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Asymptotic Distribution of Negative Eigenvalues for Three Dimensional Pauli Operators with Nonconstant Magnetic Fields

By

Akihiro SHIMOMURA*

Abstract

We study the asymptotic distribution of negative eigenvalues of three dimensional Pauli operators with a two dimensional magnetic field and a three dimensional potential which decay to zero at infinity. For $\lambda > 0$ sufficiently small, we estimate the number of eigenvalues less than $-\lambda$ of such Pauli operators.

§1. Introduction

In this paper, we study the asymptotic distribution of negative eigenvalues of three dimensional Pauli operators with a magnetic field and a potential which decay to zero at infinity. Pauli operator is the Hamiltonian of a quantum particle with spin in a magnetic field. The unperturbed Pauli operator is given by

$$H_p \ = \ (-i\nabla - A)^2 - \sigma \cdot B,$$

and it acts in $L^2(\mathbf{R}^3) \otimes \mathbf{C}^2$, where $A\colon \mathbf{R}^3 \to \mathbf{R}^3$ is a vector potential, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is a vector of 2×2 Pauli operators with components

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $B = \nabla \times A$ is a magnetic field. Throughout this paper, we assume that the direction of the magnetic field is constant. We denote the elements of $\mathbf{R}^3$

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*Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo 153-8914, Japan.

by $(x, z) = (x_1, x_2, z)$. We may assume that the direction of the magnetic field is parallel to the positive z axis. Then we can show that magnetic field B is independent of z, and that it has the form

$$B(x) = (0, 0, b(x)).$$

Let $A(x) = (a_1(x), a_2(x), 0)$ be a vector potential associated with $b(x)$. We assume that $a_j \in C^1(\mathbf{R}^2)$ $(j = 1, 2)$ is a real valued function. Namely $b(x) = \partial_1 a_2(x) - \partial_2 a_1(x)$, (where $\partial_j = \partial/\partial x_j$). The unperturbed Pauli operator has the form

$$H_p = \begin{pmatrix} H_+ - \partial_z^2 & 0 \\ 0 & H_- - \partial_z^2 \end{pmatrix},$$

where

$$H_\pm = (-i\nabla_x - A)^2 \mp b = \Pi_1^2 + \Pi_2^2 \mp b, \quad \Pi_j = -i\partial_j - a_j \quad (j = 1, 2).$$

Since $b = i[\Pi_2, \Pi_1]$, we see

$$H_\pm = (\Pi_1 + i\Pi_2)^*(\Pi_1 + i\Pi_2) \geq 0.$$

Hereafter we discuss the asymptotic distribution of negative eigenvalues of following Pauli operators;

$$H = H_+ - \partial_z^2 - V, \quad H_+ = (-i\nabla_x - A)^2 - b. \tag{1.1}$$

We assume that the magnetic field b and the potential V satisfy the following Assumptions (b) and (V), respectively:

Assumption (b). $b \in C^1(\mathbf{R}^2)$ and there exist constants $0 \leq d < 2,\ C > 1$ such that

$$\frac{1}{C}\langle x\rangle^{-d} \leq b(x) \leq C\langle x\rangle^{-d}, \quad |\nabla b(x)| \leq C\langle x\rangle^{-d-1}. \tag{1.2}$$

Assumption (V). $V \in C^1(\mathbf{R}^3)$ and there exist constants $m > 0,\ C > 1$ such that

$$\frac{1}{C}\langle x, z\rangle^{-m} \leq V(x, z) \leq C\langle x, z\rangle^{-m}, \quad |\nabla V(x, z)| \leq C\langle x, z\rangle^{-m-1}. \tag{1.3}$$

Here we denote $\langle x\rangle = (1 + |x|^2)^{1/2}$, $\langle x, z\rangle = (1 + |x|^2 + z^2)^{1/2}$.

Under these assumptions, the operator H given by (1.1) is esssentialy self-adjoint, and the essential spectrum of $H_+ - \partial_z^2$ and H are $[0, \infty)$.

For self-adjoint operator T and $c \in \mathbf{R}$, we denote the number of eigenvalues less than and greater than c of T by $N(T < c)$, $N(T > c)$, respectively.

The purpose of this work is to estimate the order of $N(H < -\lambda)$ for small λ. The next theorem is our main result.

Theorem 1.1. *Assume Assumptions* (b) *and* (V). *If* $0 < d < m < 2$, $m/2 + d < 2$, *then*

$$N(H < -\lambda) = F(\lambda)(1 + o(1)), \quad \lambda \to 0, \tag{1.4}$$

where

$$F(\lambda) = 2(2\pi)^{-2} \int_{\{(x,z) \in \mathbf{R}^3 :\, V(x,z) > \lambda\}} b(x)(V(x,z) - \lambda)^{\frac{1}{2}}\, dx\, dz. \tag{1.5}$$

In the remainder of this section, we recall several known results.

First, we consider the known results of two dimensional Pauli operators.

Assumption (V′). $V \in C^1(\mathbf{R}^2)$ and there exist constants $m > 0$, $C > 0$ such that

$$|V(x)| \le C\langle x\rangle^{-m}, \quad |\nabla V(x)| \le C\langle x\rangle^{-m-1}.$$

Let

$$H' = H_+ - V, \quad H_+ = (-i\nabla_x - A)^2 - b, \tag{1.6}$$

and we assume that it acts in $L^2(\mathbf{R}^2)$. Following theorem is proved in [5], [6].

Theorem A ([5], [6]). *Assume Assumptions* (b) *and* (V′). *Moreover suppose* V *satisfies*

$$\liminf_{\lambda \downarrow 0} \lambda^{2/m} \int_{V(x) > \lambda} dx > 0,$$

$$\limsup_{\lambda \downarrow 0} \lambda^{\frac{2-d}{m}} \int_{(1-\delta)\lambda < |V(x)| < (1+\delta)\lambda} \langle x\rangle^{-d}\, dx = o(1), \quad \delta \to 0.$$

Then

$$N(H' < -\lambda) = (2\pi)^{-1} \int_{V(x) > \lambda} b(x)\, dx (1 + o(1)), \quad \lambda \to 0.$$

Concerning three dimensional Pauli operators, following theorem is obtained in [5].

Theorem B ([5]). *Assume Assumptions* (b) *and* (V). *If* $d = 0$, $0 < m < 2$, *then*

$$N(H < -\lambda) = F(\lambda)(1 + o(1)), \quad \lambda \to 0,$$

where $F(\lambda)$ *is given by* (1.5).

§2. Preliminaries

In this section, we prepare lemmas for the proof of Theorem 1.1.

We first consider following unperturbed Pauli operators in $L^2(\mathbf{R}^2)$:

$$\tilde{H}_\pm = (-i\nabla_x - \tilde{A})^2 \mp \tilde{b} = \tilde{\Pi}_1^2 + \tilde{\Pi}_2^2 \mp \tilde{b}, \quad \tilde{\Pi}_j = -i\partial_j - \tilde{a}_j, \quad (j = 1, 2). \tag{2.1}$$

Assume that $\tilde{b} \in C^1(\mathbf{R}^2)$, and that there exist constants c, $C > 0$ such that

$$c \leq \tilde{b}(x) \leq C.$$

Then, it is known that $\tilde{H}_+$ has zero as an eigenvalue with infinite multiplicity, and that zero is an isolated point of the spectrum of $\tilde{H}_+$ ([1], [8]). As noted in Section 1, we have $\tilde{H}_\pm \geq 0$. On the other hand, we see $\tilde{H}_- \geq c > 0$ by (2.1). It is known that the non-zero spectrum of $\tilde{H}_+$ and $\tilde{H}_-$ coincide ([4], Theorem 6.4). Hence $\tilde{H}_+$ has a spectral gap above zero, and the spectral gap is greater than or equal to $c > 0$. Let P be the orthogonal projection on the zero-eigenspace, and let $Q = I - P$. Then we see $Q\tilde{H}_+Q \geq cQ > 0$.

