s/(s-1)}|\xi|^2 & ((t,\xi) \in (0,T] \times \mathbb{R}^n_\xi), \end{cases}$$

$$\tag{1.3} \max_{j=1,\dots,n} |\partial_t^k a_j(t)| \le d_1\lambda(t)^{k+2}\Lambda(t)^{-s(k+1)/(s-1)} \quad (k = 0, 1,\ 0 < t \le T),$$

where d_0, d_1 and d_2 are positive constants.

To begin with, we define the Gevrey space with exponent s (> 1)

$$L^2_s(\mathbb{R}^n) = \bigcap_{\rho>0} L^2_{s,\rho}(\mathbb{R}^n)$$

is a Fréchet space equipped with the family of the countable norms

$$\|u\|^{(\ell)}_{L^2_s(\mathbb{R}^n)} = \|u\|_{L^2_{s,\ell}(\mathbb{R}^n)} \quad (\ell = 1, 2, \cdots),$$

where

$$L^2_{s,\rho}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n); \exp(\rho\, \langle D_x\rangle^{\frac{1}{s}})u(x) \in L^2(\mathbb{R}^n_x)\}$$

is a Banach space endowed with its norm

$$\|u\|_{L^2_{s,\rho}(\mathbb{R}^n)} = \|\exp(\rho\, \langle D_x\rangle^{\frac{1}{s}})u(x)\|_{L^2(\mathbb{R}^n_x)}.$$

Here, the pseudo-differential operator $\exp(\rho\, \langle D\rangle^{1/s}) : L^2_{s,\rho}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is defined by

$$\exp(\rho\, \langle D_x\rangle^{\frac{1}{s}})u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{\sqrt{-1}(x-y)\cdot\xi + \rho\, \langle\xi\rangle^{\frac{1}{s}}} u(y)\, dy \right) d\xi,$$

while $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$. For more details of basic properties of the operator $\exp(\rho\, \langle D\rangle^{1/s})$, see Section 6 of Part I in [7].

Definition 1.1.

(i) We say that the Cauchy problem (CP) is well-posed in $L^2_s(\mathbb{R}^n)$ if for any u_0, $u_1 \in L^2_s(\mathbb{R}^n)$ and $f(t,x) \in C([0,T]; L^2_s(\mathbb{R}^n))$ there exists a unique solution $u(t,x) \in C^2([0,T]; L^2_s(\mathbb{R}^n))$ to (CP) such that for any $\rho > 0$ there is some $\rho' > 0$ satisfying the a priori estimate (1.4)

$$\begin{aligned} &\|\exp(\rho\, \langle D_x\rangle^{\frac{1}{s}})u(t,x)\|_{L^2(\mathbb{R}^n_x)} + \|\exp(\rho\, \langle D_x\rangle^{\frac{1}{s}})u_t(t,x)\|_{L^2(\mathbb{R}^n_x)} \\ &\quad \le C(T,\rho)\Big(\|\exp(\rho'\, \langle D_x\rangle^{\frac{1}{s}})u_0(x)\|_{L^2(\mathbb{R}^n_x)} + \|\exp(\rho'\, \langle D_x\rangle^{\frac{1}{s}})u_1(x)\|_{L^2(\mathbb{R}^n_x)} \\ &\qquad + \left| \int_{t_0}^t \|\exp(\rho'\, \langle D_x\rangle^{\frac{1}{s}})f(\tau,x)\|_{L^2(\mathbb{R}^n_x)}\, d\tau \right| \Big). \end{aligned}$$

(ii) Let $K_\gamma(t^0, x^0)$ denote the backward cone with vertex (t^0, x^0) and slope $\gamma > 0$:

$$K_\gamma(t^0, x^0) = \{(t, x) \in [0, T] \times \mathbb{R}^n; |x - x^0| \le \gamma(t^0 - t)\}.$$

The Cauchy problem (CP) possesses the finite propagation speed property if for every u with $u(t, x) \in C^2([0, T]; L^2_s(\mathbb{R}^n))$

$$(Lu)|_{K_\gamma(t^0,x^0)} = 0, \quad \partial_t^j u|_{K_\gamma(t^0,x^0)\cap\{t=t_0\}} = 0 \quad (j = 0, 1)$$

imply

$$u|_{K_\gamma(t^0,x^0)} = 0.$$

Then we have the following result on the Gevrey well-posedness of the Cauchy problem (CP).

Theorem 1.1. *If the conditions* (1.1), (1.2) *and* (1.3) *are satisfied for some* s (> 2) *and* T (> 0), *then the Cauchy problem* (CP) *is well-posed in* $L^2_s(\mathbb{R}^n)$. *Moreover,* (CP) *possesses the finite propagation speed property with speed*

$$\gamma \ge \max\left\{\sqrt{a_2(t, \xi)}\ ;\ t \in [0, T],\, \xi \in \mathbb{R}^n,\, |\xi| = 1\right\}.$$

However, we may say for the equations with C^∞-coefficients that Theorem 1.1 can be derived from results of [11]. Thus we are rather interested in the necessity of the Levi condition (1.3) in $L^2_s(\mathbb{R}^n)$.

Example 1.1.

(a) (finitely degenerate case) $\lambda(t) = t^\ell$ $(1 < \ell \in \mathbb{N})$.

(b) (infinitely degenerate case) Let $r > 0$. The function

$$\lambda(t) = \begin{cases} rt^{-r-1}\exp(-t^{-r}) & \text{if} \quad t > 0, \\ 0 & \text{if} \quad t = 0 \end{cases}$$

satisfies the condition (1.1) for small $t \in (0, (r/(r+1))^{1/r})$. Indeed, note that

$$\frac{\lambda(t)}{\Lambda(t)} = rt^{-r-1}, \quad \frac{\lambda'(t)}{\lambda(t)} = (r+1)t^{-r-1}\left(\frac{r}{r+1} - t^r\right).$$

Example 1.2. In [10] is actually investigated the case of $\lambda(t) = \exp(-t^{-1})$, $a_1(t, \xi) = -\sqrt{-1}\,Bt^\alpha \exp(-\beta t^{-1})\xi$ and $a_2(t, \xi) = \lambda(t)^2\xi^2$ with $B \in \mathbb{C}\setminus\{0\}$, $\alpha \in \mathbb{R}$ and $\beta \le 1$. Then he proved that (CP) is well-posed in $L^2_s(\mathbb{R})$

either for $s < (2-\beta)/(1-\beta)$ if $\beta < 1$ or for any s (> 1) if $\beta = 1$. We should remark that for the second-order equations his result is also deduced from Theorem 1.1 because $\lambda(t)^2\Lambda(t)^{-s/(s-1)} = O(t^{2s/(1-s)}\exp(-(s-2)/(s-1)t^{-1}))$ as $t \to 0$ in this case.

Historically, V. Ja. Ivrii showed in [4] that the Cauchy problem in one space dimension for a finitely degenerate hyperbolic operator

$$L_0(t,\partial_t,\partial_x) = \partial_t^2 - t^{2\ell}\partial_x^2 - \sqrt{-1}\,t^m\partial_x$$

with $0 \le m < \ell - 1$ is well-posed in a Gevrey class of order s if and only if $1 \le s < (2\ell-m)/(\ell-m-1) = \sigma$. After him, in [9] K. Shinkai and K. Taniguchi treated the degenerate hyperbolic operator on $[0,T]\times\mathbb{R}^n$

$$L_1(t,\partial_t,\partial_x) = \partial_t^2 - t^{2\ell}\sum_{j,k=1}^{n} c_{jk}(t)\partial_{x_jx_k} - \sqrt{-1}\,t^m\sum_{j=1}^{n} c_j(t)\partial_{x_j},$$

where $0 \le m < \ell - 1$, the coefficients $c_{jk}(t)$, $c_j(t)$ are analytic and there exists a positive constants C satisfying

$$\sum_{j,k=1}^{n} c_{jk}(t)\xi_j\xi_k \ge C|\xi|^2$$

for all $(t,\xi) \in [0,T]\times\mathbb{R}^n$. They proved the well-posedness for $L_1(t,\partial_t,\partial_x)$ in a Gevrey class of order s provided ($2 \le$) $s < \sigma$, which gives a generalization of Ivrii's result on the sufficient part to every space dimension n. Meanwhile, we know that the condition $s < \sigma$ is not necessary for $n \ge 2$ (see [1]).

