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Non-isotropic Gevrey Hypoellipticity for Grushin Operators

By

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Abstract

We shall determine non-isotropic Gevrey exponents for general Grushin operators based on the results given in the paper [26], where a method to determine isotropic (worst) Gevrey exponents was given. The ideas of the bracket calculus given in the paper [2] and FBI-transformation given in the paper [5] are also useful.

§1. Introduction

In the early 70s, V. V. Grushin has introduced a wide class of degenerate elliptic differential operators which are hypoelliptic in a series of the papers [10], [11] and [12]. After then, there has been investigated the problem of analytic and non-analytic hypoellipticity of the Grushin operators [1], [2], [13], [28], etc.

In the paper [26], we have tried to determine isotropic (equi-directional) Gevrey exponent of hypoellipticity for every Grushin operator. Our method given there is based on the Grushin's idea using operator-valued pseudodifferential operators [12] and our results on Gevrey calculus for pseudodifferential operators [22], [27].

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In general, we know that hypoelliptic operators may have different Gevrey exponents with respect to different variables (directions). There has been remained open the problem to determine non-isotropic (directional) Gevrey exponents precisely for each Grushin operator.

Meanwhile, by using the method of bracket calculus, A. Bove and D. Tartakoff [2] succeeded to determine precise non-isotropic Gevrey exponents for generalized Baouendi-Goulaouic operators:

$$P = \partial_y^2 + y^{2k}\partial_x^2 + y^{2l}\partial_z^2, \quad (k \geq l \geq 0, k > 0, \text{ see Example (c) in Section 2}).$$

They have proved that the operator P has $G^{\{\theta,1,d\}}_{x,z,y}$-hypoellipticity in a neighborhood of the origin, where $\theta = (1+k)/(1+l)$ and $d = (\theta+k)/(1+k)$. Here we have $1 < d < \theta$. This means the operator P is analytic hypoelliptic with respect to z but not y. The above operator P is considered to be a typical Grushin operator as well as that of L. Hörmander [17]. Their idea using bracket calculus will be also useful in this paper.

In the paper [26], we have treated Grushin operators dividing them into three groups. In this paper, we would like to treat them also dividing into three groups. We shall start from the assumption that C^∞-hypoellipticity is already proved for the Grushin operators. The other typical Grushin operators than P are given by

$$L = \partial_y^2 + (x^{2l} + y^{2k})\partial_x^2, \quad (l, k, = 1, 2, \dots),$$
$$M = \partial_y^2 + (x^{2l} + y^{2k})(\partial_x^2 + \partial_z^2), \quad (l, k = 1, 2, \dots).$$

The operator L is $G^{\{\theta,d\}}_{x,y}$-hypoelliptic in a neighborhood of the origin in $\mathbf{R}^2$, where $\theta = (l(1+k))/(l(1+k)-k)$ and $d = (\theta+k)/(1+k)$. The optimality of this exponent $\{\theta, d\}$ was already shown in the paper [28]. While, in Section 6 the operator M will be proved having $G^{\{\theta,1,d\}}_{x,z,y}$-hypoellipticity in a neighborhood of the origin in $\mathbf{R}^3$, where $\theta = (l(1+k))/(l(1+k)-k)$ and $d = (\theta+k)/(1+k)$, (cf. Theorem 2.1 and Examples). Note that we have also $1 < d < \theta$, for both operators L and M. The precise definition of the Gevrey spaces will be given in Section 2.

In this paper we shall use three basic methods. First, we shall prepare symbolic calculus for non-isotropic pseudodifferential operators of (ϱ, δ)-type in Section 3. This will be applied for Grushin operators in Section 4. Second, method of bracket calculus given in [2] will be used in Section 5. Third, method of FBI-transformation given in [5] and [6] will be developed slightly and used in Section 6 to complete the proof of our main result Theorem 2.1. It looks that both methods of Sections 5 and 6 are interesting although the result of Section

6 includes that of Section 5. Thus the original problem is almost completely solved for Grushin operators in this paper, while an interesting and challenging problem occurs, (see Remark 2.2).

§2. Main Results

We denote $x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$, and $D = (D_1, D_2, \dots, D_n)$, $D_j = -i\partial_{x_j}, j = 1, 2, \dots, n$ as usual.

First we remember the definition of Gevrey functions.

Definition 2.1. Let Ω be an open set in $\mathbf{R}^n$ and $\varphi \in C^\infty(\Omega)$. Then we say that $\varphi \in G^{\{\theta\}}(\Omega), \theta > 0$, if for any compact subset K of Ω there are positive constants C_0 and C_1 such that

$$\sup_{x \in K} |D^\alpha \varphi(x)| \le C_0 C_1^{|\alpha|} \alpha!^\theta, \quad \alpha \in \mathbf{Z}_+^n. \tag{2.1}$$

We say that $\varphi \in G^{\{d_1, d_2, \dots, d_n\}}(\Omega), 0 < d_1, d_2, \dots, d_n < \infty$, if for any compact subset K of Ω there are positive constants C_0 and C_1 such that

$$\sup_{x \in K} |D^\alpha \varphi(x)| \le C_0 C_1^{|\alpha|} \alpha_1!^{d_1} \alpha_2!^{d_2} \cdots \alpha_n!^{d_n}, \quad \alpha \in \mathbf{Z}_+^n. \tag{2.2}$$

Proposition 2.1. *Let $\varphi \in C^\infty(\Omega)$. If for any compact subset K of Ω there are positive constants C_0 and C_1 such that*

$$\sup_{x \in K} |D_j^k \varphi| \le C_0 C_1^k k!^{d_j}, \quad j = 1, 2, \dots, n, \quad k \in \mathbf{Z}_+. \tag{2.3}$$

Then we have $\varphi \in G^{\{d_1, d_2, \dots, d_n\}}(\Omega)$.

Next we remember the Grushin operators in a general form. We write $(x, y) = (x_1, \dots, x_k, y_1, \dots, y_n) \in \mathbf{R}^{k+n} = \mathbf{R}^N$. Let m be an even positive integer and let $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_k)$, $q = (q_1, q_2, \dots, q_k)$ whose elements are rational numbers such that

$$\sigma_1, \dots, \sigma_p > 0, \quad \sigma_{p+1} = \cdots = \sigma_k = 0, \quad (0 \le p \le k)$$
$$q_1 \ge q_2 \ge \cdots \ge q_k \ge 0, \quad q_1 > 0.$$

Furthermore, we assume

$$mq_j \in \mathbf{Z}, \quad j = 1, \dots, k;$$
$$\frac{mq_j}{\sigma_j} \in \mathbf{Z}, \quad j = 1, \dots, p,$$

and

$$1 + q_k > \sigma_0 = \max(\sigma_1, \sigma_2, \dots, \sigma_p).$$

We divide x into two parts such as $x = (x', x'')$ when $1 \le p < k$, where $x' = (x_1, \dots, x_p)$ and $x'' = (x_{p+1}, \dots, x_k)$. We consider $x = x'$ when $p = k$ and $x = x''$, $p = 0$ when $\sigma = (0, \dots, 0)$.

Now we consider the differential operator with polynomial coefficients:

$$(2.4) \qquad P(x', y, D_x, D_y) = \sum_{\substack{\langle\sigma,\nu\rangle+|\gamma|=\langle q,\alpha\rangle+|\alpha+\beta|-m \\ |\alpha+\beta|\le m}} a_{\alpha\beta\nu\gamma} {x'}^{\nu} y^{\gamma} D_x^{\alpha} D_y^{\beta},$$
$$a_{\alpha\beta\nu\gamma} \in \mathbf{C}, \quad \alpha, \nu \in \mathbf{Z}_+^k, \quad \beta, \gamma \in \mathbf{Z}_+^n,$$

where $a_{\alpha\beta\nu\gamma}$ can be non-zero only when $|\gamma| = \langle q, \alpha\rangle + |\alpha + \beta| - m - \langle\sigma, \nu\rangle$ is a non-negative integer and we write $|\alpha + \beta| = |\alpha| + |\beta|$. We may consider $\nu = (\nu_1, \nu_2, \dots, \nu_p, 0, \dots, 0)$.

We can see that the symbol $P(x', y, \xi, \eta)$ satisfies the following condition.

Condition 1 (quasi-homogeneity). We have

$$P(\lambda^{-\sigma} x', \lambda^{-1} y, \lambda^{1+q}\xi, \lambda\eta) = \lambda^m P(x', y, \xi, \eta), \quad \lambda > 0, \quad y, \eta \in \mathbf{R}^n, \quad x, \xi \in \mathbf{R}^k,$$

where $\lambda^{-\sigma} x' = (\lambda^{-\sigma_1} x_1, \dots, \lambda^{-\sigma_p} x_p)$ and $\lambda^{1+q}\xi = (\lambda^{1+q_1}\xi_1, \dots, \lambda^{1+q_k}\xi_k)$.

We add the following two conditions on P.

Condition 2 (ellipticity). The operator P is elliptic for $|x'| + |y| = 1$.

Condition 3 (non-zero eigenvalue). For all $\omega \in \mathbf{R}^k$, $|\omega| = 1$, the equation

$$P(x', y, \omega, D_y)v(y) = 0 \quad \text{in} \quad \mathbf{R}^n$$

has no non-trivial solution in $\mathcal{S}(\mathbf{R}_y^n)$.

We denote $\varrho_0 = (1 + q_k)/(1 + q_p) \le 1$, $\sigma_0 = \max(\sigma_1, \dots, \sigma_p) < 1 + q_k$ by assumption and $\delta = \sigma_0/(1 + q_k) < 1$. We set the Gevrey index $\theta_j = \max((1 + q_j)/(1 + q_k), 1/(1 - \varrho_0\delta)), \delta = \sigma_0/(1 + q_k)$, for $j = 1, 2, \dots, p$ and $\theta_j = (1 + q_j)/(1 + q_k)$ for $j = p + 1, \dots, k$. We set $d = (\theta_1 + q_1)/(1 + q_1) \cdot I_n = ((\theta_1 + q_1)/(1 + q_1), \dots, (\theta_1 + q_1)/(1 + q_1))$.

We shall prove the following theorem in Sections 4 through 6 under these conditions on P.

Theorem 2.1 (cf. [26]). *Let Ω be an open neighborhood of $(0,0)\in\mathbf{R}^{k+n}$ and consider the equation*

$$P(x', y, D_x, D_y)u(x,y) = f(x,y) \quad in \quad \Omega, \tag{2.5}$$

where $u(x,y) \in C^\infty(\Omega)$ and $f(x,y) \in G^{\{\theta,d\}}_{x,y}(\Omega)$. Then we have $u \in G^{\{\theta,d\}}_{x,y}(\Omega)$. Here $\theta = (\theta_1, \theta_2, \dots, \theta_k)$ and $d = (\theta_1 + q_1)/(1 + q_1) \cdot I_n$ as given above.

Remark 2.1. In the above theorem we can see that

$$\begin{aligned} &\text{(i)} \quad \theta_1 = 1 \Longleftrightarrow (\theta, d) = (1, \dots, 1), \\ &\text{(ii)} \quad \theta_1 > 1 \Longleftrightarrow 1 < d < \theta_1. \end{aligned}$$

Remark 2.2. Beyond the Conditions 1 through 3, the major hypothesis

$$1 + q_k > \sigma_0 = \max(\sigma_1, \dots, \sigma_p)$$

plays an essential role throughout the paper. Hence, a problem to weaken this hypothesis remains open.

Examples 1. (a) For the operator $L = \partial_y^2 + (x^{2l} + y^{2k})\partial_x^2$, $(l, k = 1, 2, \dots)$, given in the introduction, we have $q_1 = k, \sigma_1 = k/l, \delta = k/(l(1+k))$ and $\theta = 1/(1-\delta) = l(1+k)/(l(1+k)-k), d = (\theta+k)/(1+k)$. The optimality of the index $\{\theta, d\}$ was shown in the paper [28].