Throughout this section, we assume that the magnetic field b satisfies Assumption (b) with $0 < d < m < 2$, $m/2 + d < 2$. We use a smooth partition of unity $\{\psi_1, \ \psi_2\}$ on $\mathbf{R}^2$ such that

$$\psi_1(x)^2 + \psi_2(x)^2 = 1, \quad x \in \mathbf{R}^2, \tag{2.2}$$

$$\psi_1(x) = 1 \quad \text{if} \quad |x| \leq 1; \quad \psi_1(x) = 0 \quad \text{if} \quad |x| \geq 2. \tag{2.3}$$

We choose α so that

$$\frac{1}{m} < \alpha < \frac{1}{d}. \tag{2.4}$$

By Proposition 4.1 of [6], there exists $\phi_0 \in C^2(\mathbf{R}^2)$ such that

$$\Delta\phi_0 = b, \quad |\phi_0(x)| \leq \text{const.}\langle x\rangle^{2-p}, \quad (\forall p < d).$$

Then we set a vector potential $A(x) = (a_1(x), a_2(x))$ associated with the magnetic field b as

$$a_1(x) = -\partial_2\phi_0(x), \quad a_2(x) = \partial_1\phi_0(x).$$

Let

$$\phi_\lambda(x) = \phi_0(x) + \eta\lambda^{\alpha d}|x|^2\psi_2(\lambda^\alpha x), \tag{2.5}$$

$$A_\lambda(x) = (-\partial_2\phi_\lambda(x), \partial_1\phi_\lambda(x)), \tag{2.6}$$

$$b_\lambda(x) = \Delta\phi_\lambda(x) \;=\; \nabla \times A_\lambda(x). \tag{2.7}$$

By Assumption (b), we can choose $\eta > 0$ so small that

$$b_\lambda(x) \geq c_\alpha\lambda^{\alpha d}, \quad c_\alpha > 0. \tag{2.8}$$

We assume that a potential V satisfies Assumption (V). We consider a Pauli operator K_λ in $L^2(\mathbf{R}^2)$ with the magnetic field b_λ and the potential V:

$$K_\lambda = K_{+,\lambda} - \partial_z^2 - V, \quad K_{+,\lambda} = (-i\nabla_x - A_\lambda)^2 - b_\lambda. \tag{2.9}$$

By (2.8), $K_{+,\lambda}$ has zero as an eigenvalue with infinite multiplicity, and zero is an isolated point of the spectrum of $K_{+,\lambda}$. Moreover it has a spectral gap above zero, and the spectral gap is greater than or equal to $c_\alpha\lambda^{\alpha d} > 0$. Let P_λ be the orthogonal projection on the zero-eigenspace, and let $Q_\lambda = I - P_\lambda$. Then it follows that

$$Q_\lambda K_{+,\lambda} Q_\lambda \geq c_\alpha\lambda^{\alpha d} Q_\lambda > 0. \tag{2.10}$$

Lemma 2.1. *Assume Assumptions* (b) *and* (V). *Then for any* $\varepsilon > 0$ *small enough, there exists* $\lambda_\varepsilon > 0$ *such that*

$$N(K_\lambda < -(1+\varepsilon)\lambda) \leq N(H < -\lambda) \leq N(K_\lambda < -(1-\varepsilon)\lambda) \tag{2.11}$$

for $0 < \lambda < \lambda_\varepsilon$.

Proof. Let $\lambda > 0$, and let $\psi_1(x)$, $\psi_2(x)$ be the partition of unity defined above. Let

$$\psi_{\lambda,1}(x,z) = \psi_1(\lambda^\alpha x/2), \quad \psi_{\lambda,2}(x,z) = \psi_2(\lambda^\alpha x/2)$$

for $(x,z) \in \mathbf{R}^3$. By the IMS localization formula ([4], Theorem 3.2), we have

$$H = \psi_{\lambda,1}(H - \Psi_\lambda)\psi_{\lambda,1} + \psi_{\lambda,2}(H - \Psi_\lambda)\psi_{\lambda,2},$$

where

$$\Psi_\lambda(x,z) = |\nabla\psi_{\lambda,1}(x,z)|^2 + |\nabla\psi_{\lambda,2}(x,z)|^2 = O(\lambda^{2\alpha}) = o(\lambda), \quad \lambda \to 0. \tag{2.12}$$

By the definition of A_λ and (2.3), $A_\lambda(x) = A(x)$ for $|x| < \lambda^{-\alpha}$. Hence

$$H = \psi_{\lambda,1}(K_\lambda - \Psi_\lambda)\psi_{\lambda,1} + \psi_{\lambda,2}(H - \Psi_\lambda)\psi_{\lambda,2}. \tag{2.13}$$

By Assumption (V), $V(x,z) = O(\lambda^{m\alpha}) = o(\lambda)$ $(\lambda \to 0)$ for $x \in \mathrm{supp}\psi_{\lambda,2}$. Combining (2.12) with this estimate, for any $\varepsilon > 0$ small enough, we learn that there exists $\lambda_\varepsilon > 0$ sufficiently small such that for $0 < \lambda < \lambda_\varepsilon$,

$$\psi_{\lambda,2}(H - \Psi_\lambda)\psi_{\lambda,2} = \psi_{\lambda,2}(H_+ - V - \Psi_\lambda)\psi_{\lambda,2} \geq -\varepsilon\lambda.$$

By (2.13), it follows

$$H \geq \psi_{\lambda,1}(K_\lambda - \Psi_\lambda)\psi_{\lambda,1} - \varepsilon\lambda.$$

Therefore for any $\varepsilon > 0$ small enough, there exists $\lambda_\varepsilon > 0$ sufficiently small such that for $0 < \lambda < \lambda_\varepsilon$,

$$N(H < -\lambda) \leq N(K_{\lambda,D} < -(1-\varepsilon)\lambda) \leq N(K_\lambda < -(1-\varepsilon)\lambda), \tag{2.14}$$

where $K_{\lambda,D}$ is the operator K_λ with the Dirichlet boundary condition on the domain $\{(x,z)\colon |x| < \lambda^{-\alpha}\}$. Similarly, we obtain

$$N(K_\lambda < -(1+\varepsilon)\lambda) \leq N(H < -\lambda). \tag{2.15}$$

□

The order of $F(\lambda)$ is computed as follows:

Lemma 2.2. *Assume Assumptions* (b) *and* (V)*. Then for sufficiently small* $\lambda > 0$*,*

$$c\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{3}{m}} \leq F(\lambda) \leq C\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{3}{m}}, \tag{2.16}$$

where $c,\ C > 0$ *is constants which is independent of* λ*.*

Proof. By Assumptions (b) and (V),

$$\begin{aligned}
&\int_{V(x,z)>\lambda} b(x)(V(x,z)-\lambda)^{1/2}\,dx\,dz \\
&\quad\leq \int_{\langle x,z\rangle < \mathrm{const.}\,\lambda^{-1/m}} b(x)V(x,z)^{1/2}\,dx\,dz \\
&\quad\leq \mathrm{const.}\int_{\langle x,z\rangle < \mathrm{const.}\,\lambda^{-1/m}} \langle x\rangle^{-d}\langle x,z\rangle^{-m/2}\,dx\,dz \\
&\quad\leq \mathrm{const.}\,\lambda^{-1/m}\int_{\langle x\rangle < \mathrm{const.}\,\lambda^{-1/m}} \langle x\rangle^{-d-m/2}\,dx.
\end{aligned}$$

By simple calculation, it follows that the right hand side is $O(\lambda^{1/2+d/m-3/m})$. Therefore we obtain the second inequality of (2.16).

On the other hand, since we have

$$\begin{aligned}
&\int_{V(x,z)>\lambda} b(x)(V(x,z)-\lambda)^{1/2}\,dx\,dz \\
&\quad\geq \int_{V(x,z)>2\lambda} b(x)(V(x,z)-\lambda)^{1/2}\,dx\,dz \\
&\quad\geq \text{const.}\,\lambda^{1/2}\int_{\langle x,z\rangle<\text{const.}\,\lambda^{-1/m}} \langle x\rangle^{-d}\,dx\,dz \\
&\quad\geq \text{const.}\,\lambda^{1/2}\cdot\lambda^{d/m}\cdot\lambda^{-3/m},
\end{aligned}$$

the first inequality of (2.16) follows. □

Lemma 2.3. *To prove Theorem* 1.1*, it is sufficient to show that under the asssumptions of Theorem* 1.1*,*

$$\limsup_{\lambda\downarrow 0}\frac{N(K_\lambda<-\lambda)}{F(\lambda)}\leq 1, \tag{2.17}$$

$$\liminf_{\lambda\downarrow 0}\frac{N(K_\lambda<-\lambda)}{F(\lambda)}\geq 1. \tag{2.18}$$

Proof. Suppose (2.17) and (2.18). Then by Lemma 2.1, it follows that for any $\varepsilon>0$ small enough,

$$\limsup_{\lambda\downarrow 0}\frac{N(H<-\lambda)}{F((1-\varepsilon)\lambda)}\leq 1, \tag{2.19}$$

$$\liminf_{\lambda\downarrow 0}\frac{N(H<-\lambda)}{F((1+\varepsilon)\lambda)}\geq 1. \tag{2.20}$$

