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Multiplicity of Filtered Rings and Simple K3 Singularities of Multiplicity Two

By

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Abstract

Given a filtered ring, we give bounds of its multiplicity in terms of the data of the tangent cone using the technique of the filtered blowing-up. Applying it to each simple K3 singularity of multiplicity two, we find a good coordinate where the Newton boundary of the defining equation contains the point $(1, 1, 1, 1) \in \mathbf{R}^4$. In the course of the proof, we classify simple K3 singularities of multiplicity two into 48 weight types. Furthermore we prove that the weight type of the singularity stays the same under arbitrary one-parameter (FG)-deformations.

Contents

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References

Introduction

The study of various blowing-ups is very important in the theory of singularities. In many cases some blowing-up appears as the blowing-down of divisors of algebraic variety, and is understood naturally as a filtered blowing-up. As a continuation of [20] and [19], we will study the multiplicity of the singularity from the point of the theory of filtered rings. Let (V, p) be a germ of a projective variety at a closed point p, and (A, m) the local ring of (V, p). We consider the filtration $F = \{F^k\}$ of ideals of A which satisfies; $F^0 = A$, $F^1 = m$, $F^k \supset F^{k+1}$, $F^k \cdot F^j \subset F^{k+j}$, and $\mathcal{R} = \oplus_{k \geq 0} F^k \cdot T^k \subset A[T]$ is a finitely generated A-algebra. Here T is an indeterminate. There is a positive integer N such that $F^{kN} = (F^N)^k$ holds for $k \geq 0$, and we assume that F^N is m-primary. $G = \oplus_{k \geq 0} F^k / F^{k+1}$ and $G_+ = \oplus_{k \geq 1} F^k / F^{k+1}$ as usual. The graded ring G is also denoted by $gr_F(A)$. We set $d = \dim A$. Our first result is the following.

Theorem A.

(1) *Let a system of elements $x_1, \dots, x_s \in G_+$ be a minimal homogeneous generator system of G_+ with $\deg x_1 \leq \deg x_2 \leq \cdots \leq \deg x_s$ with $s \geq d = \dim A = \dim G$. Then we have the following*:

$$\begin{aligned}\left(\prod_{i=1}^{d} \deg x_i\right) \lim_{\lambda \to 1} (1-\lambda)^d P(G, \lambda) &\leq_{(i)} e(m, A) \\ &\leq e(G_+, G) \\ &\leq_{(ii)} (\deg x_s)^d \lim_{\lambda \to 1} (1-\lambda)^d P(G, \lambda),\end{aligned}$$

where $P(G, \lambda) = \sum_{k \geq 0} l(G_k) \lambda^k \in \mathbf{Z}[[\lambda]]$.

(2) *If the equality holds in* (i)*, then $e(m, A) = e(G_+, G)$ and there is a parameter system $y_1, \dots, y_d$ of A whose initial form gives a homogeneous parameter system $in(y_1), \dots, in(y_d)$ of G such that $\deg in(y_i) = \deg x_i$ for $i = 1, \dots, d$. Conversely, if there exists such a parameter system $y_1, \dots, y_d$ of A, then these three integers coincide.*

(3) *If the equality holds in* (ii) *and G is reduced, then $e(m, A) = e(G_+, G)$ and G is a homogeneous ring. That is* $\deg x_1 = \cdots = \deg x_s$.

As an excellent and an important application of our theory, we study the simple K3 singularities of multiplicity two. The simple K3 singularity is a class of 3-dimensional complex analytic isolated Gorenstein singularity defined as a 3-dimensional analogue of simple elliptic singularity by S. Ishii and Kimio Watanabe [10] as a 3-dimensional Gorenstein isolated purely elliptic singularity of the Hodge type $(0, 2)$ [6], [7], [22]. Related to the geometric characterization in [10], we can also characterize this class by the existence of a filtration of ideals as follows [19]:

Theorem-Definition (cf. Theorem 4.2 of [19]). *Let (A, m) be the local ring of a 3-dimensional Gorenstein isolated singularity. Then (A, m) is a simple $K3$ singularity if and only if there is a filtration of ideals $F = \{F^k\}$ as above such that $gr_F(A)$ is isomorphic to the Demazure construction $R(E, D)$ for some normal $K3$ surface E with rational double points and ample integral Weil divisor D.*

The filtration stated above is unique and the filtered blowing-up induces the canonical model of a resolution of singularity. We call this "*the canonical filtration*" of a simple K3 singularity.

In this situation, our main result is stated as follows:

Theorem B. *Let (A, m) be a simple $K3$ singularity of multiplicity two, and F the canonical filtration. Then $gr_F(A)$ is also a hypersurface of multiplicity two.*

In particular, F is induced by a weight filtration on the coordinates of a suitable minimal embedding of (A, m). In (7.1), Theorem B is proven as a corollary of Theorem A and the next theorem.

Theorem C. *Let E be a normal projective surface such that $\omega_E \cong O_E$ and $\mathrm{H}^1(E, O_E) = 0$ and assume that E has only rational double points. Let D be an ample $\mathbf{Q}$-Cartier integral Weil divisor and G the normal graded ring represented by Demazure's construction $G = R(E, D)$. Let $x_1, \dots, x_s$ be a minimal homogeneous generator of the homogeneous maximal ideal G_+ with $\deg x_1 \leq \cdots \leq \deg x_s$. Then we have the inequality $\deg x_1 {\cdot} \deg x_2 {\cdot} \deg x_3 D^2 \geq 2$.*

Further as a corollary of the proof of Theorem C, we can classify the cases where the equality holds. There are exactly 48 types, which are listed in Table (7.3). By Theorem B and its corollaries, we can define the type of simple K3 double point by weighted homogeneous type of the initial form. They are 48 classes of (7.3) as same as quasi-homogeneous isolated singularities. Furthermore we prove that the weight type of the singularity stays the same under arbitrary one-parameter (FG)-deformations (Theorem (7.5)). These corollaries are shown in Section 7.

In [9], we found simple K3 singularities of multiplicity three where G are not hypersurface. Hence we can not expect Theorem B for a simple K3 singularities of multiplicity three. For results about classification of quasi-homogeneous (or nondegenerate) simple K3 hypersurface singularities, we refer Yonemura's results [23]. Here this case is classified into the 95 weight type, which was originally discovered by M. Reid and by Fletcher [2].

In this paper, we assume that the local ring $(A, m) = (O_{V,p}, m)$ is coming from some scheme over an infinite field and analytically unramified.

Chapter I. Multiplicity of Filtered Rings

§1. A Proof of Theorem A

(1.1) As noted in Introduction, our singularity (V, p) or local ring $(A, m) = (O_{V,p}, m)$ is always coming from some scheme over an infinite field and analytically unramified. A filtration $F = \{F^k\}_{k \geq 0}$ is a decreasing sequence of ideals of A which satisfies; $F^0 = A$, $F^1 = m$, $F^k \supset F^{k+1}$, $F^k \cdot F^j \subset F^{k+j}$, and $\mathcal{R} = \oplus_{k \geq 0} F^k \cdot T^k \subset A[T]$ is a finitely generated A-algebra. Here T is an indeterminate. There is a positive integer N such that $F^{kN} = (F^N)^k$ holds for $k \geq 0$, and we assume that F^N is m-primary. $G = \oplus_{k \geq 0} F^k/F^{k+1}$ and $G_+ = \oplus_{k \geq 1} F^k/F^{k+1}$ as usual. We set $d = \dim A$. See [20] for a general information for the theory of filtered rings and filtered blowing-up.

First we shall prepare some lemmas.