(b) For the operator $M = \partial_y^2 + (x^{2l} + y^{2k})(\partial_x^2 + \partial_z^2)$, $(l, k = 1, 2, \dots)$, given in the introduction, we have $q_1 = q_2 = k, \sigma_1 = k/l, \sigma_2 = 0, x' = x_1, x'' = x_2, \delta = k/(l(1+k)), \theta_1 = 1/(1-\delta) = (l(1+k))/(l(1+k)-k), \theta_2 = 1$ and $d = (\theta_1 + k)/(1+k)$.

(c) Let $q_1, q_2, \dots, q_k$ be integers such that

$$q_1 \geq q_2 \geq \cdots \geq q_k \geq 0, \quad q_1 > q_k.$$

Then the operator

$$D_y^2 + \sum_{j=1}^{k} y^{2q_j} D_{x_j}^2 \tag{2.6}$$

has $G^{\{\theta,(\theta_1+q_1)/(1+q_1)\}}_{x,y}$-hypoellipticity in a neighborhood of the origin in $\mathbf{R}^{k+1}_{x,y}$, where $\theta = (((1+q_1)/(1+q_k)), ((1+q_2)/(1+q_k)), \dots, 1)$. Note that $\delta = 0$ in this case.

We shall show now the optimality of the exponent $\{\theta, ((\theta_1+q_1)/(1+q_1))\}$ at the origin by the method given in [2]. By the results of [32], we know that there exists a positive number a such that the ordinary differential equation

$$v''(t)-\sum_{j=1}^{k-1} t^{2q_j}v(t)+a^2t^{2q_k}v(t)=0 \tag{2.7}$$

has a non-trivial solution $v(t)\in L_2(\mathbf{R})$. Then by [26], we know $v(t)\in \mathcal{S}^{q_1/(1+q_1)}_{1/(1+q_1)}(\mathbf{R})$ which is a space of Gel'fand-Shilov, [8]. That is to say, there are positive constants C_0 and C_1 such that

$$\sup_{-\infty<t<\infty}|t^l\partial_t^j v(t)|\le C_0C_1^{l+j}l!^{\frac{1}{1+q_1}}j!^{\frac{q_1}{1+q_1}},\quad l,j=1,2,\dots. \tag{2.8}$$

Then by [8], we can see that for any small positive number ε, there are infinitely many numbers, $j_i, i=1,2,\dots$, such that

$$|\partial_t^{j_i}v(0)|\ge \varepsilon^{j_i}j_i!^{\frac{q_1}{1+q_1}-\varepsilon}. \tag{2.9}$$

Now we define the function

$$\begin{aligned}u(x,y)=&\int_0^\infty \exp\left[i\left(x_1\varrho^{\frac{1+q_1}{1+q_k}}+x_2\varrho^{\frac{1+q_2}{1+q_k}}+\cdots+x_{k-1}\varrho^{\frac{1+q_{k-1}}{1+q_k}}\right)+a\varrho x_k\right]\\ &\times v\left(\varrho^{\frac{1}{1+q_k}}y\right)e^{-\varrho}d\varrho.\end{aligned} \tag{2.10}$$

We can see the function u in (2.10) is a solution of the differential equation

$$\left(D_y^2+\sum_{j=1}^k y^{2q_j}D_{x_j}^2\right)u(x,y)=0$$

in a neighborhood of the origin. Furthermore, we can easily see that

$$|\partial_{x_j}^l u(0,0)|\sim l!^{\frac{1+q_j}{1+q_k}},\quad j=1,\dots,k$$

and

$$|\partial_y^l u(0,0)|\sim|v^{(l)}(0)|\int_0^\infty \varrho^{\frac{l}{1+q_k}}e^{-\varrho}d\varrho.$$

By (2.8) and (2.9) we have

$$|\partial_y^l u(0,0)|\sim l!^{\frac{q_1}{1+q_1}}l!^{\frac{1}{1+q_k}}.$$

We see

$$\frac{q_1}{1+q_1} + \frac{1}{1+q_k} = \frac{q_1 + \frac{1+q_1}{1+q_k}}{1+q_1} = \frac{\theta_1 + q_1}{1+q_1}.$$

This shows the optimality of the exponent $\{\theta, ((\theta_1 + q_1)/(1 + q_1))\}$ for the operator given in (2.6).

§3. Some Elementary Preparation for Non-isotropic Pseudodifferential Operators of (ϱ, δ)-type

We write $x = (x_1, x_2, \dots, x_n)$ and $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbf{R}^n$. Let $\varrho = (\varrho_1, \varrho_2, \dots, \varrho_n), 0 < \varrho_j \le 1, j = 1, 2, \dots, n$ and $0 \le \delta < 1$. We set

$$|\xi|_\varrho = |\xi_1|^{\varrho_1} + |\xi_2|^{\varrho_2} + \cdots + |\xi_n|^{\varrho_n}.$$

Let $\Omega \subset \mathbf{R}^n$ be an open set. We divide $x \in \Omega$ such as

$$x = (x_1, x_2, \dots, x_p, x_{p+1}, \dots, x_n) = (x', x'').$$

We consider $x = x'$ when $p = n$ and $x = x''$ when $p = 0$. We divide also the multi-index $\alpha = (\alpha', \alpha'') = (\alpha_1, \dots, \alpha_p, \alpha_{p+1}, \dots, \alpha_n), 1 \le p \le n$.

Definition 3.1. A function $a(x, \xi) \in C^\infty(\Omega \times \mathbf{R}^n)$ is said in the symbol class $S^m_{\varrho,\delta}(\Omega \times \mathbf{R}^n)$, if for any compact subset K of Ω there are positive constants C_0, C_1 and B such that

$$\sup_{x \in K} |a^{(\alpha)}_{(\beta)}(x, \xi)| \le C_0 C_1^{|\alpha+\beta|} \alpha! \beta! (1 + |\xi|_\varrho)^{m - |\alpha| + \delta|\beta'|}, \quad |\xi|_\varrho \ge B, \tag{3.1}$$

$$\sup_{x \in K} |a_{(\beta)}(x, \xi)| \le C_0 C_1^{|\beta|} \beta!, \quad |\xi|_\varrho \le B. \tag{3.2}$$

Here we use the notation

$$a^{(\alpha)}_{(\beta)}(x, \xi) = \partial^\alpha_\xi D^\beta_x a(x, \xi), \quad \alpha, \beta \in \mathbf{Z}^n_+.$$

For $u(x) \in C_0^\infty(\Omega)$ the pseudodifferential operator $a(x, D)$ is defined by the formula

$$\begin{aligned} a(x, D)u(x) &= (2\pi)^{-n} \iint e^{i\langle x-y, \xi\rangle} a(x, \xi) u(y) dy d\xi \\ &= (2\pi)^{-n} \int e^{i\langle x, \xi\rangle} a(x, \xi) \hat{u}(\xi) d\xi \\ &= \int K(x, y) u(y) dy, \end{aligned}$$

where the Schwartz kernel $K(x,y) \in \mathcal{D}'(\Omega \times \mathbf{R}^n)$ is an oscillatory integral defined by the formula

$$K(x,y) = (2\pi)^{-n} \int e^{i\langle x-y,\xi\rangle} a(x,\xi) d\xi. \tag{3.3}$$

Theorem 3.1. *We set* $\varrho_0 = \max(\varrho_1, \ldots, \varrho_p)$. *Then we have*

$$K(x,y) \in G_{x,y}^{\{\theta,\frac{1}{\varrho}\}}(\Omega \times \mathbf{R}^n \setminus \Delta), \tag{3.4}$$

where $\Delta = \{(x,x); x \in \Omega\}$ *and*

$$\theta = (\theta_1, \ldots, \theta_n, \theta_1, \ldots, \theta_n), \quad \frac{1}{\varrho} = \left(\frac{1}{\varrho_1}, \ldots, \frac{1}{\varrho_n}\right),$$
$$\theta_j = \max\left(\frac{1}{\varrho_j}, \frac{1}{1-\varrho_0\delta}\right), \quad j = 1, \ldots, p; \quad \theta_j = \frac{1}{\varrho_j}, \quad j = p+1, \ldots, n.$$

Proof. The idea for the proof is similar to that used in the paper [22] and in the lecture notes [27], so we shall mention briefly the essential parts of the proof.

Let U be any compact subset of $\Omega \times \mathbf{R}^n \setminus \Delta$. First we shall estimate the y-derivatives of $K(x,y)$ on U. For every $\beta \in \mathbf{Z}_+^n$ we have in the oscillatory sense:

$$D_y^\beta K(x,y) = (2\pi)^{-n} \int e^{i\langle x-y,\xi\rangle} (-\xi)^\beta a(x,\xi) d\xi.$$

We denote $r = d/n, d = dis(U, \Delta)$. For any $(x,y) \in U$ we can find some $l, 1 \le l \le n$, such that $|x_l - y_l| \ge r$. Let us write

$$\left\langle \frac{1}{\varrho}, \beta \right\rangle = \frac{1}{\varrho_1}\beta_1 + \frac{1}{\varrho_2}\beta_2 + \cdots + \frac{1}{\varrho_n}\beta_n,$$
$$N = \left[\left\langle \frac{1}{\varrho}, \beta \right\rangle\right] + 1.$$

Here we denote by $[a]$ the largest integer smaller than or equal to a.
Then we have

$$\begin{aligned} D_y^\beta K(x,y) &= (2\pi)^{-n} \int e^{i\langle x-y,\xi\rangle} (-\xi)^\beta a(x,\xi) d\xi \\ &= (2\pi)^{-n} \int_{|\xi|_\rho \le B} e^{i\langle x-y,\xi\rangle} ((-\xi)^\beta a(x,\xi)) d\xi \\ &\quad + (2\pi)^{-n} \int_{|\xi|_\rho \ge B} e^{i\langle x-y,\xi\rangle} ((-\xi)^\beta a(x,\xi)) d\xi \\ &\equiv I_1 + I_2. \end{aligned}$$

We assume $m < 0$ and $|m|$ sufficiently large for simplicity. By the assumption (3.2), we have the estimate of the form

$$|I_1| \leq MB^{|\beta|}.$$

For I_2 we have

$$\begin{aligned} I_2 &= (x_l - y_l)^{-N} \int_{|\xi|_\varrho \geq B} D_{\xi_l}^N e^{i\langle x-y,\xi\rangle} (-\xi)^\beta a(x,\xi) d\xi \\ &= \text{Boundary terms} + \int_{|\xi|_\varrho \geq B} e^{i\langle x-y,\xi\rangle} D_{\xi_l}^N (-\xi)^\beta a(x,\xi) d\xi. \end{aligned}$$

For the boundary terms we have the same type of the estimates as for I_1. By the assumption (3.1) on the symbol $a(x,\xi)$ the integrand of the last term is estimated by

$$\begin{aligned} &\left| \sum_{k=0}^{\beta_l} \frac{N!\beta_l!}{k!(N-k)!(\beta_l-k)!} \xi^\beta \xi_l^{-k} D_{\xi_l}^{N-k} a(x,\xi) \right| \\ &\leq C_0 C_1^N N! \sum_{k=0}^{\beta_l} \binom{\beta_l}{k} |\xi_1|^{\beta_1} |\xi_2|^{\beta_2} \cdots |\xi_l|^{\beta_l - k} \cdots |\xi_n|^{\beta_n} (1+|\xi|_\varrho)^{-(N-k)+m} \\ &\leq C_0' C_1'^{|\beta|} \beta!^\theta (1+|\xi|_\varrho)^m, \quad |\xi|_\varrho \geq B. \end{aligned}$$

Next we shall estimate x-derivatives of $K(x,y)$ on U. We have

$$\begin{aligned} D_x^\alpha K(x,y) &= (2\pi)^{-n} \sum_{\gamma+\tau=\alpha} \binom{\alpha}{\gamma} \int e^{i\langle x-y,\xi\rangle} \xi^\gamma a_{(\tau)}(x,\xi) d\xi \\ &= (2\pi)^{-n} \sum_{\gamma+\tau=\alpha} (-(x_l - y_l))^{-N(\tau)} \int e^{i\langle x-y,\xi\rangle} D_{\xi_l}^{N(\tau)} (\xi^\gamma a_{(\tau)}(x,\xi)) d\xi, \end{aligned}$$

where

$$N(\tau) = \left[\left\langle \frac{1}{\varrho}, \gamma \right\rangle + \delta|\tau'| \right] + 1.$$

As we can see in the proof for the isotropic case given in [22] and [27], principally we need to estimate the integrand of the last member for $|\xi|_\varrho \geq B\cdot N$. By using the assumption (3.1), we get the estimate of the form

$$|D_{\xi_l}^{N(\tau)} (\xi^\gamma a_{(\tau)}(x,\xi))| \leq C_0 C_1^{|\alpha|} \gamma!^{\frac{1}{\varrho}} \tau'!^{1+\delta} \tau''! (1+|\xi|_\varrho)^m.$$

If we assume the fact

$$1 + \delta \leq \theta_j, \quad (j = 1, 2, \ldots, p), \tag{3.5}$$

then the left-hand side of the above inequality is estimated by the quantity

$$C_0' C_1'^{|\alpha|} \gamma!^{\frac{1}{\varrho}} \tau!^{\theta} \le C_0' C_1'^{|\alpha|} \alpha!^{\theta},$$

where the constants C_0' and C_1' are taken independent of τ and α.

