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Invariants of Fold-maps via Stable Homotopy Groups

Dedicated to Professor Tatsuo Suwa on his sixtieth birthday

By

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Abstract

In the 2-jet space $J^2(n,p)$ of smooth map germs $(\mathbf{R}^n,0) \to (\mathbf{R}^p,0)$ with $n \geq p \geq 2$, we consider the subspace $\Omega^{n-p+1,0}(n,p)$ consisting of all 2-jets of regular germs and map germs with fold singularities. In this paper we determine the homotopy type of the space $\Omega^{n-p+1,0}(n,p)$. Let N and P be smooth (C^∞) manifolds of dimensions n and p. A smooth map $f : N \to P$ is called a fold-map if f has only fold singularities. We will prove that this homotopy type is very useful in finding invariants of fold-maps. For instance, by applying the homotopy principle for fold-maps in [An3] and [An4] we prove that if $n-p+1$ is odd and P is connected, then there exists a surjection of the set of cobordism classes of fold-maps into P to the stable homotopy group $\lim_{k,\ell\to\infty} \pi_{n+k+\ell}(T(\nu_P^k) \wedge T(\widehat{\gamma}^\ell_{G_{n-p+1,\ell}}))$. Here, ν_P^k is the normal bundle of P in $\mathbf{R}^{p+k}$ and $\widehat{\gamma}^\ell_{G_{n-p+1,\ell}}$ denote the canonical vector bundles of dimension ℓ over the grassman manifold $G_{n-p+1,\ell}$. We also prove the oriented version.

Introduction

Let N and P be smooth (C^∞) manifolds of dimensions n and p with $n \geq p \geq 2$. A fold-map germ $(N,x) \to (P,y)$ refers to a smooth map germ which is written as $(x_1,\dots,x_n) \mapsto (x_1,\dots,x_{p-1},\pm x_p^2 \pm \cdots \pm x_n^2)$ under suitable local coordinates systems of (N,x) and (P,y). A fold-map $N \to P$ refers to a smooth

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map which has only fold singularities. In this paper we will study the existence problem of fold-maps and homotopy-theoretic invariants for classifying fold-maps from the viewpoint of homotopy principle (the terminology used in [G2]).

Let $J^2(N,P)$ denote the 2-jet space of the manifolds N and P and let $\Omega^{n-p+1,0}(N,P)$ be the subspace of $J^2(N,P)$ associated to $\Omega^{n-p+1,0}(n,p)$, which consists of all 2-jets of regular germs and fold-map germs. We explain the motivation for studying the homotopy type of $\Omega^{n-p+1,0}(n,p)$. The existence and non-existence problem of fold-maps has been first dealt with in dimensions $(n,2)$ in [T] and [L]. A smooth map $f : N \to P$ is a fold-map if and only if the image of j^2f is contained in $\Omega^{n-p+1,0}(N,P)$ and j^2f is transverse to the Boardman submanifold $\Sigma^{n-p+1,0}(N,P)$ defined in [L] and [B] (see [Mo]). Let $C^\infty_\Omega(N,P)$ denote the space consisting of all smooth maps $f : N \to P$ such that the image of j^2f is contained in $\Omega^{n-p+1,0}(N,P)$ with the C^∞-topology. Let $\Gamma(N,P)$ denote the space consisting of all continuous sections of the fibre bundle $\pi_N|\Omega^{n-p+1,0}(N,P) : \Omega^{n-p+1,0}(N,P) \to N$ equipped with the compact-open topology. Then there exists a continuous map $j_\Omega : C^\infty_\Omega(N,P) \to \Gamma(N,P)$ defined by $j_\Omega(f) = j^2f$. In dimensions $n \geq p \geq 2$ we have the homotopy principle for fold-maps in the existence level. Namely, a continuous section s of $\Gamma(N,P)$ has a fold-map $f : N \to P$ such that j^2f and s are homotopic as sections of $\Gamma(N,P)$. As for this homotopy principle, we should refer to [G1], [G2], [E1], [E2] and [An3, Theorem 6] and [An4, Theorem 0.5] together with [An1, Theorem 2]. We will show how the homotopy type of the fibre $\Omega^{n-p+1,0}(n,p)$ is important for our purpose.

We denote, by $V^{row}_{n+1,p}$, the Stiefel manifold $(E_p \times O(n-p+1))\diagdown O(n+1)$, whose elements as $p \times n$ matrices constitute, with the canonical basis of $\mathbf{R}^n$ and $\mathbf{R}^p$, the space $\mathbf{V}(\mathbf{R}^{n+1},\mathbf{R}^p)$ of corresponding epimorphisms $\mathbf{R}^n \to \mathbf{R}^p$. We identify both spaces throughout the paper. They have the actions of $O(p) \times O(n)$ from the lefthand side through $O(p)$ and the righthand side through $O(n) \times 1$ respectively. The group $O(p) \times O(n)$ also naturally acts on $\Omega^{n-p+1,0}(n,p)$. In order to reduce our problem of finding invariants of fold-maps to the problem concerning sections of the fiber bundle $\Omega^{n-p+1,0}(N,P)$ over N, we will determine the homotopy type of $\Omega^{n-p+1,0}(n,p)$ in this paper (Theorem 2.6). As a consequence of this homotopy type, we obtain a topological embedding

$$i_{V,\Omega} : V^{row}_{n+1,p} \to \Omega^{n-p+1,0}(n,p),$$

which is equivariant with respect to the actions of $O(p) \times O(n)$. Furthermore, if $n-p+1$ is odd, then there exists an equivariant map

$$R_{\Omega,V} : \Omega^{n-p+1,0}(n,p) \to V^{row}_{n+1,p}$$

such that $R_{\Omega,V} \circ i_{V,\Omega}$ is the identity of $V^{row}_{n+1,p}$. We provide N and P with Riemannian metrics. Let $\theta_N = N \times \mathbf{R}$. Let $\mathbf{V}(TN \oplus \theta_N, TP)$ denote the fiber bundle over $N \times P$ with fiber $\mathbf{V}(T_xN \oplus \mathbf{R}, T_yP)$ associated to $\mathbf{V}(\mathbf{R}^{n+1}, \mathbf{R}^p)$, where (x,y) varies all over (N,P). By the Riemannian metrics of N and P the structure group of $J^2(N,P)$ is reduced to $O(p) \times O(n)$. Let $i_{\mathbf{V},\Omega}(N,P) : \mathbf{V}(TN \oplus \theta_N, TP) \to \Omega^{n-p+1,0}(N,P)$ and $R_{\Omega,\mathbf{V}}(N,P) : \Omega^{n-p+1,0}(N,P) \to \mathbf{V}(TN \oplus \theta_N, TP)$ be the fiber maps associated to $i_{V,\Omega}$ and $R_{\Omega,V}$ respectively. Let $\Gamma(N,P)$ and $\Gamma(\mathbf{V}(TN \oplus \theta_N, TP))$ be the space of all continuous sections of the fiber bundles $\Omega^{n-p+1,0}(N,P)$ and $\mathbf{V}(TN \oplus \theta_N, TP)$ over N respectively equipped with the compact-open topology. Let $\Gamma(i_{\mathbf{V},\Omega}) : \Gamma(\mathbf{V}(TN \oplus \theta_N, TP)) \to \Gamma(N,P)$ and $\Gamma(R_{\Omega,V}) : \Gamma(N,P) \to \Gamma(\mathbf{V}(TN \oplus \theta_N, TP))$ be the maps induced from the maps $i_{\mathbf{V},\Omega}(N,P)$ and $R_{\Omega,\mathbf{V}}(N,P)$ respectively. The first result of this paper is the following theorem.

Theorem 0.1. *Let $n \geq p \geq 2$. Let N and P be provided with Riemannian metrics. Then we have*

- (i) *the fiber map $i_{\mathbf{V},\Omega}(N,P) : \mathbf{V}(TN \oplus \theta_N, TP) \to \Omega^{n-p+1,0}(N,P)$ is a topological embedding,*
- (ii) *if $n-p+1$ is odd, then the composition $R_{\Omega,\mathbf{V}}(N,P) \circ i_{\mathbf{V},\Omega}(N,P)$ is the identity of $\mathbf{V}(TN \oplus \theta_N, TP)$.*

Let $\mathrm{Epi}(TN \oplus \theta_N, TP)$ be the fiber bundle over $N \times P$ with fiber $\mathrm{Epi}(T_xN \oplus \theta_N, T_yP)$ consisting of all epimorphisms $T_xN \oplus \theta_N \to T_yP$. Let $\Gamma(\mathrm{Epi}(TN \oplus \theta_N, TP))$ be the space of all continuous sections of the fiber bundle $\mathrm{Epi}(TN \oplus \theta_N, TP)$ over N equipped with the compact-open topology. Let $i_{\mathbf{V},Epi} : \mathbf{V}(\mathbf{R}^{n+1}, \mathbf{R}^p) \to \mathrm{Epi}(\mathbf{R}^{n+1}, \mathbf{R}^p)$ be the inclusion and let $i_{\mathbf{V},Epi}(N,P) : \mathbf{V}(TN \oplus \theta_N, TP) \to \mathrm{Epi}(TN \oplus \theta_N, TP)$ be the fiber homotopy equivalence associated to $i_{\mathbf{V},Epi}$. Let $i_{\mathbf{V},Epi}(N,P)^{-1}$ be the homotopy inverse of $i_{\mathbf{V},Epi}(N,P)$, and let $\Gamma(i^{-1}_{\mathbf{V},Epi}) : \Gamma(\mathrm{Epi}(TN \oplus \theta_N, TP)) \to \Gamma(\mathbf{V}(TN \oplus \theta_N, TP))$ be the map induced from $i_{\mathbf{V},Epi}(N,P)^{-1}$. Then Theorem 0.1, [An3, Theorem 6] and [An4, Theorem 0.5] yield the following theorem.

Theorem 0.2. *Let $n \geq p \geq 2$. Then any element $h \in \Gamma(Epi(TN \oplus \theta_N, TP))$ has a fold map $f : N \to P$ such that $\Gamma(i_{\mathbf{V},\Omega}) \circ \Gamma(i^{-1}_{\mathbf{V},Epi})(h)$ and $j^2 f$ are homotopic as sections in $\Gamma(N,P)$.*

Let P be a connected closed (resp. an oriented) smooth manifold of dimension p. For the study of invariants classifying fold-maps we define a fold-cobordism class of a fold-map between (resp. oriented) smooth manifolds. Namely, let $f_i : N_i \to P$ $(i = 0, 1)$ be two fold-maps, where N_i are closed (resp. oriented) smooth manifolds of dimension n. We say that they are (resp. *oriented-*) *fold-cobordant* when there exists a fold-map $F : (W, \partial W) \to (P \times [0,1], P \times 0 \cup P \times 1)$ such that

(i) W is a (resp. an oriented) smooth manifold of dimension $n+1$ with $\partial W = N_0 \cup (-N_1)$ and the collar of ∂W is identified with $N_0 \times [0, \varepsilon) \cup N_1 \times (1-\varepsilon, 1]$,

(ii) $F|N_0 \times [0,\varepsilon) = f_0 \times id_{[0,\varepsilon)}$ and $F|N_1 \times (1-\varepsilon, 1] = f_1 \times id_{(1-\varepsilon,1]}$,

where ε is a sufficiently small positive number. Let $\mathfrak{N}_n^{fold}(P)$ (resp. $\Omega_n^{fold}(P)$) denote the set of all (resp. oriented-) fold-cobordism classes of fold-maps into P.

Let ν_P^k be the stable normal bundle of an embedding $P \to S^{n+k}$. Let $G_{m,\ell}$ (resp. $\widetilde{G}_{m,\ell}$) be the (resp. oriented) grassmann manifold of all (resp. oriented) m-subspaces of $\mathbf{R}^{m+\ell}$. Let $\gamma^m_{G_{m,\ell}}$ and $\widehat{\gamma}^\ell_{G_{m,\ell}}$ (resp. $\gamma^m_{\widetilde{G}_{m,\ell}}$ and $\widehat{\gamma}^\ell_{\widetilde{G}_{m,\ell}}$) denote the canonical vector bundles of dimensions m and ℓ over the space $G_{m,\ell}$ (resp. $\widetilde{G}_{m,\ell}$) respectively such that $\gamma^m_{G_{m,\ell}} \oplus \widehat{\gamma}^\ell_{G_{m,\ell}}$ (resp. $\gamma^m_{\widetilde{G}_{m,\ell}} \oplus \widehat{\gamma}^\ell_{\widetilde{G}_{m,\ell}}$) is the trivial bundle $\theta^{m+\ell}_{G_{m,\ell}}$ (resp. $\theta^{m+\ell}_{\widetilde{G}_{m,\ell}}$). Let $T(\nu^k_P)$, $T(\widehat{\gamma}^\ell_{G_{m,\ell}})$ and $T(\widehat{\gamma}^\ell_{\widetilde{G}_{m,\ell}})$ be the Thom spaces of ν^k_P, $\widehat{\gamma}^\ell_{G_{m,\ell}}$ and $\widehat{\gamma}^\ell_{\widetilde{G}_{m,\ell}}$ respectively.

Theorem 0.3. *Let $n \geq p \geq 2$ and $n-p+1$ be odd. Let P be a connected closed smooth manifold of dimension p. Let $\ell \gg n$. Then there exist the surjections*

$$\omega^{\mathfrak{N}}_{n,p} : \mathfrak{N}_n^{fold}(P) \to \lim_{k\to\infty} \pi_{n+k+\ell}(T(\nu^k_P) \wedge T(\widehat{\gamma}^\ell_{G_{n-p+1,\ell}})),$$
$$\omega^{\Omega}_{n,p} : \Omega_n^{fold}(P) \to \lim_{k\to\infty} \pi_{n+k+\ell}(T(\nu^k_P) \wedge T(\widehat{\gamma}^\ell_{\widetilde{G}_{n-p+1,\ell}})).$$

Furthermore, we will give another invariant in a more general situation. Let G refer to $G_{n,\ell}$ or $\widetilde{G}_{n,\ell}$. Let $J^2(\gamma^n_G, TP)$ denote the vector bundle $\mathrm{Hom}(\gamma^n_G, TP) \oplus \mathrm{Hom}(S^2\gamma^n_G, TP)$ over $G \times P$ with projection $p_G : J^2(\gamma^n_G, TP) \to P$, where $S^2\gamma^n_G$ refers to the 2-fold symmetric product of γ^n_G (see (3.1)). Let $\Omega^{n-p+1,0}(\gamma^n_G, TP)$ denote the open subbundle of $J^2(\gamma^n_G, TP)$ with fiber $\Omega^{n-p+1,0}(n,p)$ defined in (3.2). Consider the induced bundle $p^*_G(\widehat{\gamma}^\ell_G)|_{\Omega^{n-p+1,0}(\gamma^n_G, TP)}$, the canonical bundle map $B_{\widehat{\gamma}^\ell} : p^*_{\widetilde{G}_{n,\ell}}(\widehat{\gamma}^\ell_{\widetilde{G}_{n,\ell}})|_{\Omega^{n-p+1,0}(\gamma^n_{\widetilde{G}_{n,\ell}}, TP)} \to p^*_G(\widehat{\gamma}^\ell_G)|_{\Omega^{n-p+1,0}(\gamma^n_G, TP)}$ forgetting orientations and its Thom map $T(B_{\widehat{\gamma}^\ell})$.

Theorem 0.4. *Let $n \geq p \geq 2$ and $\ell \gg n$. Let P be a connected smooth manifold of dimension p and let $f : N \to P$ be a fold-map. Let G refer to $G_{n,\ell}$ or $\widetilde{G}_{n,\ell}$, and let P and N be oriented when $G = \widetilde{G}_{n,\ell}$. Then f determines the homotopy class $\mu^G_{n,p}(f)$ defined in $\lim_{\ell\to\infty} \pi_{n+\ell}(p^*_G(\widehat{\gamma}^\ell_G)|_{\Omega^{n-p+1,0}(\gamma^n_G,TP)})$. If P and N are oriented in addition, then we have $(\lim_{\ell\to\infty} T(B_{\widehat{\gamma}^\ell}))_*(\mu^{\widetilde{G}_{n,\ell}}_{n,p}(f)) = \mu^{G_{n,\ell}}_{n,p}(f)$. Furthermore, every element α of $\lim_{\ell\to\infty} \pi_{n+\ell}(p^*_G(\widehat{\gamma}^\ell_G)|_{\Omega^{n-p+1,0}(\gamma^n_G,TP)})$ has such a fold-map $f_\alpha : N_\alpha \to P$ with $\mu^G_{n,p}(f_\alpha) = \alpha$.*

Here we give a brief definition of $\omega^\Omega_{n,p}$. By Theorem 0.1, a fold map determines an epimorphism $e_f : TN \oplus \theta_N \to TP$ covering f. Let ξ be the kernel bundle of e_f with induced orientation and let $\widetilde{c_\xi} : \xi \to \gamma^{n-p+1}_{\widetilde{G}_{n-p+1,\ell}}$ be the bundle map covering a classifying map $c_\xi : N \to \widetilde{G}_{n-p+1,\ell}$. Then the bundle map $b_f : TN \oplus \theta_N \to f^*(TP) \oplus \xi \to TP \times \gamma^{n-p+1}_{\widetilde{G}_{n-p+1,\ell}}$ covering $f \times c_\xi$ determines the homotopy class of a bundle map $\nu(b_f) : \nu^{k+\ell}_N \to \nu^k_P \times \widehat{\gamma}^\ell_{\widetilde{G}_{n-p+1,\ell}}$ covering $f \times c_\xi$ and the map $T(\nu(b_f)) : T(\nu^{k+\ell}_N) \to T(\nu^k_P \times \widehat{\gamma}^\ell_{\widetilde{G}_{n-p+1,\ell}})$ by [An2, Proposition 3.3]. Let $\alpha_N : S^{n+k+\ell} \to T(\nu^{k+\ell}_N)$ be the Pontrjagin-Thom construction of an embedding $N \to S^{n+k+\ell}$. Then $\omega^\Omega_{n,p}(f)$ is defined to be the homotopy class of the composition $T(\nu(b_f)) \circ \alpha_N$, where $T(\nu^k_P \times \widehat{\gamma}^\ell_{\widetilde{G}_{n-p+1,\ell}})$ is canonically identified with $T(\nu^k_P) \wedge T(\widehat{\gamma}^\ell_{\widetilde{G}_{n-p+1,\ell}})$.

The corresponding result for $\Omega^{fold}_n(P)$ of Theorem 0.3 in the case $n = p$ has already been described more precisely in [An2] and [An3], while the non-oriented case was not dealt with. The homotopy type $SO(n+1)$ of $\Omega^{1,0}(n,n)$ has been important in showing the relation between fold-maps and the surgery theory, or the stable homotopy groups of spheres.

As for another line of investigation concerning the existence problem of fold-maps, we refer to the results about fold-maps of special generic type due to [B-R], [Sa] and [S-S] in low dimensions $(3,2)$ and $(4,3)$, which are closely related to the differentiable structures of manifolds.

In Section 1 we will review the fundamental properties of fold singularities and explain notations. In Section 2 we will describe the homotopy types of $\Omega^{n-p+1}(n,p)$ and $\Omega^{n-p+1,0}(n,p)$ in Theorems 2.3 and 2.6 respectively without proofs. In Section 3 we will prove Theorems 0.1, 0.2, 0.3 and 0.4 by using the results in Section 2 and describe, by Theorem 0.3, differences between fold-maps and submersions. In Section 4 we will give another interpretation of $\lim_{k,\ell\to\infty} \pi_{n+k+\ell}(T(\nu^k_P) \wedge T(\widehat{\gamma}^\ell_{\widetilde{G}_{n-p+1,\ell}}))$ by using S-dual spaces and duality maps in [Spa2] to deduce many fold-cobordism invariants in $H^*(P)$. In Section 5 we will prepare lemmas, which are necessary in the proof of Theorems 2.3

and 2.6. In Section 6 we will prove Theorem 2.3. In Sections 7 and 8 we will prove Theorem 2.6.

§1. Preliminaries

Throughout the paper all manifolds are smooth of class C^∞. Maps are basically continuous, but may be smooth (of class C^∞) if so stated. We always work in dimensions $n \geq p \geq 2$. Given a fibre bundle $\pi : E \to X$ and a subset C in X, we denote $\pi^{-1}(C)$ by $E|_C$. Let $\pi' : F \to Y$ be another fibre bundle. A map $\tilde{b} : E \to F$ is called a fibre map over a map $b : X \to Y$ if $\pi' \circ \tilde{b} = b \circ \pi$ holds. The restriction $\tilde{b}|(E|_C) : E|_C \to F$ (or $F|_{b(C)}$) is denoted by $\tilde{b}|_C$. In particular, for a point $x \in X$, $E|_x$ and $\tilde{b}|_x$ are simply denoted by E_x and $\tilde{b}_x : E_x \to F_{b(x)}$ respectively. When E and F are vector bundles, a fibrewise homomorphism, epimorphism and monomorphism $E \to F$ are simply called homomorphism, epimorphism and monomorphism respectively. The trivial bundle $X \times \mathbf{R}^k$ is denoted by θ^n_X. In particular, θ^1_X is often written as θ_X.

We review the fundamental properties and notations about fold singularities (see [Bo], [L] and [Ma, Section 2]). Let $J^k(n,p)$ denote the space consisting of all k-jets $j^k_0 f$ of smooth map-germs $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$. Let $L^k(n)$ and $L^k(p)$ denotes the group of all k-jets of local diffeomorphisms of $(\mathbf{R}^n, 0)$ and $(\mathbf{R}^p, 0)$ respectively. Then $L^k(n) \times L^k(p)$ acts on $J^k(n,p)$ as follows. Let $h_1 : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and $h_2 : (\mathbf{R}^p, 0) \to (\mathbf{R}^p, 0)$ be local diffeomorphisms. Define the action $(j^k_0 h_1, j^k_0 h_2) \cdot j^k_0 f = j^k_0(h_2^{-1} \circ f \circ h_1)$.

Let $\pi^2_1 : J^2(n,p) \to J^1(n,p)$ be the canonical forgetting map. Let $\Sigma^i(n,p)$ denote the submanifold of $J^1(n,p)$ consisting of all 1-jets $z = j^1_0 f$ such that the kernel of $d_0 f$ is of dimension i. Let $\Omega^{n-p+1}(n,p)$ denote the union of $\Sigma^{n-p}(n,p)$ and $\Sigma^{n-p+1}(n,p)$ in $J^1(n,p)$. We denote $(\pi^2_1)^{-1}(\Sigma^i(n,p))$ by the same symbol $\Sigma^i(n,p)$ if there is no confusion. For a 2-jet $z = j^2_0 f$ of $\Sigma^i(n,p)$, there has been defined the second intrinsic derivative $d^2_0 f : T_0\mathbf{R}^n \to \mathrm{Hom}(\mathrm{Ker}(d_0 f), \mathrm{Cok}(d_0 f))$. Let $\Sigma^{i,j}(n,p)$ denote the submanifold of $J^2(n,p)$ consisting of all jets $z = j^2_0 f$ such that $\dim(\mathrm{Ker}(d_0 f)) = i$ and $\dim(\mathrm{Ker}(d^2_0 f|\mathrm{Ker}(d_0 f))) = j$. We say that a jet of $\Sigma^{n-p+1,0}(n,p)$ has the Boardman symbol $(n-p+1, 0)$. Let $\Omega^{n-p+1,0}(n,p)$ denote the union of $\Sigma^{n-p}(n,p)$ and $\Sigma^{n-p+1,0}(n,p)$ in $J^2(n,p)$.

We note that with the canonical bases of $\mathbf{R}^n$ and $\mathbf{R}^p$, $J^2(n,p)$ is identified with $\mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^p) \oplus \mathrm{Hom}(S^2\mathbf{R}^n, \mathbf{R}^p)$, by considering the Taylor expansion of f, where $S^2\mathbf{R}^n$ is the 2-fold symmetric product of $\mathbf{R}^n$. Furthermore, throughout the paper, we always identify $\mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^p)$ with the space $M_{p\times n}$ of all $p \times n$ matrices and identify $\mathrm{Hom}(S^2\mathbf{R}^n, \mathbf{R}^p)$ with the space of all p-tuples of $n \times n$ symmetric matrices. For subspaces V and W, $V \bigcirc W$ or S^2V denotes the symmetric product in $S^2\mathbf{R}^n$. In this paper we often express an element of $J^2(n,p)$ as (α,β) where $\alpha \in \mathrm{Hom}(\mathbf{R}^n,\mathbf{R}^p)$ and $\beta \in \mathrm{Hom}(S^2\mathbf{R}^n,\mathbf{R}^p)$. For a subspace V in $\mathbf{R}^p$, let $pr(V)$ be the orthogonal projection of $\mathbf{R}^p$ onto V. For an element $(\alpha,\beta) \in \Sigma^{n-p+1}(n,p)$, let $\beta_\alpha : S^2\mathrm{Ker}(\alpha) \to \mathrm{Im}(\alpha)^\perp$ denote a homomorphism defined by

$$\beta_\alpha = pr(\mathrm{Im}(\alpha)^\perp) \circ (\beta | S^2\mathrm{Ker}(\alpha)), \tag{1.1}$$

where the symbol $\perp$ refers to the orthogonal complement. Then $\alpha \in J^1(n,p)$ lies in $\Sigma^{n-p+1}(n,p)$ if and only if $\dim\,\mathrm{Ker}(\alpha) = n-p+1$, and $(\alpha,\beta) \in \Sigma^{n-p+1}(n,p)$ lies in $\Sigma^{n-p+1,0}(n,p)$ if and only if β_α is a non-singular quadratic form.

For a subset X and an element x, an equivalence class of x is usually expressed as $[x]$.

§2. Homotopy Types

In this section we describe the homotopy types of $\Omega^{n-p+1}(n,p)$ and $\Omega^{n-p+1,0}(n,p)$ in dimensions $n \geq p \geq 2$.

Let X and Y be spaces and let G be a Lie group. If G acts on X from the right-hand (resp. left-hand) side, then the orbit space is denoted by X/G (resp. $G\backslash X$). If G acts on X and Y from the right-hand and left-hand sides respectively, then G acts on $X \times Y$ by $g \cdot (x,y) = (xg^{-1}, gy)$. We define the twisted product of X and Y to be the orbit space $X \times_G Y$ of this action and denote its element by $[x,y]$ for $x \in X$ and $y \in Y$. Namely, we have $[x,y] = [xg^{-1}, gy]$.

Let $A_1,\ldots,A_s$ be the real square matrices of degree $i_1,\ldots,i_s$ respectively. The matrix of the form

$$\begin{pmatrix} A_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & A_s \end{pmatrix}$$

will be denoted by $A_1 \dotplus \cdots \dotplus A_s$. The diagonal matrix of degree k with diagonal components $\mathbf{d} = (d_1,\ldots,d_k)$ will be denoted by $\Delta(\mathbf{d})$. The unit matrix of degree k is denoted by E_k.

Let $O(k)$ and $SO(k)$ be the orthogonal group and the rotation group of degree k respectively. For a matrix $M = (m_{ij}) \in O(k)$, the i-th row and column vectors are denoted by $\mathbf{m}_i$ and $\overline{\mathbf{m}}_i$ respectively. Let $M(i,j)$ and $M\binom{i}{j}$ be the minor-matrices

$$(\overline{\mathbf{m}}_i, \dots, \overline{\mathbf{m}}_j) \quad \text{and} \quad \begin{pmatrix} \mathbf{m}_i \\ \vdots \\ \mathbf{m}_j \end{pmatrix}$$

respectively. Let $k \geq h$. Throughout the paper the Stiefel manifolds $(E_h \times O(k-h))\backslash O(k)$ and $O(k)/(E_h \times O(k-h))$ are canonically identified with the space consisting of all $k \times h$-matrices $M(1,h)$ and $h \times k$-matrices $M\binom{1}{h}$ respectively, where M varies in $O(k)$. Let I be the interval $[0,1]$. For $b \in I$, let $\mathbf{d}_b$ be the diagonal components $(1,\dots,1,b)$, where the degree should be relevant to the arguments. Let $\Delta(\mathbf{d}_b)$ be the diagonal matrix with diagonal components $\mathbf{d}_b$. In this paper $E_h \times O(0)$ and $O(h) \times O(0)$ refers to E_h and $O(h)$ respectively.

We consider the following action of $O(p) \times O(n)$ on $J^k(n,p)$. We regard $O \in O(p)$ and $U \in O(n)$ as linear maps, $\mathbf{R}^p \to \mathbf{R}^p$ and $\mathbf{R}^n \to \mathbf{R}^n$ respectively. Then define the action of (O,U) on a jet $z = j_0^k f$ by

$$(O,U) \cdot z = j_0^k(O \circ f \circ U^{-1}). \tag{2.1}$$

Now we describe the homotopy types of the spaces $\Omega^{n-p+1}(n,p)$ and $\Omega^{n-p+1,0}(n,p)$ in dimensions $n \geq p \geq 2$.