In contrast to the finitely degenerate case, there are not so many results on the Gevrey well-posedness for infinitely degenerate hyperbolic operators. As a result, K. Kajitani proposed in [6] a quite general Levi condition

$$\text{(KC)}\qquad \int_0^T \frac{|a_1(t,\xi)|}{\sqrt{a_2(t,\xi)+1}}\,dt \le C(T)\,\langle\xi\rangle^{\frac{1}{s}}$$

for all $\xi \in \mathbb{R}^n$. He demonstrated that the Cauchy problem for $L(t,\partial_t,\partial_x)$ with $a_{jk}(t) \in C^\infty([0,T])$ is well-posed in a Gevrey class of order s if (KC) is satisfied for some s (> 1). In our case, the condition (1.5) in [3] with $\lambda(t,\xi) = \lambda(t)^2\Lambda(t)^{-s/(s-1)}|\xi|$, which is a generalization of (KC), corresponds to our conditions (1.2) and (1.3) because

$$\int_0^{t_\xi} \lambda(t)\Lambda(t)^{-\frac{s}{2(s-1)}}|\xi|^{\frac{1}{2}}\,dt + \int_{t_\xi}^{T} \frac{\lambda(t)}{\Lambda(t)^{\frac{s}{s-1}}}\,dt \le C_{T,s}\,\langle\xi\rangle^{\frac{1}{s}}$$

for all $\xi \in \mathbb{R}^n$, will be verified in Section 2.

Next we shall examine the necessity of the Levi condition (1.3). To do so, it is convenient to introduce the real-valued function $\nu(t) \in C^1((0,T])$ fulfilling the conditions below:

$$\begin{aligned} &\nu(t) > 0 \quad (0 < t \le T), \\ 0 < &\frac{\nu'(t)}{\nu(t)} \le C_\nu \frac{\lambda(t)}{\Lambda(t)} \quad \left(0 < C_\nu \le 2c_0 - \frac{s}{s-1}\right). \end{aligned} \tag{1.5}$$

The function $\nu(t) = \Lambda(t)^\varepsilon$ $(0 < \varepsilon \le 2c_0 - s/(s-1))$ is a typical example with $\lim_{t\to 0}\nu(t) = 0$. Here we notice that if

$$\frac{\nu'(t)}{\nu(t)} = \frac{s-2}{s-1}\frac{\lambda(t)}{\Lambda(t)},$$

then for $0 < t \le T$

$$\begin{aligned} \frac{1}{\nu(t)}\lambda(t)^2\Lambda(t)^{-s/(s-1)} &= \frac{\Lambda(T)^{(s-2)/(s-1)}}{\nu(T)}\left(\frac{\lambda(t)}{\Lambda(t)}\right)^2 \\ &\ge \frac{\Lambda(T)^{(s-2)/(s-1)}}{\nu(T)} t^{-2} \to \infty \quad \text{as} \quad t \to 0. \end{aligned}$$

Hence, in this case the coefficients become unbounded, while in the present article we are interested in operators with bounded coefficients only. Now, let us consider the case of

$$\begin{aligned} a_2(t,\xi) &= \lambda(t)^2|\xi|^2, \\ a_1(t,\xi) &= -\frac{\sqrt{-1}}{\nu(t)}\lambda(t)^2\Lambda(t)^{-s/(s-1)}\sum_{j=1}^n b_j(t)\xi_j, \end{aligned} \tag{1.6}$$

where $b_j(t) \in C([0,T]) \cap C^1((0,T])$ fulfilling $|\sum_{j=1}^n b_j(0)\xi_j| > 0$ for some fixed $\xi \in \mathbb{R}^n\backslash\{0\}$. Then we make the assumption

$$\left|\partial_t \sum_{j=1}^n b_j(t)\xi_j\right| \le d_b \frac{\lambda(t)}{\Lambda(t)} \tag{1.7}$$

for all $t \in (0,T]$. In this case, we obtain the following result on the necessity of (1.3) in which the well-posedness involves the finite propagation speed property in the sense of (ii).

Theorem 1.2. *Under the conditions* (1.1), (1.5) *and* (1.7), *the Cauchy problem* (CP) *for* (1.6) *is not well-posed in* $L^2_s(\mathbb{R}^n)$ *if* $\lim_{t\to 0}\nu(t) = 0$.

Remark 1.1. For C^∞-wellposedness for the equation of more general m-th order and with coefficients depending on not only t but also the space variables x, see Theorem 5.1.4 in [12]. We further know that there are many results on (not necessarily) linear equations in the C^∞-class (cf. [2], [12] and the bibliography therein). Finally, refer to [1] and [5] for some results in the finitely degenerate case (see also [3] and [6] for the Levi condition in terms of integrals).

§2. Sufficiency of the Levi Condition

In this section we shall give the proof of Theorem 1.1. If we apply the Fourier transform in x variables to (CP), then the problem is reduced to the Cauchy problem for the ordinary differential equation

$$\begin{cases} L(t, \partial_t, -i\xi)\hat{u}(t,\xi) = \hat{f}(t,\xi), \\ \hat{u}(t_0,\xi) = \hat{u}_0(\xi),\ \hat{u}_t(t_0,\xi) = \hat{u}_1(\xi) \end{cases}$$

with $\xi \in \mathbb{R}^n$ is regarded as a parameter. First of all, under the hypotheses on λ there exists a unique root t_ξ with respect to t of the following equation

$$\Lambda(t)^s \langle\xi\rangle^{s-1} = N^{s-1}$$

with a large parameter $N \geq 1$. It is easy to see that $t_\xi \to 0$ as $|\xi| \to \infty$. Along with two large parameters M and N, we may split the strip $[0,T] \times \mathbb{R}^n$ into the following two regions:

$$\begin{aligned} Z_{pd}(M,N,s) &= \{(t,\xi) \in [0,T] \times \mathbb{R}^n\ ;\ \Lambda(t)^s \langle\xi\rangle^{s-1} \leq N^{s-1},\ \langle\xi\rangle \geq M\}, \\ Z_{hyp}(M,N,s) &= \{(t,\xi) \in [0,T] \times \mathbb{R}^n\ ;\ \Lambda(t)^s \langle\xi\rangle^{s-1} \geq N^{s-1},\ \langle\xi\rangle \geq M\}, \end{aligned}$$

according to [11], [12], will be called a *pseudodifferential zone* and *hyperbolic zone* respectively. Our main task is to derive a priori estimates in Z_{pd} and Z_{hyp} respectively which ensure the well-posedness in the Gevrey space $L^2_s(\mathbb{R}^n)$. To do so, we shall employ some reduction to a “first-order diagonal system”.

Estimates in the Pseudodifferential Zone Z_{pd}

Let us consider the root $\rho = \rho(t,\xi) \geq 1$ in $Z_{pd}(M,N,s)$ of the quadratic equation

$$\rho^2 - 1 - \langle\xi\rangle\lambda(t)^2\Lambda(t)^{-\frac{s}{s-1}} = 0.$$

In advance, we can regard the equation as the first order system in the usual way:

$$D_t\mathcal{U}(t,\xi) = \mathcal{A}(t,\xi)\mathcal{U}(t,\xi) + \mathcal{F}(t,\xi),$$

where

$$\mathcal{A}(t,\xi) = \begin{pmatrix} 0 & 1 \\ a_2(t,\xi) - ia_1(t,\xi) & 0 \end{pmatrix},\ \mathcal{U} = \begin{pmatrix} \hat{u} \\ D_t\hat{u} \end{pmatrix},\ \mathcal{F} = \begin{pmatrix} 0 \\ \hat{f} \end{pmatrix},\ D_t = -i\partial_t.$$

Now, transforming $\mathcal{U}(t,\xi)$ into $U(t,\xi) = H(t,\xi)\mathcal{U}(t,\xi)$ with the nonsingular matrix $H(t,\xi) = \begin{pmatrix} \rho(t,\xi) & 0 \\ 0 & 1 \end{pmatrix}$, we have

$$\begin{aligned} D_tU(t,\xi) &= (D_tH)\mathcal{U} + HD_t\mathcal{U} \\ &= (D_tH)H^{-1}U + H\mathcal{A}H^{-1}U + \mathcal{F}. \end{aligned}$$

For simplicity, denote

$$I = H\mathcal{A}H^{-1} = \begin{pmatrix} 0 & \rho \\ \rho^{-1}(a_2 - ia_1) & 0 \end{pmatrix},$$

$$II = (D_tH)H^{-1} = -i\frac{\rho_t}{\rho}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$A = I + II = (A_{jk}) \quad \text{and} \quad U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}.$$