Lemma 1.2. *Let the situation be as above. Then*

$$l(A/m^{l+1}) \leq l(G/(G_+)^{l+1}) \qquad \textit{for} \quad l \geq 0.$$

In particular we obtain the relations $e(m, A) \leq e(G_+, G)$ and embdim $A \leq$ embdim G.

Proof. The filtration on A/m^{l+1} induced by $F = \{F^k\}$ is given as follows:

$$0 \to m^{l+1} \cap F^k \to F^k \to F^k(A/m^{l+1}) \to 0$$

Hence we obtain $gr_F(A/m^{l+1}) = gr_F(A)/gr_F(m^{l+1})$. Here we see

$$gr_F(m^{l+1}) = \oplus_{k \geq 0} m^{l+1} \cap F^k / m^{l+1} \cap F^{k+1} \cong \oplus_{k \geq 0} \frac{F^k \cap m^{l+1} + F^{k+1}}{F^{k+1}},$$

$$(G_+)^{l+1} = \oplus_{k\geq l+1} \frac{\sum_{m_1+\cdots+m_{l+1}=k, m_i\geq 1} F^{m_1}\ldots F^{m_{l+1}} + F^{k+1}}{F^{k+1}}.$$

Clearly we have

$$\sum_{m_1+\cdots+m_{l+1}=k, m_i\geq 1} F^{m_1}\ldots F^{m_{l+1}} \subset F^k \cap m^{l+1}.$$

Hence $(G_+)^{l+1} \subset gr_F(m^{l+1})$. Therefore

$$l(A/m^{l+1}) = l(gr_F(A/m^{l+1})) = l(G/gr_F(m^{l+1})) \leq l(G/(G_+)^{l+1}).$$

Lemma 1.3. *Let L be a positive integer such that the relation $F^{mL} = \left(F^L\right)^m$ holds for any positive integer m. Then*

$$e(F^L, A) = L^d. \lim_{\lambda\to 1}(1-\lambda)^d P(G,\lambda).$$

Proof. By the assumption, $\oplus_{k\geq 0} F^{kL}/F^{(k+1)L}$ is generated by F^L/F^{2L}. Hence we obtain the equality (see Sections 13 and 14 of [12]):

$$e(F^L, A) = \lim_{\lambda\to 1}(1-\lambda)^d P(\oplus_{k\geq 0} F^{kL}/F^{(k+1)L}, \lambda).$$

Let $G^{(L,l)} = \oplus_{k\geq 0} F^{kL+l}/F^{kL+l+1}$ for $l = 0, \ldots, L-1$. Since there is an integer M such that $F^L.F^b = F^{L+b}$ holds for any $b \geq M$, $G^{(L,l)}$ is a finite $G^{(L,0)}$-module for $l = 0, \ldots, L-1$. As graded $G^{(L,0)}$-modules, we calculate the Poincare series; $P(G^{(L,l)}, \mu) \in \mathbf{Z}[[\mu]]$ for $l = 0, \ldots, L-1$. For each l, $\lim_{\mu\to 1}(1-\mu)^d P(G^{(L,l)}, \mu)$ is a finite number. Hence

$$\begin{aligned}
\lim_{\mu\to 1}(1-\mu)^d P(G,\mu) &= \lim_{\mu\to 1}(1-\mu^L)^d \sum_{l=0}^{L-1} P(G^{(L,l)}, \mu^L)\mu^l . \frac{(1-\mu)^d}{(1-\mu^L)^d} \\
&= \lim_{\nu\to 1}(1-\nu)^d \sum_{l=0}^{L-1} P(G^{(L,l)}, \nu). \frac{1}{L^d} \\
&= \lim_{\nu\to 1}(1-\nu)^d P(\oplus_{k\geq 0} F^{kL}/F^{(k+1)L}, \nu). \frac{1}{L^d} \\
&= e(F^L, A)\frac{1}{L^d}.
\end{aligned}$$

□

Next we recall a theorem of C. P. Ramanujan. We may assume that p is a closed point of a projective variety $\bar{V}$ over the field k. Let I be an m-primary ideal of $O_{V,p}$. Let $\pi : \tilde{V} \to V$ be a projective morphism such that $I.O_{\tilde{V}}$ is a locally principal $O_{\tilde{V}}$-module. Representing $I.O_{\tilde{V}} = O_{\tilde{V}}(-D(I,\pi))$ by a Cartier divisor on $\tilde{V}$, a theorem of C. P. Ramanujan on the multiplicity $e(I, O_{V,p})$ is given as;

Lemma 1.4 ([13]). *$e(I, O_{V,p}) = (-1)^{d+1} D(I,\pi)^d$. Here $d = \dim O_{V,p}$.*

Lemma 1.5. *Let $z_1, \dots, z_d \in G_+$ be a homogeneous parameter system of G. Then we have*

$$e((z_1, \dots, z_d), G) = \left(\prod_{i=1}^{d} \deg z_i\right) \lim_{\lambda \to 1} (1-\lambda)^d P(G,\lambda).$$

Proof. We introduce a natural filtration on G by the grading as follows: $F^k(G) = G|_k = \oplus_{l \geq k} G_l \subset G$ for $k \in \mathbf{Z}$. Here $\oplus_{k \geq 0} G|_k U^k \subset G[U]$ is a finitely generated G-algebra, where U is an indeterminate. Let $C = \mathrm{Proj}(\oplus_{k \geq 0} G|_k U^k) \longrightarrow \mathrm{Spec}(G)$ be the natural blowing-up by this grading. There is an integer L where the relation $G|_{Lm} = (G|_L)^m$ holds for $m \in \mathbf{N}$. Then on C we have $G|_L O_C = O_C(L) = \tilde{G(L)} \subset O_C$ and this is an invertible O_C-module sheaf. We can choose $L \in \mathbf{N}$ such that $\deg z_i | L$ for $i = 1, \dots, d$. Then $(z_1^{L/\deg z_1}, \dots, z_d^{L/\deg z_d})$ is a parameter of G and one can easily check the relation

$$(z_1^{L/\deg z_1}, \dots, z_d^{L/\deg z_d}) O_C = O_C(L).$$

By Ramanujan's Theorem 1.4, Lech's lemma (Theorem 14.12 [12]) and Lemma 1.3, we obtain the relations

$$\begin{aligned}\frac{L}{\deg z_1} \cdot \dots \cdot \frac{L}{\deg z_d} . e((z_1, \dots, z_d), G) &= e((z_1^{\frac{L}{\deg z_1}}, \dots, z_d^{\frac{L}{\deg z_d}}), G) \\ &= e(G|_L, G) = L^d . \lim_{\lambda \to 1} (1-\lambda)^d P(G,\lambda).\end{aligned}$$

(1.6) *Proof of the inequality* (i) *in* (1). Let a system of elements $x_1, \dots, x_s$ of the maximal ideal m of A whose initial forms with respect to the filtration F give the minimal homogeneous generator of G_+ as follows; $x_i \in F^{q_i} - F^{q_i+1}$ and the initial forms $in_F(x_i) = \bar{x}_i \in G_{q_i}$ satisfies the relations $G_+ = (\bar{x}_1, \dots, \bar{x}_s)G$ and $q_1 \leq \dots \leq q_s$. We can easily see the relations $m = F^n + (x_1, \dots, x_s)A$ for any positive integer n. There is an integer n such that $F^n \subset m^2$. Hence $m = (x_1, \dots, x_s)$ by NAK. There is a system of parameter $y_1, \dots, y_d$ which is a minimal reduction of m and given as linear combination of $x_1, \dots, x_s$ as follows: $y_i = \sum_{j=1}^{s} a_{i,j} x_j$, (where $a_{i,j} \in k$) with $1 \leq i \leq d$, $1 \leq j \leq s$. By the proof of Theorem 14.14 [12] (see pp. 113–114), there is a Zariski open set U of k^{sd} where $(y_1, \dots, y_d)$ in the above is a reduction of m for $(a_{i,j}) \in U$. In particular, there is a reduction in the above form with $\det(a_{i,j})_{1 \leq i,j \leq d} \neq 0$. Hence we can choose a reduction $(y_1, \dots, y_d)$ in the following form:

$$y_i = x_i + \sum_{j=d+1}^{s} a_{i,j} x_j \qquad \text{where} \quad a_{i,j} \in k$$

for $1 \leq i \leq d$ from the beginning.