Throughout the paper we denote, by $V_{n,p}^{row}$, the Stiefel manifold $(E_p \times O(n-p))\backslash O(n)$.

Case I: $\Omega^{n-p+1}(n,p)$. We first define several actions. The actions of $O(p-1)$ and $O(1)$ on $O(p)$ and $O(n)$ are defined as follows. For elements $G \in O(p-1)$, $(\delta) \in O(1)$, $S \in O(p)$ and $M \in O(n)$, we set

$$\begin{aligned} G \cdot S &= S({}^tG \dotplus (1)), & G \cdot M &= (G \dotplus E_{n-p+1})M, \\ (\delta) \cdot S &= S(E_{p-1} \dotplus (\delta)), & (\delta) \cdot M &= (E_{p-1} \dotplus (\delta) \dotplus E_{n-p})M. \end{aligned} \tag{2.2}$$

We define the twisted products $\mathfrak{k}(n,p)$, $K(n,p,b)$ for $0 \leq b \leq 1$ and $\Sigma K(n,p)$ defined by

(2.3)

$$\begin{aligned} \mathfrak{k}(n,p) &= O(p) \times_{O(p-1)\times O(1)} \{(E_p \times O(n-p))\backslash O(n)\}, \\ K(n,p,b) &= \mathfrak{k}(n,p) \times b, \\ \Sigma K(n,p) &= \{O(p)/(E_{p-1} \times O(1))\} \times_{O(p-1)} \{(E_{p-1} \times O(n-p+1))\backslash O(n)\}. \end{aligned}$$

An element of $K(n,p,b)$, $\Sigma K(n,p)$ or $V_{n,p}^{row}$ can be expressed by $[S, M\binom{1}{p}, b]$, $[S, M\binom{1}{p-1}]$ or $M\binom{1}{p}$ respectively, where $S \in O(p)$ and $M \in O(n)$.

Remark 2.1. Let $[z] = [S, M\binom{1}{p}, b]$, or $[S, M\binom{1}{p-1}]$, and $[z'] = [S', M'\binom{1}{p}, b]$, or $[S', M'\binom{1}{p-1}]$ be elements of $K(n,p,b)$, and $\Sigma K(n,p)$ respectively. Then $[z] = [z']$ if and only if there exist matrices $G \in O(p-1)$, $L \in O(n-p)$ and $L_{n-p+1} \in O(n-p+1)$ such that

(i) $S' = S({}^tG \dotplus (\delta))$ and $M' = (G \dotplus (\delta) \dotplus L)M$ for $b > 0$,

(ii) $S' = S({}^tG \dotplus (\delta))$ and $M' = (G \dotplus E_{n-p+1})(E_{p-1} \dotplus L_{n-p+1})M$ for $b = 0$.

There exist the continuous surjections

$$\rho_{n,p,\Sigma} : K(n,p,0) \to \Sigma K(n,p), \qquad \rho_{n,p,R} : K(n,p,1) \to V^{row}_{n,p} \tag{2.4}$$

defined by $\rho_{n,p,\Sigma}([S, M\binom{1}{p}, 0]) = [S, M\binom{1}{p-1}]$ and $\rho_{n,p,R}([S, M\binom{1}{p}, 1]) = SM\binom{1}{p}$. It is easily seen that these maps are well defined. We define the space $K(n,p)$ to be the quotient space obtained from the disjoint union

$$\Sigma K(n,p) \bigcup \mathfrak{k}(n,p) \times I \bigcup V^{row}_{n,p} \tag{2.5}$$

by identifying $K(n,p,0)$ with $\Sigma K(n,p)$ by $\rho_{n,p,\Sigma}$ and $K(n,p,1)$ with $V^{row}_{n,p}$ by $\rho_{n,p,R}$ respectively. Namely, we identify $[S, M\binom{1}{p}, 0] = [S, M\binom{1}{p-1}]$ and $[S, M\binom{1}{p}, 1] = SM\binom{1}{p}$. Then there exists a continuous map

$$i_{n,p} : K(n,p) \to \Omega^{n-p+1}(n,p) \tag{2.6}$$

defined by $i_{n,p}([S, M\binom{1}{p}, b]) = S\Delta(\mathbf{d}_b)M\binom{1}{p}$. We define the action of $O(p) \times O(n)$ on $K(n,p)$ by

$$(O, U) \cdot [S, M\tbinom{1}{p}, b] = [OS, M\tbinom{1}{p}U^{-1}, b].$$

Lemma 2.2. *The map $i_{n,p}$ is well defined, and is equivariant with respect to the actions of $O(p) \times O(n)$.*

Proof. Suppose that $[z] = [S, M\binom{1}{p}, b]$ and $[z'] = [S', M'\binom{1}{p}, b]$ in $K(n,p,b)$ as given in Remark 2.1. If $[z] = [z']$, then we have $S\Delta(\mathbf{d}_b)M\binom{1}{p} = S'\Delta(\mathbf{d}_b)M'\binom{1}{p}$, and hence, $i_{n,p}([\mathbf{z}]) = i_{n,p}([\mathbf{z}'])$.

If $(O, U) \in O(p) \times O(n)$, then we have by (2.1)

$$i_{n,p}((O,U) \cdot [\mathbf{z}]) = OS\Delta(\mathbf{d}_b)M\tbinom{1}{p}U^{-1} = (O,U) \cdot i_{n,p}([\mathbf{z}]).$$

This shows the lemma. □

The following theorem will be proved in Section 6.

Theorem 2.3. *The map $i_{n,p}$ is an equivariant topological embedding. There exists a deformation retraction R_λ of $\Omega^{n-p+1}(n,p)$ to $i_{n,p}(K(n,p))$ such that*

(i) *R_λ preserves $\Sigma^{n-p}(n,p)$ and $\Sigma^{n-p+1}(n,p)$ respectively,*

(ii) *the restriction $R_\lambda|\Sigma^{n-p+1}(n,p)$ is a deformation retraction of $\Sigma^{n-p+1}(n,p)$ to $i_{n,p}(\Sigma K(n,p))$.*

Case II: $\Omega^{n-p+1,0}(n,p)$. Let c, d and σ always denote the integers such that $c \geq d \geq 0$, $c+d = n-p+1$ and $\sigma = c-d$. We consider the actions in (2.2) and the actions of $O(n-p)$ on $O(n-p+1)$ and $O(n)$ defined as follows. For elements $L \in O(n-p)$, $T \in O(n-p+1)$ and $M \in O(n)$, we define

$$L \cdot T = T((1) \dotplus {}^tL), \quad L \cdot M = (E_p \dotplus L)M. \tag{2.7}$$

Next we define the action of an element $G \in O(p-1)$ on an element $[S,T,M] \in O(p) \times \{((O(c) \times O(d))\backslash O(n-p+1)) \times_{1 \times O(p-1)} O(n)\}$ by

$$G \cdot [S,T,M] = [S({}^tG \dotplus (1)), T, (G \dotplus E_{n-p+1})M]. \tag{2.8}$$

If $\sigma = 0$ and $n-p+1 = 2c$, then we consider two other actions of $O(1)$. Whenever we deal with these actions of $O(1)$, we denote $O(1)$ by $\widetilde{O(1)}$ to emphasize these exceptional actions. The action of an element $(\delta) \in \widetilde{O(1)}$ on an element $[S,T,M] \in O(p) \times_{O(p-1)} (((O(c) \times O(c))\backslash O(2c)) \times O(n))$ is defined by

(2.9)

$$(1) \cdot [S,T,M] = [S,T,M],$$
$$(-1) \cdot [S,T,M] = \left[S(E_{p-1} \dotplus (-1)), \begin{pmatrix} 0 & E_c \\ E_c & 0 \end{pmatrix} T, (E_{p-1} \dotplus (-1) \dotplus E_{n-p})M \right].$$

We define another action of $\widetilde{O(1)}$ on $O(p) \times_{O(p-1)\times 1} ((E_{p-1} \times O(c) \times O(c))\backslash O(n))$ as follows. For elements $(-1) \in \widetilde{O(1)}$ and $[S, M\binom{p}{n}] \in O(p) \times_{O(p-1)\times 1} ((E_{p-1} \times O(c) \times O(c))\backslash O(n))$, define

$$(1) \cdot [S,M] = [S,M], \tag{2.10}$$
$$(-1) \cdot [S,M] = \left[S(E_{p-1} \dotplus (-1)), \left(E_{p-1} \dotplus \begin{pmatrix} 0 & E_c \\ E_c & 0 \end{pmatrix} \right) M \right].$$

These actions in (2.9) and (2.10) are well defined. Indeed, for T_1, $T_2 \in O(c)$ we have

$$\begin{aligned}
(-1)\cdot &\left[S, \left(E_{p-1} \dotplus \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}\right) M\right] \\
&= \left[S(E_{p-1} \dotplus (-1)), \left(E_{p-1} \dotplus \begin{pmatrix} 0 & E_c \\ E_c & 0 \end{pmatrix}\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}\right) M\right] \\
&= \left[S(E_{p-1} \dotplus (-1)), \left(E_{p-1} \dotplus \begin{pmatrix} 0 & T_2 \\ T_1 & 0 \end{pmatrix}\right) M\right] \\
&= \left[S(E_{p-1} \dotplus (-1)), \left(E_{p-1} \dotplus \begin{pmatrix} T_2 & 0 \\ 0 & T_1 \end{pmatrix}\begin{pmatrix} 0 & E_c \\ E_c & 0 \end{pmatrix}\right) M\right] \\
&= \left[S(E_{p-1} \dotplus (-1)), \left(E_{p-1} \dotplus \begin{pmatrix} 0 & E_c \\ E_c & 0 \end{pmatrix}\right) M\right] \\
&= (-1)\cdot [S, M].
\end{aligned}$$

For $0 < \sigma \leq n - p + 1$ and $b \in I$, let $\mathfrak{K}(n, p, \sigma)$, $\mathcal{K}(n, p, \sigma, b)$ and $\Sigma\mathcal{K}(n, p, \sigma)$ be the spaces defined by

(2.11)

$$\begin{aligned}
\mathfrak{K}(n,p,\sigma) &= O(p) \times_{O(p-1)\times 1} \{((O(c)\times O(d))\diagdown O(n-p+1))\times_{1\times O(n-p)} O(n)\}, \\
\mathcal{K}(n,p,\sigma,b) &= \mathfrak{K}(n,p,\sigma)\times b, \\
\Sigma\mathcal{K}(n,p,\sigma) &= O(p)\times_{O(p-1)\times 1} \{(E_{p-1}\times O(c)\times O(d))\diagdown O(n)\}\,.
\end{aligned}$$

For $\sigma = 0$, $n-p+1 = 2c$ $(c = d)$ and $b \in I$, we define the spaces $\mathfrak{K}(n, p, 0)$, $\mathcal{K}(n, p, 0, b)$ and $\Sigma\mathcal{K}(n, p, 0)$ to be

(2.12)

$$\begin{aligned}
\mathfrak{K}(n,p,0) &= O(p)\times_{O(p-1)\times \widetilde{O(1)}} \{((O(c)\times O(c))\backslash O(2c))\times_{1\times O(n-p)} O(n)\}, \\
\mathcal{K}(n,p,0,b) &= \mathfrak{K}(n,p,0)\times b, \\
\Sigma\mathcal{K}(n,p,0) &= O(p)\times_{O(p-1)\times \widetilde{O(1)}} \{(E_p\times O(c)\times O(c))\backslash O(n)\}\,.
\end{aligned}$$

An element of $\mathcal{K}(n, p, \sigma, b)$ or $\Sigma\mathcal{K}(n, p, \sigma)$ will be expressed by $[S, T, M, \sigma, b]$ or $[S, M, \sigma]$ respectively, where $S \in O(p)$, $T \in O(n-p+1)$, $M \in O(n)$ and $b \in I$.

The following remark follows from (2.2) and (2.7) to (2.12).

Remark 2.4. Let $[z] = [S, T, M, \sigma, b]$, or $[S, M, \sigma]$, and $[z'] = [S', T', M', \sigma, b]$, or $[S', M', \sigma]$ be elements of $\mathcal{K}(n, p, \sigma, b)$ or $\Sigma\mathcal{K}(n, p, \sigma)$. Then $[z] = [z']$ in $\mathcal{K}(n, p, \sigma, b)$ if and only if there exist matrices $G \in O(p-1)$, $L \in O(n-p)$, $T_1 \in O(c)$ and $T_2 \in O(d)$ such that

Case (i): $\sigma > 0$ and $0 < b < 1$,

$$S' = S({}^tG \dotplus (1)), \quad T' = (T_1 \dotplus T_2)T((1) \dotplus {}^tL) \quad \text{and}$$
$$M' = (G \dotplus E_{n-p+1})(E_p \dotplus L)M,$$

Case (ii): $\sigma > 0$ and $b = 0$,

$$S' = S({}^tG \dotplus (1)) \quad \text{and} \quad M' = (G \dotplus T_1 \dotplus T_2)M,$$

Case (iii): $\sigma = 0$ and $0 < b < 1$, either

$$S' = S({}^tG \dotplus (1)), \quad T' = (T_1 \dotplus T_2)T((1) \dotplus {}^tL) \quad \text{and}$$
$$M' = (G \dotplus E_{n-p+1})(E_p \dotplus L)M,$$

or

$$S' = S({}^tG \dotplus (-1)), \quad T' = \begin{pmatrix} 0 & E_c \\ E_c & 0 \end{pmatrix} (T_1 \dotplus T_2)T((-1) \dotplus {}^tL) \quad \text{and}$$
$$M' = (G \dotplus (-1) \dotplus L)M.$$

Case (iv): $\sigma = 0$ and $b = 0$, either

$$S' = S({}^tG \dotplus (1)) \quad \text{and} \quad M' = (G \dotplus T_1 \dotplus T_2)M,$$

or

$$S' = S({}^tG \dotplus (-1)) \quad \text{and} \quad M' = \left(G \dotplus \begin{pmatrix} 0 & E_c \\ E_c & 0 \end{pmatrix}\right) (T_1 \dotplus T_2)M.$$

There exists the continuous surjections

$$\text{(2.13)} \qquad \begin{aligned} &\overline{\rho}_{n,p,\Sigma} : \mathcal{K}(n,p,\sigma,0) \to \Sigma\mathcal{K}(n,p,\sigma), \\ &\overline{\rho}_{n,p,R} : \mathcal{K}(n,p,\sigma,1) \to V^{row}_{n,p}, \end{aligned}$$

defined by

$$\overline{\rho}_{n,p,\Sigma}([S,T,M,\sigma,0]) = [S,(E_{p-1} \dotplus T)M,\sigma],$$
$$\overline{\rho}_{n,p,R}([S,T,M,\sigma,1]) = S(((E_{p-1} \dotplus T)M)\tbinom{1}{p}))$$

respectively. It is easy to see that these maps are well defined.

We define the space $\mathcal{K}(n,p,\sigma)$ to be the quotient space obtained from the disjoint union

$$\Sigma\mathcal{K}(n,p,\sigma)\bigcup\mathfrak{K}(n,p,\sigma)\times I\bigcup V^{row}_{n,p} \tag{2.14}$$

by identifying $\mathcal{K}(n,p,\sigma,0)$ with $\Sigma\mathcal{K}(n,p,\sigma)$ by $\overline{\rho}_{n,p,\Sigma}$ and $\mathcal{K}(n,p,\sigma,1)$ with $V^{row}_{n,p}$ by $\overline{\rho}_{n,p,R}$. Namely, we identify $[S,T,M,\sigma,0]=[S,(E_{p-1}\dotplus T)M,\sigma]$ and $[S,T,M,\sigma,1]=S(E_{p-1}\dotplus T)M\binom{1}{p}$. We define the space $\mathcal{K}(n,p)$ to be the quotient space obtained from the union

$$\bigcup_{d=0}^{[(n-p+1)/2]}\mathcal{K}(n,p,n-p+1-2d) \tag{2.15}$$

by the identification such that all subspaces $V^{row}_{n,p}$ in $\mathcal{K}(n,p,n-p+1-2d)$, $0\le d\le[(n-p+1)/2]$ are pasted each other by the identity of $V^{row}_{n,p}$. Furthermore, we define $\Sigma\mathcal{K}(n,p)$ to be the union

$$\bigcup_{d=0}^{[(n-p+1)/2]}\Sigma\mathcal{K}(n,p,n-p+1-2d). \tag{2.16}$$

There exists a continuous map

$$\mathcal{I}_{n,p}:\mathcal{K}(n,p)\to\Omega^{n-p+1,0}(n,p) \tag{2.17}$$

defined as follows. Let $[\mathbf{z}]$ represent an element $[S,T,M,\sigma,b]$ or $[S,M,\sigma]$ of $\mathcal{K}(n,p,\sigma)$. Let $\overline{\mathbf{s}}_p=S\mathbf{e}_p$. Define $\alpha([\mathbf{z}])$ and $\beta([\mathbf{z}])$ to be the elements of $\Omega^{n-p+1}(n,p)$ and $\mathrm{Hom}(S^2\mathbf{R}^n,\mathbf{R}^p)$ defined by

$$\begin{aligned}\alpha([\mathbf{z}])&=S\Delta(\mathbf{d}_b)M\tbinom{1}{p},\\ \beta([\mathbf{z}])(\mathbf{x},\mathbf{y})&=\sqrt{1-b^2}\{{}^t\mathbf{x}{}^tM\tbinom{p}{n}{}^tT(E_c\dotplus(-E_d))TM\tbinom{p}{n}\mathbf{y}\}\overline{\mathbf{s}}_p,\end{aligned} \tag{2.18}$$

respectively, where if $b=0$, then T should be replaced by E_{n-p+1}. We have the following properties:

(i) If $b=1$, then $\beta([\mathbf{z}])=\mathbf{0}$.

(ii) For $0\le b<1$, let $K_{\alpha([\mathbf{z}])}$ denote the subspace generated by ${}^t\mathbf{m}_p,\dots,{}^t\mathbf{m}_n$. If $\mathbf{x}\in(K_{\alpha([\mathbf{z}])})^\perp$, or $\mathbf{y}\in(K_{\alpha([\mathbf{z}])})^\perp$, then $\beta([\mathbf{z}])(\mathbf{x},\mathbf{y})=\mathbf{0}$.

(iii) $\beta([\mathbf{z}])$ is non-singular on $S^2(K_{\alpha([\mathbf{z}])})$.

If we define the map $\mathcal{I}_{n,p}$ by

$$\mathcal{I}_{n,p}([\mathbf{z}])=(\alpha([\mathbf{z}]),\beta([\mathbf{z}])), \tag{2.19}$$

then this is the map into $\Omega^{n-p+1,0}(n,p)$. We define the action of $O(p)\times O(n)$ on $\mathcal{K}(n,p)$ by

$$(O,U)\cdot[S,T,M,\sigma,b]=[OS,T,MU^{-1},\sigma,b].$$

Lemma 2.5. *The map $\mathcal{I}_{n,p}$ is well defined and equivariant with respect to the action of $O(p)\times O(n)$.*

Proof. The fact that $\alpha([\mathbf{z}])$ is well defined and equivariant is proved analogously as in the proof of Lemma 2.3.

We show that $\beta([S,T,M,\sigma,b])$ is well defined. Suppose that

(i) $[S,T,M,\sigma,b]=[S',M',T',\sigma,b]$ in $\mathcal{K}(n,p,\sigma,b)$ or

(ii) $[S,M,\sigma]=[S',M',\sigma]$ in $\Sigma\mathcal{K}(n,p,\sigma)$.

In the Case (i), by Remark 2.5, there are matrices $G\in O(p-1)$, $L\in O(n-p)$, $T_2\in O(c)$ and $T_3\in O(d)$ such that

(i-a) $$S'=S({}^tG\dotplus(1)),\quad T'=(T_2\dotplus T_3)T((1)\dotplus{}^tL)$$
$$\text{and}\quad M'=(G\dotplus E_{n-p+1})(E_p\dotplus L)M\quad \text{for } \sigma>0,$$

(i-b) $$S'=S({}^tG\dotplus(-1)),\quad T'=(T_2\dotplus T_3)\begin{pmatrix}0 & E_c\\ E_c & 0\end{pmatrix}T((-1)\dotplus{}^tL)$$
$$\text{and}\quad M'=(G\dotplus E_{n-p+1})(E_{p-1}\dotplus(-1)\dotplus L)M\quad \text{for } \sigma=0.$$

Hence, the space generated by ${}^t\mathbf{m}_p,\dots,{}^t\mathbf{m}_n$ is well defined and $S\mathbf{e}_p=S'\mathbf{e}_p$. Furthermore, we have

$${}^tM\tbinom{p}{n}{}^tT(E_c\dotplus(-E_d))TM\tbinom{p}{n}={}^tM'\tbinom{p}{n}{}^tT'(E_c\dotplus(-E_d))T'M'\tbinom{p}{n}.$$

Therefore, we have $\beta([\mathbf{z}])=\beta([\mathbf{z}'])$. The Case (ii) is a special case of the Case (i) and can be proved independently as in (i).

Next we show that $\beta:S^2\mathbf{R}^n\to\mathbf{R}^p$ is equivariant. We have

$$\begin{aligned}\beta((O,U)\cdot[\mathbf{z}])(\mathbf{x},\mathbf{y})&=\beta([OS,T,MU^{-1},\sigma,b])(\mathbf{x},\mathbf{y})\\&=\sqrt{1-b^2}\{{}^t\mathbf{x}U{}^tM\tbinom{p}{n}{}^tT(E_c\dotplus(-E_d))TM{}^tU\tbinom{p}{n}\mathbf{y}\}O\overline{\mathbf{s}}_p\\&=\sqrt{1-b^2}\{({}^t({}^tU\mathbf{x})^tM\tbinom{p}{n}{}^tT(E_c\dotplus(-E_d))TM\tbinom{p}{n}{}^tU\mathbf{y}\}O\overline{\mathbf{s}}_p\\&=O\beta([\mathbf{z}])(U^{-1}\mathbf{x},\mathbf{U}^{-1}\mathbf{y})\\&=((O,U)\beta([\mathbf{z}]))(\mathbf{x},\mathbf{y}).\end{aligned}$$

This shows the lemma. □

Now we are ready to state the follwing theorem, which will be proved in Sections 6 and 8.

Theorem 2.6. *Let $n \geq p \geq 2$. The map $\mathcal{I}_{n,p}$ is an equivariant topological embedding. There exists a deformation retraction $\mathcal{R}_\lambda$ of $\Omega^{n-p+1,0}(n,p)$ to $\mathcal{I}_{n,p}(\mathcal{K}(n,p))$ such that*

(i) *$\mathcal{R}_\lambda$ preserves $\Sigma^{n-p}(n,p)$ and $\Sigma^{n-p+1,0}(n,p)$ respectively,*

(ii) *the restriction $R_\lambda|\Sigma^{n-p+1,0}(n,p)$ is a deformation retraction of $\Sigma^{n-p+1,0}(n,p)$ to $\mathcal{I}_{n,p}(\Sigma\mathcal{K}(n,p))$.*

We consider the action of $O(p) \times O(n)$ on $V^{row}_{n+1,p}$ defined by

$$(O,U) \cdot M_{p\times(n+1)} = OM_{p\times(n+1)}(U^{-1} \dotplus (1)).$$

We now show that $\mathcal{K}(n,p,n-p+1)$ is homeomorphic to $V^{row}_{n+1,p}$.

Proposition 2.7. *Let $n \geq p \geq 2$. Then there exists a homeomorphim $j_{\mathcal{K},V} : \mathcal{K}(n,p,n-p+1) \to V^{row}_{n+1,p}$, which is equivariant with respect to the actions of $O(p) \times O(n)$.*

Proof. Let

$$j_{\mathcal{K},V} : \mathcal{K}(n,p,n-p+1,b) \to V^{row}_{n+1,p},$$

be the map defined by

$$j_{\mathcal{K},V}([S,T,M,n-p+1,b]) = S \begin{pmatrix} \mathbf{m}_1 & 0 \\ \vdots & \vdots \\ \mathbf{m}_{p-1} & 0 \\ b\mathbf{m}_p & \sqrt{1-b^2} \end{pmatrix} \quad \text{for } 0 \leq b \leq 1,$$

We note that

$$j_{\mathcal{K},V}([S,M,n-p+1]) = S \begin{pmatrix} \mathbf{m}_1 & 0 \\ \vdots & \vdots \\ \mathbf{m}_{p-1} & 0 \\ \mathbf{0}_{p-1} & 1 \end{pmatrix} \quad \text{for } b = 0.$$

This map is well defined. In fact, suppose that $[S,T,M,n-p+1,b] = [S',T',M',n-p+1,b]$ in $\mathcal{K}(n,p,n-p+1)$. Then we have $S' = S({}^tG \dotplus (1))$, $T' = T_2T((1) \dotplus {}^tL)$ and $M' = (G \dotplus E_{n-p+1})(E_p \dotplus L)M$ by Remark 2.5. Hence, we have $j_{\mathcal{K},V}([S,T,M,n-p+1,b]) = j_{\mathcal{K},V}([S',T',M',n-p+1,b])$.

We show that $j_{\mathcal{K},V}$ is a continuous injection. Suppose $j_{\mathcal{K},V}([S,T,M,n-p+1,b]) = j_{\mathcal{K},V}([S',T',M',n-p+1,b])$ for $b>0$. Then ${}^tSS'\mathbf{e}_p = \mathbf{e}_p$ and $S' = S({}^tG \dotplus (1))$. Since

$$({}^tG \dotplus (1)) \begin{pmatrix} \mathbf{m}_1 & 0 \\ \vdots & \vdots \\ \mathbf{m}_{p-1} & 0 \\ b\mathbf{m}_p & \sqrt{1-b^2} \end{pmatrix} = \begin{pmatrix} \mathbf{m}'_1 & 0 \\ \vdots & \vdots \\ \mathbf{m}'_{p-1} & 0 \\ b\mathbf{m}'_p & \sqrt{1-b^2} \end{pmatrix},$$

we have $M' = (G \dotplus E_{n-p+1})(E_p \dotplus L)M$ for some $G \in O(p-1)$ and $L \in O(n-p)$. Furthermore, we have $T' = T'((1) \dotplus {}^tL){}^tTT((1) \dotplus {}^tL)$. The proof is similar for $b=0$.

Next we show that $j_{\mathcal{K},V}$ is surjective. Let $M_{p\times(n+1)}$ be a $p\times(n+1)$-matrix in $V^{row}_{n+1,p}$. Then we have $S \in O(p)$ and $b \in [0,1]$ such that

$$M_{p\times(n+1)} = S \begin{pmatrix} \mathbf{m}_1 & 0 \\ \vdots & \vdots \\ \mathbf{m}_{p-1} & 0 \\ b\mathbf{m}_p & \sqrt{1-b^2} \end{pmatrix}.$$

Indeed, if we write $M_{p\times(n+1)} = (\overline{\mathbf{u}}_1,\dots,\overline{\mathbf{u}}_{n+1})$ and $S = (\overline{\mathbf{s}}_1,\dots,\overline{\mathbf{s}}_p)$, then we have $\overline{\mathbf{u}}_{n+1} = \sqrt{1-b^2}\,\overline{\mathbf{s}}_p$ and $b = \sqrt{1-\|\overline{\mathbf{u}}_{n+1}\|^2}$. Hence, b is determined by $M_{p\times(n+1)}$. If $b<1$, then there exists an element $S \in O(p)$ such that $S(\sqrt{1-b^2}\mathbf{e}_p) = \overline{\mathbf{u}}_{n+1}$. Then we have

$${}^tSM_{p\times(n+1)} = ({}^tS\overline{\mathbf{u}}_1,\dots,{}^tS\overline{\mathbf{u}}_n,\sqrt{1-b^2}\mathbf{e}_p),$$

which lies in $V^{row}_{n+1,p}$. Let M be any element of $O(n)$ such that $M\binom{1}{p} = {}^tS(\overline{\mathbf{u}}_1,\dots,\overline{\mathbf{u}}_n)$. Then we have

$$j_{\mathcal{K},V}([S,E_{n-p+1},M,n-p+1,b]) = M_{p\times(n+1)}.$$

If $b=1$, then $\overline{\mathbf{u}}_{n+1} = \mathbf{0}$. Let M be any element of $O(n)$ such that $M\binom{1}{p} = (\overline{\mathbf{u}}_1,\dots,\overline{\mathbf{u}}_n)$. Then we have

$$j_{\mathcal{K},V}([E_p,E_{n-p+1},M,n-p+1,1]) = M_{p\times(n+1)}.$$

Since both spaces $\mathcal{K}(n,p,n-p+1)$ and $V^{row}_{n+1,p}$ are compact, $j_{\mathcal{K},V}$ is a homeomorphism.