Then, standing for the energy function to the system $D_tU = AU + \mathcal{F}$ by

$$E(t,\xi) = \frac{1}{2}\left\{|U_1(t,\xi)|^2 + |U_2(t,\xi)|^2\right\}$$

and differentiating it in t, we get the equality

$$\begin{aligned} \frac{dE}{dt} &= \mathrm{Re}(U_{1t},\bar{U}_1) + \mathrm{Re}(U_{2t},\bar{U}_2) \\ &= \mathrm{Re}(A_{11}U_1 + A_{12}U_2,\bar{U}_1) + \mathrm{Re}(A_{21}U_1 + A_{22}U_2 + \hat{f},\bar{U}_2), \end{aligned}$$

so that

$$\begin{aligned} \left|\frac{dE}{dt}\right| &\leq |A_{11}||U_1|^2 + |A_{12}||U_2||U_1| + |A_{21}||U_1||U_2| + |A_{22}||U_2|^2 + |\hat{f}||U_2| \\ &\leq g(t,\xi)E(t,\xi) + |\hat{f}(t,\xi)|^2, \end{aligned}$$

where

$$g(t,\xi) = 2 \max_{j,k=1,\dots,n} |A_{jk}(t,\xi)|.$$

Hence, by Gronwall's inequality we obtain the energy inequality

$$E(t,\xi) \le \left(E(t_0,\xi) + \left| \int_{t_0}^{t} |\hat{f}(\tau,\xi)|^2 \, d\tau \right| \right) \exp\left(\left| \int_{t_0}^{t} g(\tau,\xi)\, d\tau \right| \right)$$

for $0 \le t_0$, $t \le t_\xi$. Here, since $\rho_t \ge 0$ in $Z_{pd}(M,N,s)$ from $c_0 > s/(2s-2)$ and

$$\rho(t,\xi) = \sqrt{1 + \langle\xi\rangle \lambda(t)^2 \Lambda(t)^{-\frac{s}{s-1}}},$$

it holds that

$$\begin{aligned}\int_0^t \|II\|\, d\tau &\le \int_0^t \frac{\rho_t}{\rho}\, d\tau = \log\rho(t,\xi) - \log\rho(0,\xi) \\ &\le \log\rho(t,\xi) \le \log\rho(t_\xi,\xi) \\ &= \frac{1}{2}\log\left(1 + \langle\xi\rangle\lambda(t_\xi)^2\Lambda(t_\xi)^{-\frac{s}{s-1}}\right).\end{aligned}$$

First we note that

$$\langle\xi\rangle\lambda(t_\xi)^2\Lambda(t_\xi)^{-\frac{s}{s-1}} = N\left(\frac{\lambda(t_\xi)}{\Lambda(t_\xi)^{\frac{s}{s-1}}}\right) \ge 1$$

according to (1.1) and $N \ge 1$. Therefore

$$\begin{aligned}\log\left(1 + \langle\xi\rangle\lambda(t_\xi)^2\Lambda(t_\xi)^{-\frac{s}{s-1}}\right) &\le \log\left(2\,\langle\xi\rangle\lambda(t_\xi)^2\Lambda(t_\xi)^{-\frac{s}{s-1}}\right) \\ &\le \log 2 + \log\langle\xi\rangle - \frac{s}{s-1}\log\Lambda(t_\xi).\end{aligned}$$

On the other hand, from the definition of t_ξ

$$-\frac{s}{s-1}\log\Lambda(t_\xi) = \log\langle\xi\rangle - \log N \le \log\langle\xi\rangle.$$

Thus for $M \ge 2$ we obtain

$$\begin{aligned}\int_0^t \|II\|\, d\tau &\le \log\left(1 + \langle\xi\rangle\lambda(t_\xi)^2\Lambda(t_\xi)^{-\frac{s}{s-1}}\right) \\ &\le \log 2 + 2\log\langle\xi\rangle \le 3\log\langle\xi\rangle.\end{aligned}$$

Next, let us evaluate $\left|\int_{t_0}^t \|I\|\, d\tau\right|$. To this end, it suffices to estimate $\left|\int_{t_0}^t \rho(\tau,\,\xi)\, d\tau\right|$ because

$$\left|\frac{a_2 - ia_1}{\rho}\right| \le O(1)\rho \quad \text{in} \quad Z_{pd}(M,N,s).$$

Indeed, on account of (1.2) and (1.3)

$$\begin{aligned}\left|\frac{a_2 - ia_1}{\rho}\right| &\le d_2\frac{\lambda^2|\xi|^2}{\rho} + nd_1\frac{\lambda^2\Lambda^{-\frac{s}{s-1}}|\xi|}{\rho},\\ \frac{\lambda^2|\xi|^2}{\rho^2} &= \frac{\lambda^2|\xi|^2}{1+\langle\xi\rangle\lambda^2\Lambda^{-\frac{s}{s-1}}} < |\xi|\Lambda(t)^{\frac{s}{s-1}}\\ &\le |\xi|\Lambda(t_\xi)^{\frac{s}{s-1}} = |\xi|\frac{N}{\langle\xi\rangle} < N,\\ \frac{\lambda^2\Lambda^{-\frac{s}{s-1}}|\xi|}{\rho^2} &= \frac{\lambda^2\Lambda^{-\frac{s}{s-1}}|\xi|}{1+\langle\xi\rangle\lambda^2\Lambda^{-\frac{s}{s-1}}} < 1.\end{aligned}$$

Then

$$\begin{aligned}\left|\int_{t_0}^{t}\rho(\tau,\xi)\,d\tau\right| &\le \int_0^{t_\xi}\sqrt{1+\langle\xi\rangle\lambda(\tau)^2\Lambda(\tau)^{-\frac{s}{s-1}}}\,d\tau\\ &\le T + \langle\xi\rangle^{\frac{1}{2}}\int_0^{t_\xi}\lambda(\tau)\Lambda(\tau)^{-\frac{s}{2(s-1)}}\,d\tau\\ &= T + \langle\xi\rangle^{\frac{1}{2}}\frac{2(s-1)}{s-2}\Lambda(t_\xi)^{\frac{s-2}{2(s-1)}}\\ &= T + C_{s,N}\,\langle\xi\rangle^{\frac{1}{s}}.\end{aligned}$$

Therefore we can deduce the inequality in $Z_{pd}(M,N,s)$

$$E(t,\xi) \le \left(E(t_0,\xi) + \left|\int_{t_0}^{t}|\hat{f}(\tau,\xi)|^2\,d\tau\right|\right)\exp\left(C\,\langle\xi\rangle^{\frac{1}{s}}\right)$$

for $0 \le t_0, t \le t_\xi$. By the way,

$$\|H(t,\xi)\| \le C\,\langle\xi\rangle^{\frac{1}{2}}, \quad \|H^{-1}(t,\xi)\| \le C.$$

Thus we conclude the desired estimate

$$\begin{aligned}&|\hat{u}(t,\xi)| + |\hat{u}_t(t,\xi)|\\ &\qquad \le C_{M,N}\exp\left(c_{M,N}\,\langle\xi\rangle^{\frac{1}{s}}\right)\left(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)| + \left|\int_{t_0}^{t}|\hat{f}(\tau,\xi)|\,d\tau\right|\right)\end{aligned}$$

for all t_0, $t \in [0,t_\xi]$, which implies (1.4) in $Z_{pd}(M,N,s)$.

Estimates in the Hyperbolic Zone Z_{hyp}

In this zone we shall adopt another regular matrix $H(t,\xi) = \begin{pmatrix} \lambda(t)|\xi| & 0 \\ 0 & 1 \end{pmatrix}$ instead of the previous one. Then

$$I = \begin{pmatrix} 0 & \lambda|\xi| \\ \frac{a_2 - ia_1}{\lambda|\xi|} & 0 \end{pmatrix},$$
$$II = \begin{pmatrix} -i\frac{\lambda_t}{\lambda} & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $\tau_j(t,\xi)$ $(j = 1, 2)$ be the characteristic roots associated to $L(t, \partial_t, \partial_x)$, that is, the ones of the quadratic equation with respect to τ:

$$L(t,\tau,\xi) = \tau^2 - a_2(t,\xi) + ia_1(t,\xi) = 0.$$

If we put $\tau = \lambda(t)|\xi|\mu$, then

$$\begin{aligned} L(t,\tau,\xi) &= (\tau - \tau_1(t,\xi))(\tau - \tau_2(t,\xi)) \\ &= \lambda(t)^2|\xi|^2(\mu - \mu_1(t,\xi))(\mu - \mu_2(t,\xi)) \\ &= \lambda(t)^2|\xi|^2 P(t,\xi;\mu). \end{aligned}$$