Let L be a positive integer divided by $L.C.M.(\deg x_1, \dots, \deg x_d)$. By Lech's lemma (Theorem 14.12 [12])

$$e\left(\left(y_1^{\frac{L}{q_1}}, \dots, y_d^{\frac{L}{q_d}}\right), A\right) = \frac{L}{q_1}. \dots .\frac{L}{q_d}.e((y_1, \dots, y_d), A).$$

Since $y_i^{L/q_i} \in F^L$ for $1 \leq i \leq d$, we have

$$e(F^L, A) \leq \frac{L}{q_1}. \dots .\frac{L}{q_d}.e((y_1, \dots, y_d), A).$$

There is an integer L as above and satisfies the relation $F^{mL} = (F^L)^m$ for any positive integer m. By Lemmas 1.3 and 1.5, we obtain the assertion.

(1.7) *Proof of* (2). Let $y_1, \dots, y_d$ be a parameter system of A as in the arguments of (1.6). First we assume the equality (i) holds. By the arguments in (1.6), we have the equality

$$e\left(\left(y_1^{\frac{L}{q_1}}, \dots, y_d^{\frac{L}{q_d}}\right), A\right) = e(F^L, A).$$

By a Theorem of Rees ([16], [5]), $(y_1^{L/q_1}, \dots, y_d^{L/q_d})$ is a reduction of F^L. That is there is an integer $r > 0$ such that $(F^L)^{r+1} = (F^L)^r(y_1^{L/q_1}, \dots, y_d^{L/q_d})$ in A. Let $\psi : X = \mathrm{Proj}(\oplus_{k\geq 0} F^k.T^k) \to \mathrm{Spec}(A)$ the filtered blowing-up of $\mathrm{Spec}(A)$ by F. On X this leads the relation

$$\left(F^L\right)^{r+1} O_X = \left(F^L\right)^r (y_1^{\frac{L}{q_1}}, \dots, y_d^{\frac{L}{q_d}})O_X \qquad \text{in} \quad O_X.$$

Here $F^L O_X = O_X(L)$ is an invertible O_X-module sheaf. Hence we obtain the relation

$$F^L O_X = O_X(L) = (y_1^{\frac{L}{q_1}}, \dots, y_d^{\frac{L}{q_d}})O_X.$$

We represent the strict transform of the scheme $\mathrm{Spec}(A/y_i)$ by ψ as $W_{y_i,\psi}$ for $i = 1, \dots, d$. Since $(y_1^{L/q_1}, \dots, y_d^{L/q_d})O_X$ is base point free, $W_{y_1,\psi} \cap \dots \cap W_{y_d,\psi} \cap E$ is empty. Here $W_{y_j,\psi} \cap E = \mathrm{Proj}(G/in(y_j)G)$, where $in(y_j)$ is the initial homogeneous element of y_j. Therefore $in(y_1), \dots, in(y_d)$ is a parameter system of G. By Lemma 1.5, we obtain the relation

$$\begin{aligned} e(G_+, G) &\leq e((in(y_1), \dots, in(y_d)), G) \\ &= \left(\prod_{i=1}^{d} \deg x_i\right) \lim_{\lambda \to 1} (1-\lambda)^d P(G, \lambda) = e(m, A). \end{aligned}$$

By Lemma 1.2 these are the same.

The proof of the converse implication follows from Lemma 1.5.

(1.8) *Proof of the inequality* (ii) *of* (1). As in (1.5), there is an integer L satisfies the relation $G|_{mL} = (G|_L)^m$ and we have $e(G|_L, G) = L^d. \lim_{\lambda\to 1}(1-\lambda)^d P(G,\lambda)$ by (1.3). We can easily see $G|_{q_sL} \subset (G_+)^L$. Hence

$$\begin{aligned} L^d e(G_+, G) = e((G_+)^L, G) \leq e(G|_{q_sL}, G) = e((G|_L)^{q_s}, G) = q_s^d e(G|_L, G) \\ = q_s^d . L^d . \lim_{\lambda\to 1}(1-\lambda)^d P(G,\lambda) \end{aligned}$$

Therefore $e(G_+, G) \leq q_s^d. \lim_{\lambda\to 1}(1-\lambda)^d P(G,\lambda)$.

(1.9) *Proof of* (3). Assume that the equality $e(G_+, G) = q_s^d. \lim_{\lambda\to 1}(1-\lambda)^d P(G,\lambda)$ holds. In the arguments of (1.8), we have the relation $e((G_+)^L, G) = e(G|_{q_sL}, G)$. We can choose a sufficiently large L where $O_C(q_sL) = G|_{q_sL}O_C \subset O_C$ is an invertible O_C-module sheaf. By a theorem of Rees ([16], [5]), $G|_{q_sL}$ is a reduction of G_+^L. As in (1.5) we obtain the relation $G_+^L O_C = O_C(q_sL) \subset O_C(q_1L)$. We will show $q_1 = q_s$. Otherwise $q_1 < q_s$. Then $0 \neq x_1^L \in G_{q_1L}$, hence $x_1^L \in G|_{q_1L} - G|_{q_1L+1}$ and $q_1L + 1 \leq q_sL$. On $D_+((x_1^L)^*)$, $O_C(q_1L)|_{D_+((x_1^L)^*)} = x_1^L O_C|_{D_+((x_1^L)^*)}$ and $x_1^L O_C \not\subset O_C(q_1L+1)$ (cf. [20] for a generality of filtered blowing). Hence $G_+^L O_C \subset_{\neq} O_C(q_sL)$. Therefore we obtain the relation $q_1 = q_s$. Now we set this integer by q. Then we obtain the relations $F^k = m^{[(k+q-1)/q]}$ for $k \geq 0$ and $G \cong gr_m(A)$ as rings. Therefore, we obtain $e(G_+, G) = e(m, A)$.

§2. Remarks on Normal Graded Rings

The purpose of this section is to collect the generalities of the number $(\prod_{i=1}^d \deg x_i) \lim_{\lambda\to 1}(1-\lambda)^d P(G,\lambda)$ in the case G is normal and represented by Demazure's construction $R(E, D) = G$. Please refer [1], [21] for the basic facts on Demazure's construction of normal graded rings. The following proposition is a joint work with K.-i. Watanabe.

Proposition 2.1. *Let $R = R(E, D)$ be a normal d-dimensional graded ring with Demazure's description. Then we have the following.*

$$D^{d-1} = \lim_{\lambda\to 1}(1-\lambda)^d P(R,\lambda)$$

where $P(R,\lambda) = \sum_{k\geq 0} l(R_k)\lambda^k \in \mathbf{Z}[[\lambda]]$, with $d = \dim R$.