Let $(O,U)\in O(p)\times O(n)$. Then we have

$$\begin{aligned}
&j_{\mathcal{K},V}((O,U)\cdot[S,T,M,n-p+1,b])\\
&\qquad = j_{\mathcal{K},V}([OS,T,MU^{-1},n-p+1,b])\\
&\qquad = OS\begin{pmatrix} \mathbf{m}_1U^{-1} & 0\\ \vdots & \vdots\\ \mathbf{m}_{p-1}U^{-1} & 0\\ b\mathbf{m}_pU^{-1} & \sqrt{1-b^2}\end{pmatrix}\\
&\qquad = OS\begin{pmatrix} \mathbf{m}_1 & 0\\ \vdots & \vdots\\ \mathbf{m}_{p-1} & 0\\ b\mathbf{m}_p & \sqrt{1-b^2}\end{pmatrix}(U^{-1}\dot{+}(1))\\
&\qquad = (O,U)\cdot j_{\mathcal{K},V}([S,T,M,n-p+1,b]).
\end{aligned}$$

Hence, $j_{\mathcal{K},V}$ is equivariant. □

§3. Stable Homotopy Groups

When $\sigma\neq 0$, we define

$$\begin{aligned}
r_{\sigma,n-p+1} &: \mathcal{K}(n,p,\sigma,b)\to \mathcal{K}(n,p,n-p+1,b),\\
r^{\Sigma}_{\sigma,n-p+1} &: \Sigma\mathcal{K}(n,p,\sigma)\to\Sigma\mathcal{K}(n,p,n-p+1)
\end{aligned}$$

to be the maps induced canonically from the inclusions $O(c)\times O(d)\to O(n-p+1)$ respectively. Furthermore, we have the canonical retraction $r^0:\mathcal{K}(n,p,0)\setminus\Sigma\mathcal{K}(n,p,0)\to V^{row}_{n,p}$. These maps canonically yield the retractions

$$\begin{aligned}
r_{\Omega,\mathcal{K}} &: \Omega^{n-p+1,0}(n,p)\to \mathcal{K}(n,p,n-p+1), && \text{when } n-p+1 \text{ is odd},\\
r^0_{\Omega,\mathcal{K}} &: \Omega^{n-p+1,0}(n,p)\setminus\Sigma\mathcal{K}(n,p,0)\to \mathcal{K}(n,p,n-p+1), && \text{when } n-p+1 \text{ is even},
\end{aligned}$$

which are equivariant with respect to the action of $O(p)\times O(n)$ satisfying that $R_{\Omega,\mathcal{K}}\circ j_{\mathcal{K},V}$ is the identity of $\mathcal{K}(n,p,n-p+1)$.

We define a topological embedding

$$i_{V,\Omega}: V^{row}_{n+1,p}\to\Omega^{n-p+1,0}(n,p)$$

and

$$\begin{aligned}
R_{\Omega,V} &: \Omega^{n-p+1,0}(n,p)\to V^{row}_{n+1,p}, && \text{when } n-p+1 \text{ is odd},\\
R^0_{\Omega,V} &: \Omega^{n-p+1,0}(n,p)\setminus\Sigma\mathcal{K}(n,p,0)\to V^{row}_{n+1,p}, && \text{when } n-p+1 \text{ is even}
\end{aligned}$$

to be the compositions $i_{\mathcal{K}(n,p,n-p+1)} \circ j_{\mathcal{K},V}^{-1}$, $j_{\mathcal{K},V} \circ r_{\Omega,\mathcal{K}}$ and $j_{\mathcal{K},V} \circ r_{\Omega,\mathcal{K}}^0$ respectively.

Let π_N and π_P be the projections of $N \times P$ onto N and P respectively. We set

$$(3.1)\quad J^2(TN,TP) = \mathrm{Hom}(\pi_N^*(TN), \pi_P^*(TP)) \oplus \mathrm{Hom}(S^2(\pi_N^*(TN)), \pi_P^*(TP))$$

over $N \times P$, where $S^2(\pi_N^*(TN))$ is the 2-fold symmetric product of $(\pi_N^*(TN))$. If we provide N and P with Riemannian metrics, then the Levi-Civita connection induces the exponential maps $\exp_N : TN \to N$ and $\exp_P : TP \to P$ ([K-N]). We define a bundle map

$$(3.2)\qquad j_{\exp} : J^2(N,P) \to J^2(TN,TP) \quad \text{over } N \times P$$

by sending $z = j_x^2 f \in J_{x,y}^2(N,P)$ to the 2-jet of $(\exp_P|T_yP)^{-1} \circ f \circ (\exp_N|T_xN)$ at $\mathbf{0} \in T_xN$, which is regarded as an element of $J^2(T_xN, T_yP)$. The structure group of $J^2(TN,TP)$ is reduced to $O(p) \times O(n)$. Set $J^2(n,p) = J_{0,0}^2(\mathbf{R}^n, \mathbf{R}^p)$ and $\Omega^{n-p+1,0}(n,p) = \Omega^{n-p+1,0}(\mathbf{R}^n,\mathbf{R}^p) \cap J^2(n,p)$. For a jet $z = j_x^2 f \in \Omega^{n-p+1,0}(\mathbf{R}^n,\mathbf{R}^p)$, we define π_Ω by $\pi_\Omega(z) = j_0^2(l(-f(x)) \circ f \circ l(x))$, where $l(a)$ denotes the parallel translation defined by $l(a)(x) = x + a$. In particular, we obtain a canonical diffeomorphism

$$(3.3)\qquad \pi_{\mathbf{R}^n}^2 \times \pi_{\mathbf{R}^p}^2 \times \pi_\Omega : \Omega^{n-p+1,0}(\mathbf{R}^n,\mathbf{R}^p) \to \mathbf{R}^n \times \mathbf{R}^p \times \Omega^{n-p+1,0}(n,p).$$

We note that $j_{\exp}(\Omega^{n-p+1,0}(N,P))$ coincides with the subbundle of $J^2(TN, TP)$ associated with $\Omega^{n-p+1,0}(n,p)$.

With the identification $V_{n+1,p}^{row} = \mathbf{V}(\mathbf{R}^{n+1},\mathbf{R}^p)$, we have the fiber maps

$$\begin{aligned} i_{\mathbf{V},\Omega}(N,P) &: \mathbf{V}(TN \oplus \theta_N, TP) \to \Omega^{n-p+1,0}(N,P), \\ R_{\Omega,\mathbf{V}}(N,P) &: \Omega^{n-p+1,0}(N,P) \to \mathbf{V}(TN \oplus \theta_N, TP), \\ R_{\Omega,\mathbf{V}}^0(N,P) &: \Omega^{w,0}(N,P) \to \mathbf{V}(TN \oplus \theta_N, TP) \end{aligned}$$

associated to the maps $i_{V,\Omega}$, $R_{\Omega,V}$ and $R_{\Omega,V}^0$ respectively. Let $\Gamma(R_{\Omega,\mathbf{V}}) : \Gamma(N,P) \to \Gamma(\mathbf{V}(TN \oplus \theta_N, TP))$ be the map induced from the map $R_{\Omega,\mathbf{V}}(N, P)$ by $\Gamma(R_{\Omega,\mathbf{V}})(s)(x) = R_{\Omega,\mathbf{V}}(N,P))(s(x))$ for $s \in \Gamma(N,P)$.

Proof of Theorems 0.1 *and* 0.2. Since $R_{\Omega,V} \circ i_{V,\Omega}$ is the identity of $V_{n+1,p}^{row} = \mathrm{Hom}(\mathbf{R}^{n+1},\mathbf{R}^p)$, we have that $R_{\Omega,\mathbf{V}}(N,P) \circ i_{\mathbf{V},\Omega}(N,P)$ is the identity of $\mathbf{V}(TN \oplus \theta_N, TP)$. This is the proof of Theorem 0.1.

Next take any element $h \in \Gamma(\mathbf{V}(TN \oplus \theta_N, TP))$. By [An4, Theorem 0.5], there exists a fold-map $f : N \to P$ such that $j^2 f$ and $\Gamma(i_{V,\Omega})(h)$ are homotopic as sections in $\Gamma(N,P)$. This is the proof of Theorem 0.2. □

As for the results concerning Theorem 0.1 we refer to [E1, 3.8 and 3.9], [Sa, Lemma 3.1] and [An2, Theorem 1]. We must refer to [E1, 3.10] as a prior work concerning Theorem 0.2. A weaker assertion of Theorem 0.2 was proved in [An4, Theorem 0.1] without using the homotopy type of $\Omega^{n-p+1,0}(n,p)$.

Remark 3.1. When $n-p+1$ is even, we have that $R^0_{\Omega,\mathbf{V}}(N,P)\circ i_{\mathbf{V},\Omega}(N,P)$ is the identity of $\mathbf{V}(TN\oplus\theta_N,TP)$.

Now we define the maps $\omega^{\mathfrak{N}}_{n,p}$ and $\omega^{\Omega}_{n,p}$ in Theorem 0.3. Let $\mathcal{G}$ refers to either $G_{n-p+1,\ell}$ or $\widetilde{G}_{n-p+1,\ell}$ and let $\omega_{n,p}$ refers to either $\omega^{\mathfrak{N}}_{n,p}$ or $\omega^{\Omega}_{n,p}$. Let $f:N\to P$ be a fold-map. Then f determines an epimorphism $\Gamma(R_{\Omega,\mathbf{V}})(j^2f):TN\oplus\theta_N\to TP$ covering f. Let ξ be the kernel bundle of $\Gamma(R_{\Omega,\mathbf{V}})(j^2f)$. Since TN has the metric, we have the orthogonal projection $TN\oplus\theta_N\to\xi$ and the splitting $TN\oplus\theta_N=f^*(TP)\oplus\xi$. For the case Ω^{fold}_n, ξ has the canonical induced orientaion. Let $\widetilde{c_\xi}:\xi\to\gamma^{n-p+1}_{\mathcal{G}}$ be the bundle map covering a classifying map $c_\xi:N\to\mathcal{G}$. Then we have the natural bundle map

$$b_f:TN\oplus\theta_N=f^*(TP)\oplus\xi\to TP\times\gamma^{n-p+1}_{\mathcal{G}}\qquad\text{covering }\ f\times c_\xi. \tag{3.4}$$

Let $\nu^{k+\ell}_N$ and ν^k_P be the normal bundles of embeddings, $N\to\mathbf{R}^{n+k+\ell}$ and $P\to\mathbf{R}^{n+k}$ with trivialization $t_N:TN\oplus\theta_N\oplus\nu^{k+\ell}_N\to\theta^{n+k+\ell+1}_N$ and $t_P:TP\oplus\nu^k_P\to\theta^{n+k}_P$ respectively (see the details in [An3, Section 2]). We have the trivialization $t_{\mathcal{G}}:\gamma^{n-p+1}_{\mathcal{G}}\oplus\widehat{\gamma}^{\ell}_{\mathcal{G}}\to\theta^{n-p+1+\ell}_{\mathcal{G}}$. By using [An2, Proposition 3.3] for trivializations t_N and

$$\begin{aligned}t_{P\times\mathcal{G}}:(TP\times\gamma^{n-p+1}_{\mathcal{G}})\oplus(\nu^k_P\times\widehat{\gamma}^{\ell}_{\mathcal{G}})&\cong(TP\oplus\nu^k_P)\times(\gamma^{n-p+1}_{\mathcal{G}}\oplus\widehat{\gamma}^{\ell}_{\mathcal{G}})\\&\xrightarrow{t_P\times t_{\mathcal{G}}}\theta^{n+k+\ell+1}_{P\times\mathcal{G}},\end{aligned}\tag{3.5}$$

b_f induces a bundle map

$$\nu(b_f):\nu^{k+\ell}_N\to\nu^k_P\times\widehat{\gamma}^{\ell}_{\mathcal{G}}\qquad\text{covering }\ f\times c_\xi \tag{3.6}$$

determined up to homotopy such that $t_{P\times\mathcal{G}}\circ(b_f\oplus\nu(b_f))\circ t_N^{-1}$ is homotopic to $(f\times c_\xi)\times id_{\mathbf{R}^{n+k+\ell+1}}$. Let $\alpha_N:S^{n+k+\ell}\to T(\nu^{k+\ell}_N)$ be the Pontrjagin-Thom construction for the embedding of N into $S^{n+k+\ell}$. Then $\omega_{n,p}(f)$ is defined to be the stable homotopy class of the composition $T(\nu(b_f))\circ\alpha_N$, where $T(\nu^k_P\times\widehat{\gamma}^{\ell}_{\mathcal{G}})$ is identified with $T(\nu^k_P)\wedge T(\widehat{\gamma}^{\ell}_{\mathcal{G}})$.

We need to show that $\omega^{\mathfrak{N}}_{n,p}(f)$ and $\omega^{\Omega}_{n,p}(f)$ are well-defined.

Lemma 3.2. *The maps $\omega^{\mathfrak{N}}_{n,p}(f)$ and $\omega^{\Omega}_{n,p}(f)$ are well-defined. Namely, they do not depend on the choices of an embedding of N, of a representative f of* *the fold-cobordism class* $[f] \in \mathfrak{N}^n_{fold}(P)$ *or* $\Omega^n_{fold}(P)$*, and Riemannian metrics of* N *and* P*.*

Proof. We first prove that $\omega_{n,p}$ does not depend on the choice of an embedding of N. Let $e'_N : N \to \mathbf{R}^{n+k+\ell}$ be another embedding with normal bundles ν'_N, the trivialization $t'_N : T_N \oplus \theta_N \oplus \nu'_N \to \theta_N^{n+k+\ell+1}$ and a bundle map $\nu(b_f)' : \nu'_N \to \nu^k_P \times \widehat{\gamma}^\ell_{\mathcal{G}}$. Let α'_N be the corresponding Pontryagin-Thom construction. Then by [An3, Remark 2.2] there exists a bundle map $b_N : \nu_N \to \nu'_N$. They yields $\nu(b_f) \circ b_N^{-1} \simeq \nu(b_f)' : \nu'_N \to \nu^k_P \times \widehat{\gamma}^\ell_{\mathcal{G}}$. Then we have

$$\begin{aligned}[T(\nu(b_f)') \circ \alpha'_N] &= [T(\nu(b_f)) \circ T(b_N^{-1}) \circ T(b_N) \circ \alpha_N] \\ &= [T(\nu(b_f)) \circ \alpha_N].\end{aligned}$$

Next we prove that $\omega_{n,p}$ does not depend on the choice of a representative f of the fold-cobordism class $[f]$. Let $f_i : N_i \to P$ $(i = 0, 1)$ be two fold-maps, where N_i are closed (resp. oriented) smooth manifolds with a (resp. an oriented-) fold-cobordism $F : (W, \partial W) \to (P \times [0, 1], P \times 0 \cup P \times 1)$ as in Introduction such that $F|N_0 = f_0$ and $F|N_1 = f_1$, for which we have the followings constructed similarly as for the fold-map f:

(i) epimorphisms $\Gamma(R_{\Omega,\mathbf{V}})(j^2 f_i) : TN_i \oplus \theta_{N_i} \to TP$ covering f_i,

(ii) the kernel bundle ξ_i of $\Gamma(R_{\Omega,\mathbf{V}})(j^2 f_i)$,

(iii) the orthogonal projection $TN_i \oplus \theta_{N_i} \to \xi_i$, the splitting $TN_i \oplus \theta_{N_i} = f^*(TP) \oplus \xi_i$, and the canonical induced orientaion of ξ_i, when $\mathcal{G}$ is $\widetilde{G}_{n-p+1,\ell}$,

(iv) the bundle map $\widetilde{c_{\xi_i}} : \xi_i \to \gamma_{\mathcal{G}}^{n-p+1}$ covering a classifying map $c_{\xi_i} : N_i \to \mathcal{G}$,

(v) the natural bundle map $b_f : TN_i \oplus \theta_{N_i} = f^*(TP) \oplus \xi_i \to TP \times \gamma_{\mathcal{G}}^{n-p+1}$ covering $f_i \times c_{\xi_i}$,

(vi) the normal bundle $\nu^{k+\ell}_{N_i}$ of embeddings, $N_i \to \mathbf{R}^{n+k+\ell}$ with trivializations $t_{N_i} : TN_i \oplus \theta_{N_i} \oplus \nu^{k+\ell}_{N_i} \to \theta^{n+k+\ell+1}_{N_i}$,

(vii) bundle maps $\nu(b_{f_i}) : \nu^{k+\ell}_{N_i} \to \nu^k_P \times \widehat{\gamma}^\ell_{\mathcal{G}}$ covering $f_i \times c_{\xi_i}$ determined up to homotopy such that $t_{P \times \mathcal{G}} \circ (b_{f_i} \oplus \nu(b_{f_i})) \circ t^{-1}_{N_i}$ is homotopic to $(f_i \times c_{\xi_i}) \times id_{\mathbf{R}^{n+k+\ell+1}}$,

(viii) the Pontrjagin-Thom construction $\alpha_{N_i} : S^{n+k+\ell} \to T(\nu^{k+\ell}_{N_i})$ for the embedding of N_i into $\mathbf{R}^{n+k+\ell}$,

(ix) the homotopy classes $\omega_{n,p}(f_i)$ of the composition $T(\nu(b_{f_i})) \circ \alpha_{N_i}$.

By Theorem 0.1, the fold map F determines an epimorphism $\Gamma(R_{\Omega,\mathbf{V}})(j^2F) : TW \oplus \theta_W \to T(P \times I)$ covering F. Let ξ_F be the kernel bundle of $\Gamma(R_{\Omega,\mathbf{V}})(j^2F)$ such that $\xi_F|_{N\times i} = \xi_i$. Since TW has the metric compatible with that of $TN_i \oplus \theta_{N_i}$, we have the orthogonal projection $TW \oplus \theta_W \to \xi_F$ and the splitting $TW \oplus \theta_W = f^*(T(P \times I)) \oplus \xi_F$. Therefore, ξ_F has the canonical induced orientation when $\mathcal{G}$ is $\widetilde{G}_{n-p+1,\ell}$. Let $\widetilde{c_{\xi_F}} : \xi_F \to \gamma_{\mathcal{G}}^{n-p+1}$ be the bundle map covering a classifying map $c_{\xi_F} : W \to \mathcal{G}$. Hence, we have the natural bundle map $b_F : TW \oplus \theta_W = f^*(T(P \times I)) \oplus \xi_F \to T(P \times I) \times \gamma_{\mathcal{G}}^{n-p+1}$ covering $F \times c_{\xi_F}$. Let $\nu_W^{k+\ell}$ and $\nu_{P\times I}^k$ be the normal bundles of embeddings, $W \to \mathbf{R}^{n+k+\ell} \times I$ and $P \times I \to \mathbf{R}^{n+k} \times I$ with trivialization $t_W : TW \oplus \theta_W \oplus \nu_W^{k+\ell} \to \theta_W^{n+k+\ell+1}$ and $t_{P\times I} : T(P \times I) \oplus \nu_{P\times I}^k \to \theta_{P\times I}^{n+k+1}$ respectively. By using [An2, Proposition 3.3] for trivializations t_W and

$$\begin{aligned} t_{(P\times I)\times\mathcal{G}} &: (T(P \times I) \times \gamma_{\mathcal{G}}^{n-p+1}) \oplus (\nu_{P\times I}^k \times \widehat{\gamma}_{\mathcal{G}}^{\ell}) \\ &\cong (T(P \times I) \oplus \nu_{P\times I}^k) \times (\gamma_{\mathcal{G}}^{n-p+1} \oplus \widehat{\gamma}_{\mathcal{G}}^{\ell}) \xrightarrow{t_{P\times I}\times t_{\mathcal{G}}} \theta_{(P\times I)\times\mathcal{G}}^{n+k+\ell+2}, \end{aligned}$$

b_F induces a bundle map $\nu(b_F) : \nu_W^{k+\ell} \to \nu_{P\times I}^k \times \widehat{\gamma}_{\mathcal{G}}^{\ell}$ covering $F \times c_{\xi_F}$ determined up to homotopy. Let $\alpha_W : S^{n+k+\ell} \times I \to T(\nu_W^{k+\ell})$ be the Pontrjagin-Thom construction for the embedding of W into $\mathbf{R}^{n+k+\ell} \times I$. Let $\omega_{n,p}(F)$ be the composition $T(\nu(b_W)) \circ \alpha_W$. If we restrict these constructions for W to N_i and $P \times i$, then we obtain the properties observed in (i)–(ix) above. Hence, $\omega_{n,p}(W)$ gives a homotopy of $\omega_{n,p}(f_0)$ and $\omega_{n,p}(f_1)$.

We show that $\omega_{n,p}(f)$ does not depend on the choices of Riemannian metrics of N and P. This follows from the fact that Riemannian metrics are all homotopic (see [Ste, 12.12]). □

Proof of Theorem 0.3. We give a proof only for the case $\Omega^n_{fold}(P)$, since the proof for the case $\mathfrak{N}^n_{fold}(P)$ is analougous.

We prove the surjectivity of $\omega^{\Omega}_{n,p}$. Let $\alpha : S^{n+k+\ell} \to T(\nu_P^k \times \widehat{\gamma}^{\ell}_{\widetilde{G}_{n-p+1,\ell}}) = T(\nu_P^k) \wedge T(\widehat{\gamma}^{\ell}_{\widetilde{G}_{n-p+1,\ell}})$. We may assume that α is transverse to the zero-section $P \times \widetilde{G}_{n-p+1,\ell}$. Set $N = \alpha^{-1}(P \times \widetilde{G}_{n-p+1,\ell})$ with normal bundle $\nu_N^{k+\ell}$ and $c_N = \alpha|N$. Then there exists a bundle map

$$h_{\nu_N} : \nu_N^{k+\ell} \to \nu_P^k \times \widehat{\gamma}^{\ell}_{\widetilde{G}_{n-p+1,\ell}} \qquad \text{covering } c_N,$$

which, by [An2, Proposition 3.3], induces a bundle map

$$\begin{aligned} h_{\tau_N} : TN \oplus \theta_N^{k'+k''+1} &\to (TP \oplus \theta_P^{k'}) \times (\gamma^{n-p+1}_{\widetilde{G}_{n-p+1,\ell}} \oplus \theta^{k''}_{\widetilde{G}_{n-p+1,\ell}}) \\ &= (TP \times \gamma^{n-p+1}_{\widetilde{G}_{n-p+1,\ell}}) \oplus \theta^{k'+k''}_{P\times\widetilde{G}_{n-p+1,\ell}} \qquad \text{covering } c_N \end{aligned}$$

such that $(t_{P\times\widetilde{G}_{n-p+1,\ell}} \oplus id_{\theta^{k'+k''}_{P\times\widetilde{G}_{n-p+1,\ell}}}) \circ (h_{\tau_N} \oplus h_{\nu_N}) \circ (t_N \oplus id_{\theta_N^{k'+k''}})^{-1}$ is homotopic to $c_N \times id_{\mathbf{R}^{n+k+\ell+k'+k''+1}}$. Let $p_P : P \times \widetilde{G}_{n-p+1,\ell} \to P$ and $p_{\widetilde{G}_{n-p+1,\ell}} : P \times \widetilde{G}_{n-p+1,\ell} \to \widetilde{G}_{n-p+1,\ell}$ be canonical projections respectively. By the dimensional reason considering $TN \oplus \theta_N^{k'+k''+1}$ and $(p_P \circ c_N)^*(TP) \oplus (p_{\widetilde{G}_{n-p+1,\ell}} \circ c_N)^*(\gamma^{n-p+1}_{\widetilde{G}_{n-p+1,\ell}}) \oplus \theta_N^{k'+k''+1}$, there exists a bundle map

$$\widetilde{h} : TN \oplus \theta_N \to TP \times \gamma^{n-p+1}_{\widetilde{G}_{n-p+1,\ell}} \qquad \text{covering} \quad c_N,$$

such that $\widetilde{h} \times id_{\mathbf{R}^{k'+k''}}$ is homotopic to h_{τ_N}. Let $p_{TP} : TP \times \gamma^{n-p+1}_{\widetilde{G}_{n-p+1,\ell}} \to TP$ be the canonical projection. Then it follows from Theorem 0.2 that $p_{TP} \circ \widetilde{h} : TN \oplus \theta_N \to TP$ has a fold-map $f : N \to P$ such that $\Gamma(R_{\Omega,\mathbf{V}})(j^2 f)$ is homotopic to $p_{TP} \circ \widetilde{h}$ in $\Gamma(\mathbf{V}(TN \oplus \theta_N, TP))$. Hence, b_f is homotopic to $\widetilde{h}$. This shows that $\nu(b_f)$ is homotopic to h_{ν_N}. By the definition of $\omega^\Omega_{n,p}$, we have that

$$\omega^\Omega_{n,p}(f) = [T(\nu(b_f)) \circ \alpha_N] = [T(h_{\nu_N}) \circ \alpha_N] = \alpha.$$

This completes the proof. □

Remark 3.3. In this remark a smooth map $f : N \to P$ is called *a quasidefinite fold-map* if f has only fold singularities of non-zero signatures. Let $\mathfrak{N}^{q.d.fold}_n(P)$ (resp. $\Omega^{q.d.fold}_n(P)$) denote the set consisiting of all quasidefinite (resp. oriented-) fold-cobordism classes of quasidefinite fold-maps into P, which are defined analogously as $\mathfrak{N}^{fold}_n(P)$ (resp. $\Omega^{fold}_n(P)$) in Introduction by replacing fold-maps with quasidefinite fold-maps. When $n-p+1$ is odd, a quasidefinite fold-map coincides with a fold-map, and hence we have $\mathfrak{N}^{q.d.fold}_n(P) = \mathfrak{N}^{fold}_n(P)$ (resp. $\Omega^{q.d.fold}_n(P) = \Omega^{fold}_n(P)$). When $n-p+1$ is even, we can define the maps

$$\overline{\omega}^{\mathfrak{N}}_{n,p} : \mathfrak{N}^{q.d.fold}_n(P) \to \lim_{k\to\infty} \pi_{n+k+\ell}(T(\nu^k_P) \wedge T(\widehat{\gamma}^\ell_{G_{n-p+1,\ell}})),$$
$$\overline{\omega}^{\Omega}_{n,p} : \Omega^{q.d.fold}_n(P) \to \lim_{k\to\infty} \pi_{n+k+\ell}(T(\nu^k_P) \wedge T(\widehat{\gamma}^\ell_{\widetilde{G}_{n-p+1,\ell}}))$$

similarly as in the case of $\mathfrak{N}^{fold}_n(P)$ (resp. $\Omega^{fold}_n(P)$). However, we cannot assert that $\overline{\omega}^{\mathfrak{N}}_{n,p}$ and $\overline{\omega}^{\Omega}_{n,p}$ are surjective, because the homotopy principle does not hold for quasidefinite fold-maps (see [An4, Theorem 0.5]).

Let $\mathbf{f} : N \to P$ be a submersion. We study the element $\overline{\omega}_{n,p}(\mathbf{f})$, where $\overline{\omega}_{n,p}$ refers to either $\overline{\omega}^{\mathfrak{N}}_{n,p}$ or $\overline{\omega}^{\Omega}_{n,p}$. Let $\mathfrak{G}$ denote either $G_{n-p,\ell}$ or $\widetilde{G}_{n-p,\ell}$ depending on whether $\mathcal{G}$ is either $G_{n-p+1,\ell}$ or $\widetilde{G}_{n-p+1,\ell}$. Let $i_{\mathfrak{G},\mathcal{G}} : \mathfrak{G} \to \mathcal{G}$ be the inclusion induced from the inclusion $\mathbf{R}^{n-p+\ell} = \mathbf{R}^{n-p+\ell} \times 0 \subset \mathbf{R}^{n-p+\ell+1}$. Then the classifying bundle maps $\widetilde{i_{\mathfrak{G},\mathcal{G}}} : \gamma_{\mathfrak{G}}^{n-p} \oplus \theta_{\mathfrak{G}} \to \gamma_{\mathcal{G}}^{n-p+1}$ and the canonical bundle map $\widehat{i_{\mathfrak{G},\mathcal{G}}} : \widehat{\gamma}_{\mathfrak{G}}^{\ell} \to \widehat{\gamma}_{\mathcal{G}}^{\ell}$ covering $i_{\mathfrak{G},\mathcal{G}}$. They induce

$$T(\widetilde{i_{\mathfrak{G},\mathcal{G}}}) : T(\gamma_{\mathfrak{G}}^{n-p} \oplus \theta_{\mathfrak{G}}) = S(T(\gamma_{\mathfrak{G}}^{n-p})) \to T(\gamma_{\mathcal{G}}^{n-p+1}),$$
$$T(\widehat{i_{\mathfrak{G},\mathcal{G}}}) : T(\widehat{\gamma}_{\mathfrak{G}}^{\ell}) \to T(\widehat{\gamma}_{\mathcal{G}}^{\ell})$$

respectively. Let

$$\mathbf{j}_{\mathfrak{G},\mathcal{G}} : \lim_{k\to\infty} \pi_{n+k+\ell}(T(\nu_P^k) \wedge T(\widehat{\gamma}_{\mathfrak{G}}^{\ell})) \to \lim_{k\to\infty} \pi_{n+k+\ell}(T(\nu_P^k) \wedge T(\widehat{\gamma}_{\mathcal{G}}^{\ell}))$$

be the map defined by sending c to $(id_{T(\nu_P^k)} \wedge T(\widehat{i_{\mathfrak{G},\mathcal{G}}}))_*(c)$. In the following proposition let L be a closed (resp. oriented) manifold of dimension $n-p$, which is embedded in $\mathbf{R}^{n-p+\ell}$. Let $\alpha_L : S^{n-p+\ell} \to T(\nu_L^{\ell})$ be the Pontrjagin-Thom construction and let $\widetilde{c_{\nu_L^{\ell}}} : \nu_L^{\ell} \to \widehat{\gamma}_{\mathfrak{G}}^{\ell}$ be the bundle map covering a claasifying map $c_{\nu_L^{\ell}} : L \to \mathfrak{G}$.