It remains to prove the inequality (3.5).

(1) The case where $1/(1-\varrho_0\delta) \le 1/\varrho_j = \theta_j, j = 1, 2, \ldots, p$.

Take $\varrho_j = \varrho_0 = \max(\varrho_1, \ldots, \varrho_p)$, then we have the inequality

$$\frac{1}{1-\varrho_0\delta} \le \frac{1}{\varrho_0},$$

from which we get the inequality

$$1+\delta \le \frac{1}{\varrho_0} \le \frac{1}{\varrho_j} = \theta_j, \quad j = 1, 2, \ldots, p.$$

(2) The case where there is some number $j, 1 \le j \le p$, such that

$$\frac{1}{\varrho_j} < \frac{1}{1-\varrho_0\delta} = \theta_j.$$

Then we have $1 - \varrho_0\delta < \varrho_j$, from which we have

$$1 < \varrho_j + \varrho_0\delta \le \varrho_0(1+\delta).$$

On the other hand, we have the equivalence relation

$$1+\delta \le \frac{1}{1-\varrho_0\delta}\ (\le \theta_j) \Longleftrightarrow 1 \le \varrho_0(1+\delta),$$

which fits the above inequality and we have (3.5). □

Next we shall consider the pseudolocal property of $a(x, D)$.

Lemma 3.1. *Let K be a compact subset of an open set $V \subset \mathbf{R}^n$. Then there is a sequence of functions $\{g_l(x)\} \subset C_0^\infty(V)$ and a constant C such that*

$$|D^\alpha g_l(x)| \le C^l \alpha!, \quad |\alpha| \le l, \quad l = 0, 1, 2, \ldots, \tag{3.6}$$

$$g_l(x) = 1, \quad x \in K, \quad l = 0, 1, 2, \ldots. \tag{3.7}$$

Lemma 3.2. *Let $\Omega' \subset \overline{\Omega'} \subset\subset \Omega$. Then for $f(x) \in C_0^\infty(\Omega')$ we have the estimate of the form*

$$\begin{aligned} |D_x^\alpha a(x,D) f(x)| \le & \sum_{\gamma+\tau=\alpha} \binom{\alpha}{\gamma} C^{1+|\tau|} \tau! Vol(\Omega') \\ & \cdot \sup_{x \in \Omega'} |D_x^\gamma (1+|D|_\varrho)^{\delta|\tau'|} f(x)|, \end{aligned} \tag{3.8}$$

where the constant C is taken independent of α and τ and the precise meaning of $(1+|D|_\varrho)^{\delta|\tau'|}$ will be given in the proof.

Proof of Lemma 3.2. We have

$$\begin{aligned} D_x^\alpha\{a(x,D)f(x)\} &= (2\pi)^{-n}\sum_{\gamma+\tau=\alpha}\binom{\alpha}{\gamma}\iint D_x^\gamma e^{i\langle x-y,\xi\rangle}a_{(\tau)}(x,\xi)f(y)dyd\xi \\ &= (2\pi)^{-n}\sum_{\gamma+\tau=\alpha}\binom{\alpha}{\gamma}\iint (-D_y)^\gamma e^{i\langle x-y,\xi\rangle}a_{(\tau)}(x,\xi)f(y)dyd\xi \\ &= (2\pi)^{-n}\sum_{\gamma+\tau=\alpha}\binom{\alpha}{\gamma}\iint e^{i\langle x-y,\xi\rangle}a_{(\tau)}(x,\xi)D_y^\gamma f(y)dyd\xi. \end{aligned}$$

We know that

$$|a_{(\tau)}(x,\xi)| \le C_0C_1^{|\tau|}\tau!(1+|\xi|_\varrho)^{\delta|\tau'|+m}.$$

We assume $m<0$ and $|m|$ sufficiently large for simplicity. We note that

$$\begin{aligned} (1+|\xi|_\varrho)^{\delta|\tau'|} &= (1+|\xi_1|^{\varrho_1}+\cdots+|\xi_n|^{\varrho_n})^{\delta|\tau'|} \\ &\le C^{\delta|\tau'|}\left(1+\xi_1^{2[\frac{\varrho_1\delta|\tau'|}{2}+1]}+\cdots+\xi_n^{2[\frac{\varrho_n\delta|\tau'|}{2}+1]}\right), \end{aligned}$$

where the constant C is taken independent of τ. We rewrite the last member of the above equality as follows:

$$\begin{aligned} &= (2\pi)^{-n}\sum_{\gamma+\tau=\alpha}\binom{\alpha}{\gamma}\iint\left(1-D_{y_1}^{2[\frac{\varrho_1\delta|\tau'|}{2}+1]}-\cdots-D_{y_n}^{2[\frac{\varrho_n\delta|\tau'|}{2}+1]}\right)e^{i\langle x-y,\xi\rangle}\,\cdot \\ &\quad\cdot\left(1+\xi_1^{2[\frac{\varrho_1\delta|\tau'|}{2}+1]}+\cdots+\xi_n^{2[\frac{\varrho_n\delta|\tau'|}{2}+1]}\right)^{-1}a_{(\tau)}(x,\xi)D_y^\gamma f(y)dyd\xi \\ &= (2\pi)^{-n}\sum_{\gamma+\tau=\alpha}\binom{\alpha}{\gamma}\iint e^{i\langle x-y,\xi\rangle}\left(1+\xi_1^{2[\frac{\varrho_1\delta|\tau'|}{2}+1]}+\cdots+\xi_n^{2[\frac{\varrho_n\delta|\tau'|}{2}+1]}\right)^{-1} \\ &\quad\cdot a_{(\tau)}(x,\xi)\left(1-D_{y_1}^{2[\frac{\varrho_1\delta|\tau'|}{2}+1]}-\cdots-D_{y_n}^{2[\frac{\varrho_n\delta|\tau'|}{2}+1]}\right)D_y^\gamma f(y)dyd\xi. \end{aligned}$$

This is the precise meaning of $(1+|D|_\varrho)^{\delta|\tau'|}$ and from where we have (3.8).

Theorem 3.2. *Let $a(x,D)$ be as above. Then we have the assertion*:

(3.9) $u\in\mathcal{E}'(\Omega)\cap G^{\{\theta\}}(\omega)\Rightarrow a(x,D)u\in\mathcal{D}'(\Omega)\cap G^{\{\theta\}}(\omega),\quad (\omega\subset\Omega),$

$$\theta_j=\max\left(\frac{1}{\varrho_j},\frac{1}{1-\varrho_0\delta}\right),\quad j=1,2,\dots,p;\quad \theta_j=\frac{1}{\varrho_j},\quad j=p+1,\dots,n.$$

Proof. Let U be a bounded open set and $\bar{U} \subset \omega$. Then there is a positive number d such that $dis(U, \omega^c) > d > 0$. By virtue of Lemma 3.1, we can take a sequence of functions $g_l(x) \in C_0^\infty(\omega)$ such that $g_l(x) = 1$ for $x \in \{x; dis(x, U) < d/4\}$ and

$$|D^\alpha g_l(x)| \leq C^l \alpha!, \quad |\alpha| \leq l, \quad l = 0, 1, 2, \dots .$$

Now take $l = 2|\alpha|$. Then we have for $x \in U$

$$\text{(3.10)} \qquad D^\alpha\{a(x,D)u(x)\} = D^\alpha\{a(x,D)g_l u(x)\} + D^\alpha \int K(x,y)\{1 - g_l(y)\}u(y)dy.$$

By using (3.8), we have

$$\begin{aligned} |D^\alpha\{a(x,D)g_l u(x)\}| &\leq \sum_{\gamma+\tau=\alpha} \binom{\alpha}{\gamma} C^{|\tau|+1}\tau! \sup_{x\in\omega} |D^\gamma(1+|D|_\varrho)^{\delta|\tau'|}\{g_l(x)u(x)\}| \\ &\leq \tilde{C}_1^{|\alpha|+1} \sum_{\gamma+\tau=\alpha} \tilde{C}_2^{|\gamma|+1} \gamma!^\theta \tau!(|\tau'|!^{\varrho_1\delta\theta_1} + \cdots + |\tau'|!^{\varrho_n\delta\theta_n}) \\ &\leq C_0' C_1'^{|\alpha|} \gamma!^\theta \tau''! \sum_{j=1}^n |\tau'|!^{1+\varrho_j\delta\theta_j} \\ &\leq C_0 C_1^{|\alpha|} \gamma!^\theta \tau!^\theta \leq C_0 C_1^{|\alpha|} \alpha!^\theta, \end{aligned}$$

where the constants C_0 and C_1 are taken independent of α. Here we need to estimate the last summation in the above inequalities. We shall show that we have

$$\begin{aligned} &1 + \varrho_k\delta\theta_k \leq \theta_j, \quad (k, j = 1, 2, \dots, p), \\ &1 + \varrho_k\delta\theta_k = 1 + \delta \leq \theta_j, \quad (k = p+1, \dots, n; j = 1, \dots, p), \end{aligned}$$

from where the last inequalities are derived completely.