On the other hand, for any $\varepsilon>0$ small enough,

(2.21)
$$\begin{aligned}
&F((1-\varepsilon)\lambda)-F(\lambda) \\
&\quad= 2(2\pi)^{-2}\int_{V(x,z)>\lambda} b(x)\{(V(x,z)-(1-\varepsilon)\lambda)^{1/2}-(V(x,z)-\lambda)^{1/2}\}\,dx\,dz \\
&\qquad+2(2\pi)^{-2}\int_{(1-\varepsilon)\lambda<V(x,z)<\lambda} b(x)(V(x,z)-(1-\varepsilon)\lambda)^{1/2}\,dx\,dz \\
&\quad=\varepsilon^{1/2}O(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{3}{m}}),
\end{aligned}$$

by Assumptions (b) and (V). Therefore there exists $C > 0$ such that for any $\varepsilon > 0$ and $\lambda > 0$ small enough,

$$\frac{F((1-\varepsilon)\lambda)}{F(\lambda)} \leq 1 + C\varepsilon^{1/2}, \tag{2.22}$$

by Lemma 2.2. Similarly, we can show that there exists $C > 0$ such that for any $\varepsilon > 0$ and $\lambda > 0$ small enough,

$$\frac{F((1+\varepsilon)\lambda)}{F(\lambda)} \geq 1 - C\varepsilon^{1/2}. \tag{2.23}$$

By (2.19) and (2.22), we obtain

$$\limsup_{\lambda\downarrow 0} \frac{N(H < -\lambda)}{F(\lambda)} \ \leq\ 1. \tag{2.24}$$

By (2.20) and (2.23), we also obtain

$$\liminf_{\lambda\downarrow 0} \frac{N(H < -\lambda)}{F(\lambda)} \ \geq\ 1. \tag{2.25}$$

These imply the conclusion of Theorem 1.1. □

§3. Proof of Theorem 1.1

Let P_λ be the orthogonal projection on the zero-eigenspace, and let $Q_\lambda = I - P_\lambda$, defined in Section 2. Let α be the constant defined in (2.4). Hereafter we assume

$$0 < d < m < 2, \quad \frac{m}{2} + d < 2. \tag{3.1}$$

To prove Theorem 1.1, we use the next proposition (see Lemma 3.3 and Section 10 of [6]).

Proposition 3.1 ([6]). *Assume that Assumption* (b), *and suppose that* $U \in C^1(\mathbf{R}^2)$ *satisfies*

$$0 < U(x) \leq C\langle x\rangle^{-m}, \quad |\nabla U(x)| \leq C\langle x\rangle^{-m-1}, \tag{3.2}$$

where $C > 0$ is a constant independent of x. Then for any $\delta > 0$ small enough, *there exists* $\lambda_\delta > 0$ *such that*

(3.3)
$$N(P_\lambda U P_\lambda > \lambda) \le (2\pi)^{-1} \int_{\{x\in\mathbf{R}^2:\, U(x)>(1-\delta)\lambda\}} b(x)\,dx + \delta C^{\frac{2-d}{m}} O(\lambda^{-\frac{2-d}{m}}),$$

(3.4)
$$N(P_\lambda U P_\lambda > \lambda) \ge 2(2\pi)^{-1} \int_{\{x\in\mathbf{R}^2:\, U(x)>(1+\delta)\lambda\}} b(x)\,dx - (2\pi)^{-1} \int_{\{x\in\mathbf{R}^2:\, U(x)>(1-\delta)\lambda\}} b(x)\,dx - \delta C^{\frac{2-d}{m}} O(\lambda^{-\frac{2-d}{m}}),$$

for $0 < \lambda < \lambda_\delta$.

§3.1. Proof of (2.17): Upper bound

In this subsection, we show some lemmas for the upper bound of Theorem 1.1.

The next Propositions 3.2 through 3.4 are obtained in [6] of Lemmas 2.1, 3.1 and 3.2, respectively.

Proposition 3.2 ([6]). *Let* T_1, T_2 *be nonnegative compact self-adjoint operators, and let* $\lambda > 0$. *Then for any* $\delta > 0$ *small enough,*

$$N(T_1 + T_2 > \lambda) \le N(T_1 > (1-\delta)\lambda) + N(T_2 > \delta\lambda).$$

Proposition 3.3 ([6]). *Assume Assumption* (b)*, and suppose* $0 < s < 1/m$. *Assume that* $U(x) = U(x, \lambda) \ge 0$ *is a function on* $\mathbf{R}^2$ *which is uniformly bounded respect to* λ*, with support in* $\{x \in \mathbf{R}^2 \colon |x| < \lambda^{-s}\}$. *Then for any* $L > 0$,

$$N(P_\lambda U P_\lambda > \lambda^L) = o(\lambda^{-\frac{2-d}{m}}), \quad \lambda \to 0.$$

Proposition 3.4 ([6]). *Assume Assumption* (b)*, and suppose that* $U \in C^1(\mathbf{R}^2)$ *satisfies*

$$|U(x)| \le const.\langle x\rangle^{-m}, \quad |\nabla U(x)| \le const.\langle x\rangle^{-m-1}.$$

Then

$$N(H_+ - U < -\lambda) = O(\lambda^{-\frac{2-d}{m}}), \quad \lambda \to 0.$$

Hereafter we identify the operator $P_\lambda \otimes I$ acting in $L^2(\mathbf{R}^3_{x,z}) = L^2(\mathbf{R}^2_x) \otimes L^2(\mathbf{R}_z)$ with the operator P_λ acting in $L^2(\mathbf{R}^2_x)$.

Lemma 3.5. *Assume Assumptions* (b), (V), *and let* $\lambda > 0$. *Then for any* $c > 0$,

$$\begin{aligned} N(K_\lambda < -\lambda) \le\, & N(-\partial_z^2 - P_\lambda(V + c\lambda^{-\alpha d}V^2)P_\lambda < -\lambda) \\ & + N(Q_\lambda(K_{+,\lambda} - V - c^{-1}\lambda^{\alpha d})Q_\lambda < -\lambda). \end{aligned}$$

Moreover if $c > 0$ *is large enough,*

$$N(Q_\lambda(K_{+,\lambda} - V - c^{-1}\lambda^{\alpha d})Q_\lambda < -\lambda) = o(\lambda^{-\frac{2-d}{m}}), \quad \lambda \to 0.$$

Proof. It is easy to see

$$-P_\lambda V Q_\lambda - Q_\lambda V P_\lambda \ge -c^{-1}\lambda^{\alpha d}Q_\lambda - c\lambda^{-\alpha d}P_\lambda V^2 P_\lambda,$$

for any $c > 0$. From this, we obtain

$$\begin{aligned} N(K_\lambda < -\lambda) \le\, & N(-\partial_z^2 - P_\lambda(V + c\lambda^{-\alpha d}V^2)P_\lambda < -\lambda) \\ & + N(Q_\lambda(K_{+,\lambda} - V - c^{-1}\lambda^{\alpha d})Q_\lambda < -\lambda). \end{aligned}$$

Therefore the first statement is proved.

Let $u_m(x) = \langle x\rangle^{-m}$. Then there exists $\beta > 0$ such that $V(x, z) \le \beta u_m(x)$. By (2.10), we can choose $c > 0$ so that

$$\begin{aligned} Q_\lambda(K_{+,\lambda} - V - c^{-1}\lambda^{\alpha d})Q_\lambda &\ge Q_\lambda(K_{+,\lambda} - \beta u_m - c^{-1}\lambda^{\alpha d})Q_\lambda \\ &\ge Q_\lambda\left(\frac{1}{2}K_{+,\lambda} - \beta u_m + c_2\lambda^{\alpha d}\right)Q_\lambda, \end{aligned}$$

for some $c_2 > 0$. Hence

$$N(Q_\lambda(K_{+,\lambda} - V - c^{-1}\lambda^{\alpha d})Q_\lambda < -\lambda) \le N(K_{+,\lambda} - 2\beta u_m < -2c_3\lambda^{\alpha d}), \tag{3.5}$$

for some $c_3 > 0$. On the other hand, as in the proof of (2.15), we obtain

$$N(K_{+,\lambda} - 2\beta u_m < -2c_3\lambda^{\alpha d}) \le N(H_+ - 2\beta u_m < -c_3\lambda^{\alpha d}).$$

By Proposition 3.4, the right hand side of the above inequality is $O(\lambda^{-\alpha d((2-d)/m)}) = o(\lambda^{-(2-d)/m})$. (Note $\alpha d < 1$ by (2.4)). From this and (3.5), the second statement follows. □

Since for any $c > 0$ large enough,

$$N(K_\lambda < -\lambda) \le N(-\partial_z^2 - P_\lambda(V + c\lambda^{-\alpha d}V^2)P_\lambda < -\lambda) + o(\lambda^{-\frac{2-d}{m}}), \quad \lambda \to 0 \tag{3.6}$$

(according to Lemma 3.5), it is sufficient to estimate $N(-\partial_z^2 - P_\lambda(V + c\lambda^{-\alpha d}V^2) \times P_\lambda < -\lambda)$. (Note Lemma 2.2 and $0 < d < m < 2$).