Proposition 3.4. *Let $\ell \gg n$.* (1) *Let $\mathbf{f} : N \to P$ be a submersion. Then $\overline{\omega}_{n,p}(\mathbf{f})$ lies in the image of $\mathbf{j}_{\mathfrak{G},\mathcal{G}}$, where $\overline{\omega}_{n,p}$ refers to either $\overline{\omega}_{n,p}^{\Omega}$ or $\overline{\omega}_{n,p}^{\mathfrak{N}}$ depending on whether N and P are provided with orientations or not.*

(2) *Let L be a manifold as above and let $p_P : L \times P \to P$ be the canonical projection. Then $\overline{\omega}_{n,p}(p_P)$ is the stable homotopy class of $\alpha_P \wedge (T(\widehat{i_{\mathfrak{G},\mathcal{G}}}) \circ T(\widetilde{c_{\nu_L^{\ell}}}) \circ \alpha_L)$.*

Proof. Let ξ' be the kernel bundle $\mathrm{Ker}(d\mathbf{f})$ over N, which is the subbundle of TN along the fibers of $\mathbf{f}$. Let $\widetilde{c_{\xi'}} : \xi' \to \gamma_{\mathfrak{G}}^{n-p}$ be the bundle map covering the classifying map $c_{\xi'} : N \to \mathfrak{G}$ and $\pi_{\xi'} : TN \to \xi'$ be the orthogonal projection. Then we have a bundle map

$$b'_{\mathbf{f}} = d\mathbf{f} \times (\widetilde{c_{\xi'}} \circ \pi_{\xi'}) : TN \to TP \times \gamma_{\mathfrak{G}}^{n-p}.$$

Let

$$t'_{TN\oplus\nu} : TN \oplus \ \nu_N^{k+\ell} \to \theta_N^{n+k+\ell},$$
$$t_{P\times\mathfrak{G}} : (TP \times \gamma_{\mathfrak{G}}^{n-p}) \oplus (\nu_P^k \times \widehat{\gamma}_{\mathfrak{G}}^{\ell}) \cong (TP \oplus \nu_P^k) \times (\gamma_{\mathfrak{G}}^{n-p} \oplus \widehat{\gamma}_{\mathfrak{G}}^{\ell}) \overset{t_P\times t_{\mathfrak{G}}}{\longrightarrow} \theta_{P\times\mathfrak{G}}^{n+k+\ell},$$

be trivializations defined similarly as in (3.5). By [An2, Propositiion 3.3] $b'_{\mathbf{f}}$ induces a bundle map $\nu(b'_{\mathbf{f}}) : \nu_N^{k+\ell} \to \nu_P^k \times \widehat{\gamma}_{\mathfrak{G}}^{\ell}$ such that $t_{P\times\mathfrak{G}} \circ (b'_{\mathbf{f}} \oplus \nu(b'_{\mathbf{f}})) \circ (t'_{TN\oplus\ \nu})^{-1}$ is homotopic to $(\mathbf{f} \times c_{\xi'}) \times id_{\mathbf{R}^{n+k+\ell}}$. By the definition of $\Gamma(R_{\Omega,\mathbf{V}})(j^2\mathbf{f})$, we know that $\Gamma(R_{\Omega,\mathbf{V}})(j^2\mathbf{f})$ is homotopic to $d\mathbf{f} \circ p_{TN} : TN \oplus \theta_N \to TN \to TP$, where p_{TN} is the canonical projection $TN \oplus \theta_N \to TN$. Since $\xi_{\mathbf{f}} = \xi' \oplus \theta_N$, we may set

$$b_{\mathbf{f}} = (id_{TP} \times \widetilde{i_{\mathfrak{G},\mathcal{G}}}) \circ \overline{b'_{\mathbf{f}}} : TN \oplus \theta_N \to TP \times (\gamma_{\mathfrak{G}}^{n-p} \oplus \theta_{\mathfrak{G}}) \to TP \times \gamma_{\mathcal{G}}^{n-p+1},$$

where $\overline{b'_{\mathbf{f}}}(\mathbf{v},t) = (b'_{\mathbf{f}}(\mathbf{v}),t)$. Hence, we may set

$$\nu(b_{\mathbf{f}}) = (id_{\nu_P^k} \times \widehat{i_{\mathfrak{G},\mathcal{G}}}) \circ \nu(b'_{\mathbf{f}}) : \nu_N^{k+\ell} \to \nu_P^k \times \widehat{\gamma}_{\mathfrak{G}}^\ell \to \nu_P^k \times \widehat{\gamma}_{\mathcal{G}}^\ell.$$

Therefore, $\overline{\omega}_{n,p}(\mathbf{f})$ is the stable homotopy class of $T(id_{\nu_P^k} \times \widehat{i_{\mathfrak{G},\mathcal{G}}}) \circ T(\nu(b'_{\mathbf{f}})) \circ \alpha_N$. This proves the assertion (1).

The differential $dp_P(= b'_{p_L}) : TL \times TP \to TP$ is the canonical projection and $\xi' = p_L{}^*(TL)$ for the canonical projection $p_L : L \times P \to L$. We have

$$\nu(dp_P) = id_{\nu_P^k} \times \widetilde{c_{\nu_L^\ell}} : \nu_{L\times P} = \nu_P^k \times \nu_L^\ell \to \nu_P^k \times \widehat{\gamma}_{\mathfrak{G}}^\ell.$$

This yields

$$\nu(b_{p_L}) = (id_{\nu_P^k} \times \widehat{i_{\mathfrak{G},\mathcal{G}}}) \circ \nu(dp_P) : \nu_P^k \times \nu_L^\ell \to \nu_P^k \times \widehat{\gamma}_{\mathcal{G}}^\ell.$$

By definition, we obtain that $\overline{\omega}_{n,p}(p_P)$ is the stable homotopy class of

$$\begin{aligned} &T(id_{\nu_P^k} \times \widehat{i_{\mathfrak{G},\mathcal{G}}}) \circ T(\nu(dp_P)) \circ \alpha_{L\times P} \\ &= (T(id_{\nu_P^k}) \wedge T(\widehat{i_{\mathfrak{G},\mathcal{G}}})) \circ (T(id_{\nu_P^k}) \wedge T(\widetilde{c_{\nu_L^\ell}})) \circ (\alpha_P \wedge \alpha_L) \\ &= \alpha_P \wedge (T(\widehat{i_{\mathfrak{G},\mathcal{G}}}) \circ T(\widetilde{c_{\nu_L^\ell}}) \circ \alpha_L). \end{aligned}$$

This proves the assertion (2). □

Let W_i and P_i be the i-th Stiefel-Whitney class and the i-th Pontrjagin class respectively. Let $I = (i_1, \dots, i_t)$, $J = (j_1, \dots, j_u)$, $W_I(\zeta) = W_{i_1}(\zeta) \cdots W_{i_t}(\zeta)$, $P_J(\zeta) = P_{j_1}(\zeta) \cdots P_{j_u}(\zeta)$ and so on. The following proposition is proved by a routine argument about characteristic classes (see [H]).

Proposition 3.5. *Let N and P be closed manifolds of dimensions n and p respectively. Let $f : N \to P$ be a quasidefinite fold-map (resp. submersion).*

(1) *Let $i_1 + \cdots + i_t + j_1 + \cdots + j_u = n$. Then the Stiefel-Whitney number $(W_I(f^*(TP))W_J(TN - f^*(TP)), [N])$ is a quasidefintite fold-cobordism invariant. Unless $i_1 + \cdots + i_t \le p$ and $j_1, \dots, j_u \le n - p + 1$ (resp. $j_1, \dots, j_u \le n - p$), then $(W_I(f^*(TP))W_J(TN - f^*(TP)), [N])$ vanishes.*

(2) *Let N and P be oriented and $4(i_1 + \cdots + i_t + j_1 + \cdots + j_u) = n$. Then the Pontrjagin number $(P_I(f^*(TP))P_J(TN - f^*(TP)), [N])$ is a quasidefintite oriented-fold-cobordism invariant. Unless $4(i_1 + \cdots + i_t) \le p$ and $4j_1, \dots, 4j_u \le n - p + 1$ (resp. $4j_1, \dots, 4j_u \le n - p$), then $(P_I(f^*(TP))P_J(TN - f^*(TP)), [N])$ vanishes.*

We now prove Theorem 0.4, which is a special case of a result in [An5].

Proof of Theroem 0.4. Let G refer to $G_{n,\ell}$ or $\widetilde{G}_{n,\ell}$. We provide N and P with Riemannian metrics. In the proof we always identify $J^2(N,P)$ and $\Omega^{n-p+1,0}(N,P)$ with $J^2(TN,TP)$ and $\Omega^{n-p+1,0}(TN,TP)$ respectively by (3.2). Let $f : N \to P$ be a fold-map. Let $B_{TN} : TN \to \gamma^n_G$ be a bundle map covering a classifying map $c_N : N \to G$. Then B_{TN} induces bundle maps $B_J : J^2(TN,TP) \to J^2(\gamma^n_G, TP)$ and $B_\Omega : \Omega^{n-p+1,0}(TN,TP) \to \Omega^{n-p+1,0}(\gamma^n_G, TP)$ covering $c_N \times id_P$. It is easy to see that $p_G \circ B_\Omega \circ j^2 f = c_N$ and $p_P \circ B_\Omega \circ j^2 f = f$. We have the commutative diagram

$$
(3.7)\qquad
\begin{array}{ccc}
\Omega^{n-p+1,0}(N,P) \cong \Omega^{n-p+1,0}(TN,TP) & \xrightarrow{B_\Omega} & \Omega^{n-p+1,0}(\gamma^n_G, TP) \\
\downarrow & & \downarrow \\
N \times P & \xrightarrow[c_N \times id_P]{} & N \times P.
\end{array}
$$

We have the trivializations $t_N : TN \oplus \nu^\ell_N \to \theta^{n+\ell}_N$ and $t_G : \gamma^n_G \oplus \widehat{\gamma}^\ell_G \to \theta^{n+\ell}_G$. Here, we should recall the definition of the bundel maps $B_{TN} : TN \to \gamma^n_G$ and $B_{\nu_N} : \nu^\ell_N \to \widehat{\gamma}^\ell_G$. For a point $x \in \mathbf{R}^{n+\ell}$, let $\ell_x : T_x\mathbf{R}^{n+\ell} \to \mathbf{R}^{n+\ell}$ be the canonical isomorphism. Then B_{TN} maps $(x, \mathbf{v}) \in T_xN$ to $(\ell_x(T_xN), \ell_x(\mathbf{v})) \in \gamma^n_G$, and B_{ν_N} maps $(x, \mathbf{w}) \in \nu^\ell_N$ to $(\ell_x((\nu^\ell_N)_x), \ell_x(\mathbf{w})) \in \widehat{\gamma}^\ell_G$. Let $B_{p^*_G(\gamma^n_G)} : p^*_G(\gamma^n_G) \to \gamma^n_G$ and $B_{p^*_G(\widehat{\gamma}^\ell_G)} : p^*_G(\widehat{\gamma}^\ell_G) \to \widehat{\gamma}^\ell_G$ be the canonical bundle maps induced from p_G. Since $p_G \circ B_\Omega \circ J_{\exp} \circ j^2 f = c_N$, B_{TN} and c_N induce bundle maps

$$B^\Omega_{TN} : TN \to p^*_G(\gamma^n_G)|_{\Omega^{n-p+1,0}(\gamma^n_G,P)} \quad \text{and} \quad B^\Omega_{\nu_N} : \nu^\ell_N \to p^*_G(\widehat{\gamma}^\ell_G)|_{\Omega^{n-p+1,0}(\gamma^n_G,P)},$$

which are defined by, for $x \in N$, $\mathbf{v} \in T_xN$, $\mathbf{w} \in (\nu^\ell_N)_x$,

$$B^\Omega_{TN}(x, \mathbf{v}) = (j^2_x f, B_{TN}(\mathbf{v})) \quad \text{and} \quad B^\Omega_{\nu_N}(x, \mathbf{w}) = (j^2_x f, B_{\nu_N}(\mathbf{w}))$$

respectively. We now define $\mu^G_{n,p}(f)$ by

$$\mu^G_{n,p}(f) = [T(B^\Omega_{\nu_N}) \circ \alpha_N].$$

Since all Riemannian metrics on a manifold are homotopic each other and $\ell \gg n$, $\mu^G_{n,p}(f)$ does not depend on choices of Riemannian metrics of N and P, and of an embedding $N \to \mathbf{R}^{n+\ell}$. It is easy to see that $(\lim_{\ell\to\infty} T(B_{\widehat{\gamma}^\ell}))_*(\mu^{\widetilde{G}_{n,\ell}}_{n,p}(f)) = \mu^{G_{n,\ell}}_{n,p}(f)$.

Next let $a : S^{n+\ell} \to T(p^*_G(\widehat{\gamma}^\ell_G)|_{\Omega^{n-p+1,0}(\gamma^n_G,P)})$ be a map. We may suppose that a is smooth around $a^{-1}(\Omega^{n-p+1,0}(\gamma^n_G, TP))$ and is transverse to

$\Omega^{n-p+1,0}(\gamma_G^n, TP)$. Let N be the submanifold $a^{-1}(\Omega^{n-p+1,0}(\gamma_G^n, TP))$ and ν_N^ℓ be the normal bundle of $N \subset \mathbf{R}^{n+\ell} = S^{n+\ell} \setminus \{\text{base point}\}$. Let $B^\Omega_{\nu_N}(a) : \nu_N^\ell \to p_G^*(\widehat{\gamma}_G^\ell)|_{\Omega^{n-p+1,0}(\gamma_G^n, P)}$ be the bundle map induced from the map a. By the definition of the structure of ν_N^ℓ as the normal bundle, we obtain the following homotopy commutative diagram of the exact sequences

$$\begin{array}{ccccccc}
0 & \longrightarrow & TN & \longrightarrow & \theta_N^{n+\ell}(\cong TN \oplus \nu_N^\ell) & \longrightarrow \\
 & & \Big\downarrow & & \Big\downarrow{\scriptstyle (\alpha|N)\times id_{\mathbf{R}^{n+\ell}}} & \\
0 & \longrightarrow & p_G^*(\gamma_G^n) & \longrightarrow & \theta^{n+\ell}_{\Omega^{n-p+1,0}(\gamma_G^n, TP)}(\cong p_G^*(\gamma_G^n \oplus \widehat{\gamma}_G^\ell)) & \longrightarrow
\end{array}$$

$$\begin{array}{ccc}
\theta_N^{n+\ell}/TN = \nu_N^\ell & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle B^\Omega_{\nu_N}(\alpha)} & & \\
\theta^{n+\ell}_{\Omega^{n-p+1,0}(\gamma_G^n, TP)}/p_G^*(\gamma_G^n) = p_G^*(\widehat{\gamma}_G^\ell) & \longrightarrow & 0.
\end{array}$$

This diagram yields the bundle map $B^\Omega_{TN}(a) : TN \to p_G^*(\gamma_G^n)$ covering $a|N$ such that $B^\Omega_{TN}(a) \oplus B^\Omega_{\nu_N}(a)$ is homotopic to $(a|N) \times id_{\mathbf{R}^{n+\ell}}$. Therefore, $p_G \circ (a|N)$ is regarded as the classifying map $c_N : N \to G$. By the commutative diagram (3.7), $a|N$ induces a section $s : N \to \Omega^{n-p+1,0}(TN, TP)(\cong \Omega^{n-p+1,0}(N, P))$ such that $B_\Omega \circ s = a|N$. By the homotopy principle for fold-maps in [An4, Theorem 0.5], we obtain a fold-map $f : N \to P$ such that $j^2 f$ and s are homotopic as sections $\Gamma(N, P)$. We should note that c_N, B_{TN} and B_{ν_N} defined for f are homotopic to $p_G \circ (a|N)$, $B_{p_G^*(\gamma_G^n)} \circ B^\Omega_{TN}(a)$ and $B_{p_G^*(\widehat{\gamma}_G^\ell)} \circ B^\Omega_{\nu_N}(a)$ respectively. Therefore, we have

$$\begin{aligned}
\mu^G_{n,p}(f) &= [T(B^\Omega_{\nu_N}) \circ \alpha_N] \\
&= [T(B^\Omega_{\nu_N}(a)) \circ \alpha_N] \\
&= [a].
\end{aligned}$$

This concludes the assertion. □

§4. Dual Spaces and Duality Isomorphisms

In this section we study $\lim_{k\to\infty} \pi_{n+k+\ell}(T(\nu_P^k) \wedge T(\widehat{\gamma}^\ell_{\widetilde{G}_{n-p+1,\ell}}))$ by using S-dual spaces and duality maps in the suspension category due to [Sp1] and [Sp2]. Let S^ℓ be the sphere with radius 1 centred at the origin in $\mathbf{R}^{\ell+1}$ with base point $(1, 0, \ldots, 0)$. We identify S^ℓ with the wedge product $S^1 \wedge \cdots \wedge S^1$ of ℓ copies of S^1. We denote the set of homotopy classes of maps $\alpha : A \to B$

by $[A,B]$. Let A be a finite polyhedron with base point. According to [Sp2], $S^\ell A$ denotes the ℓ-th suspension $A \wedge S^\ell$. Let $S^\ell(c)$ denote the ℓ-th suspension of a map c. If B is also a finite polyhedron with base point, then we denote, by $\{A,B\}$, the set of S-homotopy classes of S-maps, which preserve base points. An element of $\{A,B\}$ represented by a map $\alpha : S^\ell A \to S^\ell B (\ell \geq 0)$ is written as $\{\alpha\}$. Let $i^{\frown}_{A,B} : A \wedge B \to B \wedge A$ be the map defined by $i^{\frown}_{A,B}(x,y) = (y,x)$.

An *m-duality map* $v : A \wedge B \to S^m$ refers to a continuous map such that the map $\varphi_v : H_q(A;\mathbf{Z}) \to H^{m-q}(B;\mathbf{Z})$ defined by sending $z \in H_q(A;\mathbf{Z})$ to the slant product $(v)^*([S^m]^*)/z$ is an isomorphism. The duality map of the identification $S^k \wedge S^m \to S^{k+m}$ is denoted by $i_{\mathbf{S}}$ for any dimensions k and m.

Let $\mathcal{G} = \widetilde{G}_{n-p+1,\ell}$ and $\mathfrak{G} = \widetilde{G}_{n-p,\ell}$ in this section. Given a vector bundle ξ over X, we have that $T(\xi \oplus \theta_X)$ is canonically homeomorphic to $T(\xi) \wedge S^1$. Hence we write $T(\xi \oplus \theta_X) = T(\xi) \wedge S^1$. Under this identification, we have the following bijections for $X = \widetilde{G}_{n-p+1,\ell}$ or $\widetilde{G}_{n-p,\ell} (\ell \gg n)$.

$$\text{(4.1)} \qquad \Pi_X : \lim_{k\to\infty} \pi_{n+k+\ell}(T(\nu_P^k) \wedge T(\widehat{\gamma}_X^\ell)) \to \{S^{n+k+\ell}; T(\nu_P^k) \wedge T(\widehat{\gamma}_X^\ell)\}.$$

Let P^0 be the disjoint union of P and the base point $*_P$. By [M-S, Lemma 2] and [At, Theorem 3.3] there exist duality maps for sufficiently large numbers k, q and ℓ

$$\text{(4.2)} \qquad \begin{aligned} v_P &: (P^0) \wedge T(\nu_P^k) \to S^{p+k}, \\ v_{\mathcal{G}} &: T(\gamma_{\mathcal{G}}^{n-p+1} \oplus \nu_{\mathcal{G}}^q) \wedge T(\widehat{\gamma}_{\mathcal{G}}^\ell) \to S^{\ell(n-p+1)+\ell+q+n-p+1}, \\ v_{\mathfrak{G}} &: T(\gamma_{\mathfrak{G}}^{n-p} \oplus \theta_{\mathfrak{G}} \oplus \nu_{\mathfrak{G}}^q) \wedge T(\widehat{\gamma}_{\mathfrak{G}}^\ell) \to S^{\ell(n-p)+\ell+q+n-p+1}. \end{aligned}$$

By [Spa2, Theorem 6.8] we obtain the following duality maps

(4.3)

$$\begin{aligned} \nu_{P,\mathcal{G}} &= (v_P \wedge v_{\mathcal{G}}) \circ (id_{P^0} \wedge i^{\frown}_{T(\gamma_{\mathcal{G}}^{n-p+1} \oplus \nu_{\mathcal{G}}^q), T(\nu_P^k)} \wedge id_{T(\widehat{\gamma}_{\mathcal{G}}^\ell)}) \\ &\quad : (P^0) \wedge T(\gamma_{\mathcal{G}}^{n-p+1} \oplus \nu_{\mathcal{G}}^q) \wedge T(\nu_P^k) \wedge T(\widehat{\gamma}_{\mathcal{G}}^\ell) \to S^{\ell(n-p+1)+\ell+q+n+k+1}, \\ \nu_{P,\mathfrak{G}} &= (v_P \wedge v_{\mathfrak{G}}) \circ (id_{P^0} \wedge i^{\frown}_{T(\gamma_{\mathfrak{G}}^{n-p} \oplus \theta_{\mathfrak{G}} \oplus \nu_{\mathfrak{G}}^q), T(\nu_P^k)} \wedge id_{T(\widehat{\gamma}_{\mathcal{G}}^\ell)}) \\ &\quad : (P^0) \wedge T(\gamma_{\mathfrak{G}}^{n-p} \oplus \theta_{\mathfrak{G}} \oplus \nu_{\mathfrak{G}}^q) \wedge T(\nu_P^k) \wedge T(\widehat{\gamma}_{\mathfrak{G}}^\ell) \to S^{\ell(n-p)+\ell+q+n+k+1}. \end{aligned}$$

Let $\mathcal{D}_{\mathcal{G}}$ and $\mathcal{D}_{\mathfrak{G}}$ denote the following duality isomorphisms respectively with $m = \ell(n-p+1) + \ell + q + n + k + 1$

$$\begin{aligned} \mathcal{D}_m(i_{\mathbf{S}}, \nu_{P,\mathcal{G}}) &: \{S^{n+k+\ell}; T(\nu_P^k) \wedge T(\widehat{\gamma}_{\mathcal{G}}^\ell)\} \\ &\qquad \to \{(P^0) \wedge T(\gamma_{\mathcal{G}}^{n-p+1} \oplus \nu_{\mathcal{G}}^q); S^{\ell(n-p+1)+q+1}\}, \\ \mathcal{D}_m(i_{\mathbf{S}}, S^\ell(\nu_{P,\mathfrak{G}})) &: \{S^{n+k+\ell}; T(\nu_P^k) \wedge T(\widehat{\gamma}_{\mathfrak{G}}^\ell)\} \\ &\qquad \to \{(P^0) \wedge S^\ell T(\gamma_{\mathfrak{G}}^{n-p} \oplus \theta_{\mathfrak{G}} \oplus \nu_{\mathfrak{G}}^q); S^{\ell(n-p+1)+q+1}\}, \end{aligned}$$

which are defined as follows. Let $c : S^{n+k+\ell} \to T(\nu_P^k) \wedge T(\widehat{\gamma}_{\mathcal{G}}^\ell)$ represent a map in $\{S^{n+k+\ell}; T(\nu_P^k) \wedge T(\widehat{\gamma}_{\mathcal{G}_j}^\ell)\}$. Then $\mathcal{D}_{\mathcal{G}}(\{c\})$ is represented by the map

$$\begin{aligned}\nu_{P,\mathcal{G}} \circ (id_{(P^0)\wedge T(\gamma_{\mathcal{G}}^{n-p+1}\oplus\nu_{\mathcal{G}}^q)} \wedge c) : (P^0) \wedge T(\gamma_{\mathcal{G}}^{n-p+1} \oplus \nu_{\mathcal{G}}^q) \wedge S^{n+k+\ell} \\ \to (P_0) \wedge T(\gamma_{\mathcal{G}}^{n-p+1} \oplus \nu_{\mathcal{G}}^q) \wedge T(\nu_P^k) \wedge T(\widehat{\gamma}_{\mathcal{G}}^\ell) \\ \to S^{\ell(n-p)+\ell+q+n+k+1}.\end{aligned}$$

The definition of $\mathcal{D}_{\mathfrak{G}}$ is similar.

Let $C_0(T(\gamma_{\mathcal{G}}^{n-p+1} \oplus \nu_{\mathcal{G}}^q), S^{\ell(n-p+1)+q+1})$ and $C_0(T(\gamma_{\mathfrak{G}}^{n-p} \oplus \theta_{\mathfrak{G}} \oplus \nu_{\mathfrak{G}}^q), S^{\ell(n-p)+q+1})$ denote the space of all base point preserving continuous maps $T(\gamma_{\mathcal{G}}^{n-p+1} \oplus \nu_{\mathcal{G}}^q) \to S^{\ell(n-p+1)+q+1}$ and $T(\gamma_{\mathfrak{G}}^{n-p} \oplus \theta_{\mathfrak{G}} \oplus \nu_{\mathfrak{G}}^q) \to S^{\ell(n-p)+q+1}$ equipped with the compact-open topology respectively. With the identification $T(\xi \oplus \theta_X) = T(\xi) \wedge S^1$ we have the map

$$\begin{aligned}C_0(T(\gamma_{\mathcal{G}}^{n-p+1} \oplus \nu_{\mathcal{G}}^q), S^{\ell(n-p+1)+q+1}) \\ \to C_0(T(\gamma_{\mathcal{G}}^{n-p+1} \oplus \nu_{\mathcal{G}}^q \oplus \theta_{\mathcal{G}}), S^{\ell(n-p+1)+q+2}), \\ C_0(T(\gamma_{\mathfrak{G}}^{n-p} \oplus \theta_{\mathfrak{G}} \oplus \nu_{\mathfrak{G}}^q), S^{\ell(n-p)+q+1}) \\ \to C_0(T(\gamma_{\mathfrak{G}}^{n-p} \oplus \theta_{\mathfrak{G}} \oplus \nu_{\mathfrak{G}}^q \oplus \theta_{\mathfrak{G}}), S^{\ell(n-p)+q+2})\end{aligned}$$

defined by mapping, for example, $c_{\mathcal{G}}$ to $c_{\mathcal{G}} \wedge id_{S^1}$, where $c_{\mathcal{G}}$ is an element of $C_0(T(\gamma_{\mathcal{G}}^{n-p+1} \oplus \nu_{\mathcal{G}}^q), S^{\ell(n-p+1)+q+1})$. Let $C_0(\mathbf{T}_{\mathcal{G}}, \mathbf{S})$ and $C_0(\mathbf{T}_{\mathfrak{G}}, \mathbf{S})$ be the space defined by

$$\begin{aligned}C_0(\mathbf{T}_{\mathcal{G}}, \mathbf{S}) &= \lim_{q\to\infty} C_0(T(\gamma_{\mathcal{G}}^{n-p+1} \oplus \nu_{\mathcal{G}}^q), S^{\ell(n-p+1)+q+1}), \\ C_0(\mathbf{T}_{\mathfrak{G}}, \mathbf{S}) &= \lim_{q\to\infty} C_0(T(\gamma_{\mathfrak{G}}^{n-p} \oplus \theta_{\mathfrak{G}} \oplus \nu_{\mathfrak{G}}^q), S^{\ell(n-p)+q+1})\end{aligned} \tag{4.4}$$

respectively. Then we define the bijections

$$\begin{aligned}\mathbf{i}_{P,\mathcal{G}} &: \{(P^0) \wedge T(\gamma_{\mathcal{G}}^{n-p+1} \oplus \nu_{\mathcal{G}}^q); S^{\ell(n-p+1)+q+1}\} \to [P, C_0(\mathbf{T}_{\mathcal{G}}, \mathbf{S})], \\ \mathbf{i}_{P,\mathfrak{G}} &: \{(P^0) \wedge S^\ell T(\gamma_{\mathfrak{G}}^{n-p} \oplus \theta_{\mathfrak{G}} \oplus \nu_{\mathfrak{G}}^q); S^{\ell(n-p+1)+q+1}\} \to [P, C_0(\mathbf{T}_{\mathfrak{G}}, \mathbf{S})],\end{aligned} \tag{4.5}$$

by $\mathbf{i}_{P,\mathcal{G}}(c_{P,\mathcal{G}})(x) = [c_{P,\mathcal{G}}|(x \cup *_P) \wedge T(\gamma_{\mathcal{G}}^{n-p+1} \oplus \nu_{\mathcal{G}}^q)]$ and $\mathbf{i}_{P,\mathfrak{G}}(c_{P,\mathfrak{G}})(x) = [c_{P,\mathfrak{G}}|(x \cup *_P) \wedge S^\ell T(\gamma_{\mathfrak{G}}^{n-p} \oplus \theta_{\mathfrak{G}} \oplus \nu_{\mathfrak{G}}^q)]$, where $c_{P,\mathcal{G}}$ and $c_{P,\mathfrak{G}}$ represents elements $\{(P^0) \wedge T(\gamma_{\mathcal{G}}^{n-p+1} \oplus \nu_{\mathcal{G}}^q); S^{\ell(n-p+1)+q+1}\}$, $\{(P^0) \wedge S^\ell T(\gamma_{\mathfrak{G}}^{n-p} \oplus \theta_{\mathfrak{G}} \oplus \nu_{\mathfrak{G}}^q); S^{\ell(n-p+1)+q+1}\}$ and $x \in P$ respectively.