Proof. If R is generated by R_1, then D is Cartier. Then we have $e(R_+, R) = \lim_{\lambda\to 1}(1-\lambda)^d P(R,\lambda)$ [12], and the assertion $e(R_+, R) = D^{d-1}$

is standard. Now we will show the assertion in general. Let N be a positive integer such that $R^{(N)} = R(E, ND)$ is generated by R_N. Let $x_1, \dots, x_d$ be a homogeneous parameter system of R. Then $x_1^N, \dots, x_d^N$ is a parameter of $R^{(N)}$. Hence we have $N^d e((x_1, \dots, x_d), R) = e((x_1^N, \dots, x_d^N), R) = \mathrm{rank}_{R^{(N)}} R \cdot e((x_1^N, \dots, x_d^N), R^{(N)})$. By (1.5), $e((x_1^N, \dots, x_d^N), R^{(N)}) = (\prod_{i=1}^d \deg x_i) \times \lim_{\lambda \to 1}(1-\lambda)^d P(R^{(N)}, \lambda) = (\prod_{i=1}^d \deg x_i)(ND)^{d-1}$. Here remark that the degree of x_i^N in $R^{(N)}$ is $\deg x_i$. Hence $N^d e((x_1, \dots, x_d), R) = N^d (\prod_{i=1}^d \deg x_i) D^{d-1}$ follows. On the other hand we have the relation (1.5): $e((x_1, \dots, x_d), R) = (\prod_{i=1}^d \deg x_i) \lim_{\lambda \to 1}(1-\lambda)^d P(R, \lambda)$. This completes the proof.

Example 2.2. For a graded complete intersection

$$R = k[x_1, \dots, x_{d+s}]/(f_1, \dots, f_s),$$

where $f_1, \dots, f_s$ is a homogeneous regular sequence of $k[x_1, \dots, x_{d+s}]$, we have

$$P(R, \lambda) = \frac{(1-\lambda^{\deg f_1}).\ \dots .(1-\lambda^{\deg f_s})}{(1-\lambda^{\deg x_1}).\ \dots .(1-\lambda^{\deg x_{d+s}})}.$$

Hence

$$\lim_{\lambda \to 1}(1-\lambda)^d P(R, \lambda) = \frac{(\deg f_1).\ \dots .(\deg f_s)}{(\deg x_1).\ \dots .(\deg x_{d+s})}.$$

If R is normal and represented as $R = R(E, D)$, then (2.1) implies the following the relation

$$D^{d-1} = \frac{(\deg f_1).\ \dots .(\deg f_s)}{(\deg x_1).\ \dots .(\deg x_{d+s})}.$$

What information determines this number? Next we prove the following lemma to obtain Corollary 2.4.

Lemma 2.3. *Let a system of homogeneous elements $x_1, \dots, x_s$ of G be a minimal generator of the homogeneous maximal ideal G_+ as same as in Theorem* A. *If G is a normal domain over an algebraically closed field $k = G_0$, then any couple x_i and x_j with $i \neq j$ are algebraically independent over k.*

Proof (*Based on the idea due to Kyoji Saito*). We will show this by a contradiction. Suppose that x_i and x_j is algebraically dependent over $k = G_0$ for some pair i, j with $i \neq j$. There is a weighted homogeneous irreducible polynomial $P(s,t) \in k[s,t]$ such that

$$k[s,t]/P = k[x_i, x_j] \subset G.$$

Here $k[x_i, x_j] \cap G_+$ is the homogeneous maximal ideal of $k[x_i, x_j]$. By taking the completions we obtain the following

$$k[[s,t]]/P = k[[x_i, x_j]] \subset G^\wedge.$$

Here $x_i, x_j \in G_+ \cdot G^\wedge$. Since G is essentially of finite type over k, $G^\wedge$ is also a normal domain. There exists $u \in G_+ \cdot G^\wedge$ where the normalization of $k[[x_i, x_j]]$ is written as $k[[u]]$ and we have

$$k[[x_i, x_j]] \subset k[[u]] \subset G^\wedge.$$

There are $a, b \in k$ such that

$$x_i - a \cdot u, \quad x_j - b \cdot u \in u^2 k[[u]] \subset G_+^2 \cdot G^\wedge.$$

Hence

$$a \cdot x_j - b \cdot x_i \in G_+^2 \cdot G^\wedge \cap G = G_+^2.$$

Since x_i and x_j are linearly independent in G_+/G_+^2, we have $a = b = 0$. Hence $x_i \in G_+^2 \cdot G^\wedge \cap G = G_+^2$. This is a contradiction. □

Corollary 2.4. *Let $x_1, \dots, x_s$ be a minimal generator of the homogeneous maximal ideal G_+ as same as in Theorem* A. *If G is a normal domain over an algebraically closed field $k = G_0$, then the lower first three degrees* $\deg x_1$, $\deg x_2$, $\deg x_3$ *are completely determined from the Poincare-Hilbert series $P(G, \lambda)$. In particular, in the case* $\dim G \leq 3$,

$$\left(\prod_{i=1}^{d} \deg x_i\right) \lim_{\lambda \to 1} (1-\lambda)^d P(G, \lambda)$$

is determined by $P(G, \lambda)$.

The following example says that $e(m, A)$ and $e(G_+, G)$ are different, in general, even if we assume that G is a normal Gorenstein domain.

Example 2.5. We introduce the filtration F on the local ring A as follows:

(i) Let $A = k[[x,y,z,w]]/(w - x^2 - y^2 - z^2) \cong k[[x,y,z]]$ and F the induced filtration from the filtration on $k[[x,y,z,w]]$ by the monomial degrees as $F^k = \{x^a y^b z^c w^d \in k[x,y,z,w] \mid a+b+c+3d \geq k\}A$. One can easily see that

$$G = gr_F(A) \cong k[x,y,z,w]/x^2 + y^2 + z^2.$$

Theorem A says

$$1.1.1.\lim_{\lambda\to 1}(1-\lambda)^d P(G,\lambda) = 1.1.1.\frac{2}{1.1.1.3} \le 1 = e(m,A) \le e(G_+,G) = 2.$$

(ii) Let $A = k[[u,x,y,z,w]]/(u+y^3+z^7+w^{21}, x^2-uw) \cong k[[x,y,z,w]]/(x^2+w(y^3+z^7+w^{21}))$ and F the induced filtration from the filtration on $k[[u,x,y,z,w]]$ by the monomial degrees as $F^k = \{u^\alpha x^\beta y^\gamma z^\delta w^\varepsilon \in k[u,x,y,z,w] \mid 23\alpha+12\beta+7\gamma+3\delta+\varepsilon \ge k\}A$. One can easily see that

$$G = gr_F(A) \cong k[u,x,y,z,w]/(y^3+z^7+w^{21}, x^2-uw).$$

Theorem A says

$$\begin{aligned}1.3.7.\lim_{\lambda\to 1}(1-\lambda)^d P(G,\lambda) &= 1.3.7.\frac{21.24}{1.3.7.12.23} \le 2\\ &= e(m,A) \le e(G_+,G) = 6.\end{aligned}$$

Corollary 2.6. *Let the situation be as in Theorem* A.

(1) *If the condition*

$$\textit{the round up of the number } \left(\prod_{i=1}^{d}\deg x_i\right)\lim_{\lambda\to 1}(1-\lambda)^d P(G,\lambda) = e(G_+,G)$$

holds, then the equality $e(m,A) = e(G_+,G)$ *holds.*

(2) *If* G *is a hypersurface with the isolated singularity at* G_+*, then* $e(m,A) = e(G_+,G)$.

Proof. (1) is obvious from (1) of Theorem A. (2) Let us represent G as $G = k[x_1,\dots,x_{d+1}]/f$ by a weighted homogeneous polynomial f of the type $(q_1,\dots,q_{d+1};h)$ with $\deg x_i = q_i$ and $q_1 \le \cdots \le q_{d+1}$. By (2.2) and Theorem A, we have $h/(q_{d+1}) = q_1\dots q_d \lim_{\lambda\to 1}(1-\lambda)^d P(G,\lambda) \le e(G_+,G) = \operatorname{ord} f$. Since $\{f=0\}$ has only isolated singularity at o, a monomial of form $x_i^{m_i}x_{j(i)}$ with $j(i)\in\{1,\dots,d+1\}$ appears in f with non-zero coefficients for each i (K. Saito [17]). This implies $\operatorname{ord} f \le m_{d+1}+1 < h/q_{d+1}+1$. Hence $e(G_+,G)$ equals the round up of the rational number h/q_{d+1}.