(1) The case $\theta_k = 1/\varrho_k \geq 1/(1 - \varrho_0\delta)$, $(k = 1, \dots, p)$. In this case, we have

$$1 - \varrho_0\delta \geq \varrho_k, \quad k = 1, \dots, p.$$

Taking $\varrho_k = \varrho_0$, we have

$$\frac{1}{\varrho_0} \geq 1 + \delta$$

and

$$\theta_j = \frac{1}{\varrho_j} \geq \frac{1}{\varrho_0}, \quad j = 1, \dots, p.$$

(2) The case there is a number $k, 1 \leq k \leq p$, such that $1/\varrho_k < 1/(1-\varrho_0\delta) = \theta_k$. In this case, we have

$$1 - \varrho_0\delta < \varrho_k,$$

from where we have

$$1 < \varrho_0(1+\delta). \tag{3.11}$$

On the other hand, we have

$$1 + \varrho_0\delta\theta_k = 1 + \varrho_0\delta\frac{1}{1-\varrho_0\delta} = \frac{1}{1-\varrho_0\delta} \leq \theta_j, \quad j = 1, \dots, p.$$

If there is another number $k, 1 \leq k \leq p$, such that $\theta_k = 1/\varrho_k \leq 1/(1-\varrho_0\delta)$, by applying (3.10) we have

$$1 + \varrho_k\delta\theta_k = 1 + \delta \leq \frac{1}{1-\varrho_0\delta} \leq \theta_j, \quad j = 1, \dots, p.$$

□

In what follows we shall consider the symbolic calculus and the Gevrey hypoellipticity of the pseudodifferential operators with symbols given in Definition 3.1. The method is similar to that of [22], [27, Section 12] with some revision just like above. Therefore we omit the proof. Let Ω' be a relatively compact open subset of Ω. Let $a(x,\xi) \in S^{m'}_{\varrho,\delta}(\Omega \times \mathbf{R}^n), b(x,\xi) \in S^{m''}_{\varrho,\delta}(\Omega \times \mathbf{R}^n)$. Now consider the product

$$r(x,D) \equiv a(x,D)h(x)b(x,D),$$

where $h(x) \in C_0^\infty(\Omega)$ such that $h(x) \equiv 1$ in a neighborhood of $\overline{\Omega'}$. The symbol $r(x,\xi)$ of $r(x,D)$ is given by

$$\begin{aligned} r(x,\xi) &= a(x,D+\xi)h(x)b(x,\xi) \\ &= (2\pi)^{-n} \int e^{i\langle x-y,\xi'\rangle} a(x,\xi'+\xi)h(y)b(y,\xi)dyd\xi' \end{aligned} \tag{3.12}$$

We set

$$r^N(x,\xi) = \sum_{|\alpha|\leq N} \frac{1}{\alpha!} a^{(\alpha)}(x,\xi) b_{(\alpha)}(x,\xi) \tag{3.13}$$

We can see easily that there is a couple of constants C_0 and C_1 such that

$$\sup_{x\in K}|D_x^\beta \partial_\xi^\gamma r^N(x,\xi)| \le C_0 C_1^{N+|\beta+\gamma|} N!\gamma!\beta!(1+|\xi|_\varrho)^{m-|\gamma|+\delta|\beta'|}, \quad |\xi|_\varrho \ge B, \tag{3.14}$$

where $m = m' + m''$.

Theorem 3.3. *Each $r^N(x,D)$ is an approximation of $r(x,D)$ in the following sense: We have*

$$r(x,D) - r^N(x,D) = F^N(x,D) \quad \text{in } \Omega', \tag{3.15}$$

where $F^N(x,D)$ can be written as a sum of two operators

$$F^N(x,D) = F_1^N + F_2^N,$$

F_1^N is an integral operator with the kernel $F_1^N(x,y) \in G^{\{\theta\}}(\Omega' \times \Omega')$ and F_2^N is a pseudodifferential operator with symbol $F_2^N(x,\xi)$ satisfying the condition

$$\begin{aligned}\sup_{x\in\Omega'}|D_x^\beta \partial_\xi^\gamma F_2^N(x,\xi)| &\le C_0 C_1^{N+|\beta+\gamma|} N!\gamma!\beta!|\xi|_\varrho^{m_+ + n' - (1-\delta)N - |\gamma|} \\ &\quad \cdot \sum_{\tau\le\beta}\binom{\beta}{\tau}|\tau|^{\delta|\tau'|}|\xi|_\varrho^{\delta|\beta'-\tau'|+\delta^2|\tau'|}, \\ &|\xi|_\varrho \ge B, m_+ = \max(m,0), \quad n' = n'(n).\end{aligned} \tag{3.16}$$

Theorem 3.4. *Let $a(x,\xi) \in S^m_{\varrho,\delta}(\Omega \times \mathbf{R}^n)$ and assume there are positive constants c, B and $-\infty < m' < \infty$ such that*

$$|a(x,\xi)| \ge c|\xi|_\varrho^{m'}, \quad x \in \Omega, \quad |\xi|_\varrho \ge B. \tag{3.17}$$

Assume also that for any compact set $K \subset \Omega$, there are positive constants C_0 and C_1 such that

$$\begin{aligned}\left|a^{(\alpha)}_{(\beta)}(x,\xi)\right| &\le C_0 C_1^{|\alpha+\beta|}\alpha!\beta!|a(x,\xi)||\xi|_\varrho^{-|\alpha|+\delta|\beta'|}, \\ &x \in K, \quad |\xi|_\varrho \ge B.\end{aligned} \tag{3.18}$$

Then the operator $a(x,D)$ is Gevrey hypoelliptic of order $\{\theta\}$ given in Theorem 3.2.

Example 1. Take a differential operator considered in the paper [22]:

$$P = x_1^4(\partial_{x_2} - \partial_{x_1}^2) + 1, \quad (x_1, x_2) \in \mathbf{R}^2.$$

Let Ω be small open neighborhood of $(0,0)$. We can see $n = 2, p = 1, \varrho_1 = 1, \varrho_2 = 1/2, \delta = 1/2$ and $\{\theta\} = \{2,2\}$ for the operator P on Ω. Optimality of this exponent is shown as follows. First take a function

$$u_0(x_1,x_2) = \begin{cases} x_1 e^{-\frac{1}{x_1}}, & x_1 > 0 \\ 0, & x_1 \le 0. \end{cases}$$

The function u_0 satisfies the equation $Pu_0 = 0$ in $\mathbf{R}^2$ and it is well known the function u_0 is in $G^{\{2\}}(\Omega)$. Next take a solution $u_1 \in G^{\{1,2\}}(\Omega)$ to the heat equation

$$(\partial_{x_2} - \partial_{x_1}^2)u_1(x_1,x_2) = 0 \quad \text{in} \quad \Omega.$$

We seak a function u such that

$$P(u_1 + u) = u_1 + Pu = 0 \quad \text{in} \quad \Omega.$$

We know adjoint operator ${}^tP$ is also hypoelliptic so that such a function $u(x_1,x_2) \in G^{\{1,2\}}$ exists in Ω (shrinked if necessary) because of the solvability and Gevrey hypoellipticity of P.

§4. Proof of Theorem 2.1: I, Gevrey Regularity in x

For the proof of Gevrey hypoellipticity of the operator P given in (2.4) with respect to the variable $(x_1, x_2, \dots, x_k)$, we rely upon the method of pseudodifferential operators used in the paper [27] by making use of the preparation in Section 3. We introduce the notations

$$\begin{aligned} \langle\xi\rangle &= |\xi_1|^{\frac{1}{1+q_1}} + |\xi_2|^{\frac{1}{1+q_2}} + \dots + |\xi_k|^{\frac{1}{1+q_k}}, \\ |\xi|_\varrho &= |\xi_1|^{\frac{1+q_k}{1+q_1}} + |\xi_2|^{\frac{1+q_k}{1+q_2}} + \dots + |\xi_k|, \\ |x'|_\sigma &= |x_1|^{\frac{1}{\sigma_1}} + \dots + |x_p|^{\frac{1}{\sigma_p}}, \quad \varrho_j - \frac{1+q_k}{1+q_j}, j = 1, \dots, k. \end{aligned}$$

Under the conditions 1 through 3 on P, the following a priori estimate, called Grushin inequality, can be obtained:

Theorem 4.1 (cf. [12]). *There exists a positive constant C such that*

(4.1)

$$\sum_{|\beta|\le m} \int |(\langle\xi\rangle + (|x'|_\sigma + |y|)^{q_1}|\xi_1| + \dots + (|x'|_\sigma + |y|)^{q_k}|\xi_k|)^{m-|\beta|} D_y^\beta v(y)|^2 dy$$
$$\le C \int |P(x',y,\xi,D_y)v(y)|^2 dy, \quad \xi \in \mathbf{R}^k \setminus \{0\}, \quad v \in C_0^\infty(\mathbf{R}_y^n).$$

We denote $P^{(\mu)}_{(\lambda)}(x', y, \xi, D_y) = \partial^\mu_\xi D^\lambda_x P(x', y, \xi, D_y)$, $\mu, \lambda \in \mathbf{Z}^k_+$, as usual. Then we can derive the following estimates from (4.1).

Theorem 4.2 (cf. [12]). *There are positive constants C_0, C_1 and B such that*

$$\|P^{(\mu)}_{(\lambda)}(x', y, \xi, D_y)v(y)\| \le C_0 C_1^{|\mu+\lambda|}\mu!\lambda!|\xi|_\varrho^{-|\mu|+\delta|\lambda'|}\|P(x', y, \xi, D_y)v(y)\|, \quad v \in C_0^\infty(\mathbf{R}^n_y), \quad \mu, \lambda \in \mathbf{Z}^k_+, \quad |\xi|_\varrho \ge B. \tag{4.2}$$

Here $\|\cdot\|$ denotes the L_2-norm and $\delta = \sigma_0/(1+q_k)$ (see Section 2).

Let q be a positive rational number such that qm is an integer. Then we denote by $H_{(m,q)}(\mathbf{R}^n_y)$ a weighted Sobolev space in $L_2(\mathbf{R}^n_y)$, equipped with the norm

$$\|u\|_{H_{(m,q)}} = \left(\sum_{\substack{0\le|\gamma|\le q(m-|\beta|)\\ 0\le|\beta|\le m}} \|y^\gamma D^\beta_y u\|^2_{L_2(\mathbf{R}^n)}\right)^{\frac{1}{2}}. \tag{4.3}$$

We have the topological inclusion

$$H_{(m,q)}(\mathbf{R}^n) \subset H_m(\mathbf{R}^n) \subset L_2(\mathbf{R}^n). \tag{4.4}$$

Theorem 4.3 (cf. [26], Theorems 3.3, 6.4, 7.4, [12] and [22]).

(i) *Let $n \ge 2$ or $n = 1$ and $\sigma \ne (0, \dots, 0)$. Then there is the inverse $G(x', \xi) \in \mathcal{L}(L_2(\mathbf{R}^n_y), H_{(m,q_1)}(\mathbf{R}^n_y))$ of $P(x', y, \xi, D_y)$ such that*

$$G(x', \xi)P(x', y, \xi, D_y) = I \quad \text{in} \quad H_{(m,q_1)}(\mathbf{R}^n_y), \tag{4.5}$$

$$P(x', y, \xi, D_y)G(x', \xi) = I \quad \text{in} \quad L_2(\mathbf{R}^n_y). \tag{4.6}$$

There are constants C_0 and C_1 such that

$$\|G^{(\mu)}_{(\lambda)}\|_{(m)} \le C_0 C_1^{|\mu+\lambda|}\mu!\lambda!|\xi|_\varrho^{-|\mu|+\delta|\lambda'|}, \quad \mu, \lambda \in Z^k_+, \quad \xi \in \mathbf{R}^k \setminus \{0\}, \quad |\xi|_\varrho \ge B. \tag{4.7}$$

(ii) *In case $n = 1$ and $\sigma = (0, \dots, 0)$, let Π be the orthogonal projection on the null space of ${}^tP(y, \xi, D_y)$. Then there is a pseudoinverse $G(\xi) \in \mathcal{L}(L_2(\mathbf{R}_y), H_{(m,q_1)}(\mathbf{R}_y))$ of $P(y, \xi, D_y)$ such that*

$$G(\xi)P(y, \xi, D_y) = I \quad \text{in} \quad H_{(m,q_1)}(\mathbf{R}_y), \tag{4.8}$$

$$P(y, \xi, D_y)G(\xi) = I - \Pi(\xi) \quad \text{in} \quad L_2(\mathbf{R}_y). \tag{4.9}$$

There are constants C_0 and C_1 such that

$$\|G^{(\mu)}(\xi)\|_{(m)} \le C_0 C_1^{|\mu|}\mu!|\xi|_\varrho^{-|\mu|}, \quad \mu \in Z^k_+, \quad |\xi|_\varrho \ge B. \tag{4.10}$$

Now we can apply the results obtained in Section 3 and in the papers [12], [22] and [26]. Let U be a small neighborhood of the origin of $\mathbf{R}_x^k$ and $B_\mu = \{y \in \mathbf{R}_y^n; |y| \leq \mu\}$. Then starting with $G(x', D)$ we can construct an operator valued parametrix $\tilde{G}(x, D)$ of $Q(x, D) \equiv P(x', y, D_x, D_y)$ such that symbolically

$$\tilde{G}(x, D)Q(x, D) = I + \mathcal{R} \quad \text{in} \quad C^\infty(U : L_2(B_\mu)).$$

Here $\mathcal{R}$ is a regularizer in x. In such a manner by using the method described in Section 3, (cf. [22]), we can show the Gevrey hypoellipticity in the x-direction.