Further, by denoting

$$0 < d_0 \le \mu_0(t,\xi)^2 = \frac{a_2(t,\xi)}{\lambda(t)^2|\xi|^2} \ (\le d_2),$$
$$B(t,\xi) = \frac{-ia_1(t,\xi)}{\lambda(t)^2|\xi|^2},$$

it is represented as

$$\mu_j(t,\xi) = (-1)^j\mu_0(t,\xi) + \sum_{n=1}^{\infty} c_n^{(j)}(t,\xi)B(t,\xi)^n \qquad (j = 1, 2), \tag{2.1}$$

where

$$\begin{aligned} c_n^{(j)}(t,\xi) &= \frac{1}{2\pi i}\oint_{|z-(-1)^j\mu_0|=\varepsilon} \frac{(z - (-1)^j\mu_0(t,\xi))P_z(t,\xi;z)}{P(t,\xi;z)^{n+1}}\,dz \\ &= \frac{1}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}}\left\{\left(\frac{z-(-1)^j\mu_0(t,\xi)}{P(t,\xi;z)}\right)^{n+1} P_z(t,\xi;z)\right\}\right]_{z=(-1)^j\mu_0(t,\xi)} \end{aligned}$$

for $0 < \varepsilon < d_0$ (see Subsection 2.1.3 in [12]). Hence we have the inequalities

$$|c_n^{(j)}(t,\xi)| \le \frac{c}{2d_0}\left(\frac{c}{d_0\varepsilon}\right)^n \quad \text{for} \ \ (t,\xi) \in Z_{hyp}(M,N,s) \tag{2.2}$$

with the constant c independent of t and ξ. Thus the radius r of convergence of the series in (2.1) ($|B(t,\xi)| < r$) does not also depend on t and ξ while N is large enough. Next, noting by (1.3) that in $Z_{hyp}(M,N,s)$

$$|B(t,\xi)| \le \frac{d_1}{\Lambda(t)^{\frac{s}{s-1}}|\xi|} \le \frac{d_1}{\Lambda(t_\xi)^{\frac{s}{s-1}}|\xi|} = \frac{d_1}{N}, \tag{2.3}$$

we can see that

$$\begin{aligned} |\operatorname{Im}\mu_j(t,\xi)| &\le |c_1^{(j)}B| + |B|^2\sum_{n=2}^{\infty}|c_n^{(j)}||B|^{n-2} \\ &\le |c_1^{(j)}B| + C|B| \\ &\le \frac{C(d_1)}{\Lambda(t)^{\frac{s}{s-1}}|\xi|} \end{aligned}$$

provided N is sufficiently large. Consequently

$$|\operatorname{Im}\tau_j(t,\xi)| \le C(d_1)\lambda(t)\Lambda(t)^{-\frac{s}{s-1}} \quad \text{for} \;\; t_\xi \le t \le T.$$

So, it follows from the above inequality that

$$\begin{aligned} \int_{t_\xi}^{T} |\operatorname{Im}\tau_j(t,\xi)|\,dt &\le C\int_{t_\xi}^{T} \lambda(t)\Lambda(t)^{-\frac{s}{s-1}}\,dt \\ &\le C(s-1)\Lambda(t_\xi)^{-\frac{1}{s-1}} \\ &= C_{s,N}\,\langle\xi\rangle^{\frac{1}{s}} \end{aligned}$$

in $Z_{hyp}(M,N,s)$. Moreover, (2.1), (2.2) and (2.3) give the inequality

$$|\tau_1(t,\xi) - \tau_2(t,\xi)| \ge \delta\lambda(t)|\xi| \quad \text{for} \;\; (t,\xi) \in Z_{hyp}(M,N,s), \tag{2.4}$$

where δ is some positive constant independent of t and ξ (with a suitable modification of N, if necessary).

From now, let us similarly reduce the equation to some "first-order diagonal system" in $Z_{hyp}(M,N,s)$. For this aim, introduce the Vandermonde matrix

$$\begin{aligned} M^\sharp(t,\xi) &= \begin{pmatrix} 1 & 1 \\ \frac{\tau_1(t,\xi)}{\lambda(t)|\xi|} & \frac{\tau_2(t,\xi)}{\lambda(t)|\xi|} \end{pmatrix}, \\ M(t,\xi) = M^\sharp(t,\xi)^{-1} &= \frac{\lambda|\xi|}{\tau_2-\tau_1}\begin{pmatrix} \frac{\tau_2}{\lambda|\xi|} & -1 \\ -\frac{\tau_1}{\lambda|\xi|} & 1 \end{pmatrix} \end{aligned}$$

and use the transformation $V = MU$. Then

$$D_tV = (D_tM)M^\sharp V + MAM^\sharp V + M\mathcal{F} = \mathfrak{A}V + M\mathcal{F}.$$

Here we remark that $\|M^\sharp\|$, $\|M\| \le C$ because of (2.1)–(2.4). Since

$$MAM^\sharp = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix},$$

we can write

$$\mathfrak{A} = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} + III = (\mathfrak{A}_{jk}), \quad M\mathcal{F} = \frac{\lambda|\xi|}{\tau_2 - \tau_1} \begin{pmatrix} -\hat{f} \\ \hat{f} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

with

$$\|III\| \le C \left(\frac{\lambda_t}{\lambda} + \frac{\lambda}{\Lambda^{\frac{s}{s-1}}} \right).$$

Analogously, as in $Z_{pd}(M, N, s)$, defining the energy function to the system $D_t V = \mathfrak{A} V + M\mathcal{F}$

$$F(t,\xi) = \frac{1}{2} \left\{ |V_1(t,\xi)|^2 + |V_2(t,\xi)|^2 \right\} \quad \text{for } V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

and differentiating it with respect to t, we find

$$\begin{aligned} \frac{dF}{dt} &= \operatorname{Re}(\bar{V}_1, V_{1t}) + \operatorname{Re}(\bar{V}_2, V_{2t}) \\ &= \operatorname{Re}(\bar{V}_1, i\mathfrak{A}_{11} V_1 + i\mathfrak{A}_{12} V_2 + i f_1) + \operatorname{Re}(\bar{V}_2, i\mathfrak{A}_{21} V_1 + i\mathfrak{A}_{22} V_2 + i f_2), \end{aligned}$$

in the sequel,

$$\left| \frac{dF}{dt} \right| \le C \left\{ |\operatorname{Im} \tau_1| |V_1|^2 + |\operatorname{Im} \tau_2| |V_2|^2 + \frac{\lambda_t}{\lambda} F + |\hat{f}|^2 \right\}$$

and by virtue of Gronwall's inequality

$$\begin{aligned} F(t,\xi) \le &\left(F(t_0,\xi) + \left| \int_{t_0}^t |\hat{f}(\tau,\xi)|^2 \, d\tau \right| \right) \\ &\times \exp \left(C \left| \int_{t_0}^t \sum_{j=1}^2 |\operatorname{Im} \tau_j(\tau,\xi)| \, d\tau \right| + C \left| \int_{t_0}^t \left| \frac{\lambda_t(\tau)}{\lambda(\tau)} \right| \, d\tau \right| \right) \end{aligned}$$

for $t_\xi \le t_0$, $t \le T$, where we already knew

$$\left| \int_{t_0}^t |\operatorname{Im} \tau_j(\tau,\xi)| \, d\tau \right| \le C_{s,M,N} \langle \xi \rangle^{\frac{1}{s}}$$

and from (1.1)

$$\begin{aligned} \left| \int_{t_0}^t \left| \frac{\lambda_t(\tau)}{\lambda(\tau)} \right| \, d\tau \right| &\le \int_{t_\xi}^T \frac{\lambda_t(t)}{\lambda(t)} \, dt \le c_1 \int_{t_\xi}^T \frac{\lambda(t)}{\Lambda(t)^{\frac{s}{s-1}}} \, dt \\ &\le c_1 (s-1) \Lambda(t_\xi)^{-\frac{1}{s-1}} = C_{s,N} \langle \xi \rangle^{\frac{1}{s}}. \end{aligned}$$

Hence we have in $Z_{hyp}(M,N,s)$, that is, for all (t,ξ), $(t_0,\xi) \in Z_{hyp}(M,N,s)$

$$F(t,\xi) \le \left(F(t_0,\xi) + \left| \int_{t_0}^{t} |\hat{f}(\tau,\xi)|^2 \, d\tau \right| \right) \exp\left(C \langle \xi \rangle^{\frac{1}{s}} \right)$$

as long as N is large enough. Besides, note that

$$\|H(t,\xi)\| = \left\| \begin{pmatrix} \lambda(t)|\xi| & 0 \\ 0 & 1 \end{pmatrix} \right\| \le C|\xi|, \quad \|H^{-1}(t,\xi)\| \le C$$

and the boundedness of $M^\sharp$ and M. Therefore we arrive at the a priori estimate as required

$$\begin{aligned} &|\hat{u}(t,\xi)| + |\hat{u}_t(t,\xi)| \\ &\quad \le C_{M,N} \exp\left(c_{M,N} \langle \xi \rangle^{\frac{1}{s}} \right) \left(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)| + \left| \int_{t_0}^{t} |\hat{f}(\tau,\xi)| \, d\tau \right| \right), \end{aligned}$$

which means (1.4) in $Z_{hyp}(M,N,s)$.

Thus it remains to prove the finite propagation speed property of the Cauchy problem (CP).