Chapter II. The Inequality $q_1q_2q_3D^2 \ge 2$ for Normal K3 Surfaces

The rest of this paper is devoted to the proof of Theorem C. From here we will set $q_i = \deg x_i$ by simplicity. Our basic strategy is as follows: By

Corollary 2.4, the behavior of $b_m = \dim H^0(E, O_E(mD))$ determines q_1, q_2, q_3. Then Reid's singular Riemann-Roch formula for surfaces with rational double points is very effective. In fact, using (3.4.1), we can see that first 4 b_m determines the essential information on baskets in many cases. As a corollary of our proof, we can classify the data $\{D^2$, baskets of singularities$\}$ where the relation $q_1q_2q_3D^2 = 2$ holds.

§3. Some Easy Cases

In this section, we will derive convenient formulas from Reid's Riemann-Roch formula for normal surface and prove the assertion of Theorem C in some easy general cases. Now we recall the following famous formula.

Theorem 3.1 (M. Reid [15, (9.1)]).

(I) *There is a formula*

$$\chi(E, O_E(D)) = \chi(O_E) + \frac{1}{2}(D^2 - DK_E) + \sum_Q c_Q(D)$$

where $c_Q(D) = c_Q(O_E(D)) \in \mathbf{Q}$ is a contribution due to the singularity of $O_E(D)$ at Q, depending only on the local analytic type of $Q \in E$ and D; the sum takes place over the singularities of D (the points $Q \in E$ at which D is not Cartier).

(II) *If $P \in E$ and D is a cyclic quotient singularity of type ${}_i((1/r)(1, -1))$ then $c_P(D) = -(i(r - i))/2r$.*

(III) *For every rational double singularity $Q \in E$ and Weil divisor D on E, there exists a basket of points of $\{P_\alpha \in E_\alpha$ and $D_\alpha\}$ of type ${}_{i_\alpha}((1/r_\alpha)(1, -1))$ and with i_α coprime to r_α, such that*

$$c_Q(D) = \sum_\alpha c_{P_\alpha}(D_\alpha) = -\sum_\alpha \frac{i_\alpha(r_\alpha - i_\alpha)}{2r_\alpha}.$$

Remark 3.2. (i) This theorem says that the correction terms for the Riemann-Roch formula from the rational double points can be counted by certain numbers of set of locally poralized singularities of A_r-types. Further concerning the polarization $i_\alpha \in \{1, \ldots, r_\alpha - 1\}$ defined by D in $\mathbf{Z}/r_\alpha\mathbf{Z} \cong Cl(A_{r_\alpha - 1})$, we may assume $2i_\alpha \leq r_\alpha$ by the symmetry of the behavior of $(\overline{ki_\alpha}(r_\alpha - \overline{ki_\alpha}))/2r_\alpha$ for $k \geq 0$. (ii) In the original proof of (III) of (3.1), M. Reid used some deformation arguments. We remark that this fact also is shown by other way, e.g. by using Giraud's version of Riemann-Roch formula [4].

(3.3) Let E be a normal K3 surface and D an ample integral divisor as in the situation of Theorem C. As in (3.2) (i), we assume the conditions $2i_\alpha \leq r_\alpha$ for all α in the basket of singularities. Let $b_m = \dim H^0(E, O_E(mD))$. Then, by Kawamata's vanishing theorem [11], we have

$$b_m = \frac{m^2D^2}{2} + 2 + \sum_Q c_Q(mD) \qquad \text{for} \quad m \geq 1.$$

Lemma 3.4. *Assume the conditions $2i_\alpha \leq r_\alpha$ for all α. Then we have the following relations.*

(1) $D^2 = 2(b_1 - 2) + \sum_{\alpha=1}^N ((i_\alpha(r_\alpha - i_\alpha))/r_\alpha)$,

(2) $\sum_{\alpha=1}^N i_\alpha = b_2 - 4b_1 + 6$,

(3) $b_m = m^2 - 3m + 2 + (-m^2 + 2m)b_1 + ((m^2 - m)/2)b_2 - \sum_{s\geq 1}\sum_\alpha (mi_\alpha - sr_\alpha)_+$, *where the symbol ℓ_+ is defined as* $\ell_+ = \max\{0, \ell\}$.

Proof. (1) is nothing but (3.3) for $m = 1$, and we have $b_1 = 2 + (D^2/2) - \sum_\alpha((i_\alpha(r_\alpha - i_\alpha))/2r_\alpha)$. By the assumption $2i_\alpha \leq r_\alpha$, we have $b_2 = 2 + 2D^2 - \sum_\alpha((2i_\alpha(r_\alpha - 2i_\alpha))/2r_\alpha)$. Hence we obtain the relation $b_2 - 4b_1 = -6 + \sum_\alpha i_\alpha$ and the assertion of (2) follows. We can also see the relation $D^2 = 2 - 2b_1 + b_2 - \sum_\alpha(((i_\alpha)^2)/r_\alpha)$. By these, we obtain the relation $b_m = m^2 - 3m + 2 + (-m^2+2m)b_1 + ((m^2-m)/2)b_2 + \sum_\alpha(mi_\alpha r_\alpha - m^2(i_\alpha)^2 - \overline{mi_\alpha}\cdot r_\alpha + (\overline{mi_\alpha})^2)/2r_\alpha$. If $(\ell - 1)r_\alpha < mi_\alpha \leq \ell r_\alpha$, then $\overline{mi_\alpha} = mi_\alpha - (\ell - 1)r_\alpha$, and we can easily see the relation $(mi_\alpha r_\alpha - m^2(i_\alpha)^2 - \overline{mi_\alpha} \cdot r_\alpha + (\overline{mi_\alpha})^2)/2r_\alpha = -\{(mi_\alpha - r_\alpha) + (mi_\alpha - 2r_\alpha) + \cdots + (mi_\alpha - (\ell - 1)r_\alpha)\}$. □

We start the proof of Theorem C by the following.

Lemma 3.5. *Let $k \geq 2$ and assume that $b_1 = b_2 = \cdots = b_{k-1} = 0$ and that $b_k \geq 2$. Then we obtain the inequality $q_1q_2q_3D^2 \geq 2$. Further $q_1q_2q_3D^2 = 2$ holds if and only if $k = 2$, $N = 8$, $(i_1, \ldots, i_8) = (1, \ldots, 1)$, and $(r_1, \ldots, r_8) = (2, 2, 2, 2, 2, 2, 2, 3)$.*

Proof. First assume that $b_k \geq 3$. Then $q_1 = q_2 = q_3 = k$. By (3.3), we obtain the relations: $k^2D^2 = 2(b_k-2) + \sum_\alpha((\overline{ki_\alpha}(r_\alpha - \overline{ki_\alpha}))/r_\alpha) \geq 2(b_k-2) \geq 2$. Since $k \geq 2$, $q_1q_2q_3D^2 \geq 4$ follows.

Now assume that $b_k = 2$. By the arguments in the above, we have $q_1 = q_2 = k, q_3 \geq k+1$, and $0 < k^2D^2 = \sum_\alpha((\overline{ki_\alpha}(r_\alpha - \overline{ki_\alpha}))/r_\alpha)$. There is an index α_0 such that $\overline{ki_{\alpha_0}} \neq 0$. We have the relation $q_1q_2q_3D^2 \geq k \cdot k \cdot (k+1)D^2 \geq (k+1)((r_{\alpha_0} - 1)/r_{\alpha_0}) \geq (k+1)/2$.