Theorem 4.4. *Let P be the same operator as in Theorem* 2.1 *and consider the equation*

$$P(x', y, D_x, D_y)u(x, y) = f(x, y) \quad \text{in} \quad \Omega, \tag{4.11}$$

where $u(x, y) \in C^\infty(\Omega)$ and $f(x, y) \in G_{x,y}^{\{\theta,d\}}(\Omega)$, where $\theta = (\theta_1, \ldots, \theta_k)$ and $d = (\theta_1+k)/(1+k)\cdot I_n$ which are given in Section 2. *Then we have $u \in G_x^{\{\theta\}}(\Omega)$.*

§5. Proof of Theorem 2.1: II, Gevrey Regularity in y; Bracket Calculus

In this section, we shall prove the Gevrey hypoellipticity in the y-direction for the operators P given in (2.4) in case $\sigma = 0$, that is, for the operators of the third group by the classification in the paper [26]. In this case, the operator P is written as follows:

$$P(y, D_y, D_x) = \sum_{\substack{|\gamma| = \langle q, \alpha\rangle + |\alpha+\beta| - m \\ |\alpha+\beta| \leq m}} a_{\alpha\beta\gamma} y^\gamma D_x^\alpha D_y^\beta \tag{5.1}$$

Then we have $\delta = 0$ and

$$\begin{aligned} \theta &= (\theta_1, \ldots, \theta_j, \ldots, \theta_k), \quad \theta_j = \frac{1+q_j}{1+q_k}, \quad j = 1, \ldots, k, \\ \theta_1 &\geq \theta_2 \geq \cdots \geq \theta_k = 1, \quad \theta_1 > 1, \\ d &= \frac{\theta_1 + q_1}{1+q_1} \cdot I_n = \left(\frac{\theta_1 + q_1}{1+q_1}, \ldots, \frac{\theta_1+q_1}{1+q_1}\right). \end{aligned}$$

Let Ω be an open neighborhood of $(0, 0) \in \mathbf{R}^{k+n}$ and consider the equation

$$P(y, D_x, D_y)u(x, y) = f(x, y) \quad \text{in} \quad \Omega, \tag{5.2}$$

where $u(x,y) \in C^\infty(\Omega) \cap G_x^{\{\theta\}}(\Omega)$ and $f \in G_{x,y}^{\{\theta,d\}}(\Omega)$. Then our purpose is to prove $u(x,y) \in G_{x,y}^{\{\theta,d\}}(\Omega)$.

In case $\sigma = 0$, the estimate (4.1) yields the following one by Fourier transformation:

$$\sum_{|\beta|\le m} \int |(1+|y|^{q_1}|D_{x_1}| + \cdots + |y|^{q_k}|D_{x_k}|)^{m-|\beta|} D_y^\beta u(x,y)|^2 dxdy \tag{5.3}$$
$$\le C \int |P(y,D_x,D_y)u(x,y)|^2 dxdy, \quad u \in C_0^\infty(\Omega).$$

By an investigation of the quasi-homogeneity in ξ and y, we have the following estimate with a positive constsnt $C = C(\Omega)$:

$$\|y^\gamma D_x^{\alpha-\tilde{\alpha}} D_y^{\beta+|\tilde{\alpha}|} u(x,y)\| \le C\|P(y,D_x,D_y)u(x,y)\|, \quad u \in C_0^\infty(\Omega), \tag{5.4}$$

where $|\gamma| = \langle q,\alpha\rangle + |\alpha| + |\beta| - m, |\alpha| + |\beta| \le m$, and $0 \le \tilde{\alpha} \le \alpha$ and $D_y^{|\tilde{\alpha}|}$ denotes any derivative of the order $|\tilde{\alpha}|$ of u with respect to y.

Furthermore, we can obtain the following estimate by applying the three line theorem of complex analysis, (cf. [20]).

Theorem 5.1. *There exists a positive constant $C = C(\Omega)$ such that for any $\mu, 0 < \mu < 1$, we have*

$$\|(1+|y|^{q_1}|D_{x_1}| + \cdots + |y|^{q_k}|D_{x_k}|)^{m(1-\mu)}(1-\Delta_y)^{\frac{m}{2}\mu}u(x,y)\| \tag{5.5}$$
$$\le C\|P(y,D_x,D_y)u(x,y)\|, \quad u \in C_0^\infty(\Omega).$$

Proof. Let us write $h = |y|^{q_1}|D_{x_1}| + \cdots + |y|^{q_k}|D_{x_k}|$. Then from the estimate (5.3) we have the following inequality:

$$\|(1+h)^m u(x,y)\| + \|(1-\Delta_y)^{\frac{m}{2}}u(x,y)\| \le C\|P(y,D_x,D_y)u(x,y)\|,$$
$$u(x,y) \in C_0^\infty(\Omega).$$

For $v \in C_0^\infty(\mathbf{R}_y^n)$ consider an $L_2(\mathbf{R}_y^n)$-valued function

$$f(z) = (1+h)^{m(1-z)}(1-\Delta_y)^{\frac{m}{2}z}v, \quad z \in \mathbf{C},$$

Apparently $f(z)$ is holomorphic in $z \in \mathbf{C}$ and bounded in the strip $0 \le |\mathrm{Re}(z)| \le 1$. By applying the three line theorem with any $\mu, 0 < \mu < 1$, we have

$$\begin{aligned}\|f(\mu)\| &\le \sup_{\eta\in\mathbf{R}} \|(1+h)^{m(1-i\eta)}(1-\Delta_y)^{\frac{m}{2}(i\eta)}v\| \\ &\quad + \sup_{\eta\in\mathbf{R}} \|(1+h)^{m(i\eta)}(1-\Delta_y)^{\frac{m}{2}(1+i\eta)}v\| \\ &\le \|(1+h)^m v\| + \|(1-\Delta_y)^{\frac{m}{2}}v\|.\end{aligned}$$

We have the inequality

$$\|(1+h)^{m(1-\mu)}(1-\Delta_y)^{\frac{m}{2}\mu}v\| \le \|(1+h)^m v\| + \|(1-\Delta_y)^{\frac{m}{2}}v\|.$$

Finally by applying Fourier transformation in ξ and x, we get (5.5). □

Now in Theorem 2.1, if $q_1 = q_2 = \cdots = q_k > 0$ we have $\theta = (1, 1, \ldots, 1)$ and $d = (1+q_1)/(1+q_k) = 1$ and the operator P is analytic hypoelliptic. In fact P belongs to the first group by the classification in the paper [26] and there proved that it is analytic hypoelliptic in the space of hyperfunctions.

Therefore, we assume that $q_1 > q_k \ge 0$ in the following. Then we have $1 < d = (\theta_1+q_1)/(1+q_k) < \theta_1 = (1+q_1)/(1+q_k)$. Thus, our purpose is to show Gevrey hypoellipticity of the operator P with the exponent $d = (\theta_1+q_1)/(1+q_k)$ in $y = (y_1, y_2, \ldots, y_n)$.

Let us consider the equation

$$P(y, D_x, D_y)u(x,y) = f(x,y) \quad \text{in} \quad \Omega,$$

where $u(x,y) \in C^\infty(\Omega) \cap G_x^{\{\theta\}}(\Omega)$. Let ω be a small neighborhood of the origin such that $\bar{\omega} \subset \Omega$ and δ be a sufficiently small positive number. Then we can prepare a set of cut-off functions $\phi_j(x,y) \in C_0^\infty(\Omega)$ satisfying

$$\begin{gathered}\phi_j \equiv 1 \quad \text{on} \quad \omega,\\ \bigtriangledown_y \phi_j(x,y) \equiv 0, \quad |y| \le \delta,\\ |D_{x,y}^\alpha \phi_j(x,y)| \le C_0 C_1^{|\alpha|} j^{|\alpha|}, \quad |\alpha| \le mj,\end{gathered}$$

where positive constants C_0 and C_1 are independent of $j = 1, 2, \ldots$, (cf. [15]).

We assume that the number j is larger than m and mq_1 and let $D_y^j u$ denote any derivative of the j-th order of u in y. By the inequality (5.3) we have

$$\|D_y^m \phi_j D_y^j u\| \le C'\|P\phi_j D_y^j u\| \le C\{\|\phi_j D_y^j P u\| + \|[P, \phi_j D_y^j]u\|\}. \tag{5.6}$$

By the assumption in Theorem 2.1 with respect to $f(x,y)$, the first term in the last side is estimated by the quantity of the form $C_0 C_1^j j!^d$, where $d = (\theta_1 + q_1)/(1+q_k)$ and we have to investigate only the last term in the right-hand side. For simplicity we denote by $\phi = \phi_j$ in the following. The last term in (5.6) consists of a linear sum of the terms

$$\begin{aligned}[y^\gamma D_x^\alpha D_y^\beta, \phi D_y^j]u = &\sum_{\substack{0\le\tilde{\alpha}\le\alpha\\ 0\le\tilde{\beta}\le\beta\\ 0<\tilde{\alpha}+\tilde{\beta}}} y^\gamma \binom{\alpha}{\tilde{\alpha}}\binom{\beta}{\tilde{\beta}}(D_x^{\tilde{\alpha}} D_y^{\tilde{\beta}}\phi) D_x^{\alpha-\tilde{\alpha}} D_y^{\beta-\tilde{\beta}} D_y^j u\\ &- \phi \sum_{0<\nu\le\gamma} \binom{j}{\nu}(D_y^\nu y^\gamma) D_x^\alpha D_y^{\beta+j-\nu} u,\end{aligned} \tag{5.7}$$

where $|\gamma| = \langle q, \alpha\rangle + |\alpha+\beta| - m$ and $|\alpha+\beta| \le m$.

We can see that there is almost no problem with the first summation in the right-hand side. In fact, since the operator P is uniformly elliptic for $|y| \ge \delta, L_2$-norm of the terms with $\tilde\beta \ne 0$ are estimated by the quantity of the kind $C_0 C_1^j j!^d$, where the constants C_0 and C_1 are taken independent of j. We shall call such terms non-disturbing. Therefore, we need to investigate the L^2-norm of the terms with $\tilde\beta = 0, 0 < \tilde\alpha \le \alpha, 0 \le |\beta| < m$ and $|\alpha+\beta| \le m$ in the first summation:

$$\binom{\alpha}{\tilde\alpha} \|y^\gamma D_x^{\tilde\alpha}\phi D_x^{\alpha-\tilde\alpha} D_y^\beta D_y^j u\|.$$

First we shall consider the case where $\tilde\alpha = \alpha$. Denoting by $D_y^{|\alpha|}$ any derivative of the order $|\alpha|$ with respect to y and so on, we see this is equal to

$$\begin{aligned} \|y^\gamma D_x^\alpha\phi \cdot D_y^\beta D_y^j u\| &= \|y^\gamma D_x^\alpha \phi \cdot D_y^{\beta+|\alpha|} D_y^{j-|\alpha|} u\| \\ &= \|y^\gamma D_y^{\beta+|\alpha|}(D_x^\alpha\phi \cdot D_y^{j-|\alpha|}u)\| + \text{non-disturbing terms.} \end{aligned}$$

By the estimates (5.4), we have

$$\|y^\gamma D_y^{\beta+|\alpha|}(D_x^\alpha\phi \cdot D_y^{j-|\alpha|}u)\| \le C\|P(D_x^\alpha\phi \cdot D_y^{j-|\alpha|}u)\|,$$

where the constant C can be taken independent of j and α. In the right-hand side of the above inequality, we see the order of the derivative in y decreases: $D_y^j u \longrightarrow D_y^{j-|\alpha|}u$, and the same times of derivation of ϕ with respect to x increases: $\phi \longrightarrow D_x^\alpha\phi$.