At first, if $t_0 > 0$, then the problem enjoys the finite propagation speed property. Because the operator $L(t,\partial_t,\partial_x)$ for $t > 0$ is strictly hyperbolic, it is well-known that

$$u(t,t_0,x) = 0 \quad \text{for all } \ (t,x) \in K_\gamma(t^0,x^0),\ t \ge t_0,$$

if $t_0 > 0$ and $t^0 > 0$ (see, for instance, Section 12 of Chapter 6 in [8]). Further, the values of the solution $u(0,x)$ for $(0,x) \in K_\gamma(t^0,x^0)$ can be obtained as limit of the values in $K_\gamma(t^0,x^0) \cap \{t > 0\}$, so that $u(0,x)$ vanishes.

Next, we shall consider the case $t_0 = 0$, $t^0 > 0$, $\gamma > 0$, and suppose that

$$(Lu)|_{K_\gamma(t^0,x^0)} = 0,\ \partial_t^j u|_{K_\gamma(t^0,x^0)\cap\{t=0\}} = 0 \qquad (j = 0,1).$$

To this end, introduce the approximate operators $L_\varepsilon(t,\partial_t,\partial_x)$ for $\varepsilon \in (0,\varepsilon_0]$ with $\varepsilon_0 \in (0, T - t^0)$, by means of

$$L_\varepsilon(t,\partial_t,\partial_x) = L(t+\varepsilon,\partial_t,\partial_x), \quad (t,x) \in [0, T-\varepsilon_0] \times \mathbb{R}^n.$$

Then, let us consider the following Cauchy problems

$$\begin{cases} L_\varepsilon(t,\partial_t,\partial_x) v_\varepsilon(t,x) = f(t,x) \quad \text{on } \ [0,T-\varepsilon_0] \times \mathbb{R}^n, \\ v_\varepsilon(0,x) = u_0(x),\ \partial_t v_\varepsilon(0,x) = u_1(x). \end{cases}$$

It is evident that $v_\varepsilon(t,x)=0$ for all $(t,x)\in K_\gamma(t^0,x^0)$. Now, according to the already proved statements of Theorem 1.1, for every $\rho>0$ there exists some ρ' $(>\rho)$ such that the a priori estimate

$$\begin{aligned}&\sum_{j=0}^{1}\|\exp(\rho\langle D_x\rangle^{\frac{1}{s}})\partial_t^j v_\varepsilon(t,x)\|_{L^2(\mathbb{R}^n_x)}\\&\le C_\rho\left(\sum_{j=0}^{1}\|\exp(\rho'\langle D_x\rangle^{\frac{1}{s}})u_j(x)\|_{L^2(\mathbb{R}^n_x)}+\int_0^t\|\exp(\rho'\langle D_x\rangle^{\frac{1}{s}})f(\tau,x)\|_{L^2(\mathbb{R}^n_x)}\,d\tau\right)\end{aligned}$$

holds for all $t\in[0,T-\varepsilon_0]$, where the constants C_ρ and ρ' are independent of ε. In addition, we have

$$\begin{cases}L_{\varepsilon_1}(v_{\varepsilon_1}-v_{\varepsilon_2})=(L_{\varepsilon_2}-L_{\varepsilon_1})v_{\varepsilon_2} & \text{on}\quad [0,T-\varepsilon_0]\times\mathbb{R}^n,\\ \partial_t^j(v_{\varepsilon_1}-v_{\varepsilon_2})(0,x)=0 & (j=0,1).\end{cases}$$

As for the above problem, we get

$$\begin{aligned}&\sum_{j=0}^{1}\|\exp(\rho\langle D_x\rangle^{\frac{1}{s}})\partial_t^j(v_{\varepsilon_1}-v_{\varepsilon_2})(t,x)\|_{L^2(\mathbb{R}^n_x)}\\&\le C_\rho\int_0^t\|\exp(\rho'\langle D_x\rangle^{\frac{1}{s}})(L_{\varepsilon_2}-L_{\varepsilon_1})v_{\varepsilon_2}(\tau,x)\|_{L^2(\mathbb{R}^n_x)}\,d\tau\\&\le C_\rho\int_0^t\Bigg\{\sum_{j,k=1}^{n}\|\exp(\rho'\langle D_x\rangle^{\frac{1}{s}})(a_{jk}(\tau+\varepsilon_1)-a_{jk}(\tau+\varepsilon_2))\partial_{x_jx_k}v_{\varepsilon_2}(\tau,x)\|_{L^2(\mathbb{R}^n_x)}\\&\quad+\sum_{j=1}^{n}\|\exp(\rho'\langle D_x\rangle^{\frac{1}{s}})(a_j(\tau+\varepsilon_1)-a_j(\tau+\varepsilon_2))\partial_{x_j}v_{\varepsilon_2}(\tau,x)\|_{L^2(\mathbb{R}^n_x)}\Bigg\}\,d\tau\\&\le C_\rho\int_0^t\Bigg\{\max_{j,k}\left|\int_{\tau+\varepsilon_2}^{\tau+\varepsilon_1}\partial_t a_{jk}(\sigma)\,d\sigma\right|\sum_{j,k=1}^{n}\|\exp(\rho'\langle D_x\rangle^{\frac{1}{s}})\partial_{x_jx_k}v_{\varepsilon_2}(\tau,x)\|_{L^2(\mathbb{R}^n_x)}\\&\quad+\max_{j}\left|\int_{\tau+\varepsilon_2}^{\tau+\varepsilon_1}\partial_t a_j(\sigma)\,d\sigma\right|\sum_{j=1}^{n}\|\exp(\rho'\langle D_x\rangle^{\frac{1}{s}})\partial_{x_j}v_{\varepsilon_2}(\tau,x)\|_{L^2(\mathbb{R}^n_x)}\Bigg\}\,d\tau\\&\le C'_\rho|\varepsilon_1-\varepsilon_2|\max_{j,k}\sup_{t\in[0,T]}(|\partial_t a_{jk}(t)|+|\partial_t a_j(t)|)\\&\quad\times\int_0^t\|\langle D_x\rangle^2\exp(\rho'\langle D_x\rangle^{\frac{1}{s}})v_{\varepsilon_2}(\tau,x)\|_{L^2(\mathbb{R}^n_x)}\,d\tau\end{aligned}$$

with the constant C'_ρ independent of ε_1 and ε_2. Hence, if $\varepsilon_j\downarrow 0$, then $\{v_{\varepsilon_j}\}$ is a Cauchy sequence in the space $C^1([0,T];L^2_s(\mathbb{R}^n))$. In view of the uniqueness of the solution, we know $u = \lim_{j\to\infty} v_{\varepsilon_j}$ in that space and a fortiori in the distribution space $\mathcal{D}'(K_\gamma(t^0, x^0))$. In particular, the equalities

$$\langle u, \varphi\rangle = \lim_{j\to\infty} \langle v_{\varepsilon_j}, \varphi\rangle = 0$$

for every test function $\varphi \in C_0^\infty(K_\gamma(t^0, x^0))$, induce

$$u|_{K_\gamma(t^0,x^0)} = 0.$$

§3. Necessity of the Levi Condition

In this section we shall prove Theorem 1.2. For this purpose it is enough to construct a sequence of the solutions which violate the a priori estimate (1.4) in the Gevrey space $L^2_s(\mathbb{R}^n)$. We are going to look for these solutions in the form

$$u_\xi(t, x) = e^{ix\cdot\xi}\varphi(x)\tilde{u}(t, \xi),$$

where $\varphi(x) \in L^2_s(\mathbb{R}^n_x)$, $\operatorname{supp}\varphi \subset \{x \in \mathbb{R}^n; |x| \le 2\gamma\}$, $\varphi(x) = 1$ when $|x| \le \gamma$, and $\xi \in \mathbb{R}^n$ is a parameter with large $|\xi|$, while $\tilde{u}(t, \xi)$ is determined by the solution to the ordinary differential equation $L(t, \partial_t, -i\xi)\tilde{u}(t, \xi) = 0$ with parameter ξ. Here γ comes from (ii) of Definition 1.1. Then $u_\xi(t, x) \in C^2([0, T]; L^2_s(\mathbb{R}^n))$ for every $\xi \in \mathbb{R}^n$. This function solves the equation

$$L(t, \partial_t, \partial_x)u_\xi(t, x) = f_\xi(t, x),$$

where

$$\begin{aligned} f_\xi(t, x) = &- e^{ix\cdot\xi} \sum_{j,k=1}^{n} a_{jk}(t)\left\{2i\xi_j(\partial_{x_k}\varphi(x)) + (\partial_{x_j x_k}\varphi(x))\right\}\tilde{u}(t, \xi) \\ &- e^{ix\cdot\xi}\sum_{j=1}^{n} a_j(t)(\partial_{x_j}\varphi(x))\tilde{u}(t, \xi). \end{aligned}$$