Hence if $k \geq 4$, we obtain $q_1q_2q_3D^2 \geq (5/2) > 2$.

Next assume $k = 3$. We have $b_1 = b_2 = 0$ and $b_3 = 2 - \sum_{\alpha=1}^{N}(3i_\alpha - r_\alpha)_+$ by (3.4.3). The condition $b_3 = 2$ implies $3 \leq 3i_\alpha \leq r_\alpha$ for all α. Since $\overline{3i_{\alpha_0}} \neq 0$, $r_{\alpha_0} \geq 4$. Hence, $q_1q_2q_3D^2 \geq 4((r_{\alpha_0} - 1)/r_{\alpha_0}) \geq 3$.

Finally assume $k = 2$. Since $\overline{2i_{\alpha_0}} \neq 0$, we have $r_{\alpha_0} \geq 3$. Hence $q_1q_2q_3D^2 \geq 3((r_{\alpha_0} - 1)/r_{\alpha_0}) \geq 2$. Further the equality $q_1q_2q_3D^2 = 2$ implies $r_{\alpha_0} = 3$, $i_{\alpha_0} = 1$ and $\overline{2i_\alpha} = 0$ for $\alpha \neq \alpha_0$. Since $(i_\alpha, r_\alpha) = 1$, we have $(r_\alpha, i_\alpha) = (2,1)$ for $\alpha \neq \alpha_0$. Hence $N = \sum_{\alpha=1}^{N} i_\alpha = 6 + b_2 - 4b_1 = 8$ and we obtain the assertions. □

(3.6) **In the case $b_1 \geq 3$.** We have $q_1 = q_2 = q_3 = 1$. By (3.3) for b_1, we have $D^2 \geq 2 + \sum_{\alpha=1}^{N}((i_\alpha(r_\alpha - i_\alpha))/r_\alpha) \geq 2$ (cf. the proof of (3.5)). Hence $q_1q_2q_3D^2 \geq 2$, and the equality holds if and only if $N = 0$ and $D^2 = 2$.

(3.7) **In the case $b_1 = 2$.** We have $q_1 = q_2 = 1, q_3 \geq 2$. Let $R_1 = \mathbf{C}x_1 + \mathbf{C}x_2$, then x_1, x_2 are algebraically independent over R_0 by Lemma 2.3. Hence $b_k \geq k + 1$.

First assume $b_2 \geq 4$. Then we have $q_3 = 2$. By (3.3) for b_2, we obtain the relation $q_1q_2q_3D^2 = 2D^2 \geq 2 + \sum_{\alpha=1}^{N}((2i_\alpha(r_\alpha - 2i_\alpha))/2r_\alpha) \geq 2$. Here the equality $2D^2 = 2$ holds if and only if $b_2 = 4$ and $r_\alpha = 2$ for all α. Then $i_\alpha = 1$ for all α, and we obtain $N = 2$ by (3.4.2).

Next assume $b_2 = 3$. Then we have $q_3 \geq 3$. By (3.4.2), we obtain $\sum_{\alpha=1}^{N} i_\alpha = 1$. Hence $N = 1$ and $i_1 = 1$. By (3.4), we obtain $D^2 = (r_1 - 1)/r_1$ and $b_3 = 5 - (3 - r_1)_+$. If $r_1 \geq 3$, then $b_3 = 5$ and $q_3 = 3$. We obtain $q_1q_2q_3D^2 = 3((r_1 - 1)/r_1) \geq 2$. The equality holds if and only if $r_1 = 3$. Finally assume $r_1 = 2$. Then $b_3 = 4, b_4 = 6$ and $q_3 = 4$. In this case we have the relation $q_1q_2q_3D^2 = 2$.

In the rest of this section, we shall study the case that there is $k \geq 2$ where $b_l = 0$ for $0 < l < k - 1$, $b_{k-1} = 1$, and $b_k \geq 2$.

We begin with the following:

Lemma 3.8. *Let $k \geq 3$. Assume that $b_l = 0$ for $1 \leq l \leq k-2$, $b_{k-1} = 1$ and that $b_k \geq 2$. Then $q_1q_2q_3D^2 > 2$.*

Proof. We have $q_1 = k-1, q_2 = q_3 = k$. By (3.3) we obtain the following $k^2D^2 = 2(b_k - 2) + \sum_\alpha((\overline{ki_\alpha}(r_\alpha - \overline{ki_\alpha}))/r_\alpha)$. If $b_k \geq 3$, we obtain the relations $q_1q_2q_3D^2 = (k-1)k^2D^2 \geq (k-1)2 \geq 4$.

Now assume $b_k = 2$. We have $0 < k^2D^2 = \sum_\alpha((\overline{ki_\alpha}(r_\alpha - \overline{ki_\alpha}))/r_\alpha)$. There is an index α_0 such that $\overline{ki_{\alpha_0}} \neq 0$. We have the relation $q_1q_2q_3D^2 = (k-1)k^2D^2 \geq (k-1)((r_{\alpha_0} - 1)/r_{\alpha_0}) \geq ((k-1)/2)$.

Hence if $k \geq 6$, we obtain $q_1q_2q_3D^2 \geq (5/2) > 2$.

Next assume $k = 5$. We will show the following.

Claim 3.8.1. $5^2D^2 = \sum_\alpha((\overline{5i_\alpha}(r_\alpha - \overline{5i_\alpha}))/r_\alpha) > (1/2)$.

Proof of 3.8.1. If not, we have $\sum_\alpha((\overline{5i_\alpha}(r_\alpha - \overline{5i_\alpha}))/r_\alpha) = (1/2)$. Then $(r_{\alpha_0}, i_{\alpha_0}) = (2,1)$ and $\overline{5i_\alpha} = 0$ for $\alpha \neq \alpha_0$. Hence $r_\alpha = 5$ for $\alpha \neq \alpha_0$. On the other hand, we have the relation $b_5 = 12 - \sum_\alpha(5i_\alpha - r_\alpha)_+ - \sum_\alpha(5i_\alpha - 2r_\alpha)_+$. Then we can easily show $b_5 \equiv 3 \pmod 5$. This contradicts to our assumption $b_5 = 2$.

Next we assume $k = 4$. We will show the following.

Claim 3.8.2. $4^2D^2 = \sum_\alpha((\overline{4i_\alpha}(r_\alpha - \overline{4i_\alpha}))/r_\alpha) > (2/3)$.

Proof of 3.8.2. If $\overline{4i_\alpha} \neq 0$, then $r_\alpha = 3$ or $r_\alpha \geq 5$. Then we have $(r_\alpha - 1)/r_\alpha = (2/3)$ or $\geq 4/5$ respectively. Hence we have $\sum_\alpha((\overline{4i_\alpha}(r_\alpha - \overline{4i_\alpha}))/r_\alpha) \geq (2/3)$. Suppose the equality holds here. Then there exists α_0 where $(r_{\alpha_0}, i_{\alpha_0}) = (3,1)$ and $\overline{4i_\alpha} = 0$ for $\alpha \neq \alpha_0$. Then $(r_\alpha, i_\alpha) = (4,1)$ or $(2,1)$. On the other hand, we have the relation $b_4 = 6 - \sum_\alpha(4i_\alpha - r_\alpha)_+$. Then we can easily show b_4 is odd. This contradicts to our assumption $b_4 = 2$.

Finally we assume $k = 3$. We will show the following.

Claim 3.8.3. $3^2D^2 = \sum_\alpha((\overline{3i_\alpha}(r_\alpha - \overline{3i_\alpha}))/r_\alpha) > 1$.