For each term with $0 < \tilde\alpha < \alpha$, using a finite times (independent of j) of commutation, we see it is essentially (except non-disturbing terms) estimated by a constant times of

$$\sum_{0<\tilde\alpha<\alpha} \|P(D_x^{\tilde\alpha}\phi \cdot D_y^{j-|\tilde\alpha|}u)\|$$

by applying (5.3) and (5.4). By virtue of this procedure, we see y-derivation of u decreases: $D_y^j u \longrightarrow D_y^{j-|\tilde\alpha|}u$. The same times of the derivation of ϕ with respect to x increases: $\phi \longrightarrow D_x^{j-|\tilde\alpha|}\phi$. Continuing finite times (at most j-times) of these steps, we can see finally L_2-norm of the first summation in the right-hand side of (5.7) is estimated by the quantity of the kind $C_0 C_1^j j!^d$, where C_0 and C_1 are independent of j.

It remains to treat the last summation in (5.7). We need to estimate the terms

$$\binom{j}{\nu} \|\phi y^{\gamma-\nu} D_x^\alpha D_y^\beta D_y^{j-\nu} u\|, \quad 0 < \nu \le \gamma, \tag{5.8}$$

$$|\gamma| = \langle q, \alpha\rangle + |\alpha+\beta| - m, \quad |\alpha+\beta| \le m.$$

First we consider the term with $\nu = \gamma$:

$$\binom{j}{\gamma}\|\phi D_x^\alpha D_y^\beta D_y^{j-\gamma}u\| = \binom{j}{\gamma}\|\phi D_y^m D_y^\beta D_y^{j-m-\gamma}D_x^\alpha u\|$$
$$= \binom{j}{\gamma}\|D_y^m(\phi D_y^{j-m+\beta-\gamma}D_x^\alpha u)\| + \text{non-disturbing terms.}$$

By the estimate (5.3), we have

$$\binom{j}{\gamma}\|D_y^m(\phi D_y^{j-m+\beta-\gamma}D_x^\alpha u)\| \leq C\binom{j}{\gamma}\|P\phi D_y^{j-m+\beta-\gamma}D_x^\alpha u\|. \tag{5.9}$$

We consider the procedure from (5.6) to (5.9) as a typical part of the first cycle of the total procedure (cf. [2]). That is, we started from (5.6) and we see that the order of derivative in y of u decreases with multiplication by $\binom{j}{\gamma}$ and the order of the derivative in x of u increases:

$$D_y^j u \longrightarrow \binom{j}{\gamma}D_y^{j-(m-\beta+\gamma)}D_x^\alpha u.$$

Since we have $\binom{j}{\gamma} \leq j^{|\gamma|}$, we may consider that, in such a typical cycle, for every loss of the power of D_y there corresponds to the effect of multiplication of the kind

$$j^{\frac{|\gamma|+\langle\theta,\alpha\rangle}{m-|\beta|+|\gamma|}}.$$

By using the assumption $|\gamma| = \langle q,\alpha\rangle + |\alpha+\beta| - m \leq \langle q,\alpha\rangle$ and $\theta = (1+q)/(1+q_k)$, we can see such exponent of j is always smaller than or equal to d:

$$\frac{|\gamma| + \langle\theta,\alpha\rangle}{m-|\beta|+|\gamma|} \leq \frac{\theta_1+q_1}{1+q_1} = d.$$

It remains finally to investigate the terms with $0 < \nu < \gamma$ in (5.8). By the cut-off function method, we may consider $u \in C_0^\infty(\Omega)$. We have

$$\begin{aligned}
&j^{|\nu|}\|y^{\gamma-\nu}D_x^\alpha\, D_y^\beta\, D_y^{j-\nu}u(x,y)\| \\
&\quad\leq\ j^{|\nu|}\|(|y|^{|\gamma|}|D_x^\alpha|)^{1-\frac{|\nu|}{|\gamma|}}|D_x^\alpha|^{\frac{|\nu|}{|\gamma|}}D_y^{j+\beta-\nu}u(x,y)\| \\
&\quad\leq\ j^{|\nu|}\|(1+h)^{(m-|\beta|)\left(1-\frac{|\nu|}{|\gamma|}\right)}|D_x^\alpha|^{\frac{|\nu|}{|\gamma|}}D_y^{j+\beta-\nu}u(x,y)\| \\
&\quad=\ j^{|\nu|}\|(1+h)^{(m-|\beta|)\left(1-\frac{|\nu|}{|\gamma|}\right)}\langle D_y\rangle^{m\frac{|\nu|}{|\gamma|}+|\beta|\left(1-\frac{|\nu|}{|\gamma|}\right)} \\
&\qquad\cdot D_y^{j+\beta-\nu}\langle D_y\rangle^{-m\frac{|\nu|}{|\gamma|}-|\beta|\left(1-\frac{|\nu|}{|\gamma|}\right)}|D_x^\alpha|^{\frac{|\nu|}{|\gamma|}}u(x,y)\| \\
&\quad\leq\ Cj^{|\nu|}\|P(y,D_x,D_y)\langle D_y\rangle^{j-\frac{|\nu|}{|\gamma|}(m-|\beta|+|\gamma|)}|D_x^\alpha|^{\frac{|\nu|}{|\gamma|}}u(x,y)\|.
\end{aligned}$$

Here $\langle D_y\rangle = (1-\Delta_y)^{1/2}$ has only a symbolical meaning of the first-order derivation in y and it may be justified and efficient at the end of the cycles.

Thus, we may consider in this cycle for every loss of the power of D_y there corresponds to the multiplication of the kind

$$j^{\frac{|\nu|+\langle\theta,\alpha\frac{|\nu|}{|\gamma|}\rangle}{\frac{|\nu|}{|\gamma|}(m-|\beta|+|\gamma|)}}.$$

Again we have the same inequality as before:

$$\frac{|\nu|+\langle\theta,\alpha\frac{|\nu|}{|\gamma|}\rangle}{\frac{|\nu|}{|\gamma|}(m-|\beta|+|\gamma|)}=\frac{|\gamma|+\langle\theta,\alpha\rangle}{m-|\beta|+|\gamma|}\leq\frac{\theta_1+q_1}{1+q_1}=d.$$

Thus, we can finally obtain the estimate of the form

$$\|D_y^m\phi D_y^j u\|_{L_2(\omega)}\leq C_0C_1^j j^{jd},$$

where C_0 and C_1 are independent of j. □

§6. Proof of Theorem 2.1: III, Gevrey Regularity in y; FBI-transformation

It remains to determine the Gevrey exponent with respect to y-variables for the operators P given in (2.4) in case $\sigma\neq 0$, that is, for the operators of the second group by the classification in the paper [26]. As was seen in Section 5 or in the paper [5], for those operators which hold the strong inequalities like (5.3) with $\sigma=0$ the method of bracket calculus is efficient, but it seems that it does not work well in case $\sigma\neq 0$. We shall apply the method of FBI-transformation used in the result of M. Christ [5], [6] etc. to overcome this difficulty.

At first we shall mention a non-isotropic version of the result by M. Christ, (cf. [5], Theorem 2.3). We refer to the paper [5] for the precise explanation of FBI-transformation.

We use the notation

$$\langle\xi\rangle_s=(1+\xi_1^2)^{\frac{1}{2s_1}}+\cdots+(1+\xi_n^2)^{\frac{1}{2s_n}}.$$

For $u(x)\in C_0^\infty(\mathbf{R}^n)$ and $(x,\xi)\in\mathbf{R}^n\times\mathbf{R}^n$, FBI-transformation of u is defined by

$$\mathcal{F}_s u(x,\xi)=\int u(y)e^{i\langle x-y,\xi\rangle-\langle\xi\rangle_s(x-y)^2}\alpha_s(x-y,\xi)dy, \tag{6.1}$$

where

$$\alpha_s(x-y,\xi)=\prod_{j=1}^n\left(1+\frac{i}{s_j}(x_j-y_j)\xi_j(1+\xi_j^2)^{\frac{1}{2s_j}-1}\right).$$

Then the following inversion formula holds:

$$(6.2) \qquad u(x) = (2\pi)^{-n} \int \mathcal{F}_s u(x,\xi) d\xi, \quad u(x) \in C_0^\infty(\mathbf{R}^n).$$

Theorem 6.1 (cf. [5], Theorem 2.3). *Let $s = (s_1, s_2, \dots, s_n), s_j \geq 1$, $j = 1, 2, \dots, n$, and $u(x) \in C_0^\infty(\mathbf{R}^n)$. Then the following four assertions are mutually equivalent*:

(a) $u(x) \in G^{\{s\}}$ *in a neighborhood of* $x_0 \in \mathbf{R}^n$.

(b) *There exist* $C, \delta \in \mathbf{R}_+$ *and a neighborhood* V *of* x_0 *such that*

$$|\mathcal{F}u(x,\xi)| \leq Ce^{-\delta \sum_{j=1}^n |\xi_j|^{\frac{1}{s_j}}} .(x,\xi) \in V \times \mathbf{R}^n,$$

(c) *There exist an open neighborhood* $U = U(x_0) \subset \mathbf{C}^n$ *of* x_0 *and* $C, \delta \in \mathbf{R}_+$ *such that, for each* $\lambda \in \mathbf{R}^n_+, \sum \lambda_j \geq 1$, *there exists a decomposition*

$$u = g_\lambda + h_\lambda \quad in \quad U \cap \mathbf{R}^n$$

such that g_λ *is holomorphic in* U,

$$|g_\lambda(z)| \leq Ce^{C|\mathrm{Im}\langle\lambda, z\rangle|}, \quad z \in U$$

and

$$|h_\lambda(x)| \leq Ce^{-\delta \sum_{j=1}^n \lambda_j^{\frac{1}{s_j}}}, \quad x \in U \cap \mathbf{R}^n.$$

(d) *There exist an open neighborhood* U *of* x_0 *and* $C, \delta \in \mathbf{R}_+$ *such that for each* $\lambda \in \mathbf{R}^n_+, \sum_{j=1}^n \lambda_j \geq 1$, *there exists a decomposition*

$$u = g_\lambda + h_\lambda \quad in \ U \cap \mathbf{R}^n$$

such that g_λ *is holomorphic in* $\{z \in U : |\mathrm{Im}(z_j)| \leq \lambda_j^{\gamma_j - 1}\} = U_\lambda$,

$$|g_\lambda(z)| \leq C \quad z \in \ U_\lambda,$$

and

$$|h_\lambda(x)| \leq Ce^{-\delta \sum_{j=1}^n \lambda_j^{\frac{1}{s_j}}}, \quad x \in U \cap \mathbf{R}^n.$$

Now we come back to consider the operator given in (2.4):

$$(2.4) \quad P(x', y, D_x, D_y) = \sum_{\substack{\langle\sigma,\nu\rangle + |\gamma| = \langle q,\alpha\rangle + |\alpha+\beta| - m \\ |\alpha+\beta| \leq m}} a_{\alpha\beta\nu\gamma} {x'}^{\nu} y^\gamma D_x^\alpha D_y^\beta, \quad a_{\alpha\beta\nu\gamma} \in \mathbf{C}$$

We assume that $\sigma \neq 0$ under the same conditions given in Section 2. Let Ω be an open neighborhood of the origin $(0,0) \in \mathbf{R}^k_x \times \mathbf{R}^n_y$ and let $u, f \in C_0^\infty(\Omega)$ and we consider the equation

$$P(x', y, D_x, D_y)u(x,y) = f(x,y) \quad \text{in} \quad \Omega. \tag{6.3}$$

Our purpose is to prove u is in $G^{\{\theta,d\}}_{x,y}$ in a neighborhood of $(0,0)$ if f is so. By the result of Section 4, we may assume that $u(x,y)$ is already in $G^{\{\theta\}}_x$ in a neighborhood of $(0,0)$. Therefore, it is sufficient to prove that u is in $G^{\{d\}}_y$ in a neighborhood of $(0,0)$, where $d = (\theta_1 + q_1)/(1+q_1)$.