Here we remark that $f_\xi(t, x) \equiv 0$ for all $|x| \le \gamma$. If we now consider the another Cauchy problem

$$\begin{cases} L(t, \partial_t, \partial_x)v_\xi(t, x) = 0, \\ \partial_t^j v_\xi(0, x) = \partial_t^j u_\xi(0, x) \qquad (j = 0, 1), \end{cases}$$

then due to the finite propagation speed property we get

$$v_\xi(t, x) \equiv u_\xi(t, x) \quad \text{for all } \ x \in \mathbb{R}^n,\ |x| \le \gamma/2,\ t \le T < 1.$$

On the other hand, if the above Cauchy problem is well-posed in $L^2_s(\mathbb{R}^n)$, then $v_\xi(t,x) \in C^2([0,T]; L^2_s(\mathbb{R}^n))$ and according to (1.4) we obtain (3.1)

$$\begin{aligned}
&\| \exp(\rho \langle D_x\rangle^{\frac{1}{s}}) v_\xi(t,x)\|_{L^2(\mathbb{R}^n_x)} + \| \exp(\rho \langle D_x\rangle^{\frac{1}{s}}) \partial_t v_\xi(t,x)\|_{L^2(\mathbb{R}^n_x)} \\
&\le C(T,\rho) \left(\| \exp(\rho' \langle D_x\rangle^{\frac{1}{s}}) u_\xi(0,x)\|_{L^2(\mathbb{R}^n_x)} + \| \exp(\rho' \langle D_x\rangle^{\frac{1}{s}}) \partial_t u_\xi(0,x)\|_{L^2(\mathbb{R}^n_x)} \right).
\end{aligned}$$

Furthermore, we have by Sobolev's imbedding theorem

$$\begin{aligned}
|\tilde{u}(t,\xi)| = |u_\xi(t,0)| &\le C \sum_{|\alpha|\le n} \|D_x^\alpha u_\xi(t,x)\|_{L^2(\{|x|\le \gamma/2\})} \\
&= C \sum_{|\alpha|\le n} \|D_x^\alpha v_\xi(t,x)\|_{L^2(\{|x|\le \gamma/2\})} \le C \sum_{|\alpha|\le n} \|D_x^\alpha v_\xi(t,x)\|_{L^2(\mathbb{R}^n_x)} \\
&\le C(T,\rho) \| \exp(\rho \langle D_x\rangle^{\frac{1}{s}}) v_\xi(t,x)\|_{L^2(\mathbb{R}^n_x)}.
\end{aligned}$$

So, if we apply (3.1), then

$$\begin{aligned}
|\tilde{u}(t,\xi)| \le C(T,\rho) \Big(&\| \exp(\rho' \langle D_x\rangle^{\frac{1}{s}}) u_\xi(0,x)\|_{L^2(\mathbb{R}^n_x)} \\
&+ \| \exp(\rho' \langle D_x\rangle^{\frac{1}{s}}) \partial_t u_\xi(0,x)\|_{L^2(\mathbb{R}^n_x)} \Big).
\end{aligned}$$

Similarly,

$$\begin{aligned}
|\partial_t \tilde{u}(t,\xi)| \le C(T,\rho) \Big(&\| \exp(\rho' \langle D_x\rangle^{\frac{1}{s}}) u_\xi(0,x)\|_{L^2(\mathbb{R}^n_x)} \\
&+ \| \exp(\rho' \langle D_x\rangle^{\frac{1}{s}}) \partial_t u_\xi(0,x)\|_{L^2(\mathbb{R}^n_x)} \Big).
\end{aligned}$$

Thus we can sum up the estimate (3.2)

$$\begin{aligned}
&|\tilde{u}(t,\xi)| + |\partial_t \tilde{u}(t,\xi)| \\
&\quad \le C_\rho \left(\| \exp(\rho' \langle D_x\rangle^{\frac{1}{s}}) u_\xi(0,x)\|_{L^2(\mathbb{R}^n_x)} + \| \exp(\rho' \langle D_x\rangle^{\frac{1}{s}}) \partial_t u_\xi(0,x)\|_{L^2(\mathbb{R}^n_x)} \right)
\end{aligned}$$

with the constant C_ρ independent of $\xi \in \mathbb{R}^n$.

If we put $u_\xi(t,x)$ into the left hand side of (1.4), then for every $\rho' > 0$

$$\begin{aligned}
\| \exp(\rho' \langle D_x\rangle^{\frac{1}{s}}) u_\xi(t,x)\|_{L^2(\mathbb{R}^n_x)} &= \| \exp(\rho' \langle \zeta\rangle^{\frac{1}{s}}) \hat{u}_\xi(t,\zeta)\|_{L^2(\mathbb{R}^n_\zeta)} \\
&= \| \exp(\rho' \langle \zeta\rangle^{\frac{1}{s}}) \hat{\varphi}(\zeta-\xi) \tilde{u}(t,\xi)\|_{L^2(\mathbb{R}^n_\zeta)} \\
&= |\tilde{u}(t,\xi)| \| \exp(\rho' \langle \zeta+\xi\rangle^{\frac{1}{s}}) \hat{\varphi}(\zeta)\|_{L^2(\mathbb{R}^n_\zeta)} \\
&\le e^{2\rho' \langle \xi\rangle^{\frac{1}{s}}} |\tilde{u}(t,\xi)| \| \exp(2\rho' \langle \zeta\rangle^{\frac{1}{s}}) \hat{\varphi}(\zeta)\|_{L^2(\mathbb{R}^n_\zeta)}.
\end{aligned}$$

In particular,

$$\begin{aligned}
(3.3)\quad &\| \exp(\rho' \langle D_x\rangle^{\frac{1}{s}}) u_\xi(0,x)\|_{L^2(\mathbb{R}^n_x)} + \| \exp(\rho' \langle D_x\rangle^{\frac{1}{s}}) (\partial_t u_\xi)(0,x)\|_{L^2(\mathbb{R}^n_x)} \\
&\le e^{2\rho' \langle \xi\rangle^{\frac{1}{s}}} \left(|\tilde{u}(0,\xi)| + |\tilde{u}_t(0,\xi)| \right) \| \exp(2\rho' \langle D_x\rangle^{\frac{1}{s}}) \varphi(x)\|_{L^2(\mathbb{R}^n_x)}.
\end{aligned}$$

Hence we shall prove that (3.2) and (3.3) cannot hold simultaneously for large $|\xi|$ and all $t \in [0, t_\xi^{(2)}]$ with the sequence $\{t_\xi^{(2)}\}$ of positive numbers depending on ξ.

Let $\rho = \rho(t, \xi)$ be the positive root of the quadratic equation with respect to ρ

$$\rho^2 - 1 - \frac{1}{\nu(t)}\lambda(t)^2\Lambda(t)^{-\frac{s}{s-1}}|\xi| = 0.$$

We note that $\rho_t(t) \geq 0$ near $t = 0$ because of (1.1) and (1.5). Also, set $t_\xi^{(1)}$ by the unique root of the following equation in t:

$$|\xi|^{\frac{1}{2}}\frac{1}{\sqrt{\nu(t)}}\Lambda(t)^{\frac{s-2}{2(s-1)}} = |\xi|^{\frac{1}{s}}.$$

Then we shall first establish the inequality below

$$\int_0^{t_\xi^{(1)}}\left(\rho(t) + \frac{\rho_t(t)}{\rho(t)}\right)dt \leq C\,\langle\xi\rangle^{\frac{1}{s}}. \tag{3.4}$$

Since the inequality

$$\frac{1}{\sqrt{\nu(t)}}\lambda(t)\Lambda(t)^{-\frac{s}{2(s-1)}} \leq \frac{2(s-1)}{(s-2)-(s-1)C_\nu}\frac{d}{dt}\left(\frac{1}{\sqrt{\nu(t)}}\Lambda(t)^{\frac{s-2}{2(s-1)}}\right)$$

is derived from (1.5), we get by integrating it from 0 to $t_\xi^{(1)}$

$$\int_0^{t_\xi^{(1)}}\frac{1}{\sqrt{\nu(t)}}\lambda(t)\Lambda(t)^{-\frac{s}{2(s-1)}}\,dt \leq C_s\frac{1}{\sqrt{\nu(t_\xi^{(1)})}}\Lambda(t_\xi^{(1)})^{\frac{s-2}{2(s-1)}} = C_s|\xi|^{\frac{1}{s}-\frac{1}{2}}.$$

So the inequality

$$\begin{aligned}\int_0^{t_\xi^{(1)}}\rho(t)\,dt &\leq \int_0^{t_\xi^{(1)}}dt + |\xi|^{\frac{1}{2}}\int_0^{t_\xi^{(1)}}\frac{1}{\sqrt{\nu(t)}}\lambda(t)\Lambda(t)^{-\frac{s}{2(s-1)}}\,dt \\ &\leq C_{s,M}\,\langle\xi\rangle^{\frac{1}{s}}\end{aligned}$$

is valid. Meanwhile,

$$\int_0^{t_\xi^{(1)}}\frac{\rho_t(t)}{\rho(t)}\,dt \leq \log\rho(t_\xi^{(1)}) = \frac{1}{2}\log\left(1 + \frac{1}{\nu(t_\xi^{(1)})}\lambda(t_\xi^{(1)})^2\Lambda(t_\xi^{(1)})^{-\frac{s}{s-1}}|\xi|\right).$$