Since $m < 2$, we can choose constants r and α such that they satisfy (2.4) and following relations:

$$mr - \alpha d = 0, \tag{3.7}$$

$$\frac{1}{2} < r < \frac{1}{m}. \tag{3.8}$$

(For example, we may get $\alpha = (1/2d)(m/2 + 1)$ and $r = (1/2)(1/2 + 1/m)$ if $d < m^2$; $\alpha = (1/2)(1/m + 1/d)$ and $r = (1/2m)(d/m + 1)$ if $d \geq m^2$). Hereafter we fix $\delta > 0$ small enough. Let $\{I_k\}_{k\in\mathbf{Z}}$ be a sequence of disjoint open intervals satisfying $|I_k| = \lambda^{-r}$, $\mathbf{R} = \bigcup_{k\in\mathbf{Z}} \overline{I_k}$. Let z_k be the center of I_k, and let

$$J_k = \left\{ z \in \mathbf{R} \colon |z - z_k| < \frac{1+\delta}{2}\lambda^{-r} \right\}.$$

Let $\{\varphi_k\}_{k\in\mathbf{Z}}$ be a smooth partition of unity which satisfies following properties:

$$\sum_{k\in\mathbf{Z}} \varphi_k(z)^2 = 1, \quad z \in \mathbf{R}, \tag{3.9}$$

$$\operatorname{supp}\varphi_k \subset J_k, \tag{3.10}$$

$$\left| \frac{d}{dz}\varphi_k(z) \right| \leq C\delta^{-1}\lambda^r. \tag{3.11}$$

Let

$$W_\lambda(x, z) = V(x, z) + c\lambda^{-\alpha d}V(x, z)^2. \tag{3.12}$$

Let $N_k(t)$ be the number of eigenvalues less than $-t$ of the operator $-\partial_z^2 - P_\lambda W_\lambda P_\lambda$ in $L^2(\mathbf{R}_x^2 \times J_k)$ with the Dirichlet boundary condition.

Lemma 3.6.

$$N(K_\lambda < -\lambda) \leq \sum_{k\in\mathbf{Z}} N_k((1-\delta)\lambda) + o(\lambda^{-\frac{2-d}{m}}), \quad \lambda \to 0. \tag{3.13}$$

Proof. According to (3.11) and (3.8),

$$\sum_{k\in\mathbf{Z}} \left| \frac{d}{dz}\varphi_k(z) \right|^2 = O(\delta^{-2})\lambda^{2r} \leq \delta\lambda$$

holds for $\lambda > 0$ small enough. Hence we have

$$\begin{aligned}-\partial_z^2 - P_\lambda W_\lambda P_\lambda &= \sum_{k\in\mathbf{Z}} \varphi_k \left(-\partial_z^2 - P_\lambda W_\lambda P_\lambda - \sum_{j\in\mathbf{Z}} \left| \frac{d}{dz}\varphi_j \right|^2 \right) \varphi_k \\ &\geq \sum_{k\in\mathbf{Z}} \varphi_k(-\partial_z^2 - P_\lambda W_\lambda P_\lambda)\varphi_k - \delta\lambda,\end{aligned} \tag{3.14}$$

by the IMS localization formula. (3.13) follows from (3.6) and (3.14). □

To estimte $N_k((1-\delta)\lambda)$ $(k \in \mathbf{Z})$, we decompose $\mathbf{R}_z$ into three parts. According to Assumption (V) and the fact $\alpha d < 1$, we can choose $M > 0$ so large that if $|z_k| > M\lambda^{-1/m}$,

$$\sup_{x\in\mathbf{R}^2} W_\lambda(x, z) \leq \frac{\lambda}{2}, \tag{3.15}$$

uniformly in $z \in J_k$. Then let

(3.16)

$$\Omega_{1,\lambda} = \{k \in \mathbf{Z}\colon |z_k| \leq \delta^{-1}\lambda^{-r}\}, \quad \Omega_{2,\lambda} = \{k \in \mathbf{Z}\colon \delta^{-1}\lambda^{-r} < |z_k| < M\lambda^{-\frac{1}{m}}\},$$
$$\Omega_{3,\lambda} = \{k \in \mathbf{Z}\colon |z_k| \geq M\lambda^{-\frac{1}{m}}\}.$$

Lemma 3.7. *Assume Assumption* (b), *and suppose that* U_1, $U_2 \in C^1(\mathbf{R}^2)$ *satisfy*

$$|U_1(x)| \leq C\langle x\rangle^{-m}, \quad |\nabla U_1(x)| \leq C\langle x\rangle^{-m-1}, \tag{3.17}$$

$$|U_2(x)| \leq C\langle x\rangle^{-2m}, \quad |\nabla U_2(x)| \leq C\langle x\rangle^{-2m-1}, \tag{3.18}$$

where $C > 0$ *is a constant independent of* x. *Then for any* $\delta > 0$ *small enough, there exists* $\lambda_\delta > 0$ *such that for* $0 < \lambda < \lambda_\delta$

$$\begin{aligned}&N(P_\lambda(U_1 + c\lambda^{-\alpha d}U_2)P_\lambda > \lambda) \\ &\quad \leq (2\pi)^{-1} \int_{\{x\in\mathbf{R}^2\colon U_1(x)>(1-\delta)\lambda\}} b(x)\,dx + \delta C^{\frac{2-d}{m}} O(\lambda^{-\frac{2-d}{m}}), \quad (c > 0).\end{aligned}$$

Proof. We choose s such that $(\alpha d+1)/2m < s < 1/m$. By the assumption on U_2, we have

$$\lambda^{-\alpha d}U_2(x) = O(\lambda^{-\alpha d+2ms}) = o(\lambda), \quad \lambda \to 0$$

for $|x| > \lambda^{-s}$. Applying Proposition 3.3 to $\chi_{\{|x|\leq\lambda^{-s}\}}U_2$, we learn

(3.19)

$$N(P_\lambda(c\lambda^{-\alpha d}U_2)P_\lambda > \lambda) = N(P_\lambda U_2 P_\lambda > c^{-1}\lambda^{1+\alpha d}) = o(\lambda^{-\frac{2-d}{m}}), \quad \lambda \to 0,$$

where $\chi_{\{|x|\le\lambda^{-s}\}}$ is the characteristic function of the set $\{|x| \le \lambda^{-s}\}$. Since $P_\lambda U_1 P_\lambda$ and $P_\lambda U_2 P_\lambda$ are compact operators, by Proposition 3.2, we see

$$N(P_\lambda(U_1 + c\lambda^{-\alpha d}U_2)P_\lambda > \lambda) \le N(P_\lambda U_1 P_\lambda > (1-\delta)\lambda) + N(P_\lambda(c\lambda^{-\alpha d}U_2)P_\lambda > \delta\lambda)$$

for $\delta > 0$ small enough. By (3.19), the second term of the right hand side is $o(\lambda^{-(2-d)/m})$. Applying Proposition 3.1 (3.3) to the first term, we complete the proof. □

We begin with the cases $k \in \Omega_{1,\lambda}$ and $k \in \Omega_{3,\lambda}$.

Lemma 3.8.

$$\sum_{k\in\Omega_{1,\lambda}} N_k((1-\delta)\lambda) = o(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{3}{m}}), \quad (\lambda \to 0), \tag{3.20}$$

$$\sum_{k\in\Omega_{3,\lambda}} N_k((1-\delta)\lambda) = 0. \tag{3.21}$$

Proof. For $k \in \Omega_{3,\lambda}$, we have by (3.15),

$$-\partial_z^2 - P_\lambda W_\lambda P_\lambda \ge -\frac{\lambda}{2} > -(1-\delta)\lambda.$$

These operators are considered in $L^2(\mathbf{R}_x^2 \times J_k)$ with the Dirichlet boundary condition. From this, we learn $N_k((1-\delta)\lambda) = 0$, and hence (3.21) follows.