Let $(x,y), (\xi,\eta) \in \mathbf{R}^k \times \mathbf{R}^n$. We need to rewrite the definition of FBI-transformation as follows:

$$\begin{aligned}\mathcal{F}_\theta u(x,y,\xi,\eta) = \int & u(x',y')e^{\{i(x-x')\xi + i(y-y')\eta - \langle\xi\rangle_\theta (x-x')^2 - \langle\eta\rangle_d (y-y')^2\}} \\ & \cdot \alpha_\theta(x-x', y-y', \xi, \eta)dx'dy',\end{aligned}$$

where

$$\begin{aligned}\langle\xi\rangle_\theta &= (1+\xi_1^2)^{\frac{1}{2\theta_1}} + \cdots + (1+\xi_k^2)^{\frac{1}{2\theta_k}}, \\ \langle\eta\rangle_d &= (1+\eta_1^2 + \cdots + \eta_n^2)^{\frac{1}{2d}}, \quad d = \frac{\theta_1+q_1}{1+q_1}, \\ \alpha_\theta &= \prod_{j=1}^k \left(1 + \frac{i}{\theta_j}(x_j - x_j')\xi_j(1+\xi_j^2)^{\frac{1}{2\theta_j}-1}\right) \\ &\quad \cdot \prod_{j=1}^n \left(1 + \frac{i}{d}(y_j - y_j')\eta_j(1+\eta_1^2 + \cdots + \eta_n^2)^{\frac{1}{2d}-1}\right).\end{aligned}$$

Then by applying Theorem 6.1 (b) for $\mathcal{F}_\theta u$, we can find an open neighborhood V of $(0,0)$, and $C, \delta > 0$ such that

$$\begin{aligned}|\mathcal{F}_\theta u(x,y,\xi,\eta)| \le Ce^{-\delta\sum_{j=1}^k |\xi_j|^{\frac{1}{\theta_j}}} \le Ce^{-\delta|\xi|^{\frac{1}{\theta_1}}}, \\ (x,y) \in V, \quad (\xi,\eta) \in \mathbf{R}^k \times \mathbf{R}^n.\end{aligned} \tag{6.4}$$

Here we note that $\theta_1 \ge \theta_2 \ge \cdots \ge \theta_k \ge 0$ and $\theta_1 \ge (1+q_1)/(1+q_k) > 1$.

Let $c > 0$ be a small constant determined later. Then from (6.4) we can find another couple of constants $C, \delta > 0$ depending on c such that

$$\begin{aligned}|\mathcal{F}_\theta u(x,y,\xi,\eta)| \le Ce^{-\delta\left(|\eta|^{\frac{1}{d}} + |\xi|^{\frac{1}{\theta_1}}\right)}, \\ (x,y) \in V, \quad (\xi,\eta) \in \mathbf{R}^k \times \mathbf{R}^n, \quad c|\eta|^{\frac{1}{d}} \le |\xi|^{\frac{1}{\theta_1}}\end{aligned} \tag{6.5}$$

Thus, the final problem left to prove is that we have the same type of the inequality as in (6.5) in the domain

$$(x,y)\in V,\quad (\xi,\eta)\in \mathbf{R}^k\times\mathbf{R}^n,\quad c|\eta|^{\frac{1}{d}}\geq|\xi|^{\frac{1}{\theta_1}}$$

for V shrunk if necessary.

Now we use the notation like in [5]. We set

$$E(x,y,\xi,\eta)=e^{i(\langle\tilde{x}-x,\xi\rangle+\langle\tilde{y}-y,\eta\rangle)-\langle\xi\rangle_\theta(\tilde{x}-x)^2-\langle\eta\rangle_d(\tilde{y}-y)^2}$$

and

$$\Psi(x,y,\xi,\eta)=\alpha_\theta(\tilde{x}-x,\tilde{y}-y,\xi,\eta)\cdot E(x,y,\xi,\eta),$$

where $(\tilde{x},\tilde{y})\in V$ and $(\xi,\eta)\in\mathbf{R}^k\times\mathbf{R}^n$ are considered to be parameters.

Lemma 6.1. *Let P^* be the formal adjoint operator of P:*

$$P^*=\sum_{\substack{\langle\sigma,\nu\rangle+|\gamma|=\langle q,\alpha\rangle+|\alpha|+|\beta|-m\\|\alpha|+|\beta|\leq m}}(-1)^{|\alpha|+|\beta|}a_{\alpha\beta\nu\gamma}D_x^\alpha D_y^\beta(x'^\nu y^\gamma)\tag{6.6}$$

Then there exist a small polydisk $D=\{y\in\mathbf{C}^n;|y_j|<r\}\cup\{x\in\mathbf{C}^k;|x_j|<r\}$ and $\delta,c>0$ such that for each $(\tilde{x},\tilde{y})\in D\cap\mathbf{R}^{k+n}$ and for each $(\xi,\eta)\in\mathbf{R}^k\times\mathbf{R}^n, c|\eta|^{1/d}\geq|\xi|^{1/\theta_1}$, there exists $g\in C^\infty(D\cap\mathbf{R}^{k+n})$ satisfying the following conditions:

$$\begin{aligned}P^*(Eg)&=\Psi(x,y,\xi,\eta)+O(e^{-\delta(|\eta|^{\frac{1}{d}}+|\xi|^{\frac{1}{\theta_1}})}),\\&\quad(x,y)\in D\cap\mathbf{R}^{k+n},\end{aligned}\tag{6.7}$$

where g extends to a holomorphic function of (x,y) and $g(x,y,\xi,\eta)=O(1)$ in

$$U=D\cap\{|\mathrm{Im}(y)|\leq|\eta|^{\frac{1}{d}-1},|\mathrm{Im}(x_j)|\leq|\eta|^{\sigma_j(\frac{1}{d}-1)}\},\quad c|\eta|^{\frac{1}{d}}\geq|\xi|^{\frac{1}{\theta_1}}.\tag{6.8}$$

Before giving a proof of Lemma 6.1, we shall show how to use Lemma 6.1 to establish the inequality of the type (6.5) in the domain

$$(x,y)\in V,\quad(\xi,\eta)\in\mathbf{R}^k\times\mathbf{R}^n,\quad c|\eta|^{\frac{1}{d}}\geq\sum_{j=1}^k|\xi_j|^{\frac{1}{\theta_j}},$$

which completes the proof of Gevrey regularity of u at $(0,0)$ by Theorem 6.1 and Theorem 4.4.

We may suppose $u,f\in C_0^\infty(V)$ satisfying the equation (2.5) and $f\in G_{x,y}^{\{\theta,d\}}$ at $(0,0)$. Then we have

$$\int_V P^*Egu(x,y)dxdy=\int_V EgPudxdy=\int_V Egf(x,y)dxdy.$$

On the other hand, by (6.7), this is equal to

$$\int \Psi(x,y)u(x,y)dxdy + O(e^{-\delta(|\eta|^{\frac{1}{d}}+|\xi|^{\frac{1}{\theta_1}})})$$
$$= \mathcal{F}_\theta u(\tilde{x},\tilde{y},\xi,\eta) + O(e^{-\delta(|\eta|^{\frac{1}{d}}+|\xi|^{\frac{1}{\theta_1}})}).$$

By applying Theorem 6.1 (d) for $f(x,y)$, we can see that there exist a small complex neighborhood $U_{\xi,\eta}$ of $(0,0)$ and $\delta > 0$ such that for each $(\xi,\eta) \in \mathbf{R}^k \times \mathbf{R}^n$, there exists a decomposition

$$f(x,y) = G(x,y,\xi,\eta) + O(e^{-\delta(|\eta|^{\frac{1}{d}}+|\xi|^{\frac{1}{\theta_1}})}),$$

where G extends to a holomorphic function with respect to y and x and $O(1)$ in U. Of course we may assume that $U \cap \mathbf{R}^{k+n} \subset V$.

Now we have for all $(\tilde{x},\tilde{y})$ in a compact subset of V

$$\int_V Egf dxdy = \int_V EgG dxdy + O(e^{-\delta(|\eta|^{\frac{1}{d}}+|\xi|^{\frac{1}{\theta_1}})}).$$

Let $r > 0$ be a sufficiently small number and fix $\varphi \in C^1(\mathbf{R})$ such that $0 \le \varphi \le 1$, $\varphi = 1$ for $|t| \le r$, $\varphi = 0$ for $|t| \ge 2r$. Let $\varepsilon > 0$ be small and shift the contour of integration

$$y \longrightarrow \phi(y) = (y_1 + i\varepsilon|\eta|^{\frac{1}{d}-1}\varphi(y_1), y_2, \dots, y_n), \quad |\eta| \ge 1.$$

Then we obtain

$$\int_V EgG dxdy = \int_{\mathbf{R}^{k+n}} e^{(i\langle\tilde{x}-x,\xi\rangle + i\langle\tilde{y}-\phi(y),\eta\rangle - (\langle\theta\rangle(\tilde{x}-x)^2 + \langle\eta\rangle_d(\tilde{y}-\phi(y))^2)}$$
$$\cdot\, \alpha_\theta(\tilde{x}-x, \tilde{y}-\phi(y), \xi, \eta)(1 + i\varepsilon|\eta|^{\frac{1}{d}-1}\varphi'(y_1))dxdy$$
$$= O(e^{-\delta|\eta|^{\frac{1}{d}}}) = O(e^{-\delta'(|\eta|^{\frac{1}{d}}+|\xi|^{\frac{1}{\theta_1}})}),$$

uniformly for

$$(\xi,\eta) \in \mathbf{R}^{k+n}, \quad c|\eta|^{\frac{1}{d}} \ge |\xi|^{\frac{1}{\theta_1}}, \quad |\eta| \ge 1.$$

□

Proof of Lemma 6.1. Let $U \subset D \subset \mathbf{C}^{k+n}$ be the same type of the set given in Lemma 6.1 and define the space $H_\infty(U)$ of functions of $(x,y) \in U$ that are bounded and holomorphic with respect to (x,y) in D. Here we consider D a $2(k+n)$-dimensional measurable set.
Let

$$E(x,y) = e^{i(\langle\tilde{x}-x,\xi\rangle + \langle\tilde{y}-y,\eta\rangle) - \langle\xi\rangle_\theta(\tilde{x}-x)^2 - \langle\eta\rangle_d(\tilde{y}-y)^2}.$$

Then we write

(6.9)

$$
\begin{aligned}
E^{-1}P^*E = & \sum_{\substack{\langle\sigma,\nu\rangle+|\gamma|=\langle q,\alpha\rangle+|\alpha|+|\beta|-m \\ |\alpha|+|\beta|\le m}} (-1)^{|\alpha|+|\beta|} \\
& \cdot\{a_{\alpha\beta\nu\gamma}(\xi+2\langle\xi\rangle_\theta(\tilde{x}-x)+D_x)^\alpha(\eta+2\langle\eta\rangle_d(\tilde{y}-y)+D_y)^\beta\}(x'^\nu y^\gamma) \\
& \equiv \left(\sum_{|\beta|=m} a'_\beta\eta^\beta + \sum_{\substack{\langle\sigma,\nu\rangle+|\gamma|=\langle q,\alpha\rangle \\ |\alpha|+|\beta|=m \\ |\beta|<m}} a_{\alpha\beta\nu\gamma}x'^\nu y^\gamma\xi^\alpha\eta^\beta\right)+\mathcal{R} \\
& \equiv \mathcal{A}+\mathcal{R},
\end{aligned}
$$

where P^* is the operator given by (6.6) which has the same properties as in P. We consider

$$
\mathcal{A} = \left(\sum_{|\beta|=m} a'_\beta\eta^\beta + \sum_{\substack{\langle\sigma,\nu\rangle+|\gamma|=\langle q,\alpha\rangle \\ |\alpha|+|\beta|=m \\ |\beta|<m}} a_{\alpha\beta\nu\gamma}x'^\nu y^\gamma\xi^\alpha\eta^\beta\right)
$$

as a multiple operator from $H_\infty(U)$ to $H_\infty(U)$ and so on.