Set $\mathcal{D}_{\mathcal{G},\mathfrak{G}} = \mathcal{D}_{\ell(n-p+1)+\ell+q+n-p+1}(v_{\mathcal{G}}, S^\ell(v_{\mathfrak{G}}))$. Let $\mathcal{D}_{\mathcal{G},\mathfrak{G}}(\{T(\widehat{i_{\mathfrak{G},\mathcal{G}}})\}) \in \{T(\nu_{\mathcal{G}}^q \oplus \gamma_{\mathcal{G}}^{n-p+1}); S^\ell T(\gamma_{\mathfrak{G}}^{n-p} \oplus \theta_{\mathfrak{G}} \oplus \nu_{\mathfrak{G}}^q)\}$ be the dual map of $T(\widehat{i_{\mathfrak{G},\mathcal{G}}}) : T(\widehat{\gamma}_{\mathfrak{G}}^\ell) \to T(\widehat{\gamma}_{\mathcal{G}}^\ell)$. We define the map

$$\mathcal{D}_{\mathcal{G},\mathfrak{G}}(\{T(\widehat{i_{\mathfrak{G},\mathcal{G}}})\})_* : [P, C_0(\mathbf{T}_{\mathfrak{G}}, \mathbf{S})] \to [P, C_0(\mathbf{T}_{\mathcal{G}}, \mathbf{S})].$$

Let $C_{\mathfrak{G},\mathcal{G}} : T(\gamma_{\mathcal{G}}^{n-p+1} \oplus \nu_{\mathcal{G}}^q) \to S^\ell T(\gamma_{\mathfrak{G}}^{n-p} \oplus \theta_{\mathfrak{G}} \oplus \nu_{\mathfrak{G}}^q)$ represent $\mathcal{D}_{\mathcal{G},\mathfrak{G}}(\{T(\widehat{i_{\mathfrak{G},\mathcal{G}}})\})$. For an element $c_{\mathfrak{G}} \in [P, C_0(T(\gamma_{\mathfrak{G}}^{n-p} \oplus \theta_{\mathfrak{G}} \oplus \nu_{\mathfrak{G}}^q), S^{\ell(n-p)+q+1})]$ we set $\mathcal{D}_{\mathcal{G},\mathfrak{G}}(\{T(\widehat{i_{\mathfrak{G},\mathcal{G}}})\})_*(c_{\mathfrak{G}})(x) = [c_{\mathfrak{G}}(x) \circ C_{\mathcal{G},\mathfrak{G}}]$, where $x \in P$. It is obvious that this definition is well defined.

We have the following proposition.

Proposition 4.1. *Let $\ell \gg n$ and let $\mathcal{G} = \widetilde{G}_{n-p+1,\ell}$ and $\mathfrak{G} = \overline{G}_{n-p,\ell}$. Then we have the commutative diagram*

$$\begin{array}{ccc}
\lim_{k\to\infty} \pi_{n+k+\ell}(T(\nu_P^k) \wedge T(\widehat{\gamma}_{\mathfrak{G}}^\ell)) & \xrightarrow{\mathbf{i}_{P,\mathfrak{G}} \circ \mathcal{D}_{\mathfrak{G}} \circ \Pi_{\mathfrak{G}}} & [P, C_0(\mathbf{T}_{\mathfrak{G}}, \mathbf{S})] \\
{\scriptstyle (id_{T(\nu_P^k)} \wedge T(\widehat{i_{\mathfrak{G},\mathcal{G}}}))_*}\downarrow & & \downarrow{\scriptstyle \mathcal{D}_{\mathcal{G},\mathfrak{G}}(\{T(\widehat{i_{\mathfrak{G},\mathcal{G}}})\})_*} \\
\lim_{k\to\infty} \pi_{n+k+\ell}(T(\nu_P^k) \wedge T(\widehat{\gamma}_{\mathcal{G}}^\ell)) & \xrightarrow[\mathbf{i}_{P,\mathcal{G}} \circ \mathcal{D}_{\mathcal{G}} \circ \Pi_{\mathcal{G}}]{} & [P, C_0(\mathbf{T}_{\mathcal{G}}, \mathbf{S})],
\end{array}$$

where $\mathbf{i}_{P,\mathfrak{G}} \circ \mathcal{D}_{\mathfrak{G}} \circ \Pi_{\mathfrak{G}}$ and $\mathbf{i}_{P,\mathcal{G}} \circ \mathcal{D}_{\mathcal{G}} \circ \Pi_{\mathcal{G}}$ are bijective.

Proof. We set $\mathcal{D}_P = \mathcal{D}_{p+k}(v_P, v_P) : \{T(\nu_P^k); T(\nu_P^k)\} \to \{P^0; P^0\}$. By (4.1) we have

$$\begin{aligned}
&(\mathcal{D}_{\mathcal{G},\mathfrak{G}}(\{T(\widehat{i_{\mathfrak{G},\mathcal{G}}})\})_* \circ \mathbf{i}_{P,\mathfrak{G}} \circ \mathcal{D}_{\mathfrak{G}} \circ \Pi_{\mathfrak{G}}(c))(x) \\
&= [\mathcal{D}_{\mathfrak{G}}(c) \circ (id_{P^0} \wedge \mathcal{D}_{\mathcal{G},\mathfrak{G}}(\{T(\widehat{i_{\mathfrak{G},\mathcal{G}}})\})|(x \cup *_P) \wedge T(\nu_{\mathcal{G}}^q \oplus \gamma_{\mathcal{G}}^{n-p+1})],
\end{aligned}$$

and

$$\begin{aligned}
&(\mathbf{i}_{P,\mathcal{G}} \circ \mathcal{D}_{\mathcal{G}} \circ \Pi_{\mathcal{G}} \circ (id_{T(\nu_P^k)} \wedge T(\widehat{i}_{\mathfrak{G},\mathcal{G}}))_*(c))(x) \\
&= [\mathcal{D}_{\mathcal{G}}(\{(id_{T(\nu_P^k)} \wedge T(\widehat{i_{\mathfrak{G},\mathcal{G}}})) \circ c\})|(x \cup *_P) \wedge T(\nu_{\mathcal{G}}^q \oplus \gamma_{\mathcal{G}}^{n-p+1})].
\end{aligned}$$

Since we have

$$\begin{aligned}
&\mathcal{D}_{\mathfrak{G}}(\{c\}) \circ (id_{P^0} \wedge \mathcal{D}_{\mathcal{G},\mathfrak{G}}(\{T(\widehat{i_{\mathfrak{G},\mathcal{G}}})\}) \\
&= \mathcal{D}_{\mathfrak{G}}(\{c\}) \circ (\mathcal{D}_P(\{id_{T(\nu_P^k)}) \wedge \mathcal{D}_{\mathcal{G},\mathfrak{G}}(\{T(\widehat{i_{\mathfrak{G},\mathcal{G}}})\})\}) \\
&= \mathcal{D}_{\mathfrak{G}}(\{c\}) \circ \mathcal{D}_{\mathcal{G}}(\{id_{T(\nu_P^k)} \wedge T(\widehat{i_{\mathfrak{G},\mathcal{G}}})\}) \\
&= \mathcal{D}_{\mathcal{G}}(\{(id_{T(\nu_P^k)} \wedge T(\widehat{i_{\mathfrak{G},\mathcal{G}}})) \circ c\})
\end{aligned}$$

by [Spa2, Theorems 5.11 and 6.3], it follows that maps representing $\mathcal{D}_{\mathfrak{G}}(\{c\}) \circ (id_{P^0} \wedge \mathcal{D}_{\mathcal{G},\mathfrak{G}}(\{T(\widehat{i_{\mathfrak{G},\mathcal{G}}})\})$ and $\mathcal{D}_{\mathcal{G}}(\{(id_{T(\nu_P^k)} \wedge T(\widehat{i_{\mathfrak{G},\mathcal{G}}})) \circ c\})$ are homotopic. This fact shows the commutativity of the diagram. ☐

Corollary 4.2. *Let $\ell \gg n$. Let $f : N \to P$ be a (resp. quasidefinite) fold-map. Given an element $a \in H^*(C_0(\mathbf{T}_{\mathcal{G}}, \mathbf{S}))$, the class $(\mathbf{i}_{P,\mathcal{G}} \circ \mathcal{D}_{\mathcal{G}} \circ \Pi_{\mathcal{G}} \circ \overline{\omega}_{n,p}^{\Omega})^*(a) \in H^*(P)$ depends only on the oriented-fold-cobordism class of f.*

By Corollary 4.2 it is important to study the structure of the algebra $H^*(C_0(\mathbf{T}_{\mathcal{G}}, \mathbf{S}))$ for $n > p$.

Remark 4.3. Let $n = p \geq 2$. This case has been dealt with more precisely in [An3], where $\mathcal{G}$ is regarded as a single point. Then we have

$$C_0(\mathbf{T}_{\mathcal{G}}, \mathbf{S}) = F = \lim_{q \to \infty} F(q+1),$$

where $F(q+1)$ is the space of all base point preserving maps of S^q equipped with the compact-open topology (see [At], [M-M] and [Tsu]). In our case $\mathcal{G} = S^\ell$, we have $\gamma^1_{S^\ell} = \theta_{S^\ell}$ and $\nu^q_{S^\ell} = \theta^q_{S^\ell}$. Since $T(\gamma^1_{S^\ell} \oplus \nu^q_{S^\ell})$ is homeomorphic to $(S^\ell)^0 \wedge S^{q+1}$, $C_0(\mathbf{T}_{S^\ell}, \mathbf{S})$ is weakly homotopy equivalent to F.

The following proposition follows from Propositions 3.3 and 3.4.

Proposition 4.4. *Let $\ell \gg n$ and let $\mathcal{G} = \widetilde{G}_{n-p+1,\ell}$ and $\mathfrak{G} = \widetilde{G}_{n-p,\ell}$.*

(1) *Let $f : N \to P$ be a submersion. Then $\mathbf{i}_{P,\mathcal{G}} \circ \mathcal{D}_{\mathcal{G}} \circ \Pi_{\mathcal{G}} \circ \overline{\omega}_{n,p}(f)$ lies in the image of $\mathcal{D}_{\mathcal{G},\mathfrak{G}}(\{T(\widehat{i_{\mathfrak{G},\mathcal{G}}})\})_*$.*

(2) *Let L and $p_P : L \times P \to P$ be as in Proposition 3.4. Then $\mathbf{i}_{P,\mathcal{G}} \circ \mathcal{D}_{\mathcal{G}} \circ \Pi_{\mathcal{G}} \circ \overline{\omega}_{n,p}(p_P)$ is homotopic to the constant map with value $\mathcal{D}_{\mathcal{G}}(\{T(\widehat{i_{\mathfrak{G},\mathcal{G}}}) \circ T(\widetilde{c_{\nu^k_L}}) \circ \alpha_L\})$ in $C_0(\mathbf{T}_{\mathcal{G}}, \mathbf{S})$.*

§5. Lemmas

Let A be a $p \times n$ matrix, where $n \geq p$. Then A^tA is a symmetric and non-negative definite $p \times p$ matrix. Hence, A^tA is triangulated by an orthogonal matrix T as $T(A^tA)^tT = \Delta(d_1^2, \dots, d_p^2)$, where $d_1, \dots, d_p$ are non-negative real numbers. Suppose that TA is written as $\begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_p \end{pmatrix}$ by the row vectors $\mathbf{a}_i$ $(1 \leq i \leq p)$. Then we have that $(\mathbf{a}_i, \mathbf{a}_j) = 0$ for $i \neq j$ and $(\mathbf{a}_i, \mathbf{a}_i) = d_i^2$. If $\mathbf{a}_i \neq \mathbf{0}$, then set $\mathbf{f}_i = \mathbf{a}_i / \|\mathbf{a}_i\|$. By choosing row vectors $\mathbf{f}_j$ of degree n for numbers j such that $\mathbf{a}_j = \mathbf{0}$ properly, we can find orthonormal vectors $\mathbf{f}_1, \dots, \mathbf{f}_p$. Then it follows that

$$TA = \Delta(\|\mathbf{a}_1\|, \dots, \|\mathbf{a}_p\|) \begin{pmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_p \end{pmatrix}.$$

Hence, we have

$$A = {}^tT\Delta(\|\mathbf{a}_1\|, \dots, \|\mathbf{a}_p\|)\begin{pmatrix}\mathbf{f}_1\\ \vdots\\ \mathbf{f}_p\end{pmatrix}. \tag{5.1}$$

Lemma 5.1. *Let $n \geq p \geq 2$. Let A be a $p \times p$ matrix of rank m $(0 \leq m \leq p)$. Then there exist matrices $S \in O(p)$, $M \in O(n)$ and real numbers $d_1, \dots, d_p$ such that*

(1) $d_1 \geq \cdots \geq d_m > 0$ *and* $d_{m+1} = \cdots = d_p = 0$,

(2) $A = S\Delta(\mathbf{d})M\binom{1}{p} = S(1,m)\Delta(d_1, \dots, d_m)M\binom{1}{m}$,

(3) $d_1^2, \dots, d_p^2$ *are eigen-values of* A^tA.

Proof. By (5.1) we can find matrices $S \in O(p)$ and $M \in O(n)$ such that A is expressed by $S\Delta(\mathbf{d})M\binom{1}{p}$. Suppose that $d_{i_1} \geq \cdots \geq d_{i_p} \geq 0$. Let $P(i_1, \dots, i_p)$ be the permutation matrix in $O(p)$ such that $P(i_1, \dots, i_p)(\mathbf{e}_j) = \mathbf{e}_{i_j}$. Then we have that

$$\begin{aligned} A &= S\Delta(\mathbf{d})M\tbinom{1}{p} \\ &= SP(i_1, \dots, i_p)\Delta(d_{i_1}, \dots, d_{i_p})^tP(i_1, \dots, i_p)M\tbinom{1}{p} \end{aligned}$$

since $P(i_1, \dots, i_p)\Delta(d_{i_1}, \dots, d_{i_p})^tP(i_1, \dots, i_p) = \Delta(d_1, \dots, d_p)$. □

We say that the diagonal components $\mathbf{d} = (d_1, \dots, d_p)$ are *non-negative* if $d_i \geq 0$ for all i and are *decreasing* if $d_1 \geq \cdots \geq d_p$. The expression $A = S\Delta(\mathbf{d})M\binom{1}{p}$ will be called a *diagonalization* of A.

Lemma 5.2. *Let $\mathbf{d}$ and $\mathbf{d}'$ be decreasing diagonal components of degree ℓ. Suppose that ${}^tT\Delta(\mathbf{d})T = \Delta(\mathbf{d}')$ for $T \in O(\ell)$. Then we have the following.*

(1) *We have* $\mathbf{d} = \mathbf{d}'$.

(2) *Suppose that $\Delta(\mathbf{d})(= \Delta(\mathbf{d}'))$ is written as $a_1E_{i_1} \dotplus a_2E_{i_2} \dotplus \cdots \dotplus a_sE_{i_s}$, where $a_1, \dots, a_s$ are all distinct and $\ell = i_1 + \cdots + i_s$. Then T is also a matrix of the form $T_1 \dotplus \cdots \dotplus T_s$, where T_j is of rank i_j for every j.*

Proof. The assertion (1) follows from the fact that the set of eigen values of ${}^tT\Delta(\mathbf{d})T$ is $\{d_1, \dots, d_p\}$. We write $T = (t_{iq}) = (\overline{\mathbf{t}}_1, \dots, \overline{\mathbf{t}}_\ell)$. By the assumption ${}^tT\Delta(\mathbf{d})T = \Delta(\mathbf{d})$, we have

$$({}^t(d_1t_{1q}, \dots, d_\ell t_{\ell q}), \overline{\mathbf{t}}_m) = d_q\delta_{qm} = d_q(\overline{\mathbf{t}}_q, \overline{\mathbf{t}}_m).$$

In other words,

$$({}^t(d_1t_{1q},\dots,d_\ell t_{\ell q}) - d_q\overline{\mathbf{t}}_q, \overline{\mathbf{t}}_m) = 0 \qquad (m = 1,\dots,\ell).$$

Since $\overline{\mathbf{t}}_1,\dots,\overline{\mathbf{t}}_\ell$ are orthonormal basis of $\mathbf{R}^\ell$, it follows that ${}^t(d_1t_{1q},\dots,d_\ell t_{\ell q}) - d_q\overline{\mathbf{t}}_q = \mathbf{0}$ for each q. Therefore, if $i_1+\dots+i_{j-1} < q \le i_1+\dots+i_j$ and r does not satisfy $i_1+\dots+i_{j-1} < r \le i_1+\dots+i_j$, then we have $t_{rq} = 0$. This implies the assertion (2). □

Lemma 5.3. *Let $\mathbf{d}$ be decreasing diagonal components of degree ℓ given in Lemma 5.2 (2). For a sequence $\{T^k\}$ in $O(\ell)$ and a sequence of decreasing diagonal components $\{\mathbf{d}^k\}$, assume that the sequence $\{{}^tT^k\Delta(\mathbf{d}^k)T^k\}$ converges to $\Delta(\mathbf{d})$. Then we have the following.*

(1) *$\{\mathbf{d}^k\}$ converges to $\mathbf{d}$.*

(2) *If a pair (r,q) of numbers does not satisfy the inequality*

$$i_1+\dots+i_{j-1} < r, q \le i_1+\dots+i_j$$

for every integer j with $1 \le j \le s$ ($i_0 = 0$), then every sequence $\{t^k_{rq}\}$ made of (r,q) components of T^k converges to 0.

(3) *Let $\delta(T^k) = \delta(T^k)_1 \dotplus \dots \dotplus \delta(T^k)_s$ be a matrix made of T^k by replacing all (r,q) components described in (2) with 0, where $\delta(T^k)_j$ is of rank i_j. Then for all numbers j with $a_j \ne 0$, $\{{}^t\delta(T^k)_j\delta(T^k)_j\}$ converges to E_{i_j}.*

Proof. The assertion (1) follows from the fact that the set of eigen values of a matrix is continuous with respect to components of matrices ([W, Appendix V, Section 4]). For any positive real number ε, there is a number k_0 such that if $k > k_0$, then we have

$$\|{}^tT^k\Delta(\mathbf{d}^k)T^k - \Delta(\mathbf{d})\| < \varepsilon. \tag{5.2}$$

We write $T^k = (t^k_{iq}) = (\overline{\mathbf{t}}^k_1,\dots,\overline{\mathbf{t}}^k_\ell)$. Let Υ_{qm} be the (q,m) component of ${}^tT^k\Delta(\mathbf{d}^k)T^k - \Delta(\mathbf{d})$. Then we have

$$\Upsilon_{qm} = ({}^t(d^k_1t^k_{1q},\dots,d^k_\ell t^k_{\ell q}), \overline{\mathbf{t}}^k_m) - d_q\delta_{qm} = ({}^t(d^k_1t^k_{1q},\dots,d^k_\ell t^k_{\ell q}) - d_q\overline{\mathbf{t}}^k_q, \overline{\mathbf{t}}^k_m).$$

By (5.2), we have $\sum_{m=1}^\ell \Upsilon^2_{qm} < \varepsilon^2$. Since $\overline{\mathbf{t}}^k_1,\dots,\overline{\mathbf{t}}^k_\ell$ is an orthonormal basis, we have that

$$\sum_{q=1}^\ell \|{}^t(d^k_1t^k_{1q},\dots,d^k_\ell t^k_{\ell q}) - d_q\overline{\mathbf{t}}^k_q\|^2 < \varepsilon^2,$$

namely

$$\sum_{m=1}^{\ell}(d_m^k-d_q)^2(t_{mq}^k)^2<\varepsilon^2.$$

Setting $V=\min\{|a_m-a_q||m\neq q\}$, and replacing k_0 by a larger one, we may suppose that $d_m^k-d_q\geq V/2$. Then we deduce

$$(t_{1q}^k)^2+\cdots+(t_{(i_1+\cdots+i_{j-1})q}^k)^2+(t_{(i_1+\cdots+i_j+1)q}^k)^2+\cdots+(t_{\ell q}^k)^2<\frac{4\varepsilon^2}{V^2}.$$

If r and q are such numbers given in (2), then the sequence $\{t_{rq}^k\}$ converges to 0. This is what we want to prove. □

Lemma 5.4. *Let n,p be integers with $n\geq p\geq 2$. Let $S,S'\in O(p)$ and $M,M'\in O(n)$ and let $\mathbf{d}=(d_1,\ldots,d_p)$ be non-negative and decreasing diagonal components with $d_{p-1}>0$ such that $\Delta(\mathbf{d})$ is written as $a_1E_{i_1}\dotplus a_2E_{i_2}\dotplus\cdots\dotplus a_sE_{i_s}$, where $a_1,\ldots,a_s$ are all distinct and $p=i_1+\cdots+i_s$. Assume that $S\Delta(\mathbf{d})M\binom{1}{p}=S'\Delta(\mathbf{d})M'\binom{1}{p}$. Then we have the following.*

(1) *If $d_p>0$, then there exist matrices $G_j\in O(i_j)$ $(1\leq j\leq s)$ such that $S'=S({}^tG_1\dotplus\cdots\dotplus{}^tG_{s-1}\dotplus{}^tG_s)$ and $M'\binom{1}{p}=(G_1\dotplus\cdots\dotplus G_{s-1}\dotplus G_s)M\binom{1}{p}$.*

(2) *If $d_p=0$ and $i_s=1$, then there exist matrices $G_j\in O(i_j)$ $(1\leq j\leq s)$ such that $S'=S({}^tG_1\dotplus\cdots\dotplus{}^tG_{s-1}\dotplus{}^tG_s)$ and $M'\binom{1}{p-1}=(G_1\dotplus\cdots\dotplus G_{s-1})M\binom{1}{p-1}$.*

Proof. We prove the case $d_p=0$ and leave the proof for the case $d_p>0$ to the reader, since it is similar and easier. So let $d_{p-1}>0$ and $d_p=0$.

By the assumption of $S\Delta(\mathbf{d})M\binom{1}{p}=S'\Delta(\mathbf{d})M'\binom{1}{p}$, we have

$${}^tSS'\Delta(d_1,\ldots,d_p)M'\tbinom{1}{p}{}^tM=(\Delta(d_1,\ldots,d_p),\mathbf{0}_{p\times(n-p)}).$$

Writing both terms A and calculating A^tA we deduce

$${}^tSS'\Delta(d_1^2,\ldots,d_p^2){}^tS'S=\Delta(d_1^2,\ldots,d_p^2).$$

Since $\Delta(\mathbf{d})$ is written as $a_1E_{i_1}\dotplus a_2E_{i_2}\dotplus\cdots\dotplus a_sE_{i_s}$, it follows that there exists a decomposition of ${}^tS'S$ into $G_1\dotplus\cdots\dotplus G_{s-1}\dotplus G_s$ with the properties described in Lemma 5.2 (2), where G_j is of rank i_j $(1\leq j\leq s)$. Hence, we have $S'=S({}^tG_1\dotplus\cdots\dotplus{}^tG_{s-1}\dotplus{}^tG_s)$.

Furthermore, we obtain that

$$\begin{aligned}
&{}^tSS'\Delta(d_1,\dots,d_p)M'\binom{1}{p}{}^tM \\
&\quad= ({}^tG_1 \dotplus \cdots \dotplus {}^t G_{s-1} \dotplus {}^t G_s)\Delta(d_1,\dots,d_p)M'\binom{1}{p}{}^tM \\
&\quad= \Delta(d_1,\dots,d_p)({}^tG_1 \dotplus \cdots \dotplus {}^t G_{s-1} \dotplus {}^t G_s)M'\binom{1}{p}{}^tM \\
&\quad= (\Delta(d_1,\dots,d_p), \mathbf{0}_{p\times(n-p)}).
\end{aligned}$$

This induces

$$({}^tG_1 \dotplus \cdots \dotplus {}^tG_{s-1})M'\binom{1}{p-1}{}^tM = (E_{p-1}, \mathbf{0}_{(p-1)\times(n-p+1)}).$$

Hence, we have $({}^tG_1 \dotplus \cdots \dotplus {}^tG_{s-1})M'\binom{1}{p-1} = (E_{p-1}, \mathbf{0}_{(p-1)\times(n-p+1)})M = M\binom{1}{p-1}$. ☐

Lemma 5.5. *Let $n \geq p \geq 2$ and let c, d be non-negative integers with $n-p+1 = c+d$. Let $(\mathbf{v},\mathbf{w}) = (v_1,\dots,v_c,w_1,\dots,w_d)$ be diagonal components with $v_1 \geq \cdots \geq v_c > 0 > w_1 \geq \cdots \geq w_d$ and let M, M' be elements of $O(n)$.*

(1) *If ${}^tM\binom{p}{n}\Delta(\mathbf{v},\mathbf{w})M\binom{p}{n} = {}^tM'\binom{p}{n}\Delta(\mathbf{v},\mathbf{w})M'\binom{p}{n}$, then there exist matrices $T_1 \in O(c)$, $T_2 \in O(d)$ such that*

$$M'\binom{p}{n} = (T_1 \dotplus T_2)M\binom{p}{n}.$$

(2) *If $c = d$ and ${}^tM\binom{p}{n}\Delta(\mathbf{v},\mathbf{w})M\binom{p}{n} = {}^tM'\binom{p}{n}\Delta(\mathbf{w},\mathbf{v})M'\binom{p}{n}$, then there exist matrices T_1, $T_2 \in O(c)$ such that*

$$M'\binom{p}{n} = \begin{pmatrix} \mathbf{0} & E_c \\ E_c & \mathbf{0} \end{pmatrix}(T_1 \dotplus T_2)M\binom{p}{n}.$$

Proof.

(1) Since $M\binom{p}{n}{}^tM\binom{p}{n} = M'\binom{p}{n}{}^tM'\binom{p}{n} = E_{n-p+1}$, we have

$$M'\binom{p}{n}{}^tM\binom{p}{n}\Delta(\mathbf{v},\mathbf{w})M\binom{p}{n}{}^tM'\binom{p}{n} = \Delta(\mathbf{v},\mathbf{w}).$$

Since

$$\begin{aligned}
M'\binom{1}{p-1}{}^tM'\binom{p}{n}\Delta(\mathbf{v},\mathbf{w})M'\binom{p}{n} &= \mathbf{0}_{(p-1)\times(n-p+1)} \\
&= M'\binom{1}{p-1}{}^tM\binom{p}{n}\Delta(\mathbf{v},\mathbf{w})M\binom{p}{n},
\end{aligned}$$

we have $M'\binom{1}{p-1}{}^tM\binom{p}{n} = \mathbf{0}_{(p-1)\times(n-p+1)}$. Furthermore, we have ${}^tM'M' = E_n = {}^tM'\binom{1}{p-1}M'\binom{1}{p-1} + {}^tM'\binom{p}{n}M'\binom{p}{n}$. We show $M\binom{p}{n}{}^tM'\binom{p}{n} \in O(n-$

$p+1)$. Indeed, we have

$$\begin{aligned}
&{}^t\big(M'\tbinom{p}{n}{}^tM\tbinom{p}{n}\big)M'\tbinom{p}{n}{}^tM\tbinom{p}{n}\\
&\qquad = M\tbinom{p}{n}{}^tM'\tbinom{p}{n}M'\tbinom{p}{n}{}^tM\tbinom{p}{n}\\
&\qquad = M\tbinom{p}{n}(E_n - {}^tM'\tbinom{1}{p-1}M'\tbinom{1}{p-1}){}^tM\tbinom{p}{n}\\
&\qquad = M\tbinom{p}{n}E_n^tM\tbinom{p}{n} - M\tbinom{p}{n}{}^tM'\tbinom{1}{p-1}M'\tbinom{1}{p-1}{}^tM\tbinom{p}{n}\\
&\qquad = E_{n-p+1}.
\end{aligned}$$

Hence, it follows from Lemma 5.2 that there exist matrices $T_1 \in O(c)$, $T_2 \in O(d)$ such that $M'\tbinom{p}{n}{}^tM\tbinom{p}{n} = T_1 \dotplus T_2$. Thus we have $M'\tbinom{p}{n} = (T_1 \dotplus T_2)M\tbinom{p}{n}$.