Next we consider the case $k \in \Omega_{1,\lambda}$. Let $u_m(x) = \langle x\rangle^{-m}$. Since

$$W_\lambda(x,z) \le \beta(u_m(x) + c\lambda^{-\alpha d}u_m(x)^2)$$

for $z \in J_k$, it follows that

$$-\partial_z^2 - P_\lambda W_\lambda P_\lambda \ge -\partial_z^2 - \beta P_\lambda(u_m + c\lambda^{-\alpha d}u_m^2)P_\lambda, \tag{3.22}$$

in $L^2(\mathbf{R}_x^2 \times J_k)$ with the Dirichlet boundary condition. Let $\mu_j^{(\lambda)}$ be the j-th eigenvalue of the operator $P_\lambda(u_m + c\lambda^{-\alpha d}u_m^2)P_\lambda \in B(L^2(\mathbf{R}_x^2))$, where $B(L^2(\mathbf{R}_x^2))$ is the set of all bounded operators acting in $L^2(\mathbf{R}_x^2)$. Then the eigenvalues of the operator in the right hand side of (3.22) are

$$\frac{l^2\pi^2}{|J_k|^2} - \beta\mu_j^{(\lambda)}, \quad l \in \mathbf{N}.$$

Let $\mu > 1/2\beta$. Applying Proposition 3.7 with $U_1 = u_m/\mu$, $U_2 = u_m^2/\mu$, for any $\varepsilon > 0$, we learn that there exists $\lambda > 0$ so small that

$$\begin{aligned} &N(P_\lambda(u_m + c\lambda^{-\alpha d}u_m^2)P_\lambda > \mu\lambda) \\ &\quad\le (2\pi)^{-1}\int_{\{x\in\mathbf{R}^2:\ u_m(x)>(1-\varepsilon)\mu\lambda\}} b(x)\,dx + \varepsilon(\mu\lambda)^{-\frac{2-d}{m}} \\ &\quad\le \text{const.}(\mu\lambda)^{-\frac{2-d}{m}}, \quad (c>0). \end{aligned}$$

Since, this implies that $\mu_j^{(\lambda)} \le \text{const.}j^{-m/(2-d)}$, there exists $p > 0$ such that

$$\frac{l^2\pi^2}{|J_k|^2} - \beta\mu_j^{(\lambda)} > -(1-\delta)\lambda \tag{3.23}$$

if $j > p\lambda^{-(2-d)/m}$. Therefore, by the fact $|J_k| = (1+\delta)\lambda^{-r}$ and (3.22), we obtain

$$\begin{aligned} N_k((1-\delta)\lambda) &\le \sum_{j=1}^{p\lambda^{-\frac{2-d}{m}}} \left(\frac{|J_k|}{\pi}(\beta\mu_j^{(\lambda)} - (1-\delta)\lambda)^{1/2} + 1\right) \\ &\le O(\lambda^{\frac{1}{2}-\frac{2-d}{m}-r}) \end{aligned}$$

uniformly in δ. Since the number of the elements of $\Omega_{1,\lambda}$ is $\sharp\Omega_{1,\lambda} = O(\delta^{-1})$, it follows that

$$\sum_{k\in\Omega_{1,\lambda}} N_k((1-\delta)\lambda) = \delta^{-1}O(\lambda^{\frac{1}{2}-\frac{2-d}{m}-r}) = o(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{3}{m}}).$$

(Here we note $r < 1/m$). □

Next we consider the case $k \in \Omega_{2,\lambda}$. Let $v_k(x) = V(x, z_k)$. By Assumption (V) (3.1), and the relation $mr - \alpha d = 0$, it follows that

$$W_\lambda(x,z) = v_k(x)(1 + O(\delta))$$

for $z \in J_k$, uniformly in $x \in \mathbf{R}^2$. Thus there exists $\beta > 0$ such that

$$-\partial_z^2 - P_\lambda W_\lambda P_\lambda \ge -\partial_z^2 - (1+\beta\delta)P_\lambda v_k P_\lambda, \tag{3.24}$$

in $L^2(\mathbf{R}_x^2 \times J_k)$ with the Dirichlet boundary condition.

Lemma 3.9.

$$\sum_{k\in\Omega_{2,\lambda}} N_k((1-\delta)\lambda) \le \sum_{k\in\Omega_{2,\lambda}} G_{k,1}(\lambda) + \delta O(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{3}{m}}), \tag{3.25}$$

where

$$G_{k,1}(\lambda) = -\frac{|J_k|}{\pi}\int_{\eta_\delta(\lambda)}^{R\lambda^{rm}} ((1+\beta\delta)\nu - (1-\delta)\lambda)^{1/2}\, dg_k((1-\delta)\nu), \tag{3.26}$$

$$\eta_\delta(\lambda) = \frac{1-\delta}{1+\beta\delta}\lambda = \lambda(1+O(\delta)), \tag{3.27}$$

$$g_k(\nu) = (2\pi)^{-1}\int_{v_k(x)>\nu} b(x)\, dx. \tag{3.28}$$

Proof. Let $\nu_{k,j}^{(\lambda)}$ be the j-th eigenvalue of the operator $P_\lambda v_k P_\lambda \in B(L^2(\mathbf{R}_x^2))$. Then the eigenvalues of the right hand side of (3.24) are

$$\frac{l^2\pi^2}{|J_k|^2} - (1+\beta\delta)\nu_{k,j}^{(\lambda)}, \quad l \in \mathbf{N}.$$

Let $\nu > 1/(\beta+2)$. We apply (3.3) in Proposition 3.1 to v_k/ν. Then for any $\varepsilon > 0$, we can choose $\lambda_\varepsilon > 0$ such that for $0 < \lambda < \lambda_\varepsilon$,

$$\begin{aligned} N(P_\lambda v_k P_\lambda > \nu\lambda) &\le (2\pi)^{-1}\int_{\{x\in\mathbf{R}^2:\ v_k(x)>(1-\varepsilon)\nu\lambda\}} b(x)\, dx + \varepsilon O((\nu\lambda)^{-\frac{2-d}{m}}) \\ &= O((\nu\lambda)^{-\frac{2-d}{m}}). \end{aligned} \tag{3.29}$$

Thus we see that $\nu_{k,j}^{(\lambda)} \le \text{const}.j^{-m/(2-d)}$, and that there exists $p > 0$ such that

$$\frac{\ell^2\pi^2}{|J_k|^2} - (1+\beta\delta)\nu_{k,j}^{(\lambda)} > -(1-\delta)\lambda \tag{3.30}$$

for $j > p\lambda^{-(2-d)/m}$. Therefore

$$N_k((1-\delta)\lambda) \le \sum_{j=1}^{p\lambda^{-\frac{2-d}{m}}} \left(\frac{|J_k|}{\pi}((1+\beta\delta)\nu_{k,j}^{(\lambda)} - (1-\delta)\lambda)^{1/2} + 1\right).$$

Let $m_k^{(\lambda)}(\nu) = N(P_\lambda v_k P_\lambda > \nu\lambda)$. Since, by Assumption (V), there exists $R > 0$ large enough such that $m_k^{(\lambda)}(\nu) = 0$ for all $\nu > R\lambda^{-(1-rm)}$, we see that

$$\begin{aligned} &N_k((1-\delta)\lambda) \\ &\quad \le -\frac{|J_k|}{\pi}\int_{\zeta_\delta}^{R\lambda^{-(1-rm)}} ((1+\beta\delta)\nu\lambda - (1-\delta)\lambda)^{1/2}\, dm_k^{(\lambda)}(\nu) + O(\lambda^{-\frac{2-d}{m}}), \end{aligned} \tag{3.31}$$

where

$$\zeta_\delta = \frac{1-\delta}{1+\beta\delta} = 1 + O(\delta). \tag{3.32}$$

Let

$$G_k(\lambda) = -\frac{|J_k|}{\pi}\int_{\zeta_\delta}^{R\lambda^{-(1-rm)}} ((1+\beta\delta)\nu\lambda - (1-\delta)\lambda)^{1/2}\, dm_k^{(\lambda)}(\nu). \tag{3.33}$$

Since $\sharp\Omega_{2,\lambda} = O(\lambda^{-(1/m)+r})$ and $r > 1/2$, we see

$$\begin{aligned}\sum_{k\in\Omega_{2,\lambda}} N_k((1-\delta)\lambda) &\le \sum_{k\in\Omega_{2,\lambda}} G_k(\lambda) + O(\lambda^{\frac{d}{m}-\frac{3}{m}+r})\\ &= \sum_{k\in\Omega_{2,\lambda}} G_k(\lambda) + o(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{3}{m}}),\end{aligned} \tag{3.34}$$

by (3.31). It follows that

$$m_k^{(\lambda)}(\nu) \le g_k((1-\delta)\lambda\nu) + \delta O((\nu\lambda)^{-\frac{2-d}{m}}), \tag{3.35}$$

by (3.29). Let

$$f(\nu) = -\frac{|J_k|}{\pi}((1+\beta\delta)\nu\lambda - (1-\delta)\lambda)^{1/2}.$$