We have assumed that the coefficients of $a_1(t,\xi)/|\xi|$ are bounded, so that

$$b(t) := \frac{1}{\nu(t)}\lambda(t)^2\Lambda(t)^{-\frac{s}{s-1}} < \infty \quad \text{near} \quad t = 0.$$

On the other hand, by the definition of $t_\xi^{(1)}$

$$\frac{s-2}{2(s-1)}\log\Lambda(t_\xi^{(1)}) = \left(\frac{1}{s}-\frac{1}{2}\right)\log|\xi| + \frac{1}{2}\log\frac{1}{\nu(t_\xi^{(1)})},$$

so that

$$\frac{s-2}{2(s-1)}\log\Lambda(t_\xi^{(1)}) \geq \left(\frac{1}{s}-\frac{1}{2}\right)\log|\xi|$$

implies

$$-\frac{s}{s-1}\log\Lambda(t_\xi^{(1)}) \leq \log|\xi|.$$

At the same time,

$$\frac{\lambda(t)}{\Lambda(t)} \geq \frac{1}{t} \geq C > 0 \quad \text{near} \quad t=0,$$

meanwhile, by the definition of $t_\xi^{(1)}$

$$b(t_\xi^{(1)})|\xi| = |\xi|^{\frac{2}{s}}\frac{\lambda(t_\xi^{(1)})^2}{\Lambda(t_\xi^{(1)})^2}.$$

Consequently, $b(t_\xi^{(1)})|\xi| \geq 1$ for $|\xi| \gg 1$. In addition, according to (1.5), we have

$$\frac{1}{\nu(t)} \leq c\frac{1}{\Lambda(t)^{C_\nu}}, \quad 0 < C_\nu < \frac{s-2}{s-1}.$$

Therefore, taking into account all these estimates, we obtain

$$\begin{aligned}
\int_0^{t_\xi^{(1)}} \frac{\rho_t(t)}{\rho(t)}\,dt &\leq \frac{1}{2}\log 2b(t_\xi^{(1)})|\xi| \\
&= \frac{1}{2}\log\frac{1}{\nu(t_\xi^{(1)})} + \log\lambda(t_\xi^{(1)}) \\
&\quad -\frac{1}{2}\frac{s}{s-1}\log\Lambda(t_\xi^{(1)}) + \frac{1}{2}\log|\xi| + \frac{1}{2}\log 2 \\
&\leq -\frac{1}{2}C_\nu\log\Lambda(t_\xi^{(1)}) + \frac{1}{2}\log c \\
&\quad -\frac{1}{2}\frac{s}{s-1}\log\Lambda(t_\xi^{(1)}) + \frac{1}{2}\log|\xi| + \frac{1}{2}\log 2 \\
&\leq C\log|\xi| \leq o(1)|\xi|^{\frac{1}{s}}.
\end{aligned}$$

Now we conclude the inequality (3.4). Thus the following estimate from above is established:

$$|\tilde{u}(0,\xi)| + |\tilde{u}_t(0,\xi)| \leq C\left(|\tilde{u}(t_\xi^{(1)},\xi)| + |\tilde{u}_t(t_\xi^{(1)},\xi)|\right)\exp\left(C\langle\xi\rangle^{\frac{1}{s}}\right)$$

for any solution $\tilde{u}$ of the ordinary differential equation $L(t, \partial_t, -i\xi)\tilde{u}(t,\xi) = 0$ with parameter $\xi \in \mathbb{R}^n$. Here, if we particularly choose the initial data

$$\tilde{u}(t_\xi^{(1)}, \xi) = 1, \quad \tilde{u}_t(t_\xi^{(1)}, \xi) = 0,$$

then the above estimate turns out

$$|\tilde{u}(0,\xi)| + |\tilde{u}_t(0,\xi)| \le C \exp(C_0 \langle \xi \rangle^{\frac{1}{s}}). \tag{3.5}$$

Next we shall reduce the equation to a "first-order diagonal system" similar as in Section 2. Let us denote the characteristic roots by $\tau_1 = \tau_1(t,\xi)$, $\tau_2 = \tau_2(t,\xi)$ of the subprincipal part defined by

$$\tau^2 + \frac{i}{\nu(t)} \lambda(t)^2 \Lambda(t)^{-\frac{s}{s-1}} \sum_{j=1}^n b_j(t)\xi_j = 0.$$

That is to say,

$$\tau_k = (-1)^k \frac{1}{\sqrt{\nu(t)}} \lambda(t) \Lambda(t)^{-\frac{s}{2(s-1)}} \left(-i \sum_{j=1}^n b_j(t)\xi_j \right)^{\frac{1}{2}} \qquad (k = 1, 2),$$

where their branches are taken as $\operatorname{Im} \tau_1 < 0$ and $\operatorname{Im} \tau_2 > 0$. Now, transform $\mathcal{V}(t,\xi) = \left(\begin{smallmatrix} \tilde{u}(t,\xi) \\ D_t \tilde{u}(t,\xi) \end{smallmatrix} \right)$ into $V(t,\xi) = M(t,\xi)\mathcal{V}(t,\xi)$ with the nonsingular matrix $M = \frac{1}{\tau_2 - \tau_1} \left(\begin{smallmatrix} \tau_2 & -1 \\ -\tau_1 & 1 \end{smallmatrix} \right)$. Then

$$\begin{aligned} D_t V &= (D_t M) M^{-1} V + M \mathcal{A} M^{-1} V \\ &= III \cdot V + (I + II)V, \end{aligned}$$

where

$$\begin{aligned} I &= M \begin{pmatrix} 0 & 1 \\ -\frac{i}{\nu} \lambda^2 \Lambda^{-\frac{s}{s-1}} \sum_{j=1}^n b_j \xi_j & 0 \end{pmatrix} M^{-1} = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \\ II &= M \begin{pmatrix} 0 & 0 \\ \lambda^2 |\xi|^2 & 0 \end{pmatrix} M^{-1} = \frac{\lambda^2 |\xi|^2}{2\tau_2} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \\ III &= (D_t M) M^{-1} = \frac{\tau_{2t}}{2i\tau_2} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}. \end{aligned}$$

Moreover, let us consider the following Cauchy problem on the interval $[t_\xi^{(1)}, t_\xi^{(2)}]$ ($t_\xi^{(2)}$ is not yet defined, but will be determined later)

$$\begin{cases} D_t V = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} V + (II + III)V, \quad V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}, \\ V_1(t_\xi^{(1)}) = 1, \quad V_2(t_\xi^{(1)}) = 0. \end{cases}$$

For the sake of brevity we indicate $C = (C_{jk}) = II + III$. Now we introduce the Lyapunov function

$$W(t) = \frac{1}{2}\left(|V_1(t)|^2 - |V_2(t)|^2\right).$$

Differentiating $W(t)$ in t, we find an absolute constant $\delta \in (0,1)$ such that

$$\begin{aligned}
\frac{dW}{dt} &= \mathrm{Re}(iD_tV_1, V_1) - \mathrm{Re}(iD_tV_2, V_2)\\
&= \mathrm{Re}\left(i\left(\tau_1 V_1 + \sum_{j=1}^{2} C_{1j}V_j\right), V_1\right) - \mathrm{Re}\left(i\left(\tau_2 V_2 + \sum_{j=1}^{2} C_{2j}V_j\right), V_2\right)\\
&= (-\operatorname{Im}\tau_1)|V_1|^2 - (-\operatorname{Im}\tau_2)|V_2|^2 + \sum_{j=1}^{2} iC_{1j}V_j\bar{V}_1 - \sum_{j=1}^{2} iC_{2j}V_j\bar{V}_2\\
&\geq \frac{\operatorname{Im}\tau_2}{2}\left(|V_1|^2 - |V_2|^2\right) + \left\{\frac{\operatorname{Im}\tau_2}{2} - \left(\max_{j,k}|C_{jk}|\right)\right\}\left(|V_1|^2 - |V_2|^2\right)\\
&\quad + 2\left\{(\operatorname{Im}\tau_2) - \left(\max_{j,k}|C_{jk}|\right)\right\}|V_2|^2\\
&= (\operatorname{Im}\tau_2 + G)\left(|V_1|^2 + |V_2|^2\right) \geq \frac{\delta}{2}(\operatorname{Im}\tau_2)(|V_1|^2 - |V_2|^2) = \delta(\operatorname{Im}\tau_2)W
\end{aligned}$$