Proof of 3.8.3. Since $b_1 = 0, b_2 = 1$, we have $b_3 = 5 - \sum_\alpha(3i_\alpha - r_\alpha)_+$. Hence $\sum_\alpha(3i_\alpha - r_\alpha)_+ = 3$. Remark that $2i_\alpha \leq r_\alpha$ and $3i_\alpha < 2r_\alpha$ by assumption. So if $(3i_\alpha - r_\alpha)_+ \neq 0$, we have $\overline{3i_\alpha} = (3i_\alpha - r_\alpha) = (3i_\alpha - r_\alpha)_+$, and $2 \cdot \overline{3i_\alpha} \leq r_\alpha$.

If there is α_0 with $(3i_{\alpha_0} - r_{\alpha_0})_+ = 3$, then $3 \mid r_{\alpha_0}$ and we can see $r_{\alpha_0} \geq 6$. Hence $((\overline{3i_{\alpha_0}}(r_{\alpha_0} - \overline{3i_{\alpha_0}}))/r_{\alpha_0}) \geq (3/2) > 1$.

If there are α_1 and α_2 with $(3i_{\alpha_1} - r_{\alpha_1})_+ = 2$ and $(3i_{\alpha_2} - r_{\alpha_2})_+ = 1$, then we have $r_{\alpha_1} \geq 4$ and $r_{\alpha_2} \geq 2$. Hence we have $\sum_\alpha((\overline{3i_\alpha}(r_\alpha - \overline{3i_\alpha}))/r_\alpha) \geq (3/4) + (1/2) > 1$.

If there α's with $(3i_\alpha - r_\alpha)_+ \neq 0$, then we can see $\sum_\alpha((\overline{3i_\alpha}(r_\alpha - \overline{3i_\alpha}))/r_\alpha) \geq 3 \cdot (1/2) > 1$. □

Now we will assume $k = 2$.

(3.9) **In the case $b_1 = 1, b_2 \geq 3$.** We have $q_1 = 1, q_2 = q_3 = 2$. By (3.3) for b_2, we have $2D^2 \geq 1 + \sum_{\alpha=1}^N((2i_\alpha(r_\alpha - 2i_\alpha))/2r_\alpha) \geq 1$. Here the equality $q_1q_2q_3D^2 = 4D^2 = 2$ holds when $b_2 = 3$, $r_\alpha = 2$ for all α. Then we have $i_\alpha = 1$ for all α and $N = 5$ follows by (3.4.2).

(3.10) In the rest of this section, we assume that $\underline{b_1 = 1, b_2 = 2}$. We have $q_1 = 1, q_2 = 2, q_3 \geq 3$ the relations $\sum_{\alpha=1}^{N} i_\alpha = 4$, and $D^2 = 2 - \sum_{\alpha=1}((i_\alpha)^2/r_\alpha)$. We have the formulas: $b_3 = 5 - \sum_\alpha (3i_\alpha - r_\alpha)_+, b_4 = 10 - \sum_\alpha (4i_\alpha - r_\alpha)_+$. By the algebraic independence of two homogeneous elements (2.3), we have $b_3 \geq 2, b_4 \geq 3$. In the below we separate the arguments into cases depending on the type $(i_1, \ldots, i_N)$. Here we remark that if $D^2 > 1/3$, then $q_1 q_2 q_3 D^2 > 2$ by assumptions.

Case 3.10.1. $N = 4, (i_1, i_2, i_3, i_4) = (1, 1, 1, 1)$, with $2 \leq r_1 \leq r_2 \leq r_3 \leq r_4$.

We have $b_3 = 5 - \sum_{\alpha=1}^{4} (3 - r_\alpha)_+$.

(3.10.1.1) If $b_3 \geq 3$, then $r_3 \geq 3$ and $q_3 = 3$. Hence $D^2 = 2 - (1/r_1) - (1/r_2) - (1/r_3) - (1/r_4) \geq 2 - (1/2) \times 2 - (1/3) \times 2 = 1/3$ and we obtain $q_1 q_2 q_3 D^2 \geq 2$. The equality holds if and only if $r_1 = r_2 = 2, r_3 = r_4 = 3$.

(3.10.1.2) Next assume $b_3 = 2$. Then $r_1 = r_2 = r_3 = 2$ and $r_4 \geq 3$. Here we have $b_4 = 4 - (4 - r_4)_+$.

(3.10.1.2.1) If $b_3 = 2, b_4 = 4$, then $r_4 \geq 4$ and $q_3 = 4$. Hence we obtain $D^2 = (1/2) - (1/r_4) \geq 1/4$ and $q_1 q_2 q_3 D^2 \geq 2$. Here the equality holds if and only if $r_4 = 4$.

(3.10.1.2.2) If $b_3 = 2, b_4 = 3$ then $r_4 = 3$ and $D^2 = 1/6$. Now a direct computation shows $q_3 = 6$, hence $q_1 q_2 q_3 D^2 = 2$ holds in this case.

Case 3.10.2. $N = 3, (i_1, i_2, i_3) = (2, 1, 1)$ with $r_1 \geq 5$, $(r_1, 2) = 1$, and $2 \leq r_2 \leq r_3$.

We have $b_3 = 5 - (6 - r_1)_+ - (3 - r_2)_+ - (3 - r_3)_+$.

(3.10.2.1) If $b_3 \geq 3$, then we have $r_1 \geq 7$ or $r_3 \geq 3$. In these two cases, we can easily show $D^2 > 1/3$ and $q_1 q_2 q_3 D^2 > 2$.

(3.10.2.2) If $b_3 = 2$, then $r_1 = 5, r_2 = r_3 = 2$, and $D^2 = 1/5$. Now a direct computation shows $q_3 = 5$, hence $q_1 q_2 q_3 D^2 = 2$ holds.

Case 3.10.3. $N = 2, (i_1, i_2) = (3, 1)$, with $r_1 \geq 7, (r_1, 3) = 1, r_2 \geq 2$.

We have $b_3 = 5 - (9 - r_1)_+ - (3 - r_2)_+$.

(3.10.3.1) If $b_3 \geq 3$, then $r_1 \geq 8$ or $r_2 \geq 3$. In these two cases, we can easily show $D^2 > 1/3$ and $q_1 q_2 q_3 D^2 > 2$.

(3.10.3.2) If $b_3 = 2$, then $r_1 = 7, r_2 = 2$ and $D^2 = 3/14$. Now a direct computation shows $q_3 = 5$, hence $q_1 q_2 q_3 D^2 > 2$.

Case 3.10.4. $N = 2, (i_1, i_2) = (2, 2)$, with $5 \leq r_1 \leq r_2, (r_i, 2) = 1, i = 1, 2$.

We have $D^2 \geq 2 - (4/5) \times 2 = (2/5) > (1/3)$, hence $q_1 q_2 q_3 D^2 > 2$.

Case 3.10.5. $N = 1, i_1 = 4, r_1 \geq 9, (r_1, 4) = 1$.
We have $b_3 = 5 - (12 - r_1)_+$.

(3.10.5.1) If $b_3 \geq 3$, then $r_1 \geq 10$ and $D^2 \geq 2 - (16/11) = (6/11) > (1/3)$. Hence $q_1 q_2 q_3 D^2 > 2$.
(3.10.5.2) If $b_3 = 2$, then $r_1 = 9$ and $D^2 = 2/9$. Now a direct computation shows $q_3 = 5$, hence $q_1 q_2 q_3 D^2 > 2$.