Lemma 6.2. *Let $r = diam D$ and $c > 0$ be sufficiently small. Then $\mathcal{A}$ is considered to be an invertible operator from $H_\infty(U)$ to $H_\infty(U)$, where U is given above.*

Proof. By Conditions 1 and 2 given in Section 2, there is a positive constant c_0 such that

$$
\text{(6.10)}\quad |\mathcal{A}| \ge c_0(|\eta| + (|x'|_\sigma+|y|)^{q_1}|\xi_1| + \cdots + (|x'|_\sigma+|y|)^{q_k}|\xi_k|)^m \ge c_0|\eta|^m,
$$
$$
(x,y)\in\mathbf{R}^{k+n},\quad (\xi,\eta)\in\mathbf{R}^{k+n},\quad |\eta|\ge 1,
$$

where

$$
|x'|_\sigma = |x_1|^{\frac{1}{\sigma_1}} + \cdots + |x_p|^{\frac{1}{\sigma_p}}
$$

as was given in Section 4.

Next, for $(x,y)\in\mathbf{C}^{k+n}$, considering the method of quasi-homogeneity, we have

$$
\begin{aligned}
|\mathcal{A}| \ge & \, {c_0}'(|\eta| + (|x'|_\sigma+|\mathrm{Re}(y)|)^{q_1}|\xi_1| + \cdots + (|x'|_\sigma+|\mathrm{Re}(y)|)^{q_k}|\xi_k|)^m \\
& - {c_0}''(|\mathrm{Im}(y)|^{q_1}|\xi_1| + \cdots + |\mathrm{Im}(y)|^{q_k}|\xi_k|)^m
\end{aligned}
$$

for some constants ${c_0}', {c_0}'' > 0$.

Now for any $\varepsilon > 0$, we use the assumption $|\mathrm{Im}(x_j)| \le |\eta|^{\sigma_j((1/d)-1)}$, $|\mathrm{Im}(y)| \le |\eta|^{(1/d)-1}$ and $c|\eta|^{1/d} \ge \sum_{j=1}^k |\xi_j|^{1/\theta_j}$, $|\eta| \ge 1$, to obtain the estimate of the form

$$(|\mathrm{Im}(x')|_\sigma + |Im(y)|)^{q_j}|\xi_j| \le \varepsilon|\eta|, \quad j = 1, \dots, k,$$

if the constant c is taken sufficiently small. Thus we have with some $c_0 > 0$ the inequality

$$|\mathcal{A}| \ge c_0(|\eta| + (|\mathrm{Re}(x')|_\sigma + |\mathrm{Re}(y)|)^{q_1}|\xi_1| + \cdots + (|\mathrm{Re}(x')|_\sigma + |\mathrm{Re}(y)|)^{q_k}|\xi_k|)^m,$$
$$(x, y) \in U, \quad c|\eta|^{\frac{1}{d}} \ge \textstyle\sum_{j=1}^k |\xi_j|^{\frac{1}{\theta_j}}, \quad |\eta| \ge 1.$$

□

Lemma 6.3 (cf. [5]). *Let $D' \subset D$ be bounded open domains in $\mathbf{C}_z$ with distance$(D', \partial D) \ge \varepsilon > 0$. Then the norm of the operator*

$$D_z : H_\infty(D) \longrightarrow H_\infty(D')$$

is $O(\varepsilon^{-1})$.

We omit the proof which is obtained by using Cauchy's integral formula.

Let D_1 and D_∞ be the open polydisks with center at $(0, 0)$ and $diam D_1 = r$ and $diam D_\infty = (1/2)r$. We write

$$U_1 = D_1 \cap \Bigg\{ |\mathrm{Im}(y)| \le |\eta|^{\frac{1}{d}-1}, |\mathrm{Im}(x_j)| \le |\eta|^{\sigma_j(\frac{1}{d}-1)} \quad \text{if} \quad \sigma_j \ge 1 \quad \text{or}$$
$$|\mathrm{Im}(x_j)| \le |\eta|^{\frac{1}{d}-1} \quad \text{if} \quad \sigma_j < 1, c|\eta|^{\frac{1}{d}} \ge \sum_{j=1}^k |\xi_j|^{\frac{1}{\theta_j}} \Bigg\},$$

$$U_\infty = D_\infty \cap \Bigg\{ |\mathrm{Im}(y)| \le \frac{1}{2}|\eta|^{\frac{1}{d}-1}, |\mathrm{Im}(x_j)| \le \frac{1}{2}|\eta|^{\sigma_j(\frac{1}{d}-1)} \quad \text{if} \quad \sigma_j \ge 1 \quad \text{or}$$
$$|\mathrm{Im}(x_j)| \le \frac{1}{2}|\eta|^{\frac{1}{d}-1} \quad \text{if} \quad \sigma_j \le 1, c|\eta|^{\frac{1}{d}} \ge \sum_{j=1}^k |\xi_j|^{\frac{1}{\theta_j}} \Bigg\}.$$

Let $\Lambda > 0$ be a large constant to be chosen later. Given a large η, choose an integer $N = [\Lambda^{-1}|\eta|^{1/d}]$, where $[a]$ denotes the largest integer less than or equal to a. For $2 \le j \le mN$ construct open sets $U_\infty = U_{mN} \subset\subset U_{mN-1} \subset\subset \cdots \subset\subset U_1$ satisfying

$$distance(U_{j+1}|_{\mathbf{C}_y^k}, \partial U_j|_{\mathbf{C}_y^k}) \ge \varepsilon\Lambda|\eta|^{-1},$$

$$distance(U_{j+1}|_{x_l}, \partial U_j|_{x_l}) \geq \varepsilon\Lambda|\eta|^{-\sigma_l} \quad \text{if} \quad \sigma_l \geq 1, \ \varepsilon\Lambda|\eta|^{-1} \quad \text{if} \quad \sigma_l \leq 1.$$

Here $U_j|_{\mathbf{C}^k_y} = U_j \cap \mathbf{C}^k_y$, etc. and ε is a small constant, independent of η, Λ, j. This is possible because for large η we have

$$1 \leq \sigma_j \Longrightarrow |\eta|^{\frac{1}{d}-\sigma_j} \leq \eta|^{\sigma_j(\frac{1}{d}-1)}.$$

Now we can complete the proof of Lemma 6.1. Considering $\mathcal{R}$ an operator from $H_\infty(U_j)$ to $H_\infty(U_{j+m})$, we assume that $\mathcal{A}^{-1}\mathcal{R}$ is sufficiently small for the moment. In order to solve the equation $(\mathcal{A}+\mathcal{R})g = \alpha$, we define

$$g = \sum_{j=0}^{N} (-1)^j (\mathcal{A}^{-1}\mathcal{R})^j \mathcal{A}^{-1}\alpha. \tag{6.11}$$

Thus we have

$$(\mathcal{A}+\mathcal{R})g = \alpha + (-1)^{N+1}(\mathcal{R}\mathcal{A}^{-1})^{N+1}\alpha,$$
$$(\mathcal{R}\mathcal{A}^{-1})^{N+1} = O(\exp(-\varepsilon N)) = O(\exp(-\varepsilon'|\eta|^{\frac{1}{d}})).$$

From where we have the estimation of the form (6.7) in the domain U_∞.

It remains to estimate $\mathcal{A}^{-1}\mathcal{R}$ in the sense of a linear operator from $H_\infty(U_j)$ to $H_\infty(U_{j+m})$. We can see the inequality (6.10) holds for $(x, y) \in U_1$. Therefore with some positive constant C, we have the following estimation in U_1:

$$|\mathcal{A}^{-1}\mathcal{R}| \leq \frac{C|\mathcal{R}|}{(|\eta| + (|x'|_\sigma + |y|)^{q_1}|\xi_1| + \cdots + (|x'|_\sigma + |y|)^{q_k}|\xi_k|)^m}. \tag{6.12}$$

The right-hand side is composed of the terms of the form

$$\frac{\xi^{\alpha_1}(\langle\xi\rangle_\theta(\tilde{x}-x))^{\alpha_2} x'^{\nu-\alpha_3} D_x^{\alpha_4} \cdot \eta^{\beta_1}(\langle\eta\rangle_d(\tilde{y}-y))^{\beta_2} y^{\gamma-\beta_3} D_y^{\beta_4}}{(|\eta| + (|x'|_\sigma + |y|)^{q_1}|\xi_1| + \cdots + (|x'|_\sigma + |y|)^{q_k}|\xi_k|)^m},$$

$$\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \quad |\alpha_2 + \alpha_3 + \alpha_4| > 0 \quad \text{if} \quad \alpha > 0,$$

$$\beta = \beta_1 + \beta_2 + \beta_3 + \beta_4, \quad |\beta_2 + \beta_3 + \beta_4| > 0 \quad \text{if} \quad \beta > 0, \quad |\alpha + \beta| \leq m,$$

$$\langle\sigma, \nu\rangle + |\gamma| = \langle q, \alpha\rangle + |\alpha + \beta| - m.$$

The quasi-homogeneous order (cf. the condition 1 given in Section 2) of the denominator is m. In the numerator, D_x and D_y are operators with norms $O(\Lambda^{-1}|\eta|^{\sigma_j})$ and $O(\Lambda^{-1}|\eta|)$. Hence we may consider the quasi-homogeneous orders of them are σ_j or 1. The terms $\langle\xi\rangle_\theta(\tilde{x}-x)$ and $\langle\eta\rangle_d(\tilde{y}-y)$ are estimated by $r|\eta|$, hence we may also consider the quasi-homogeneous orders of them are

1. Thus with parameter $\lambda > 0$, we may consider that the quasi-homogeneous order of (6.12) is estimated by

$$O(\lambda^{-\langle q,\alpha_2\rangle-\langle 1+q-\sigma,\alpha_3\rangle-\langle 1+q-\tilde{\sigma},\alpha_4\rangle})\cdot O(\Lambda^{-|\alpha_4|}\cdot r^{|\alpha_2|}),$$

where

$$\tilde{\sigma}=(\tilde{\sigma_1},\dots,\tilde{\sigma_p}),\tilde{\sigma_j}=\sigma_j \quad \text{if} \quad \sigma_j\geq 1 \quad \text{and} \quad \tilde{\sigma_j}=1 \quad \text{if} \quad \sigma_j<1.$$

The hypothesis $1+q_k > \max(\sigma_1,\dots,\sigma_p)$ given in Section 2 with the condition $|\alpha_2+\alpha_3+\alpha_4|>0$ (if $\alpha>0$) assures the negative order of the above quantity, and finally we have

$$|\mathcal{A}^{-1}\mathcal{R}|<1$$

in the sense of a linear operator from $H_\infty(U_j)$ to $H_\infty(U_{j+m})$ if c and r are taken sufficiently small and Λ and $|\eta|$ sufficiently large.

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