when $\max_{j,k}|C_{jk}| = o(\operatorname{Im}\tau_2)$ as $|\xi| \to \infty$. So, by Gronwall's inequality

$$W(t_\xi^{(2)}) \geq W(t_\xi^{(1)})\exp\left(\delta\int_{t_\xi^{(1)}}^{t_\xi^{(2)}} \operatorname{Im}\tau_2(t,\xi)\,dt\right) = \frac{1}{2}\exp\left(\delta\int_{t_\xi^{(1)}}^{t_\xi^{(2)}} \operatorname{Im}\tau_2(t,\xi)\,dt\right)$$

holds. Here, if we define $t_\xi^{(2)}$ $(> t_\xi^{(1)})$ satisfying

$$|\xi|^{\frac{1}{2}}\frac{1}{\sqrt{\nu(t_\xi^{(2)})}}\Lambda(t_\xi^{(2)})^{\frac{s-2}{2(s-1)}} = (N+1)|\xi|^{\frac{1}{s}}$$

with a large parameter $N > 0$, then we can show that

$$\int_{t_\xi^{(1)}}^{t_\xi^{(2)}} \operatorname{Im}\tau_2(t,\xi)\,dt \geq C(N)|\xi|^{\frac{1}{s}},$$

where $C(N)$ tends to ∞ as N does to ∞. In fact,

$$\begin{aligned}
&\int_{t_\xi^{(1)}}^{t_\xi^{(2)}} \operatorname{Im}\tau_2(t,\xi)\,dt \\
&\geq C_b|\xi|^{\frac{1}{2}} \int_{t_\xi^{(1)}}^{t_\xi^{(2)}} \frac{1}{\sqrt{\nu(t)}}\lambda(t)\Lambda(t)^{-\frac{s}{2(s-1)}}\,dt \\
&= \frac{2(s-1)}{s-2}C_b|\xi|^{\frac{1}{2}}\left(\frac{1}{\sqrt{\nu(t_\xi^{(2)})}}\Lambda(t_\xi^{(2)})^{\frac{s-2}{2(s-1)}} - \frac{1}{\sqrt{\nu(t_\xi^{(1)})}}\Lambda(t_\xi^{(1)})^{\frac{s-2}{2(s-1)}}\right) \\
&\quad + \frac{s-1}{s-2}C_b|\xi|^{\frac{1}{2}} \int_{t_\xi^{(1)}}^{t_\xi^{(2)}} \frac{1}{\sqrt{\nu(t)}}\frac{\nu_t(t)}{\nu(t)}\Lambda(t)^{\frac{s-2}{2(s-1)}}\,dt \\
&\geq \frac{2(s-1)}{s-2}C_b|\xi|^{\frac{1}{2}}\left(\frac{1}{\sqrt{\nu(t_\xi^{(2)})}}\Lambda(t_\xi^{(2)})^{\frac{s-2}{2(s-1)}} - \frac{1}{\sqrt{\nu(t_\xi^{(1)})}}\Lambda(t_\xi^{(1)})^{\frac{s-2}{2(s-1)}}\right).
\end{aligned}$$

Finally, we must verify that $G(t,\xi) = o(\operatorname{Im}\tau_2)$ as $|\xi| \to \infty$. To this end, it is sufficient to estimate the two quantities:

$$\frac{\lambda^2|\xi|^2}{\tau_2} = o(\operatorname{Im}\tau_2), \qquad \frac{\tau_{2t}}{\tau_2} = o(\operatorname{Im}\tau_2).$$

As for the first one, since

$$|\tau_2|^2 \geq C_b\frac{1}{\nu}\lambda^2\Lambda^{-\frac{s}{s-1}}|\xi|,$$

we have

$$\frac{\lambda^2|\xi|^2}{|\tau_2|^2} \leq \frac{1}{C_b}\nu\Lambda^{\frac{s}{s-1}}|\xi| = o(1) \quad \text{ on } \ [t_\xi^{(1)}, t_\xi^{(2)}].$$

In addition, as to the second one, we have to check that

$$\frac{\nu_t}{\nu},\ \frac{\lambda_t}{\lambda},\ \frac{\lambda}{\Lambda},\ \frac{\sum\partial_t b_j\xi_j}{\sum b_j\xi_j} = o(\operatorname{Im}\tau_2) \quad \text{ on } \ [t_\xi^{(1)}, t_\xi^{(2)}]$$

as $|\xi| \to \infty$. From now, we shall only give a proof of

$$\frac{\lambda}{\Lambda} = o(\operatorname{Im}\tau_2) \quad \text{ on } \ [t_\xi^{(1)}, t_\xi^{(2)}]$$

because the proofs of the remaining ones are completely similar due to (1.1), (1.5) and (1.7). For this aim, it is enough to verify that

$$\frac{\lambda(t)}{\Lambda(t)} = o(1)\frac{1}{\sqrt{\nu(t)}}\lambda(t)\Lambda(t)^{-\frac{s}{2(s-1)}}|\xi|^{\frac{1}{2}} \quad \text{ on } \ [t_\xi^{(1)}, t_\xi^{(2)}].$$

This is equivalent to

$$\frac{\sqrt{\nu(t)}}{\Lambda(t)^{\frac{s-2}{2(s-1)}}} = o(1)|\xi|^{\frac{1}{2}} \quad \text{on} \quad [t_\xi^{(1)}, t_\xi^{(2)}].$$

Since (1.5) implies that $\sqrt{\nu(t)}\,/\Lambda(t)^{(s-2)/(2(s-1))}$ is non-increasing,

$$\frac{\sqrt{\nu(t)}}{\Lambda(t)^{\frac{s-2}{2(s-1)}}} \leq \frac{\sqrt{\nu(t_\xi^{(1)})}}{\Lambda(t_\xi^{(1)})^{\frac{s-2}{2(s-1)}}}$$

for all $t \in [t_\xi^{(1)}, t_\xi^{(2)}]$. On the other hand, according to the definition of $t_\xi^{(1)}$,

$$|\xi| = \nu(t_\xi^{(1)})^{\frac{s}{s-2}} \Lambda(t_\xi^{(1)})^{-\frac{s}{s-1}},$$

so that it suffices to prove

$$\Lambda(t_\xi^{(1)})^{\frac{s-2}{s-1}} = o(1)\nu(t_\xi^{(1)}).$$

This follows from (1.5). Actually, let us choose a positive number ε with $C_\nu + \varepsilon \leq (s-2)/(s-1)$. Then $\nu(t)\Lambda(t)^{\varepsilon-(s-2)/(s-1)}$ is non-increasing. Therefore

$$\nu(t)\Lambda(t)^{\varepsilon-\frac{s-2}{s-1}} \geq \nu(T)\Lambda(T)^{\varepsilon-\frac{s-2}{s-1}} =: \delta > 0,$$

that is,

$$\nu(t)\Lambda(t)^{\varepsilon} \geq \delta\Lambda(t)^{\frac{s-2}{s-1}}.$$

Now we just set $t = t_\xi^{(1)}$ in the last inequality.

Thus we can deduce that

$$\exp\left(\delta C(N)\langle\xi\rangle^{\frac{1}{s}}\right) \leq 2W(t_\xi^{(2)}) \leq (|V_1|^2 + |V_2|^2)|_{t=t_\xi^{(2)}},$$

which means the following estimate from below

$$\exp\left(\delta' C(N)\langle\xi\rangle^{\frac{1}{s}}\right) \leq C\left(|\tilde{u}(t_\xi^{(2)},\xi)| + |\tilde{u}_t(t_\xi^{(2)},\xi)|\right). \tag{3.6}$$

Hence, reminding that $C(N)$ goes to ∞ as N does to ∞, we gain a contradiction thanks to the inequalities (3.5) and (3.6). Indeed, if the Cauchy problem for (1.6) on $[0, t_\xi^{(2)}] \times \mathbb{R}^n$ is well-posed in $L_s^2(\mathbb{R}^n)$, then we have already found that (3.1) implies (3.2) at $t = t_\xi^{(2)}$ and for $|x| \leq \gamma/2$ and $|\xi| \gg 1$. Here, by recalling (3.2) and (3.3), (3.5) and (3.6) lead us to the inequality

$$\exp\left(\delta' C(N)\langle\xi\rangle^{\frac{1}{s}}\right) \leq C(T,\rho,\varphi)\exp\left((C_0 + 2\rho')\langle\xi\rangle^{\frac{1}{s}}\right)$$

has to be satisfied for large $|\xi|$, but it fails to be valid when $C(N) > (C_0 + 2\rho')/\delta'$. Therefore the sequence $\{u_\xi\}$ for large $|\xi|$ breaks down the a priori estimate (1.4). Thus we now complete the proof of Theorem 1.2.

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