By integration by parts, we see

$$\begin{aligned}G_k(\lambda) \le& -\frac{|J_k|}{\pi}\int_{\zeta_\delta}^{R\lambda^{-(1-rm)}} ((1+\beta\delta)\nu\lambda - (1-\delta)\lambda)^{1/2}\, dg_k((1-\delta)\lambda\nu)\\ &- O(\delta)\lambda^{-\frac{2-d}{m}}\int_{\zeta_\delta}^{R\lambda^{-(1-rm)}} \nu^{-\frac{2-d}{m}}\, df(\nu)\\ \le& G_{k,1}(\lambda) - O(\delta)\lambda^{-\frac{2-d}{m}}\lambda^{(1-rm)\frac{2-d}{m}} f(R\lambda^{-(1-rm)})\\ &+ O(\delta)\lambda^{-\frac{2-d}{m}}\int_{\zeta_\delta}^{R\lambda^{-(1-rm)}} f(\nu)\, d\nu^{-\frac{2-d}{m}}\\ =& I + II + III,\end{aligned} \tag{3.36}$$

from (3.35). Since $m/2 + d < 2$ and $r < 1/m$,

$$\begin{aligned}II &= O(\delta)\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{2}{m}-r}\lambda^{-(2-d-\frac{m}{2})r+\frac{2-d}{m}}\\ &= \delta O(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{2}{m}-r}),\\ III &= O(\delta)\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{2}{m}-r}\int_{\zeta_\delta}^{R\lambda^{-(1-rm)}} \nu^{1/2}\nu^{-\frac{2-d}{m}-1}\, d\nu\\ &= \delta O(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{2}{m}-r}).\end{aligned}$$

Hence we obtain

$$G_k(\lambda) \le G_{k,1}(\lambda) + \delta O(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{2}{m}-r}). \tag{3.37}$$

Since $\sharp\Omega_{2,\lambda} = O(\lambda^{-1/m+r})$, (3.25) follows from (3.34) and (3.37). □

Lemma 3.10.

$$\sum_{k\in\Omega_{2,\lambda}} G_{k,1}(\lambda) \le F(\lambda)(1+O(\delta^{1/2})). \tag{3.38}$$

Proof. By (3.28), (3.26) and the definition of Stieltjes integral, we see

$$G_{k,1}(\lambda) = 2(2\pi)^{-2}|J_k|c_\delta \int_{v_k(x)>(1-\delta)\eta_\delta(\lambda)} b(x)(v_k(x)-(1-\delta)\eta_\delta(\lambda))^{1/2}\,dx, \tag{3.39}$$

where

$$c_\delta = \left(\frac{1+\beta\delta}{1-\delta}\right)^{1/2} = 1+O(\delta). \tag{3.40}$$

To estimate the integral in the right hand side of (3.39), we decompose the integral as follows;

$$\begin{aligned}
&\int_{v_k(x)>(1-\delta)\eta_\delta(\lambda)} b(x)(v_k(x)-(1-\delta)\eta_\delta(\lambda))^{1/2}\,dx\\
&\quad= \int_{v_k(x)>\lambda} b(x)(v_k(x)-\lambda)^{1/2}\,dx\\
&\qquad+\int_{v_k(x)>\lambda} b(x)\left\{(v_k(x)-(1-\delta)\eta_\delta(\lambda))^{1/2}-(v_k(x)-\lambda)^{1/2}\right\}dx\\
&\qquad+\int_{(1-\delta)\eta_\delta(\lambda)<v_k(x)\le\lambda} b(x)(v_k(x)-(1-\delta)\eta_\delta(\lambda))^{1/2}\,dx\\
&\quad= I'+II'+III'.
\end{aligned} \tag{3.41}$$

By (3.27),

$$\begin{aligned}
II' &\le \int_{v_k(x)>\lambda} b(x)\left\{(v_k(x)-(1-C\delta)\lambda)^{1/2}-(v_k(x)-\lambda)^{1/2}\right\}dx\\
&= \int_{v_k(x)>\lambda} b(x)\left(\int_0^1 \frac{C\delta\lambda}{2(v_k(x)-(1-C\delta t)\lambda)^{1/2}}\,dt\right)dx\\
&\le C\delta\lambda\int_{v_k(x)>\lambda} b(x)\left(\int_0^1 \frac{1}{2(C\delta\lambda t)^{1/2}}\,dt\right)dx\\
&= O(\delta^{1/2})\lambda^{1/2}\int_{v_k(x)>\lambda} b(x)\,dx\\
&= \delta^{1/2}O(\lambda^{\frac12+\frac{d}{m}-\frac{2}{m}}).
\end{aligned}$$

We also have

$$\begin{aligned} III' &\leq \int_{(1-C\delta)\lambda<v_k(x)\leq\lambda} b(x)(v_k(x)-(1-C\delta)\lambda)^{1/2}\,dx \\ &= O(\delta^{1/2})\lambda^{1/2}\int_{(1-C\delta)\lambda<v_k(x)\leq\lambda} b(x)\,dx \\ &= \delta^{1/2}O(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{2}{m}}). \end{aligned}$$

Here we used the estimate

$$\int_{v_k(x)>\lambda} b(x)\,dx = O(\lambda^{-\frac{2-d}{m}}). \tag{3.42}$$

which follows from Assumptions (b) and (V). Hence we see

$$(3.41) \leq \int_{v_k(x)>\lambda} b(x)(v_k(x)-\lambda)^{1/2}\,dx + \delta^{1/2}O(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{2}{m}}).$$

Since $|J_k| = \lambda^{-r}(1+O(\delta))$, we obtain

(3.43)

$$G_{k,1}(\lambda) \leq 2(2\pi)^{-2}|I_k|\int_{v_k(x)>\lambda} b(x)(v_k(x)-\lambda)^{1/2}\,dx + \delta^{1/2}O(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{2}{m}-r})$$

from (3.39) and (3.40).

Next we estimate the right hand side of (3.43). It follows from Assumption (V) that

$$v_k(x) = V(x,z)(1+O(\delta))$$

for $z \in I_k$. Thus we see

(3.44)

$$\begin{aligned} &|I_k|\int_{v_k(x)>\lambda} b(x)(v_k(x)-\lambda)^{1/2}\,dx \\ &= \int_{I_k}\int_{v_k(x)>\lambda} b(x)(v_k(x)-\lambda)^{1/2}\,dx\,dz \\ &\leq \iint_{\{V(x,z)>\frac{\lambda}{1+C\delta},z\in I_k\}} b(x)((1+C\delta)V(x,z)-\lambda)^{1/2}\,dx\,dz \\ &= \iint_{\{V(x,z)>\lambda,z\in I_k\}} b(x)(V(x,z)-\lambda)^{1/2}\,dx\,dz \\ &\quad + \iint_{\{V(x,z)>\lambda,z\in I_k\}} b(x)\Big\{((1+C\delta)V(x,z)-\lambda)^{1/2}-(V(x,z)-\lambda)^{1/2}\Big\}\,dx\,dz \\ &\quad + \iint_{\{\frac{\lambda}{1+C\delta}<V(x,z)\leq\lambda,z\in I_k\}} b(x)((1+C\delta)V(x,z)-\lambda)^{1/2}\,dx\,dz. \end{aligned}$$

Since

$$\int_{V(x,z)>\lambda} b(x)V(x,z)^{1/2}\,dx = O(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{2}{m}-r}) \tag{3.45}$$

uniformly in $z \in I_k$ (this is easily seen from Assumptions (b) and (V)), the second and the third term in the right hand side of (3.44) are bounded $\delta^{1/2} \times O(\lambda^{1/2+d/m-2/m-r})$ from above, by the same computation as in the estimate of the second and the third term in the right hand side of (3.41). Therefore

$$\begin{aligned}(\text{the LHS of (3.44)}) \le \iint_{\{V(x,z)>\lambda, z\in I_k\}} & b(x)(V(x,z)-\lambda)^{1/2}\,dx\,dz \\ &+ \delta^{1/2}O(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{2}{m}-r}).\end{aligned}$$

Combining (3.43) with this estimate, we obtain

$$\begin{aligned}G_{k,1}(\lambda) \le 2(2\pi)^{-2} & \iint_{\{V(x,z)>\lambda, z\in I_k\}} b(x)(V(x,z)-\lambda)^{1/2}\,dx\,dz \\ &+ \delta^{1/2}O(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{2}{m}-r}).\end{aligned} \tag{3.46}$$

Since $\sharp\Omega_{2,\lambda} = O(\lambda^{-1/m+r})$, we see

(3.47)

$$\begin{aligned}\sum_{k\in\Omega_{2,\lambda}} G_{k,1}(\lambda) \le \sum_{k\in\Omega_{2,\lambda}} & 2(2\pi)^{-2} \iint_{\{V(x,z)>\lambda, z\in I_k\}} b(x)(V(x,z)-\lambda)^{1/2}\,dx\,dz \\ &+ \delta^{1/2}O(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{3}{m}}).\end{aligned}$$

On the other hand, it is easily seen

$$\begin{aligned}&\sum_{k\in\Omega_{1,\lambda}} \iint_{\{V(x,z)>\lambda, z\in I_k\}} b(x)(V(x,z)-\lambda)^{1/2}\,dx\,dz = o(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{3}{m}}), \\ &\sum_{k\in\Omega_{3,\lambda}} \iint_{\{V(x,z)>\lambda, z\in I_k\}} b(x)(V(x,z)-\lambda)^{1/2}\,dx\,dz = 0.\end{aligned} \tag{3.48}$$

Therefore from (3.47) and Lemma 2.2, it follows

$$\begin{aligned}\sum_{k\in\Omega_{2,\lambda}} G_{k,1}(\lambda) &\le F(\lambda) + \delta^{1/2}O(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{3}{m}}) \\ &\le F(\lambda)(1+O(\delta^{1/2})).\end{aligned}$$

□

The next lemma follows immediately from Lemmas 3.9 and 3.10.