(2) The assertion follows from (1) and the fact that

$$\begin{pmatrix} \mathbf{0} & E_c \\ E_c & \mathbf{0} \end{pmatrix} \Delta(\mathbf{v},\mathbf{w}) \begin{pmatrix} \mathbf{0} & E_c \\ E_c & \mathbf{0} \end{pmatrix} = \Delta(\mathbf{w},\mathbf{v}).$$

□

Lemma 5.6. *Let* $\mathbf{d}$ *be non-negative and decreasing diagonal components given in Proposition* 5.4. *For two sequences* $\{S^k\}$ *in* $O(p)$, $\{T^k\}$ *in* $O(n)$ *and a sequence of non-negative and decreasing diagonal components* $\{\mathbf{d}^k\}$ *of degree* p, *assume that the sequence* $\{S^k\Delta(\mathbf{d}^k)M^k\tbinom{1}{p}\}$ *converges to* $(\Delta(\mathbf{d}), \mathbf{0}_{p\times(n-p)})$. *Then we have the following.*

(1) $\{\mathbf{d}^k\}$ *converges to* $\mathbf{d}$.

(2) *If a pair* (r,q) *of numbers does not satisfy the inequality*

$$i_1+\cdots+i_{j-1} < r,q \le i_1+\cdots+i_j$$

for every integer j *with* $1\le j\le s$, *then every sequence* $\{s^k_{rq}\}$ *made of* (r,q) *components of* S^k *converges to* 0.

(3) *Let* $\delta(S^k) = \delta(S^k)_1 \dotplus \cdots \dotplus \delta(S^k)_s$ *be a matrix made of* S^k *by replacing every* (r,q) *components described in* (2), *in turn with* 0, *where* $\delta(S^k)_j$ *is of rank* i_j. *Then*

(3-i) *if* $a_j \ne 0$ *for every number* j, *then* $\{\delta(S^k)M^k\tbinom{1}{p}\}$ *converges to* $(E_p, \mathbf{0}_{p\times(n-p)})$,

(3-ii) *if* $a_s = 0$, *then* $\{\delta(S^k)_1 \dotplus \cdots \dotplus \delta(S^k)_{s-1}M^k\tbinom{1}{p-1}\}$ *converges to* $(E_{p-1}, \mathbf{0}_{(p-1)\times(n-p+1)})$.

Proof. Setting $A^k = S^k\Delta(\mathbf{d}^k)M^k\binom{1}{p}$, we have $A^k({}^tA^k) = S^k\Delta((d_1^k)^2, \dots, (d_p^k)^2)^tS^k$ and

$$\lim_{k\to\infty} S^k\Delta((d_1^k)^2, \dots, (d_p^k)^2)^tS^k = \lim_{k\to\infty} A^k({}^tA^k) = A({}^tA) = \Delta(d_1^2, \dots, d_p^2).$$

The assertion (1) follows from Lemma 5.3 (1). By Lemma 5.3 (2) and (3) there exist matrices $\delta(S^k) = \delta(S^k)_1 \dotplus \cdots \dotplus \delta(S^k)_s$ with the property

$$\lim_{k\to\infty} S^k - \delta(S^k) = \mathbf{0}_{p\times p}.$$

Then we have

$$\begin{aligned}\lim_{k\to\infty} S^k\Delta(\mathbf{d}^k)M^k\tbinom{1}{p} &= \lim_{k\to\infty} S^k(\Delta(\mathbf{d}) - \Delta(\mathbf{d}-\mathbf{d}^k))M^k\tbinom{1}{p} \\ &= \lim_{k\to\infty} S^k\Delta(\mathbf{d})M^k\tbinom{1}{p} \\ &= \lim_{k\to\infty} \delta(S^k)\Delta(\mathbf{d})M^k\tbinom{1}{p} \\ &= \lim_{k\to\infty} \Delta(\mathbf{d})\delta(S^k)M^k\tbinom{1}{p} \\ &= \Delta(\mathbf{d})(\lim_{k\to\infty} \delta(S^k)M^k\tbinom{1}{p}) \\ &= (\Delta(\mathbf{d}), \mathbf{0}_{p\times(n-p)}).\end{aligned}$$

Hence, we have (3-i) and (3-ii). □

§6. Homotopy Type of $\Omega^{n-p+1}(n,p)$

For an integer $p \geq 2$, let $\Delta^p(\Omega)$ be the subspace in $\mathbf{R}^p$ consisting of all points $(d_1, \dots, d_p)$ such that $d_1 \geq \cdots \geq d_{p-1} > 0$ and $d_p \geq 0$ and let $\Delta^p(1)$ be the subspace consisting of all points $(1, d_2, \dots, d_p) \in \Delta^p(\Omega)$. Let I^p_Δ be the subspace in $\Delta^p(1)$ consisting of all points $(1, \dots, 1, b)$ with $0 \leq b \leq 1$ and let Δ^p_Σ be the subspace consisting of all points $(1, d_2, \dots, d_{p-2}, 0, 0)$ with $1 \geq d_2 \geq \cdots \geq d_{p-2} \geq 0$. It is clear that $\Delta^p(1)$ is a deformation retract of $\Delta^p(\Omega)$ by a deformation retraction $(d_1, \dots, d_p) \longmapsto ((1-\lambda) + \lambda d_1)^{-1}(d_1, \dots, d_p)$ with $0 \leq \lambda \leq 1$. We show that $\Delta^p(1)$ is homeomorphic to $(I^p_\Delta * \Delta^p_\Sigma) \setminus \Delta^p_\Sigma$, where $*$ refers to the join. Indeed, suppose that an element $(1, d_2, \dots, d_p) \in \Delta^p(1)$ is expressed by

$$(1, d_2, \dots, d_p) = s(1, \dots, 1, b) + (1-s)(1, f_2, \dots, f_{p-2}, 0, 0).$$

Then we have $d_{p-1} = s$, $d_p = sb$ and $d_i = s + (1-s)f_i$ $(2 \leq i \leq p-2)$. Hence, if $s < 1$, then we have $s = d_{p-1}$, $b = d_p/d_{p-1}$ and $f_i = (d_i - d_{p-1})/(1 - d_{p-1})$ $(2 \leq i \leq p-2)$ and vice versa.

Let α be an element of $\Omega^{n-p+1}(n,p)$ with diagonalization $S\Delta(\mathbf{d})M\binom{1}{p}$, where $S \in O(p)$, $M \in O(n)$ and $\mathbf{d} = (d_1, \ldots, d_p)$ is a decreasing diagonal components with $d_{p-1} > 0$ and $d_p \geq 0$. Let Ω_Δ (resp. Σ_Δ) denote the subset consisting of all elements α with diagonalization $S\Delta(\mathbf{d})M\binom{1}{p}$ such that $\mathbf{d} \in \Delta^p(1)$ (resp. $\mathbf{d} \in \Delta^p(1)$ with $d_p = 0$). We define a homotopy $R'_\lambda : \Omega^{n-p+1}(n,p) \to \Omega^{n-p+1}(n,p)$ by

$$R'_\lambda(S\Delta(\mathbf{d})M\tbinom{1}{p}) = ((1-\lambda) + \lambda d_1)^{-1} S\Delta(\mathbf{d})M\tbinom{1}{p}. \tag{6.1}$$

The following lemma is obvious.

Lemma 6.1. *The homotopy R'_λ is a deformation retraction of $\Omega^{n-p+1}(n,p)$ to Ω_Δ such that*

(1) *R'_λ preserves $\Sigma^{n-p}(n,p)$ and $\Sigma^{n-p+1}(n,p)$ respectively,*

(2) *$R'_\lambda | \Sigma^{n-p+1}(n,p)$ induces a deformation retraction of $\Sigma^{n-p+1}(n,p)$ to Σ_Δ.*

Let $K'(n,p,b)$ for $0 < b < 1$, $\Sigma K'(n,p)$ and $R'(n,p)$ denote the subsets consisting of all elements α with diagonalization $S\Delta(\mathbf{d}_b)M\binom{1}{p}$ such that $\mathbf{d}_b \in I^p_\Delta$ with $0 < b < 1$, $\mathbf{d}_0 \in I^p_\Delta$ and $\mathbf{d}_1$ respectively. Let $K'(n,p)$ denote the union

$$\Sigma K'(n,p) \bigcup (\cup_{b\in(0,1)} K'(n,p,b)) \bigcup R'(n,p).$$

By definition, we have that $K'(n,p,b)$, $\Sigma K'(n,p)$ and $R'(n,p)$ coincide with $i_{n,p}(K(n,p,b))$, $i_{n,p}(\Sigma K(n,p))$ and $i_{n,p}(V^{row}_{n,p})$ respectively.

We prove that $i_{n,p}$ induces a homeomorphism of $K(n,p)$ onto $K'(n,p)$. Let $D : \Omega_\Delta \to K(n,p)$ be the map defined as follows. For an element $\alpha = S\Delta(\mathbf{d})M\binom{1}{p} \in \Omega_\Delta$, let $b(\alpha)$ denote the real number d_p/d_{p-1}. Then we set

$$D(\alpha) = [S, M\tbinom{1}{p}, b(\alpha)] \in K(n,p). \tag{6.2}$$

We show that D is well defined. Suppose that $\Delta(\mathbf{d})$ is written as $a_1E_{i_1} \dotplus a_2E_{i_2} \dotplus \cdots \dotplus a_sE_{i_s}$, where $a_1, \ldots, a_s$ are all distinct. Take another diagonalization $S'\Delta(\mathbf{d})M'\binom{1}{p}$ of α. If $d_p > 0$, then there exist matrices $G_j \in O(i_j)$ $(1 \leq j \leq s)$ such that $S' = S({}^tG_1 \dotplus \cdots \dotplus {}^tG_{s-1} \dotplus {}^tG_s)$ and $M'\binom{1}{p} = (G_1 \dotplus \cdots \dotplus G_{s-1} \dotplus G_s)$ $M\binom{1}{p}$ by Lemma 5.4. If $d_{p-1} = d_p > 0$, then $b(\alpha) = 1$ and $SM\binom{1}{p} = S'M'\binom{1}{p} \in (E_p \times O(n-p))\backslash O(n)$. If $d_{p-1} > d_p > 0$, then $i_s = 1$ and so $G_s \in O(1)$. Hence, we have $[S, M\binom{1}{p}, b(\alpha)] = [S', M'\binom{1}{p}, b(\alpha)]$ in $K(n,p)$ by Remark 2.1. If $d_p = 0$, then by Lemma 5.4 there exist matrices $G_j \in O(i_j)$ with $i_s = 1$ such that $S'\binom{1}{p} = S\binom{1}{p}({}^tG_1 \dotplus \cdots \dotplus {}^tG_s)$ and $M'\binom{1}{p-1} = (G_1 \dotplus \cdots \dotplus G_{s-1})M\binom{1}{p-1}$. This implies that $[S, M\binom{1}{p-1}] = [S', M'\binom{1}{p-1}]$ in $\Sigma K(n,p)$ by Remark 2.1. Thus D

is well defined. The fact that D is continuous will be proved in Proposition 6.3 below.

Now we have the following lemma.

Lemma 6.2.

(1) *The map* $i_{n,p} \circ D : \Omega_\Delta \to K'(n,p)$ *is a retraction which maps* Σ_Δ *and* $\Omega_\Delta \setminus \Sigma_\Delta$ *onto* $\Sigma K'(n,p)$ *and* $K'(n,p) \setminus \Sigma K'(n,p)$ *respectively.*

(2) *The maps* $i_{n,p} : K(n,p) \to K'(n,p)$ *and* $i_{n,p}|\Sigma K(n,p) : \Sigma K(n,p) \to \Sigma K'(n,p)$ *are homeomorphisms.*

Proof. Since $K(n,p)$ is a compact space, it is enough to prove that $D \circ i_{n,p} = id_{K(n,p)}$ and $i_{n,p} \circ D|K'(n,p) = id_{K'(n,p)}$ and that the map $i_{n,p} \circ D$ preserves Σ_Δ and $\Omega_\Delta \setminus \Sigma_\Delta$.

Let $[S, M(^1_p), b]$ be an element of $K(n,p)$. Then we have

$$D \circ i_{n,p}([S, M(^1_p), b]) = D(S\Delta(1, \dots, 1, b)M(^1_p)) = [S, M(^1_p), b].$$

On the other hand, let $\alpha = S\Delta(1, \dots, 1, b)M(^1_p) \in K'(n,p)$. Then we have

$$i_{n,p} \circ D(\alpha) = i_{n,p}([S, M(^1_p), b]) = S\Delta(1, \dots, 1, b)M(^1_p) = \alpha.$$

If $\alpha = S\Delta(\mathbf{d})M(^1_p) \in \Sigma_\Delta$, namely $d_p = 0$, then $b(\alpha) = 0$ and $i_{n,p} \circ D(\alpha) \in \Sigma K'(n,p)$ and vice versa. This proves the lemma. □

Let $r_\lambda : \Delta^p(1) \to \Delta^p(1)$ be the deformation retraction of $\Delta^p(1)$ to I^p_Δ defined by

$$r_\lambda(1, d_2, \dots, d_p) = (1 - \lambda)(1, d_2, \dots, d_p) + \lambda(1, \dots, 1, d_p/d_{p-1}).$$

We should note that if $d_i = d_j$, then we have that $r_\lambda(d_i) = r_\lambda(d_j)$ for $0 \le \lambda \le 1$. For an element $\alpha = S\Delta(\mathbf{d})M(^1_p) \in \Omega_\Delta$, we define $D_\lambda(\alpha)$ by

$$D_\lambda(\alpha) = (1 - \lambda)\alpha + \lambda i_{n,p} \circ D(\alpha) = S\Delta(r_\lambda(\mathbf{d}))M(^1_p). \tag{6.3}$$

Then we have the following proposition.

Proposition 6.3. *The homotopy* $D_\lambda : \Omega_\Delta \to \Omega_\Delta$ *is a deformation retraction of* Ω_Δ *to* $K'(n,p)$ *such that* D_λ *preserves* Σ_Δ *and* $\Omega_\Delta \setminus \Sigma_\Delta$ *respectively. In particular,* $D_\lambda|\Sigma_\Delta$ *induces a deformation retraction of* Σ_Δ *to* $\Sigma K'(n,p)$.

Proof. We first show that $D(\alpha)$ is continuous. Take a sequence $\{\alpha_k\}$ converging to $\alpha \in \Omega_\Delta$. We consider the sequence $\{{}^tS\alpha_k({}^tM)\}$ in place of α_k. By (6.2), it is clear that $D({}^tS\alpha({}^tM)) = {}^tSD(\alpha)({}^tM)$. Furthermore, $\lim_{k\to\infty} D(\alpha_k) = D(\alpha)$ holds if and only if $\lim_{k\to\infty} D({}^tS\alpha_k({}^tM)) = D({}^tS\alpha \times({}^tM))$ holds. Therefore, it is enough for the continuity to prove the last equality. For this, let $\alpha_k = S^k\Delta(\mathbf{d}^k)M^k\binom{1}{p}$ be diagonalizations. We note ${}^tS\alpha({}^tM) = (\Delta(\mathbf{d}), \mathbf{0}_{p\times(n-p)})$. If $d_p = 0$, then we have $\lim_{k\to\infty} d_p^k = 0$ by Lemma 5.6.

Considering the expressions ${}^tSS^k\Delta(\mathbf{d}^k)(M^k\binom{1}{p}{}^tM)$, we have

$$\lim_{k\to\infty} {}^tSS^k\Delta(\mathbf{d}^k)(M^k\tbinom{1}{p}{}^tM) = (\Delta(\mathbf{d}), \mathbf{0}_{p\times(n-p)}).$$

By Lemma 5.6, we have $\delta({}^tSS^k) = \delta({}^tSS^k)_1 \dotplus \cdots \dotplus \delta({}^tSS^k)_s$ such that

(1) if $d_p \neq 0$, then $\lim_{k\to\infty} \delta({}^tSS^k)M^k\binom{1}{p}{}^tM = \lim_{k\to\infty}(E_p, \mathbf{0}_{p\times(n-p)})$,

(2) if $d_p = 0$, then $\lim_{k\to\infty}(\delta({}^tSS^k)_1 \dotplus \cdots \dotplus \delta({}^tSS^k)_{s-1})M^k\binom{1}{p-1}{}^tM = (E_{p-1}, \mathbf{0}_{(p-1)\times(n-p+1)})$.

Since $i_{n,p}$ is continuous bijection, we have

$$\begin{aligned} i_{n,p}(\lim_{k\to\infty} D({}^tS\alpha_k({}^tM))) &= \lim_{k\to\infty} i_{n,p} \circ D({}^tS\alpha_k({}^tM)) \\ &= \lim_{k\to\infty} {}^tSS^k\Delta(r_1(\mathbf{d}^k)){}^tM\tbinom{1}{p}{}^tM \\ &= \lim_{k\to\infty} (\delta({}^tSS^k)\Delta(r_1(\mathbf{d}^k)){}^tM\tbinom{1}{p}{}^tM \\ &= \lim_{k\to\infty} (\Delta(r_1(\mathbf{d}^k))\delta({}^tSS^k){}^tM\tbinom{1}{p}{}^tM \\ &= (\Delta(r_1(\mathbf{d}))(E_p, \mathbf{0}_{p\times(n-p)}) \\ &= (\Delta(r_1(\mathbf{d})), \mathbf{0}_{p\times(n-p)}) \\ &= i_{n,p} \circ D({}^tS\alpha({}^tM)). \end{aligned}$$

Hence, D is continuous. This yields by (6.3) that $D_\lambda(\alpha)$ is continuous with respec to α and λ.

We next prove that $D_\lambda : \Omega_\Delta \to \Omega_\Delta$ is a deformation retraction of Ω_Δ to $K'(n,p)$. Since D_1 coincides with $i_{n,p} \circ D$, the image of D_1 is $K'(n,p)$. We have by Lemma 6.2 (1) that $D_\lambda | K'(n,p) = id_{K'(n,p)}$ and that D_λ preserves Σ_Δ and $\Omega_\Delta \setminus \Sigma_\Delta$. Indeed, if $\alpha = S\Delta(\mathbf{d}_b)M\binom{1}{p} \in K'(n,p)$, then we have $D_\lambda(\alpha) = \alpha$, since $r_\lambda(\mathbf{d}_b) = \mathbf{d}_b$. Furthermore, $d_p = 0$ in the expression $\alpha = S\Delta(\mathbf{d})M\binom{1}{p}$ if and only if the p-th component of $r_\lambda(\mathbf{d})$ is also equal to 0. This completes the proof. □

Proof of Theorem 2.3. We define the homotopy $R_\lambda : \Omega^{n-p+1}(n,p) \to \Omega^{n-p+1}(n,p)$ by

$$R_\lambda = \begin{cases} R'_{2\lambda} & \text{for} \quad 0 \le \lambda \le 1/2, \\ D_{2\lambda-1} & \text{for} \quad 1/2 \le \lambda \le 1. \end{cases}$$

Then the assertion of Theorem 2.4 follows from Lemma 6.1 and Proposition 6.3. □

§7. Homotopy Type of $\Omega^{n-p+1,0}(n,p)$

For a subspace C in $\mathbf{R}^p$, let $pr(C)$ be the orthogonal projection of $\mathbf{R}^p$ onto C. Let V be a subspace of $\mathbf{R}^n$. Let C be of dimension 1 and $q : S^2V \to C$ be a quadratic form. Then we say that q is a quadratic form *with eigen values* $\pm a$ if every eigen value of q is equal to either a or $-a$.

We begin by studying the image $\mathcal{I}_{n,p}(\mathcal{K}(n,p,\sigma,b))$. The following observation of this image will be helpful in understanding the arguments in Sections 7 and 8. By definition, it is clear that $\mathcal{I}_{n,p}(V^{row}_{n,p}) = R'(n,p) \times \mathbf{0}^p_{n\times n}$, where $\mathbf{0}^p_{n\times n}$ refers to the null-homomorphism in $\mathrm{Hom}(S^2\mathbf{R}^n, \mathbf{R}^p)$, $\mathcal{I}_{n,p}(\mathcal{K}(n,p,\sigma,b)) \subset K'(n,p,b) \times \mathrm{Hom}(S^2\mathbf{R}^n, \mathbf{R}^p)$ and $\mathcal{I}_{n,p}(\Sigma\mathcal{K}(n,p,\sigma)) \subset \Sigma K'(n,p) \times \mathrm{Hom}(S^2\mathbf{R}^n, \mathbf{R}^p)$.

Let $0 \le b < 1$. For an element $\alpha \in K'(n,p,b)$ with diagonalization $\alpha = S\Delta(\mathbf{d}_b)M\binom{1}{p}$, we denote, by C_α, the subspace of dimension 1 in $\mathbf{R}^p$ generated by $\bar{\mathbf{s}}_p$ and by K_α, the subspace of dimension $n-p+1$ in $\mathbf{R}^n$ generated by ${}^t\mathbf{m}_p, \dots, {}^t\mathbf{m}_n$ respectively. Since $b < 1$, it follows from Lemma 5.4 that C_α and K_α are independently defined from the choice of a diagonalization. Let $K_\alpha^\perp$ and $C_\alpha^\perp$ be the orthogonal complements of K_α in $\mathbf{R}^n$ and of C_α in $\mathbf{R}^p$ respectively. If $0 < b < 1$, then we have that $\alpha^{-1}(C_\alpha) = K_\alpha$, and the orthogonal complement of $\mathrm{Ker}(\alpha)$ in K_α is generated by the vector ${}^t\mathbf{m}_p$, which is invariantly determined by α. If $b = 0$, then K_α coincides with $\mathrm{Ker}(\alpha)$ and C_α is identified with $\mathbf{R}^p/\mathrm{Im}(\alpha)$ through the canonical isomorphism $C_\alpha \subset \mathbf{R}^p \xrightarrow{projection} \mathbf{R}^p/\mathrm{Im}(\alpha)$.

Let (α,β) be an element of $K'(n,p,b) \times \mathrm{Hom}(S^2\mathbf{R}^n, \mathbf{R}^p)$. Let β_α be the quadratic form defined by $\beta_\alpha = pr(\mathrm{Im}(\alpha)^\perp) \circ (\beta | S^2K_\alpha)$ as in (1.1). We define the spaces $\mathcal{K}'(n,p,\sigma,b)$ for any b with $0 < b < 1$ and $\Sigma\mathcal{K}'(n,p,\sigma)$ for $b = 0$ to be the subspaces of $K'(n,p,b) \times \mathrm{Hom}(S^2\mathbf{R}^n, \mathbf{R}^p)$ and $\Sigma K'(n,p) \times \mathrm{Hom}(S^2\mathbf{R}^n, \mathbf{R}^p)$ consisting of all elements (α,β) such that

(C-1) $\beta | S^2(\mathbf{R}^n \bigcirc K_\alpha^\perp)$ and $pr(C_\alpha^\perp) \circ \beta$ vanish,

(C-2) β_α is a non-singular quadratic form with eigen values $\pm\sqrt{1-b^2}$, (C-3) β_α has the signature $\pm\sigma$,

respectively. For $b = 1$, we set $\mathcal{R}'(n,p) = R'(n,p) \times \mathbf{0}^p_{n\times n}$. We define $\mathcal{K}'(n,p,\sigma)$, $\mathcal{K}'(n,p)$ and $\Sigma\mathcal{K}'(n,p)$ to be the union

$$\mathcal{K}'(n,p,\sigma) = \Sigma\mathcal{K}'(n,p) \bigcup (\cup_{b\in(0,1)} \mathcal{K}'(n,p,\sigma,b)) \bigcup \mathcal{R}'(n,p),$$

$$\mathcal{K}'(n,p) = \bigcup_{d=0}^{[(n-p+1)/2]} \mathcal{K}'(n,p,n-p+1-2d),$$

$$\Sigma\mathcal{K}'(n,p) = \bigcup_{d=0}^{[(n-p+1)/2]} \Sigma\mathcal{K}'(n,p,n-p+1-2d),$$

respectively. We first prove that the map $\mathcal{I}_{n,p}$ induces a homeomorphism of $\mathcal{K}(n,p)$ onto $\mathcal{K}'(n,p)$.

Theorem 7.1. *Let σ be a signature as above. Then $\mathcal{I}_{n,p}|\mathcal{K}(n,p,\sigma,b)$ for $0 < b < 1$, $\mathcal{I}_{n,p}|\Sigma\mathcal{K}(n,p,\sigma)$ and $\mathcal{I}_{n,p}|V^{row}_{n,p}$ are topological embeddings of $\mathcal{K}(n,p,\sigma,b)$ onto $\mathcal{K}'(n,p,\sigma,b)$, of $\Sigma\mathcal{K}(n,p,\sigma)$ onto $\Sigma\mathcal{K}'(n,p,\sigma)$, and of $V^{row}_{n,p}$ onto $\mathcal{R}'(n,p)$ respectively.*

Proof. The assertion for $\mathcal{I}_{n,p}|V^{row}_{n,p}$ follows from the fact that the map $\mathcal{I}_{n,p}|V^{row}_{n,p}$ coincides with the composition of the map $i_{n,p}$ and the inclusion $\mathcal{R}'(n,p) \subset R'(n,p) \times \mathrm{Hom}(S^2\mathbf{R}^n, \mathbf{R}^p)$.

Let $0 < b < 1$. Let $[\mathbf{z}]$ be $[S,T,M,\sigma,b]$. By the definition (2.18) of $\alpha([\mathbf{z}])$, it is clear that $\alpha([\mathbf{z}]) = S\Delta(\mathbf{d}_b)M(^1_p) \in K'(n,p,b)$. By the definition (2.18) of $\beta([\mathbf{z}])$ it follows that $\beta([\mathbf{z}])|S^2(\mathbf{R}^n \bigcirc K^\perp_\alpha)$ vanishes, since $K^\perp_\alpha$ is generated by ${}^t\mathbf{m}_1, \ldots, {}^t\mathbf{m}_{p-1}$. Furthermore, $pr(C^\perp_\alpha) \circ \beta([\mathbf{z}])$ vanishes, since $\mathrm{Im}\beta([\mathbf{z}]) \subset C_\alpha$. If $\sigma > 0$, then the vectors ${}^t\mathbf{m}_p$ and $\overline{\mathbf{s}}_p$ are determined by Remark 2.4 Case (i) and $\beta([\mathbf{z}])_{\alpha([\mathbf{z}])}$ is a non-singular quadratic form with index d and eigen values $\sqrt{1-b^2}$ by (2.18). If $\sigma = 0$, then the pair of the vectors $({}^t\mathbf{m}_p, \overline{\mathbf{s}}_p)$ are determined up to sign by Remark 2.4 Case (iii) and $\beta([\mathbf{z}])_{\alpha([\mathbf{z}])}$ is a non-singular quadratic form with index $(n-p+1)/2$ and eigen values $\pm\sqrt{1-b^2}$. Hence, $\mathcal{I}_{n,p}([\mathbf{z}])$ lies in $\mathcal{K}'(n,p,\sigma,b)$. It is similar to prove that $\mathrm{Im}(\mathcal{I}_{n,p}|\Sigma\mathcal{K}(n,p,\sigma)) \subset \Sigma\mathcal{K}'(n,p,\sigma)$.

We show the surjectivity. Let (α,β) be an element of $\mathcal{K}'(n,p,\sigma,b)$ or $\Sigma\mathcal{K}'(n,p,\sigma)$. In a diagonalization $\alpha = S\Delta(\mathbf{d}_b)M(^1_p)$, we have seen that K_α and C_α have the orthonormal basis ${}^t\mathbf{m}_p, \ldots, {}^t\mathbf{m}_n$ and $\overline{\mathbf{s}}_p$ respectively. With these basis there is a $(n-p+1)\times(n-p+1)$ matrix $B = (b_{ij})$ $(p \le i,j \le n)$ defined by

$$\beta_\alpha({}^t\mathbf{m}_i, {}^t\mathbf{m}_j) = pr(C_\alpha) \circ \beta({}^t\mathbf{m}_i, {}^t\mathbf{m}_j) = b_{ij}\overline{\mathbf{s}}_p.$$

By the properties (C-1) to (C-3), B is symmetric and non-singular of signature $\pm(c-d)$ with eigen values $\pm\sqrt{1-b^2}$. Suppose that B has the signature $\delta(c-d)$ with $\delta=\pm 1$. Then there exists a matrix $T \in O(n-p+1)$ such that

$$TB^tT = \delta\sqrt{1-b^2}(E_c \dotplus (-E_d)) \tag{7.1}$$

with $c \geq d$. Hence, we have

$$\beta_\alpha({}^t\mathbf{m}_i, {}^t\mathbf{m}_j) = \sqrt{1-b^2}\{{}^t\mathbf{m}_i{}^tM\binom{p}{n}{}^tT(E_c \dotplus (-E_d))TM\binom{p}{n}\mathbf{m}_j\}(\delta\overline{\mathbf{s}}_p).$$

This induces

$$\beta_\alpha(\mathbf{x},\mathbf{y}) = \sqrt{1-b^2}\{{}^t\mathbf{x}{}^tM\binom{p}{n}{}^tT(E_c \dotplus (-E_d))TM\binom{p}{n}\mathbf{y}\}(\delta\overline{\mathbf{s}}_p).$$

Let $b>0$. If we set $S' = S(E_{p-1} \dotplus (\delta))$ and $M' = (E_{p-1} \dotplus (\delta) \dotplus E_{n-p})M$, then we have that $\beta_\alpha(\mathbf{x},\mathbf{y})$ coincides with

$$\beta([S',T',M',\sigma,b])(\mathbf{x},\mathbf{y}) = \sqrt{1-b^2}\{{}^t\mathbf{x}{}^tM'\binom{p}{n}{}^tT(E_c \dotplus (-E_d))TM'\binom{p}{n}\mathbf{y}\}\overline{\mathbf{s}}'_p.$$

Since $\alpha = S\Delta(\mathbf{d}_b)M\binom{1}{p} = S'\Delta(\mathbf{d}_b)M'\binom{1}{p}$ in $K(n,p,b)$, we have that $\alpha([S,T,M,\sigma,b]) = \alpha([S',T',M',\sigma,b])$. Thus we concludes $\mathcal{I}_{n,p}([S',T',M',\sigma, b]) = (\alpha,\beta)$.

Let $b=0$. If we set $S' = S(E_{p-1} \dotplus (\delta))$ and $M' = (E_{p-1} \dotplus T)M$, then we have that $\beta(\mathbf{x},\mathbf{y})$ coincides with

$$\beta([S',M',\sigma])(\mathbf{x},\mathbf{y}) = \{{}^t\mathbf{x}{}^tM'\binom{p}{n}(E_c \dotplus (-E_d))M'\binom{p}{n}\mathbf{y}\}\overline{\mathbf{s}}'_p.$$

Since $M\binom{1}{p-1} = M'\binom{1}{p-1}$ and $\alpha = S\Delta(\mathbf{d}_0)M\binom{1}{p} = S'\Delta(\mathbf{d}_0)M'\binom{1}{p}$ in $\Sigma K(n,p)$, we have that $\alpha([S,M,\sigma]) = \alpha([S',M',\sigma])$. Thus we concludes $\mathcal{I}_{n,p}([S',M',\sigma]) = (\alpha,\beta)$.

It remains to prove the injectivity. Let $[\mathbf{z}] = [S,T,M,\sigma,b]$, $[\mathbf{z}'] = [S',T', M',\sigma,b]$ in $\mathcal{K}'(n,p,\sigma,b)$, or $[\mathbf{z}] = [S,M,\sigma]$ and $[\mathbf{z}'] = [S',M',\sigma]$ in $\Sigma\mathcal{K}'(n,p,\sigma)$ respectively. Suppose that $\mathcal{I}_{n,p}([\mathbf{z}]) = \mathcal{I}_{n,p}([\mathbf{z}'])$. This implies that

$$\alpha = S\Delta(\mathbf{d}_b)M\binom{1}{p} = S'\Delta(\mathbf{d}_b)M'\binom{1}{p}, \tag{7.2}$$

and for $\mathbf{x},\mathbf{y} \in \mathbf{R}^n$,

$$\begin{aligned}&\sqrt{1-b^2}\{{}^t\mathbf{x}{}^tM\binom{p}{n}{}^tT(E_c \dotplus (-E_d))TM\binom{p}{n}\mathbf{y}\}\overline{\mathbf{s}}_p \\ &\quad = \sqrt{1-b^2}\{{}^t\mathbf{x}{}^tM'\binom{p}{n}{}^tT'(E_c \dotplus (-E_d))T'M'\binom{p}{n}\mathbf{y}\}\overline{\mathbf{s}}'_p,\end{aligned} \tag{7.3}$$

where if $b=0$, then $T = T' = E_{n-p+1}$. For $b<1$ we need deal with the following four cases.

Case (i): $\sigma > 0$ and $0 < b < 1$. By (7.2) and Lemma 5.4 there exist $G \in O(p-1)$ and $(\delta) \in O(1)$ such that $S' = S({}^tG \dotplus (\delta))$ and $M'\binom{1}{p} = (G \dotplus (\delta))M\binom{1}{p}$. In this case a unit basis of C_α is uniquely selected so that β_α has the index d, and hence we have $\overline{\mathbf{s}}_p = \overline{\mathbf{s}}'_p$, namely $\delta = 1$ by (7.3). Since $\alpha({}^t\mathbf{m}_p) = S(b\mathbf{e}_p) = b\overline{\mathbf{s}}_p$ and $\alpha({}^t\mathbf{m}'_p) = S'(b\mathbf{e}_p) = b\overline{\mathbf{s}}'_p$ and $b > 0$, we have $\mathbf{m}_p = \mathbf{m}'_p$. Furthermore, it follows from (7.3) and Lemma 5.5 that there exist matrices $T_1 \in O(c)$ and $T_2 \in O(d)$ such that $T'M'\binom{p}{n} = (T_1 \dotplus T_2)TM\binom{p}{n}$. This induces $M'\binom{p}{n} = {}^tT'(T_1 \dotplus T_2)TM\binom{p}{n}$. Setting $L' = {}^tT'(T_1 \dotplus T_2)T$, we have that $T' = (T_1 \dotplus T_2)T{}^tL'$ and $M'\binom{p}{n} = L'M\binom{p}{n}$. Since $\mathbf{m}_p = \mathbf{m}'_p$, we have $L' = ((1) \dotplus L)$ for some $L \in O(n-p)$. This implies

$$\begin{aligned}[S', T', M', \sigma, b] &= [S({}^tG \dotplus (1)), (T_1 \dotplus T_2)T((1) \dotplus {}^tL), (G \dotplus (1) \dotplus L)M, \sigma, b] \\ &= [S, T, M, \sigma, b]\end{aligned}$$

in $\mathcal{K}(n, p, \sigma, b)$ by Remark 2.4 Case (i).

Case (ii): $\sigma > 0$ and $b = 0$. By (7.2) and Lemma 5.4 there exist $G \in O(p-1)$ and $(\delta) \in O(1)$ such that $S' = S({}^tG \dotplus (\delta))$ and $M'\binom{1}{p-1} = GM\binom{1}{p-1}$. By (7.3) and Lemma 5.5 there exist matrices $T_1 \in O(c)$ and $T_2 \in O(d)$ such that $M'\binom{p}{n} = (T_1 \dotplus T_2)M\binom{p}{n}$. This implies

$$[S', M', \sigma] = [S({}^tG \dotplus (\delta)), (G \dotplus T_1 \dotplus T_2)M, \sigma] = [S, M, \sigma]$$

in $\Sigma\mathcal{K}(n, p, \sigma)$ by Remark 2.4 Case (ii).

Case (iii): $\sigma = 0$ and $0 < b < 1$. By (7.2) and Lemma 5.4, there exist $G \in O(p-1)$ and $(\delta) \in O(1)$ such that $S' = S({}^tG \dotplus (\delta))$ and $M'\binom{1}{p} = (G \dotplus (\delta))M\binom{1}{p}$. In this case we have $\overline{\mathbf{s}}_p = \delta\overline{\mathbf{s}}'_p$ and $\mathbf{m}_p = \delta\mathbf{m}'_p$. If $\delta = 1$, then, by (7.3) and Lemma 5.5, there exist matrices T_1, $T_2 \in O(c)$ such that $T'M'\binom{p}{n} = (T_1 \dotplus T_2)TM\binom{p}{n}$. This induces $M'\binom{p}{n} = {}^tT'(T_1 \dotplus T_2)TM\binom{p}{n}$. Setting $L' = {}^tT'(T_1 \dotplus T_2)T$, we have $T' = (T_1 \dotplus T_2)T{}^tL'$ and $M'\binom{p}{n} = L'M\binom{p}{n}$. Since $\mathbf{m}_p = \mathbf{m}'_p$, we have $L' = ((1) \dotplus L)$ for some $L \in O(n-p)$. This implies $[S', T', M', 0, b] = [S, T, M, 0, b]$ in $\mathcal{K}(n, p, 0, b)$ as in the Case (i). If $\delta = -1$, then we have $\overline{\mathbf{s}}_p = -\overline{\mathbf{s}}'_p$ and $\mathbf{m}_p = -\mathbf{m}'_p$. By (7.2) and Lemma 5.5 it follows that

$$\begin{aligned}&{}^tM'\tbinom{p}{n}{}^tT'(E_c \dotplus (-E_c))T'M'\tbinom{p}{n} \\ &= {}^tM\tbinom{p}{n}{}^tT((-E_c) \dotplus E_c)TM\tbinom{p}{n} \\ &= {}^tM\tbinom{p}{n}{}^tT\begin{pmatrix}0 & E_c \\ E_c & 0\end{pmatrix}(E_c \dotplus (-E_c))\begin{pmatrix}0 & E_c \\ E_c & 0\end{pmatrix}TM\tbinom{p}{n}.\end{aligned}$$

By Lemma 5.5 there exist matrices $T_1 \in O(c)$ and $T_2 \in O(c)$ such that

$$T'M'\binom{p}{n} = \begin{pmatrix} 0 & E_c \\ E_c & 0 \end{pmatrix} (T_1 \dotplus T_2) T M\binom{p}{n}.$$

Hence, we have

$$M'\binom{p}{n} = {}^tT' \begin{pmatrix} 0 & E_c \\ E_c & 0 \end{pmatrix} (T_1 \dotplus T_2) T M\binom{p}{n}.$$

Setting

$$L' = {}^tT' \begin{pmatrix} 0 & E_c \\ E_c & 0 \end{pmatrix} (T_1 \dotplus T_2) T,$$

we have

$$T' = \begin{pmatrix} 0 & E_c \\ E_c & 0 \end{pmatrix} (T_1 \dotplus T_2) T {}^tL'$$

and $M'\binom{p}{n} = L'M\binom{p}{n}$. Since $\mathbf{m}_p = -\mathbf{m}'_p$, we have $L' = ((-1) \dotplus L)$ for some $L \in O(n-p)$. This implies

$$\begin{aligned}
&[S', T', M', 0, b] \\
&= \left[S({}^tG \dotplus (-1)), \begin{pmatrix} 0 & E_c \\ E_c & 0 \end{pmatrix} (T_1 \dotplus T_2) T((-1) \dotplus {}^tL), (G \dotplus (-1) \dotplus L)M, 0, b \right] \\
&= ((-1), L) \cdot [S({}^tG \dotplus (1)), (T_1 \dotplus T_2)T, (G \dotplus E_{n-p+1})M, 0, b] \\
&= ((-1), L) \cdot [S, T, M, 0, b] \\
&= [S, T, M, 0, b]
\end{aligned}$$

in $\mathcal{K}(n, p, 0, b)$ by Remark 2.4 Case (iii).

Case (iv): $\sigma = 0$ and $b = 0$. By (7.2) and Lemma 5.4 there exist $G \in O(p-1)$ and $(\delta) \in O(1)$ such that $S' = S({}^tG \dotplus (\delta))$ and $M'\binom{1}{p-1} = GM\binom{1}{p-1}$. Since $b = 0$, we have $\mathrm{Ker}(\alpha) = \{{}^t\mathbf{m}_p, \dots, {}^t\mathbf{m}_n\} = \{{}^t\mathbf{m}'_p, \dots, {}^t\mathbf{m}'_n\}$. If $\delta = 1$, then $\bar{\mathbf{s}}_p = \bar{\mathbf{s}}'_p$. By (7.3) and Lemma 5.5 we have matrices $T_1,\ T_2 \in O(c)$ such that $M'\binom{p}{n} = (T_1 \dotplus T_2)M\binom{p}{n}$. This gives

$$[S', M', 0] = [S({}^tG \dotplus (1)), (G \dotplus T_1 \dotplus T_2)M, 0] = [S, M, 0]$$

in $\Sigma\mathcal{K}(n, p, 0)$ by Remark 2.4 Case (iv). If $\delta = -1$, then $\bar{\mathbf{s}}_p = -\bar{\mathbf{s}}'_p$. By using Lemma 5.5 similarly as in the Case (iii), we can show that there exist matrices

T_1, $T_2 \in O(c)$ such that

$$M'(\tbinom{p}{n}) = \begin{pmatrix} 0 & E_c \\ E_c & 0 \end{pmatrix} (T_1 \dotplus T_2) M(\tbinom{p}{n}).$$

Hence, we have

$$\begin{aligned} [S', M', 0] &= \left[S({}^tG \dotplus (-1)), (G \dotplus \begin{pmatrix} 0 & E_c \\ E_c & 0 \end{pmatrix} (T_1 \dotplus T_2)) M, 0 \right] \\ &= (-1) \cdot [S({}^tG \dotplus (1)), (G \dotplus T_1 \dotplus T_2) M, 0] \\ &= (-1) \cdot [S, M, 0] \\ &= [S, M, 0] \end{aligned}$$

in $\Sigma\mathcal{K}(n, p, 0)$ by Remark 2.4 Case (iv).

This completes the proof. □

§8. Deformation Retraction of $\Omega^{n-p+1,0}(n,p)$ to $\mathcal{K}'(n,p)$

In this section we complete the proof of Theorem 2.6. Let $C = (c_{ij})$ $(1 \le i, j \le n)$ be an $n \times n$ matrix. The norm $\|C\|$ is defined to be $(\sum_{i=1}^n \sum_{j=1}^n c_{ij}^2)^{1/2}$. If $L, U \in O(n)$, then we have $\|LCU\| = \|C\|$. We canonically identify an element $\beta \in \mathrm{Hom}(S^2\mathbf{R}^n, \mathbf{R}^p)$ with the p-tuple $(C_1, \ldots, C_p)$ of symmetric $n \times n$ matrices. Then the norm $\|\beta\|$ is defined to be $(\sum_{i=1}^p \|C_i\|^2)^{1/2}$. In particular, we have

$$\begin{aligned} \|\beta([S, T, M, \sigma, b])\| &= \sqrt{1 - b^2} \|{}^tM(\tbinom{p}{n}) {}^tT(E_c \dotplus (-E_d)) T M(\tbinom{p}{n})\| \\ &= \sqrt{1 - b^2} \|{}^tM(\mathbf{0}_{(p-1)\times(p-1)} \dotplus {}^tT(E_c \dotplus (-E_d))T)M\| \\ &= \sqrt{1 - b^2} \|(\mathbf{0}_{(p-1)\times(p-1)} \dotplus {}^tT(E_c \dotplus (-E_d))T)\| \\ &= \sqrt{1 - b^2} \|(E_c \dotplus (-E_d))\| \\ &= \sqrt{(1 - b^2)(n - p + 1)}. \end{aligned}$$

If an element $\alpha \in K'(n,p)$ is written as $S\Delta(\mathbf{d}_b)M(\tbinom{1}{p})$, then we define the continuous functions $b(\alpha)$ and $\|x(\alpha)\|$ to be b and $\sqrt{(1 - b(\alpha)^2)(n - p + 1)}$ respectively.

Proof of Theorem 2.6. Using the deformation retraction R_λ of $\Omega^{n-p+1} \times (n,p)$ to $K'(n,p)$ in Theorem 2.3, we first define a deformation retraction H_λ of $\Omega^{n-p+1,0}(n,p)$ to $(\pi_1^2|\Omega^{n-p+1,0}(n,p))^{-1}(K'(n,p))$ by $H_\lambda(\alpha, \beta) = (R_\lambda(\alpha), \beta)$

for $0 \leq \lambda \leq 1$. Actually, $H_\lambda(\alpha,\beta)$ lies in $\Omega^{n-p+1,0}(n,p)$. For, if $\alpha \in \Sigma^{n-p}(n,p)$, then $b(\alpha) > 0$, namely $R_\lambda(\alpha) \in \Sigma^{n-p}(n,p)$ by Theorem 2.3. If $(\alpha,\beta) \in \Sigma^{n-p+1,0}(n,p)$, namely $b(\alpha)=0$, then $\mathrm{Ker}(R_\lambda(\alpha)) = \mathrm{Ker}(\alpha)$ and $\mathrm{Cok}(R_\lambda(\alpha)) = \mathrm{Cok}(\alpha)$ for any λ by (6.1) and (6.3), and hence $\beta_{R_\lambda(\alpha)}$ coincides with β_α for any λ by (1.1). This implies $H_\lambda(\alpha,\beta) \in \Sigma^{n-p+1,0}(n,p)$. If $\alpha \in K'(n,p)$, then $H_\lambda(\alpha,\beta) = (\alpha,\beta)$ for $0 \leq \lambda \leq 1$, since $R_\lambda(\alpha) = \alpha$. The image of H_1 clearly coincides with $(\pi_1^2|\Omega^{n-p+1,0}(n,p))^{-1}(K'(n,p))$.

Next let

$$h_\lambda : (\pi_1^2|\Omega^{n-p+1,0}(n,p))^{-1}(K'(n,p)) \to (\pi_1^2|\Omega^{n-p+1,0}(n,p))^{-1}(K'(n,p))$$

be the homotopy defined by

$$h_\lambda(\alpha,\beta) = \begin{cases} (\alpha, ((1-\lambda)+\lambda\|x(\alpha)\|)(\|\beta\| - 2\|x(\alpha)\|)\frac{\beta}{\|\beta\|} + 2\|x(\alpha)\|\frac{\beta}{\|\beta\|}, \\ \qquad\qquad\qquad \text{if } \ \|\beta\| \geq 2\|x(\alpha)\| \ \text{and} \ \ \|\beta\| \neq 0, \\ (\alpha,\beta) \qquad\qquad \text{if } \ \|\beta\| \leq 2\|x(\alpha)\|. \end{cases}$$

Then the image of h_1 coincides with the union

$$(\pi_1^2|\Omega^{n-p+1,0}(n,p))^{-1}(K'(n,p) \setminus R'(n,p)) \bigcup R'(n,p) \times \mathbf{0}^p_{n\times n}.$$

If $(\alpha,\beta) \in \mathcal{K}'(n,p)$, then we have $\|\beta\| = \sqrt{(1-b(\alpha)^2)(n-p+1)} \leq 2\|x(\alpha)\|$, and hence $h_\lambda(\alpha,\beta) = (\alpha,\beta)$ by the definition of h_λ. It is clear that h_0 is the identity. On the other hand, by Proposition 8.1 below we have a deformation retraction $\mathcal{D}_\lambda$ of $\mathrm{Im}(h_1)$ to $\mathcal{K}'(n,p)$. Thus we obtain a deformation retraction $\mathcal{R}_\lambda$ of $\Omega^{n-p+1,0}(n,p)$ to $K'(n,p)$ defined by

$$\mathcal{R}_\lambda(\alpha,\beta) = \begin{cases} H_{3\lambda}(\alpha,\beta) & 0 \leq \lambda \leq 1/3, \\ h_{3\lambda-1}(\alpha,\beta) & 1/3 \leq \lambda \leq 2/3, \\ \mathcal{D}_{3\lambda-2}(\alpha,\beta) & 2/3 \leq \lambda \leq 1. \end{cases}$$

This is what we want to prove. □

Proposition 8.1. *There exists a deformation retraction $\mathcal{D}_\lambda$ of* $\mathrm{Im}(h_1)$ *to $\mathcal{K}'(n,p)$ such that $\mathcal{D}_\lambda$ preserves $(\pi_1^2|\mathrm{Im}(h_1))^{-1}(K'(n,p) \setminus \Sigma K'(n,p))$ and $(\pi_1^2|\Sigma^{n-p+1,0}(n,p))^{-1}(\Sigma K'(n,p))$ respectively. In particular, the restriction $\mathcal{D}_\lambda|(\pi_1^2|\Sigma^{n-p+1,0}(n,p))^{-1}(\Sigma K'(n,p))$ is a deformation retraction of $(\pi_1^2|\Sigma^{n-p+1,0}(n,p))^{-1}(\Sigma K'(n,p))$ to $\Sigma\mathcal{K}'(n,p)$.*

The proof of this proposition is rather long. Let (α,β) be an element of $\mathrm{Im}(h_1)$. With the basis ${}^t\mathbf{m}_p, \ldots, {}^t\mathbf{m}_n$ of K_α and $\overline{\mathbf{s}}_p$ of C_α, let $B = (b_{ij}(\alpha,\beta))$ $(p \leq i,j \leq n)$ be the matrix defined by $\beta_\alpha({}^t\mathbf{m}_i, {}^t\mathbf{m}_j) = b_{ij}(\alpha,\beta)\overline{\mathbf{s}}_p$. This satisfies that for any $\mathbf{x}, \mathbf{y} \in K_\alpha$, $\beta_\alpha(\mathbf{x}, \mathbf{y}) = \{{}^t\mathbf{x}^t M\binom{p}{n} B M\binom{p}{n}\mathbf{y}\}\bar{\mathbf{s}}_p$. Let $a(\alpha, \beta)$ denote the absolute value of $\det B$, which is well defined for (α, β). Furthermore, $a(\alpha, \beta)$ is a continuous function. Indeed, it is easy to prove that $a(\alpha, \beta)$ is continuous at (α, β) with $b(\alpha) < 1$ (use Lemma 8.4 and Corollary 8.5 below if necessary). If $b(\alpha) = 1$ and (α', β') converges to $(\alpha, \mathbf{0}^p_{n\times n})$, then $a(\alpha', \beta')$ converges to 0, whatever $\bar{\mathbf{s}}_p$ varies. We define the non-negative real number $b(\alpha, \beta)$ by

$$b(\alpha, \beta) = \frac{b(\alpha)}{\sqrt{a(\alpha, \beta)^2 + b(\alpha)^2}}. \tag{8.1}$$

If $b(\alpha) = 0$, then α lies in $\Sigma K'(n, p)$, and hence $a(\alpha, \beta)$ is not equal to 0 by (C-2) in Section 7. If $b(\alpha) = 1$, then $\beta = \mathbf{0}^p_{n\times n}$ and hence, $b(\alpha, \beta) = 1$. Therefore, $b(\alpha)$, $a(\alpha, \beta)$ and $b(\alpha, \beta)$ are all continuous functions on $\mathrm{Im}(h_1)$.

We define maps $A : \mathrm{Im}(h_1) \to K'(n, p)$ and $B : \mathrm{Im}(h_1) \to \mathrm{Hom}(S^2\mathbf{R}^n, \mathbf{R}^p)$, which yields a retraction $\mathcal{D} : \mathrm{Im}(h_1) \to \mathcal{K}'(n, p)$ defined by

$$\mathcal{D}(\alpha, \beta) = (A(\alpha, \beta), B(\alpha, \beta)).$$

Let (α, β) be an element of $\mathrm{Im}(h_1)$ with a diagonalization $\alpha = S\Delta(\mathbf{d}_{b(\alpha)})M\binom{1}{p}$. If $a(\alpha, \beta) = 0$, then define $A(\alpha, \beta) = SM\binom{1}{p}$ and $B(\alpha, \beta) = \mathbf{0}^p_{n\times n}$. It is clear that $\mathcal{D}(\alpha, \beta)$ lies in $\mathcal{R}'(n, p)$. Next let $a(\alpha, \beta) \neq 0$. Then β_α is non-singular. Suppose that the signature of the matrix B associated to β_α is $\delta\sigma$ ($\delta = \pm 1$) as in (C-3) in Section 7. Since σ is invariantly defined for (α, β), we may write $\sigma(\alpha, \beta)$ for σ. We define $c(\alpha, \beta)$ and $d(\alpha, \beta)$ by $c(\alpha, \beta) = (n - p + 1 + \sigma(\alpha, \beta))/2$ and $d(\alpha, \beta) = (n - p + 1 - \sigma(\alpha, \beta))/2$ so that $c(\alpha, \beta) \geq d(\alpha, \beta)$. If $c(\alpha, \beta) > d(\alpha, \beta)$, then we can uniquely determine the unit vector $\bar{\mathbf{s}}_p \in C_\alpha$ in the expression $S\Delta(\mathbf{d}_{b(\alpha)})M\binom{1}{p}$ so that the index of B is $d(\alpha, \beta)$. If $c(\alpha, \beta) = d(\alpha, \beta)$, then we have no canonical method to determine the orientation of C_α in the expression $S\Delta(\mathbf{d}_{b(\alpha)})M\binom{1}{p}$. There exists a matrix $T \in O(n - p + 1)$ such that

$${}^tTBT = \Delta(\mathbf{v}(\alpha, \beta), \mathbf{w}(\alpha, \beta)),$$

where $\mathbf{v}(\alpha, \beta) = (v_1, \ldots, v_{c(\alpha,\beta)})$, $\mathbf{w}(\alpha, \beta) = (w_1, \ldots, w_{d(\alpha,\beta)})$ and $v_1 > \cdots > v_{c(\alpha,\beta)} > 0 > w_1 > \cdots > w_{d(\alpha,\beta)}$. When $a(\alpha, \beta) \neq 0$, we define $A(\alpha, \beta)$ and $B(\alpha, \beta)$ by

$$A(\alpha, \beta) = S\Delta(\mathbf{d}_{b(\alpha,\beta)})M\tbinom{1}{p}, \tag{8.2}$$

$$\begin{aligned} &B(\alpha, \beta)(\mathbf{x}, \mathbf{y}) \\ &\quad = \sqrt{1 - b(\alpha, \beta)^2}\{{}^t\mathbf{x}^t M\tbinom{p}{n}{}^tT(E_{c(\alpha,\beta)} \dotplus (-E_{d(\alpha,\beta)}))TM\tbinom{p}{n}\mathbf{y}\}\bar{\mathbf{s}}_p. \end{aligned} \tag{8.3}$$

Lemma 8.2. *Let* $(\alpha,\beta) \in \mathrm{Im}(h_1)$. *Then the elements* $A(\alpha,\beta)$ *and* $B(\alpha,\beta)$ *are well-defined.*

Proof. Suppose that $\alpha = S\Delta(\mathbf{d}_b)M\binom{1}{p} = S'\Delta(\mathbf{d}_b)M'\binom{1}{p}$. Let $b(\alpha) = 1$. Then we have $SM\binom{1}{p} = S'M'\binom{1}{p}$. Since $\beta = \mathbf{0}^p_{n\times n}$, we have $b(\alpha,\beta) = 1$. Hence, $A(\alpha,\beta) = SM\binom{1}{p}$ and $B(\alpha,\beta) = \mathbf{0}^p_{n\times n}$ are well-defined. Let $0 \leq b(\alpha) < 1$. Then by Lemma 5.4 there exist matrices $G \in O(p-1)$ and $(\delta) \in O(1)$ such that $S' = S({}^tG \dotplus (\delta))$ and $M'\binom{1}{p} = (G \dotplus (\delta))M\binom{1}{p}$. Hence, we have $S\Delta(\mathbf{d}_{b(\alpha,\beta)})M\binom{1}{p} = S'\Delta(\mathbf{d}_{b(\alpha,\beta)})M'\binom{1}{p}$. This implies that $A(\alpha,\beta)$ is well-defined by (8.2).