Lemma 3.11.

$$\sum_{k\in\Omega_{2,\lambda}} N_k((1-\delta)\lambda) \le F(\lambda)(1+O(\delta^{1/2})) \tag{3.49}$$

Proof of the upper bound (2.17). It follows from Lemmas 3.6, 3.8 and 3.11 that

$$N(K_\lambda < -\lambda) \le F(\lambda)(1+O(\delta^{1/2})) + o(\lambda^{-\frac{2-d}{m}}), \quad \lambda \to 0. \tag{3.50}$$

Since δ is arbitrary, (2.17) follows from Lemma 2.2. □

§3.2. Proof of (2.18): Lower bound

In this subsection, we prove the lower bound in a similar way as the upper bound. The proof of the lower bound is simpler than the upper bound.

Let $\delta > 0$ be fixed. Let r be a constant and let $\{I_k\}$ be a sequence of open intervals defined in the proof of upper bound.

By Assumption (V), we can choose $M > 0$ so large that

$$\sup_{x\in\mathbf{R}^2} V(x,z) \le \frac{\lambda}{2}$$

for $|z_k| > M\lambda^{-1/m}$, uniformly in $z \in I_k$. Then let

(3.51)
$$\Omega_{1,\lambda} = \{k \in \mathbf{Z}\colon |z_k| \le \delta^{-1}\lambda^{-r}\}, \quad \Omega_{2,\lambda} = \{k \in \mathbf{Z}\colon \delta^{-1}\lambda^{-r} < |z_k| < M\lambda^{-\frac{1}{m}}\},$$
$$\Omega_{3,\lambda} = \{k \in \mathbf{Z}\colon |z_k| \ge M\lambda^{-\frac{1}{m}}\}.$$

We note

$$\begin{aligned} N(K_\lambda < -\lambda) &\ge N(P_\lambda K_\lambda P_\lambda < -\lambda) \\ &= N(P_\lambda(-\partial_z^2 - V)P_\lambda < -\lambda) \\ &= N(-\partial_z^2 - P_\lambda V P_\lambda < -\lambda). \end{aligned}$$

Let $N_k(t)$ be the number of eigenvalues less than $-t$ of the operator $-\partial_z^2 - P_\lambda V P_\lambda$ in $L^2(\mathbf{R}_x^2 \times I_k)$ with the Dirichlet boundary condition.

Lemma 3.12.

$$N(K_\lambda < -\lambda) \ge \sum_{k\in\Omega_{2,\lambda}} N_k(\lambda). \tag{3.52}$$

Let $v_k(x) = V(x, z_k)$. Then we have

Lemma 3.13.

$$\sum_{k\in\Omega_{2,\lambda}} N_k(\lambda) \geq 2 \sum_{k\in\Omega_{2,\lambda}} G_{k,0}(\lambda) - F(\lambda) - \delta^{1/2} O(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{3}{m}}), \tag{3.53}$$

where

$$G_{k,0}(\lambda) = -\frac{|I_k|}{\pi} \int_{\xi_\delta(\lambda)}^{R\lambda^{rm}} ((1-\beta\delta)\nu - \lambda)^{1/2}\, dg_k((1+\delta)\nu), \tag{3.54}$$

$$\xi_\delta(\lambda) = \frac{1}{1-\beta\delta}\lambda = \lambda(1+O(\delta)), \tag{3.55}$$

$$g_k(\nu) = (2\pi)^{-1} \int_{v_k(x)>\nu} b(x)\, dx. \tag{3.56}$$

Proof. Let $k \in \Omega_{2,\lambda}$. By Assumption (V), it is easily seen that

$$V(x, z) = v_k(x)(1 + O(\delta))$$

for $z \in I_k$, uniformly in $x \in \mathbf{R}^2$. Thus there exists $\beta > 0$ such that

$$-\partial_z^2 - P_\lambda V P_\lambda \leq -\partial_z^2 - (1-\beta\delta) P_\lambda v_k P_\lambda, \tag{3.57}$$

in $L^2(\mathbf{R}_x^2 \times I_k)$ with the Dirichlet boundary condition. Let $\nu_{k,j}^{(\lambda)}$ be the j-th eigenvalue of the operator $P_\lambda v_k P_\lambda \in B(L^2(\mathbf{R}_x^2))$. Then the eigenvalues of the right hand side of (3.57) are

$$\frac{l^2\pi^2}{|I_k|^2} - (1-\beta\delta)\nu_{k,j}^{(\lambda)}, \quad l \in \mathbf{N}.$$

Applying (3.29) to v_k/ν, we see that $\nu_{k,j}^{(\lambda)} \leq \text{const.} j^{-m/(2-d)}$ and that there exists $p > 0$ such that

$$\frac{l^2\pi^2}{|I_k|^2} - (1-\beta\delta)\nu_{k,j}^{(\lambda)} > -\lambda \tag{3.58}$$

for $j > p\lambda^{-(2-d)/m}$. Hence we have

$$N_k(\lambda) \geq \sum_{j=1}^{p\lambda^{-\frac{2-d}{m}}} \left(\frac{|I_k|}{\pi} ((1-\beta\delta)\nu_{k,j}^{(\lambda)} - \lambda)^{1/2} - 1 \right).$$

Let $m_k^{(\lambda)}(\nu) = N(P_\lambda v_k P_\lambda > \lambda\nu)$. By Assumption (V), there exists $R > 0$ large enough such that $m_k^{(\lambda)}(\nu) = 0$ for all $\nu > R\lambda^{-(1-rm)}$. Hence we obtain

$$N_k(\lambda) \geq -\frac{|I_k|}{\pi} \int_{\theta_\delta}^{R\lambda^{-(1-rm)}} ((1-\beta\delta)\lambda\nu - \lambda)^{1/2}\, dm_k^{(\lambda)}(\nu) - O(\lambda^{-\frac{2-d}{m}}), \tag{3.59}$$

where

$$\theta_\delta = \frac{1}{1-\beta\delta} = 1 + O(\delta). \tag{3.60}$$

Let

$$G_k(\lambda) = -\frac{|I_k|}{\pi} \int_{\theta_\delta}^{R\lambda^{-(1-rm)}} ((1-\beta\delta)\lambda\nu - \lambda)^{1/2}\, dm_k^{(\lambda)}(\nu). \tag{3.61}$$

Since $\sharp\Omega_{2,\lambda} = O(\lambda^{-1/m+r})$ and $r > 1/2$, we see that

$$\begin{aligned} \sum_{k\in\Omega_{2,\lambda}} N_k(\lambda) &\geq \sum_{k\in\Omega_{2,\lambda}} G_k(\lambda) - O(\lambda^{-\frac{d}{m}-\frac{3}{m}+r}) \\ &= \sum_{k\in\Omega_{2,\lambda}} G_k(\lambda) - o(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{3}{m}}), \end{aligned} \tag{3.62}$$

by (3.59). Let $\nu > 1$. We apply (3.4) in Proposition 3.1 to v_k/ν $(\nu > 1)$. Then for any $\delta > 0$, we can choose $\lambda_\delta > 0$ such that for $0 < \lambda < \lambda_\delta$,

$$m_k^{(\lambda)}(\nu) \geq 2g_k((1+\delta)\lambda\nu) - g_k((1-\delta)\lambda\nu) - \delta O((\nu\lambda)^{-\frac{2-d}{m}}). \tag{3.63}$$

Therefore by integration by parts and the same computation as in (3.36), we obtain

$$\begin{aligned} G_k(\lambda) \geq & -\frac{2|I_k|}{\pi} \int_{\theta_\delta}^{R\lambda^{-(1-rm)}} ((1-\beta\delta)\lambda\nu - \lambda)^{1/2}\, dg_k((1+\delta)\lambda\nu) \\ & + \frac{|I_k|}{\pi} \int_{\theta_\delta}^{R\lambda^{-(1-rm)}} ((1-\beta\delta)\lambda\nu - \lambda)^{1/2}\, dg_k((1-\delta)\lambda\nu) \\ & - \delta\lambda^{-\frac{2-d}{m}} \int_{\theta_\delta}^{R\lambda^{-(1-rm)}} \nu^{-\frac{2-d}{m}}\, df(\nu) \\ \geq & \, 2G_{k,0}(\lambda) - G_{k,1}(\lambda) - \delta O(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{2}{m}-r}), \end{aligned} \tag{3.64}$$

where

$$G_{k,1}(\lambda) = -\frac{|I_k|}{\pi} \int_{\xi_\delta(\lambda)}^{R\lambda^{rm}} ((1-\beta\delta)\nu - \lambda)^{1/2}\, dg_k((1-\delta)\nu), \tag{3.65}$$

$$f(\nu) = -\frac{|I_k|}{\pi}((1-\beta\delta)\lambda\nu - \lambda)^{1/2}. \tag{3.66}$$