Next we deal with $B(\alpha,\beta)$ in the case $0 \leq b(\alpha) < 1$. In the proof we write c, d, $\mathbf{v}$ and $\mathbf{w}$ for $c(\alpha,\beta)$, $d(\alpha,\beta)$, $\mathbf{v}(\alpha,\beta)$ and $\mathbf{w}(\alpha,\beta)$ for simplicity. Suppose that $\alpha = S\Delta(\mathbf{d}_b)M\binom{1}{p} = S'\Delta(\mathbf{d}_b)M'\binom{1}{p}$, where S and S' are chosen so that if $c > d$, then $\overline{\mathbf{s}}_p = \overline{\mathbf{s}}'_p$. Let $B' = (b'_{ij})$ be the matrix defined by

$$\beta_\alpha({}^t\mathbf{m}'_i, {}^t\mathbf{m}'_j) = b'_{ij}\overline{\mathbf{s}}'_p = \{\mathbf{m}'^t_i M'\tbinom{p}{n}B'M'\tbinom{p}{n}{}^t\mathbf{m}'_j\}\overline{\mathbf{s}}'_p$$

and let B' be diagonalized as $B' = {}^tT'\Delta(\mathbf{v},\mathbf{w})T'$ by a matrix $T' \in O(n-p+1)$. It is easy to see that

$$\beta_\alpha(\mathbf{x},\mathbf{y}) = \{{}^t\mathbf{x}{}^tM\tbinom{p}{n}BM\tbinom{p}{n}\mathbf{y}\}\overline{\mathbf{s}}_p = \{{}^t\mathbf{x}{}^tM'\tbinom{p}{n}B'M'\tbinom{p}{n}\mathbf{y}\}\overline{\mathbf{s}}'_p.$$

Hence, if $\overline{\mathbf{s}}_p = \delta\overline{\mathbf{s}}'_p$, then we have

$${}^tM\tbinom{p}{n}BM\tbinom{p}{n} = \delta{}^tM'\tbinom{p}{n}B'M'\tbinom{p}{n}.$$

Let $a(\alpha,\beta) = 0$, and hence $b(\alpha,\beta) = 1$. Then $B(\alpha,\beta)$ is well defined since $B(\alpha,\beta) = \mathbf{0}^p_{n\times n}$ by (8.3).

Let $a(\alpha,\beta) \neq 0$, $0 \leq b(\alpha) < 1$ and $\sigma(\alpha,\beta) > 0$. In this case we have chosen so that $\overline{\mathbf{s}}'_p = \overline{\mathbf{s}}_p$. If $b(\alpha) > 0$, we have $\mathbf{m}'_p = \mathbf{m}_p$ and the subspace $\{{}^t\mathbf{m}_{p+1},\ldots,{}^t\mathbf{m}_n\}$ coincides with $\{{}^t\mathbf{m}'_{p+1},\ldots,{}^t\mathbf{m}'_n\}$. If $b(\alpha) = 0$, the subspace $\{{}^t\mathbf{m}_p,\ldots,{}^t\mathbf{m}_n\}$ coincides with $\{{}^t\mathbf{m}'_p,\ldots,{}^t\mathbf{m}'_n\}$. Whether $b(\alpha) > 0$ or $b(\alpha) = 0$, we have ${}^tM\binom{p}{n}BM\binom{p}{n} = {}^tM'\binom{p}{n}B'M'\binom{p}{n}$. This gives

$${}^tM\tbinom{p}{n}{}^tT\Delta(\mathbf{v},\mathbf{w})TM\tbinom{p}{n} = {}^tM'\tbinom{p}{n}{}^tT'\Delta(\mathbf{v},\mathbf{w})T'M'\tbinom{p}{n}.$$

By Lemma 5.5 there exist matrices $T_1 \in O(c)$ and $T_2 \in O(d)$ such that $T'M'\binom{p}{n} = (T_1 \dotplus T_2)TM\binom{p}{n}$. Hence we have

$${}^tM\tbinom{p}{n}{}^tT(E_c \dotplus (-E_d))TM\tbinom{p}{n} = {}^tM'\tbinom{p}{n}{}^tT'(E_c \dotplus (-E_d))T'M'\tbinom{p}{n}.$$

Thus $B(\alpha,\beta)$ is well defined by (8.3) in this case.

Let $a(\alpha,\beta) \neq 0$, $0 \leq b(\alpha) < 1$ and $\sigma(\alpha,\beta) = 0$. In this case we need to consider the cases where δ is 1 or -1. The proof of the case $\delta = 1$ is just the same as above. So let $\delta = -1$. Then we have

$$\begin{aligned}
&{}^tM\tbinom{p}{n}{}^tT\Delta(\mathbf{v},\mathbf{w})TM\tbinom{p}{n}\\
&\quad = {}^tM'\tbinom{p}{n}{}^tT'\Delta(-\mathbf{v},-\mathbf{w})T'M'\tbinom{p}{n}\\
&\quad = {}^tM'\tbinom{p}{n}{}^tT'\begin{pmatrix}0 & E_c\\ E_c & 0\end{pmatrix}\Delta(-\mathbf{w},-\mathbf{v})\begin{pmatrix}0 & E_c\\ E_c & 0\end{pmatrix}T'M'\tbinom{p}{n}.
\end{aligned}$$

By Lemmas 5.2 and 5.5 we have $\mathbf{v} = -\mathbf{w}$ and there exists T_1, $T_2 \in O(c)$ such that $T'M'\binom{p}{n} = \left(\begin{smallmatrix}0 & E_c\\ E_c & 0\end{smallmatrix}\right)(T_1 \dotplus T_2)TM\binom{p}{n}$. Hence, we have

$$\begin{aligned}
&\{{}^t\mathbf{x}{}^tM'\tbinom{p}{n}{}^tT'(E_c \dotplus (-E_c))T'M'\tbinom{p}{n}\mathbf{y}\}\bar{\mathbf{s}}'_p\\
&\qquad = -\left\{{}^t\mathbf{x}{}^tM\tbinom{p}{n}{}^tT({}^tT_1 \dotplus {}^tT_2)\begin{pmatrix}0 & E_c\\ E_c & 0\end{pmatrix}(E_c \dotplus (-E_c))\right.\\
&\qquad\quad \left.\times\begin{pmatrix}0 & E_c\\ E_c & 0\end{pmatrix}(T_1 \dotplus T_2)TM\tbinom{p}{n}\mathbf{y}\right\}\bar{\mathbf{s}}'_p\\
&\qquad = \{{}^t\mathbf{x}{}^tM\tbinom{p}{n}{}^tT(E_c \dotplus (-E_c))TM\tbinom{p}{n}\mathbf{y}\}\bar{\mathbf{s}}_p.
\end{aligned}$$

Thus $B(\alpha,\beta)$ is well defined by (8.3). □

We here state the properties of $\mathcal{D}(\alpha,\beta)$, which are easily proved from Remark 2.4.

Proposition 8.3. *Let $(\alpha,\beta) \in \mathrm{Im}(h_1)$. Then we have the following properties.*

(1) *If $(\alpha,\beta) \in \mathcal{K}'(n,p)$, then $\mathcal{D}(\alpha,\beta) = (\alpha,\beta)$.*

(2) *The image of $\mathcal{D}$ coincides with $\mathcal{K}'(n,p)$.*

(3) *If $a(\alpha,\beta) = 0$, then $\mathcal{D}(\alpha,\beta) \in \mathcal{R}'(n,p)$.*

(4) *If $\alpha \in \Sigma K'(n,p)$, $(\alpha,\beta) \in \Sigma^{n-p+1,0}(n,p)$ with $\sigma(\alpha,\beta)$, then $\mathcal{D}(\alpha,\beta) \in \Sigma\mathcal{K}'(n,p,\sigma(\alpha,\beta))$.*

(5) *If $a(\alpha,\beta) \neq 0$ and $0 < b(\alpha) < 1$, then we have $0 < b(\alpha,\beta) < 1$.*

Let $G_{\ell,m-\ell}$ be the grassman manifold of ℓ-dimensional subspaces of $\mathbf{R}^m$. An element of $G_{\ell,m-\ell}$ is expressed by an ℓ-dimensional subspace V of $\mathbf{R}^m$. The proof of the following lemma is left to the reader.

Lemma 8.4. *Let $\{\alpha^k\}$ be a sequence which converges to α in $K'(n,p)$. Assume that if $0 < b(\alpha) < 1$, then $0 < b(\alpha^k) < 1$ for all k. Then we have the followings.*

(1) *The sequence $\{C_{\alpha^k}\}$ converges to C_α in RP^{p-1}.*

(2) *If $0 < b(\alpha) < 1$, then the sequence $\{\mathrm{Ker}(\alpha^k)\}$ converges to $\mathrm{Ker}(\alpha)$ in $G_{n-p,p}$.*

(3) *The sequence $\{K_{\alpha^k}\}$ converges to K_α in $G_{n-p+1,p-1}$.*

Corollary 8.5. *Let $\{\alpha^k\}$ be a sequence which converges to α in $K'(n,p)$ such that $0 < b(\alpha) < 1$, and $0 < b(\alpha^k) < 1$ for all k. Let $\mathbf{m}$ be a unit vector of K_α with $\mathbf{m} \perp \mathrm{Ker}(\alpha)$. Then for sufficiently large k there exists a unit vector $\mathbf{m}^k$ of K_{α^k} with $\mathbf{m}^k \perp \mathrm{Ker}(\alpha^k)$ such that $\lim_{k\to\infty} \mathbf{m}^k = \mathbf{m}$.*

Proposition 8.6. *The map $\mathcal{D} = (A, B) : \mathrm{Im}(h_1) \to \mathcal{K}'(n,p)$ is continuous.*

Proof. Let $\{(\alpha^k, \beta^k)\}$ be a sequence which converges to (α, β) in $\mathrm{Im}(h_1)$ with diagonalizations

$$\alpha^k = S^k \Delta(\mathbf{d}_{b(\alpha^k)}) M^k \tbinom{1}{p} \qquad \text{and} \quad \alpha = S\Delta(\mathbf{d}_{b(\alpha)}) M\tbinom{1}{p}.$$

Since $\lim_{k\to\infty} {}^tS\alpha_k {}^tM = {}^tS\alpha {}^tM$, we have

$$\begin{aligned}\lim_{k\to\infty} {}^tSS^k\Delta(\mathbf{d}_{b(\alpha^k)})M\tbinom{1}{p}{}^tM &= \Delta(\mathbf{d}_{b(\alpha)})(M\tbinom{1}{p}{}^tM\tbinom{1}{p}, M\tbinom{1}{p}{}^tM\tbinom{p+1}{n}) \\ &= \Delta(\mathbf{d}_{b(\alpha)})(E_p, \mathbf{0}_{p\times(n-p)}).\end{aligned}$$

By Lemma 5.6 we have matrices $\delta({}^tSS^k)$ which, if $b(\alpha) < 1$, is written as $G^k \dotplus (x)$ such that $\lim_{k\to\infty}({}^tSS^k - \delta({}^tSS^k)) = \mathbf{0}_{p\times p}$. Furthermore, if $0 < b(\alpha) < 1$, then $\lim_{k\to\infty} \delta({}^tSS^k)M^k\tbinom{1}{p}{}^tM = (E_p, \mathbf{0}_{p\times(n-p)})$, and if $b(\alpha) = 0$, then $\lim_{k\to\infty} G^kM^k\tbinom{1}{p-1}{}^tM = (E_{p-1}, \mathbf{0}_{(p-1)\times(n-p+1)})$.

Case (i): Suppose $a(\alpha, \beta) = 0$.

We note that $b(\alpha) \neq 0$. Since the set of eigen values of a matrix is continuous with respect to components of matrices, we have $\lim_{k\to\infty} a(\alpha^k, \beta^k) = a(\alpha, \beta) = 0$. By (8.1) we have

$$\lim_{k\to\infty} b(\alpha^k, \beta^k) = \lim_{k\to\infty} \frac{b(\alpha^k)}{\sqrt{a(\alpha^k, \beta^k)^2 + b(\alpha^k)^2}} = 1.$$

Hence, we have

$$\begin{aligned}\lim_{k\to\infty} {}^tSA(\alpha^k,\beta^k){}^tM &= \lim_{k\to\infty} {}^tSS^k\Delta(\mathbf{d}_{b(\alpha^k,\beta^k)})M^k\tbinom{1}{p}{}^tM \\ &= \lim_{k\to\infty} {}^tS(S^kM^k\tbinom{1}{p} + S^k(\Delta(\mathbf{d}_{b(\alpha^k,\beta^k)} - E_p)M^k\tbinom{1}{p})){}^tM \\ &= \lim_{k\to\infty} {}^tSS^kM^k\tbinom{1}{p}{}^tM \\ &= \lim_{k\to\infty} {}^t\delta({}^tSS^k))M^k\tbinom{1}{p}{}^tM \\ &= \lim_{k\to\infty} (E_p, \mathbf{0}_{p\times(n-p)}) \\ &= {}^tSA(\alpha,\beta){}^tM.\end{aligned}$$

Since $\lim_{k\to\infty} b(\alpha^k,\beta^k) = 1$ and the norm $\|\beta^k_{\alpha^k}\|$ converges to 0, it follows that $\lim_{k\to\infty} B(\alpha^k,\beta^k) = \mathbf{0}$. Therefore, if $a(\alpha,\beta) = 0$, then $\mathcal{D}$ is continuous at (α,β).

Case (ii): Suppose $a(\alpha,\beta) \neq 0$.

Since we are working in $\mathrm{Im}(h_1)$, this yields $0 \leq b(\alpha) < 1$. Then we have

$$\begin{aligned}\lim_{k\to\infty} {}^tSA(\alpha^k,\beta^k){}^tM &= \lim_{k\to\infty} {}^tSS^k\Delta(\mathbf{d}_{b(\alpha^k,\beta^k)})M^k\tbinom{1}{p}{}^tM \\ &= \lim_{k\to\infty} \delta({}^tSS^k)\Delta(\mathbf{d}_{b(\alpha^k,\beta^k)})M^k\tbinom{1}{p}{}^tM \\ &= \lim_{k\to\infty} \Delta(\mathbf{d}_{b(\alpha^k,\beta^k)})\delta({}^tSS^k)M^k\tbinom{1}{p}{}^tM \\ &= \lim_{k\to\infty} \Delta(\mathbf{d}_{b(\alpha^k,\beta^k)})(E_p, \mathbf{0}_{p\times(n-p)}) \\ &= {}^tSA(\alpha,\beta){}^tM.\end{aligned}$$

Thus we have proved $\lim_{k\to\infty} A(\alpha^k,\beta^k) = A(\alpha,\beta)$.

We prove the continuity of $B(\alpha,\beta)$. We note that if $\sigma(\alpha,\beta) > 0$, then we have chosen a unit basis $\overline{\mathbf{s}}_p$ so that the index of B is less than $(n-p+1)/2$ and that if $\sigma(\alpha,\beta) = 0$, then we chose $\overline{\mathbf{s}}_p$ arbitrarily. For a sufficiently large number k we set $\overline{\mathbf{s}}^k_p = pr(C_{\alpha^k})(\overline{\mathbf{s}}_p)/\|pr(C_{\alpha^k})(\overline{\mathbf{s}}_p)\|$. If $0 < b(\alpha) < 1$, then it follows from Corollary 8.5 that for the vector ${}^t\mathbf{m}_p$, there exists a unit vector ${}^t\mathbf{m}^k_p$ for a sufficiently large number k with ${}^t\mathbf{m}^k_p \in K_{\alpha^k}$ and ${}^t\mathbf{m}^k_p \perp \mathrm{Ker}(\alpha^k)$ such that $\lim_{k\to\infty} {}^t\mathbf{m}^k_p = {}^t\mathbf{m}_p$. For the orthonormal basis ${}^t\mathbf{m}_p, \ldots, {}^t\mathbf{m}_n$ of K_α, we set $\mathbf{a}^k_j = pr(K_{\alpha^k})({}^t\mathbf{m}_j)$ $(j = p+1, \ldots, n)$. There is a large number k_0 such that if $k > k_0$, then ${}^t\mathbf{m}^k_p, {}^t\mathbf{a}^k_{p+1}, \ldots, {}^t\mathbf{a}^k_n$ are linearly independent. By applying the Gram-Schmidt orthonormalization process to ${}^t\mathbf{m}^k_p, {}^t\mathbf{a}^k_{p+1}, \ldots, {}^t\mathbf{a}^k_n$ putted in this order, we obtain an orthonormal basis, say ${}^t\mathbf{m}^k_p, \ldots, {}^t\mathbf{m}^k_n$. It is easily verified that $\lim_{k\to\infty} {}^t\mathbf{m}_j^k = {}^t\mathbf{m}_j$ for $j = p, \dots, n$. If $b(\alpha) = 0$, then there exists an orthonormal basis ${}^t\mathbf{m}_p, \dots, {}^t\mathbf{m}_n$ of $K_\alpha = \mathrm{Ker}(\alpha)$. We set $\mathbf{a}_j^k = pr(K_{\alpha^k})({}^t\mathbf{m}_j)$ $(j = p, \dots, n)$. By the similar arguments we obtain an orthonormal basis, say ${}^t\mathbf{m}_p^k, \dots, {}^t\mathbf{m}_n^k$ such that $\lim_{k\to\infty} {}^t\mathbf{m}_j^k = {}^t\mathbf{m}_j$ for $j = p, \dots, n$. Suppose that S^k, $S \in O(p)$ and M^k, $M \in O(n)$ in the expressions (8.2) and (8.3) are chosen to have these column and row vectors.

For (α^k, β^k) we define the matrix B^k by $\beta_{\alpha^k}^k({}^t\mathbf{m}_i^k, {}^t\mathbf{m}_j^k) = b_{ij}^k \bar{\mathbf{s}}_p^k$, namely

$$\beta_{\alpha^k}^k(\mathbf{x}, \mathbf{y}) = \{{}^t\mathbf{x}{}^t M^k\tbinom{p}{n} B^k M^k\tbinom{p}{n}\mathbf{y}\}\bar{\mathbf{s}}_p^k.$$

Then we have

$$\begin{aligned}
b_{ij}\bar{\mathbf{s}}_p &= pr(C_\alpha) \circ \beta({}^t\mathbf{m}_i, {}^t\mathbf{m}_j) \\
&= \lim_{k\to\infty} pr(C_{\alpha^k}) \circ \beta^k({}^t\mathbf{m}_i^k, {}^t\mathbf{m}_j^k) \\
&= \lim_{k\to\infty} \beta_{\alpha^k}^k({}^t\mathbf{m}_i^k, {}^t\mathbf{m}_j^k) \\
&= \lim_{k\to\infty} b_{ij}^k \bar{\mathbf{s}}_p^k \\
&= (\lim_{k\to\infty} b_{ij}^k)\bar{\mathbf{s}}_p.
\end{aligned}$$

Hence, we have $\lim_{k\to\infty} B^k = B$.

Since $a(\alpha, \beta) \neq 0$, β_α is non-singular. By the choice of $\bar{\mathbf{s}}_p$, we have $c(\alpha, \beta) \geq d(\alpha, \beta)$. Therefore, we can assert that if k is sufficiently large, then $\beta_{\alpha^k}^k$ is non-singular, and $c(\alpha^k, \beta^k) = c(\alpha, \beta)$, $d(\alpha^k, \beta^k) = d(\alpha, \beta)$ and $\sigma(\alpha^k, \beta^k) = \sigma(\alpha, \beta)$. Suppose that B^k is diagonalized, by a matrix T^k, as $T^k B^k({}^tT^k) = \Delta(\mathbf{v}, \mathbf{w})$ with $v_1^k \geq \cdots \geq v_c^k > 0 > w_1^k \geq \cdots \geq w_d^k$ for large k. Since $\lim_{k\to\infty} B^k = B$, we have $\lim_{k\to\infty} {}^tT^k\Delta(\mathbf{v}, \mathbf{w})T^k = {}^tT\Delta(\mathbf{v}, \mathbf{w})T$. Hence,

$$\lim_{k\to\infty} T({}^tT^k)\Delta(\mathbf{v}, \mathbf{w})T^k({}^tT) = \Delta(\mathbf{v}, \mathbf{w}).$$

Then we have matrices $\delta(T({}^tT^k))$ described in Lemma 5.3. Thus, we have

$$\begin{aligned}
\lim_{k\to\infty} T({}^tT^k)(E_c \dot{+} (-E_d))T^k({}^tT) &= \lim_{k\to\infty} \delta(T({}^tT^k))(E_c \dot{+} (-E_d))^t\delta(T({}^tT^k)) \\
&= \lim_{k\to\infty} (E_c \dot{+} (-E_d))\delta(T({}^tT^k))^t\delta(T({}^tT^k)) \\
&= (E_c \dot{+} (-E_d)).
\end{aligned}$$

Therefore, we have $\lim_{k\to\infty} {}^tT^k(E_{c(\alpha^k,\beta^k)} \dot{+} (-E_{d(\alpha^k,\beta^k)}))T^k = {}^tT(E_c \dot{+} (-E_d))$ T. Since $\lim_{k\to\infty} {}^t\mathbf{m}_j^k = {}^t\mathbf{m}_j$ for $j = p, \dots, n$, we have

$$\lim_{k\to\infty} {}^tM^k\tbinom{p}{n}{}^tT^k(E_c \dot{+} (-E_d))T^k M^k\tbinom{p}{n} = {}^t M\tbinom{p}{n}{}^tT(E_c \dot{+} (-E_d))TM\tbinom{p}{n}.$$

For $\mathbf{x}$, $\mathbf{y} \in \mathbf{R}^n$, set $\mathbf{x}^k = pr(K_{\alpha^k})(\mathbf{x})$, $\mathbf{y}^k = pr(K_{\alpha^k})(\mathbf{y})$, $\mathbf{x}^0 = pr(K_\alpha)(\mathbf{x})$ and $\mathbf{y}^0 = pr(K_\alpha)(\mathbf{y})$. By the definition (8.3) we have

$$\begin{aligned}
&B(\alpha,\beta)(\mathbf{x},\mathbf{y}) \\
&\qquad = B(\alpha,\beta)(\mathbf{x}^0,\mathbf{y}^0) \\
&\qquad = \sqrt{1-b(\alpha,\beta)^2}\{({}^t\mathbf{x}^0)^t M\tbinom{p}{n}^t T(E_c \dotplus (-E_d)) T M\tbinom{p}{n}\mathbf{y}^0\}\bar{\mathbf{s}}_p \\
&\qquad = \lim_{k\to\infty} \sqrt{1-b(\alpha^k,\beta^k)^2}\{({}^t\mathbf{x}^k)^t M^k\tbinom{p}{n}^t T^k(E_c \dotplus (-E_d)) T^k M^k\tbinom{p}{n}\mathbf{y}^k\}\bar{\mathbf{s}}_p^k \\
&\qquad = \lim_{k\to\infty} B(\alpha^k,\beta^k)(\mathbf{x}^k,\mathbf{y}^k) \\
&\qquad = \lim_{k\to\infty} B(\alpha^k,\beta^k)(\mathbf{x},\mathbf{y}).
\end{aligned}$$

This shows $\lim_{k\to\infty} B(\alpha^k,\beta^k) = B(\alpha,\beta)$. Therefore, $B(\alpha,\beta)$ is continuous at a point (α,β) with $a(\alpha,\beta) \neq 0$.

This completes the proof. □

Proof of Proposition 8.1. We define a deformation retraction $\mathcal{D}_\lambda$ of $\mathrm{Im}(h_1)$ to $\mathcal{K}'(n,p)$ by

$$\mathcal{D}_\lambda(\alpha,\beta) = (1-\lambda)(\alpha,\beta) + \lambda\mathcal{D}(\alpha,\beta) = (A_\lambda(\alpha,\beta), B_\lambda(\alpha,\beta)),$$

where

$$A_\lambda(\alpha,\beta) = (1-\lambda)\alpha + \lambda A(\alpha,\beta) = S\Delta(\mathbf{d}_{(1-\lambda)b(\alpha)+\lambda b(\alpha,\beta)})M\tbinom{1}{p},$$

$$B_\lambda(\alpha,\beta) = (1-\lambda)\beta + \lambda B(\alpha,\beta).$$

By Propositin 8.6, $\mathcal{D}_\lambda(\alpha,\beta)$ is continuous with respect to α, β and λ. We first prove that $\mathcal{D}_\lambda$ is a map into $\mathrm{Im}(h_1)$. In fact, if $b(\alpha) = 1$ and $\beta = \mathbf{0}^p_{n\times n}$, then $\mathcal{D}(\alpha,\beta) = (\alpha,\beta)$, and hence $\mathcal{D}_\lambda(\alpha,\beta) = (\alpha,\beta) = (\alpha,\mathbf{0}^p_{n\times n})$ by Proposition 8.3 (1).

If $b(\alpha) = 0$, then $b(\alpha,\beta) = 0$, and hence $(1-\lambda)b(\alpha) + \lambda b(\alpha,\beta) = 0$. This implies that $A_\lambda(\alpha,\beta)$ is always equal to α for such (α,β). We have that if $b(\alpha) = 0$, then $B_\lambda(\alpha,\beta)$ is non-singular, since $(1-\lambda)\Delta(\mathbf{v}(\alpha,\beta),\mathbf{w}(\alpha,\beta)) + \lambda\sqrt{1-b(\alpha,\beta)^2}(E_c + (-E_d))$ is non-singular. This shows that $\mathcal{D}_\lambda(\alpha,\beta)$ lies in $\mathrm{Im}(h_1)$. If $0 < b(\alpha) \leq 1$, then we have $0 < (1-\lambda)b(\alpha) + \lambda b(\alpha,\beta) \leq 1$.

We have that $\mathcal{D}_0 = id_{\mathrm{Im}(h_1)}$ by definition, $\mathrm{Im}\mathcal{D}_1 = \mathcal{K}'(n,p)$ and $\mathcal{D}_\lambda|\mathcal{K}'(n,p) = id_{\mathcal{K}'(n,p)}$ by Proposition 8.3 (1) and (3). This completes the proof. □

References

[An1] Ando, Y., On the elimination of Morin singularities, *J. Math. Soc. Japan*, **37** (1985), 471–487.
[An2] ———, Folding maps and the surgery theory on manifolds, *J. Math. Soc. Japan*, **53** (2001), 357–382.
[An3] ———, Fold-maps and the space of base point preserving maps of spheres, *J. Math. Kyoto. Univ.*, **41** (2002), 691–735.
[An4] ———, Existence theorems of fold-maps, submitted.
[An5] ———, An invariant of maps with prescribed singularities, in preparation.
[At] Atiyah, M. F., Thom complexes, *Proc. London Math. Soc.*, **11** (1961), 291–310.
[Bo] Boardman, J. M., Singularities of differentiable maps, *IHES Publ. Math.*, **33** (1967), 21–57.
[B-R] Burlet, O. and de Rham, G., Sur certaines applications génériques d'une variété close à 3 dimensions dans le plan, *Enseign. Math.*, **20** (1974), 275–292.
[E1] Èliašberg, J. M., On singularities of folding type, *Math. USSR Izv.*, **4** (1970), 1119–1134.
[E2] ———, Surgery of singularities of smooth mappings, *Math. USSR Izv.*, **6** (1972), 1302–1326.
[G1] Gromov, M., Stable mappings of foliations into manifolds, *Math. USSR Izv.*, **3** (1969), 671–694.
[G2] ———, *Partial Differential Relations*, Springer-Verlag, 1986.
[H] Hirzebruch, F., *Topological Methods in Algebraic Geometry*, Springer-Verlag, 1956.
[K-N] Kobayashi, S. and Nomizu, K., *Foundations of Differential Geometry*, Vol.1, Interscience Publishers, 1963.
[L] Levine, H. I., Singularities of differentiable maps, *Proc. Liverpool Singularities Symposium, Springer Lecture Notes in Math.*, **192** (1971), 1–85.
[M-M] Madsen, I. and Milgram, R. J., *The Classifying Spaces for Surgery and Cobordism of Manifolds*, Princeton, 1979.
[Ma] Mather, J., Stability of C^∞ mappings: VI. The nice dimensions, *Proceedings of Liverpool Singularities, Lecture Notes in Math.*, **192** (1971), 207–253.
[M-S] Milnor, J. and Spanier, E., Two remarks on fiber homotopy type, *Pacific J. Math.*, **10** (1960), 585–590.
[Mo] Morin, B., Formes canoniques des singularités d'une application différentiable, *C. R. Acad. Sci. Paris*, **260** (1965), 5662–5665; 6503–6506.
[Sa] Saeki, O., Notes on the topology of folds, *J. Math. Soc. Japan*, **44** (1992), 551–566.
[S-S] Saeki, O. and Sakuma, K., Special generic maps of 4-manifolds and compact complex analytic surfaces, *Math. Ann.*, **313** (1999), 617–633.
[Spa1] Spanier, E., Infinite symmetric products, function spaces, and duality, *Ann. of Math.*, **69** (1959), 142–198.
[Spa2] ———, Function spaces and duality, *Ann. of Math.*, **70** (1959), 338–378.
[T] Thom, R., Les singularités des applications différentiables, *Ann. Inst. Fourier*, **6** (1955–56), 43–87.
[Tsu] Tsuchiya, A., Characteristic classes for spherical fiber spaces, *Nagoya Math. J.*, **43** (1971), 1–39.
[W] Whitney, H., *Complex Analytic Varieties*, Addison-Wesley, 1972.