By the same computation in the proof of upper bound, we learn

$$G_{k,1}(\lambda) \le 2(2\pi)^{-2} \iint_{\{V(x,z)>\lambda, z\in I_k\}} b(x)(V(x,z)-\lambda)^{1/2}\,dx\,dz \\ + \delta^{1/2} O(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{2}{m}-r}).$$

Therefore, since $\sharp\Omega_{2,\lambda} = O(\lambda^{-1/m+r})$, (3.53) follows from (3.62) and (3.64). □

Lemma 3.14.

$$\sum_{k\in\Omega_{2,\lambda}} G_{k,0}(\lambda) \ge F(\lambda)(1-O(\delta^{1/2})). \tag{3.67}$$

Proof. By (3.56), (3.54) and definition of Stieltjes integral, we have

$$G_{k,0}(\lambda) = 2(2\pi)^{-2}|I_k| c_\delta \int_{v_k(x)>(1+\delta)\xi_\delta(\lambda)} b(x)(v_k(x)-(1+\delta)\xi_\delta(\lambda))^{1/2}\,dx, \tag{3.68}$$

where

$$c_\delta = \left(\frac{1-\beta\delta}{1+\delta}\right)^{1/2} = 1 + O(\delta). \tag{3.69}$$

In order to estimate the integral in the right hand side of (3.68) from below, we decompose it as follows:

$$\begin{aligned} &\int_{v_k(x)>(1+\delta)\xi_\delta(\lambda)} b(x)(v_k(x)-(1+\delta)\xi_\delta(\lambda))^{1/2}\,dx \\ &= \int_{v_k(x)>\lambda} b(x)(v_k(x)-\lambda)^{1/2}\,dx \\ &\quad - \int_{v_k(x)>(1+\delta)\xi_\delta(\lambda)} b(x)\left\{(v_k(x)-\lambda)^{1/2} - (v_k(x)-(1+\delta)\xi_\delta(\lambda))^{1/2}\right\} dx \\ &\quad - \int_{\lambda\le v_k(x)<(1+\delta)\xi_\delta(\lambda)} b(x)(v_k(x)-\lambda)^{1/2}\,dx \\ &= I - II - III. \end{aligned} \tag{3.70}$$

Recalling (3.55), we see that

$$
\begin{aligned}
II &\le \int_{v_k(x)>(1+C\delta)\lambda} b(x)\left\{(v_k(x)-\lambda)^{1/2}-(v_k(x)-(1-C\delta)\lambda)^{1/2}\right\}dx \\
&= \int_{v_k(x)>(1+C\delta)\lambda} b(x)\left(\int_0^1 \frac{C\delta\lambda}{2(v_k(x)-(1+C\delta t)\lambda)^{1/2}}\,dt\right)dx \\
&\le C\delta\lambda \int_{v_k(x)>(1+C\delta)\lambda} b(x)\left(\int_0^1 \frac{1}{2(C\delta\lambda(1-t))^{1/2}}\,dt\right)dx \\
&= O(\delta^{1/2})\lambda^{1/2}\int_{v_k(x)>(1+C\delta)\lambda} b(x)\,dx \\
&= \delta^{1/2}O(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{2}{m}}),
\end{aligned}
$$

and we also have

$$
\begin{aligned}
III &\le \int_{\lambda\le v_k(x)<(1+C\delta)\lambda} b(x)(v_k(x)-\lambda)^{1/2}\,dx \\
&= O(\delta^{1/2})\lambda^{1/2}\int_{\lambda\le v_k(x)<(1+C\delta)\lambda} b(x)\,dx \\
&= \delta^{1/2}O(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{2}{m}}),
\end{aligned}
$$

where we have used (3.42). Hence we obtain

$$
(3.70) \ge \int_{v_k(x)>\lambda} b(x)(v_k(x)-\lambda)^{1/2}\,dx - \delta^{1/2}O(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{2}{m}}).
$$

Noting $|I_k| = \lambda^{-r}$, (3.42) and (3.69), we obtain

(3.71)
$$
G_{k,0}(\lambda) \ge 2(2\pi)^{-2}|I_k|\int_{v_k(x)>\lambda} b(x)(v_k(x)-\lambda)^{1/2}\,dx - \delta^{1/2}O(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{2}{m}-r}).
$$

Next we estimate the right hand side of (3.71). It follows from Assumption (V) that

$$
V(x,z) = v_k(x)(1+O(\delta))
$$

for $z \in I_k$. Thus we see that

$$
\begin{aligned}
&|I_k| \int_{v_k(x)>\lambda} b(x)(v_k(x)-\lambda)^{1/2}\,dx \\
&= \int_{I_k}\int_{v_k(x)>\lambda} b(x)(v_k(x)-\lambda)^{1/2}\,dx\,dz \\
&\geq \iint_{\{V(x,z)>(1+C\delta)\lambda, z\in I_k\}} b(x)((1+C\delta)^{-1}V(x,z)-\lambda)^{1/2}\,dx\,dz \\
&= \iint_{\{V(x,z)>\lambda, z\in I_k\}} b(x)(V(x,z)-\lambda)^{1/2}\,dx\,dz \\
&\quad - \iint_{\{V(x,z)>(1+C\delta)\lambda, z\in I_k\}} \\
&\quad \times b(x)\left\{(V(x,z)-\lambda)^{1/2} - ((1+C\delta)^{-1}V(x,z)-\lambda)^{1/2}\right\}\,dx\,dz \\
&\quad - \iint_{\lambda\leq V(x,z)<(1+C\delta)\lambda, z\in I_k\}} b(x)(V(x,z)-\lambda)^{1/2}\,dx\,dz.
\end{aligned}
\tag{3.72}
$$

Now we recall (3.45). Then the second and the third term in the right hand side of (3.72) are bounded $\delta^{1/2}O(\lambda^{1/2+d/m-2/m-r})$ from above, by the same computation as in the estimate of (3.70). Therefore

$$
\begin{aligned}
\text{(the LHS of (3.72))} \geq &\iint_{\{V(x,z)>\lambda, z\in I_k\}} \\
&\times b(x)(V(x,z)-\lambda)^{1/2}\,dx\,dz - \delta^{1/2}O(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{2}{m}-r}).
\end{aligned}
$$

Combining this estimate with (3.71), we obtain

$$
\begin{aligned}
G_{k,0}(\lambda) \geq 2(2\pi)^{-2} &\iint_{\{V(x,z)>\lambda, z\in I_k\}} \\
&\times b(x)(V(x,z)-\lambda)^{1/2}\,dx\,dz - \delta^{1/2}O(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{2}{m}-r}).
\end{aligned}
\tag{3.73}
$$

Since $\sharp\Omega_{2,\lambda} = O(\lambda^{-1/m+r})$, we see that

$$
\begin{aligned}
\sum_{k\in\Omega_{2,\lambda}} G_{k,0}(\lambda) \geq \sum_{k\in\Omega_{2,\lambda}} 2(2\pi)^{-2} &\iint_{\{V(x,z)>\lambda, z\in I_k\}} \\
&\times b(x)(V(x,z)-\lambda)^{1/2}\,dx\,dz - \delta^{1/2}O(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{3}{m}}).
\end{aligned}
\tag{3.74}
$$

Therefore it follows from (3.48), (3.74) and Lemma 2.2 that

$$
\begin{aligned}
\sum_{k\in\Omega_{2,\lambda}} G_{k,0}(\lambda) &\geq F(\lambda) - \delta^{1/2}O(\lambda^{\frac{1}{2}+\frac{d}{m}-\frac{3}{m}}) \\
&= F(\lambda)(1-O(\delta^{1/2}))
\end{aligned}
$$

□

The next lemma follows immediately from Lemmas 3.13 and 3.14.

Lemma 3.15.

$$\sum_{k\in\Omega_{2,\lambda}} N_k(\lambda) \geq F(\lambda)(1-O(\delta^{1/2})). \tag{3.75}$$

Proof of the lower bound (2.18). It follows from Lemmas 3.12 and 3.15 that

$$N(K_\lambda < -\lambda) \geq F(\lambda)(1-O(\delta^{1/2})), \quad \lambda \to 0. \tag{3.76}$$

Since $\delta > 0$ is arbitrary, (2.18) follows from Lemma 2.2. □

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