§4. The Case $b_1 = b_2 = 0$ and $b_3 \leq 1$, $b_4 \leq 1$

(4.1) In this section, we assume that $\underline{b_1 = 0, b_2 = 0, b_3 \leq 1, b_4 \leq 1}$. By (3.4) we have $\sum_{\alpha=1}^{N} i_\alpha = 6$. Hence we have the following basic cases for $\mathcal{I} = (i_1, \ldots, i_N)$ and r_i's:

I. $N = 6$, $\mathcal{I} = (1,1,1,1,1,1)$ with $2 \leq r_1 \leq \cdots \leq r_6$.

II. $N = 5$, $\mathcal{I} = (2,1,1,1,1)$ with $r_1 \geq 5$, $(r_1, 2) = 1$, $2 \leq r_2 \leq \cdots \leq r_5$.

III. $N = 4$, $\mathcal{I} = (3,1,1,1)$ with $r_1 \geq 7$, $(r_1, 3) = 1$, $2 \leq r_2 \leq \cdots \leq r_4$.

IV. $N = 4$, $\mathcal{I} = (2,2,1,1)$ with $5 \leq r_1 \leq r_2$, $(r_1, 2) = (r_2, 2) = 1$, $2 \leq r_3 \leq r_4$.

V. $N = 3$, $\mathcal{I} = (4,1,1)$ with $r_1 \geq 9$, $(r_1, 4) = 1$, $2 \leq r_2 \leq r_3$.

VI. $N = 3$, $\mathcal{I} = (3,2,1)$ with $r_1 \geq 7$, $(r_1, 3) = 1$, $r_2 \geq 5$, $(r_2, 2) = 1$, $2 \leq r_3$.

VII. $N = 3$, $\mathcal{I} = (2,2,2)$ with $5 \leq r_1 \leq r_2 \leq r_3$, $(r_i, 2) = 1$ for $i = 1,2,3$.

VIII. $N = 2$, $\mathcal{I} = (5,1)$ with $r_1 \geq 11$, $(r_1, 5) = 1$, $r_2 \geq 2$.

IX. $N = 2$, $\mathcal{I} = (4,2)$ with $r_1 \geq 9$, $(r_1, 4) = 1$, $r_2 \geq 5$, $(r_2, 2) = 1$.

X. $N = 2$, $\mathcal{I} = (3,3)$ with $7 \leq r_1 \leq r_2$, $(r_i, 3) = 1$ for $i = 1, 2$.

XI. $N = 1$, $\mathcal{I} = (6)$ with $r_1 \geq 13$, $(r_1, 6) = 1$.

In these cases, we can show $q_1 q_2 q_3 D^2 \geq 2$ for I–VI and $q_1 q_2 q_3 D^2 > 2$ for VII–XI. In the below we will discuss the cases I–VI according to the types of b_3, b_4. We have the relations (3.4.3): $b_3 = 2 - \sum_\alpha (3i_\alpha - r_\alpha)_+, b_4 = 6 - \sum_\alpha (4i_\alpha - r_\alpha)_+$. By using these we can show the following.

Lemma 4.2. *Assume $b_3 = b_4 = 1$. Then, for the cases* I–VI*, one of the followings occurs.*

$I_{1,1}1 : r_1 = r_2 = r_3 = r_4 = 3, 4 \leq r_5 \leq r_6$.

$II_{1,1}1 : r_1 = 5, r_2 = r_3 = 3, 4 \leq r_4 \leq r_5$, $II_{1,1}2 : r_1 = 7, r_2 = 2, r_3 = r_4 = 3, r_5 \geq 4$, $II_{1,1}3 : r_1 \geq 9, r_2 = 2, r_3 = r_4 = r_5 = 3$.

$III_{1,1}1 : r_1 = 8, r_2 = 3, 4 \leq r_3 \leq r_4$, $III_{1,1}2 : r_1 = 10, r_2 = 2, r_3 = 3, r_4 \geq 4$, $III_{1,1}3 : r_1 = 11, r_2 = 2, r_3 = r_4 = 3$.

$IV_{1,1}1 : r_1 = 5, r_2 = 7, r_3 = 3, 4 \leq r_4$, $IV_{1,1}2 : r_1 = 5, r_2 \geq 9, r_3 = r_4 = 3$, $IV_{1,1}3 : r_1 = r_2 = 7, r_3 = 2, r_4 = 3$.

$V_{1,1}1 : r_1 = 11, 4 \leq r_2 \leq r_3$, $V_{1,1}2 : r_1 = 13, r_2 = 2, r_3 \geq 4$.

$VI_{1,1}1 : r_1 = 8, r_2 = 7, r_3 \geq 4$, $VI_{1,1}2 : r_1 = 8, r_2 \geq 9, r_3 = 3$, $VI_{1,1}3 : r_1 = 10, r_2 = 5, r_3 \geq 4$, $VI_{1,1}4 : r_1 = 11, r_2 = 5, r_3 = 3$, $VI_{1,1}5 : r_1 = 10, r_2 = 7, r_3 = 2$.

(4.3) *Proof of the Theorem* C *for the cases of* 4.2. Here we have $q_1 = 3, q_2 = 4, q_3 \geq 5$, and $D^2 = 2 - \sum_\alpha((i_\alpha)^2/r_\alpha)$. We can easily see $D^2 > (1/30)$ for $\mathrm{I}_{1,1}1$, $\mathrm{II}_{1,1}3$, $\mathrm{III}_{1,1}1$, $\mathrm{IV}_{1,1}1$, $\mathrm{IV}_{1,1}2$, $\mathrm{V}_{1,1}1$, $\mathrm{VI}_{1,1}1$, $\mathrm{VI}_{1,1}2$, $\mathrm{VI}_{1,1}3$, $\mathrm{VI}_{1,1}4$. Hence $q_1q_2q_3D^2 > 2$ holds for these cases.

($\mathrm{II}_{1,1}1$) We have $D^2 = (8/15) - (1/r_4) - (1/r_5) \geq 1/30$. Hence $q_1q_2q_3D^2 \geq 2$. Here the equality holds if and only if $r_4 = r_5 = 4$.

($\mathrm{II}_{1,1}2$) We have $D^2 = (11/42) - (1/r_5)$. If $r_5 \geq 5$, then $D^2 > (1/20)$ and $q_1q_2q_3D^2 > 2$. Now assume $r_5 = 4$. Then $D^2 = 1/84$. A computation shows $q_3 = 14$. Hence $q_1q_2q_3D^2 = 2$ follows.

($\mathrm{III}_{1,1}2$) We have $D^2 = (4/15) - (1/r_4)$. If $r_4 \geq 5$, then $D^2 > (1/30)$ and $q_1q_2q_3D^2 > 2$. Now assume $r_4 = 4$. Then $D^2 = 1/60$. A computation shows $q_3 = 10$. Hence $q_1q_2q_3D^2 = 2$ follows.

($\mathrm{III}_{1,1}3$) We have $D^2 = 1/66$. A computation shows $q_3 = 11$. Hence $q_1q_2q_3D^2 = 2$ follows.

($\mathrm{IV}_{1,1}3$) We have $D^2 = 1/42$. A computation shows $q_3 = 7$. Hence $q_1q_2q_3D^2 = 2$ follows.

($\mathrm{V}_{1,1}2$) We have $D^2 = (7/26) - (1/r_3)$. If $r_3 \geq 5$, then $D^2 > (1/30)$ and $q_1q_2q_3D^2 > 2$. Now assume $r_3 = 4$. Then $D^2 = 1/52$. A computation shows $q_3 = 10$. Hence $q_1q_2q_3D^2 > 2$.

($\mathrm{VI}_{1,1}5$) We have $D^2 = 1/35$. A computation shows $q_3 = 7$. Hence $q_1q_2q_3D^2 > 2$.

Lemma 4.4. *Assume $b_3 = 1, b_4 = 0$. Then, for the cases* I–VI*, one of the followings occurs.*

$I_{1,0}1 : r_1 = 2, r_2 = r_3 = r_4 = r_5 = 3, r_6 \geq 4$.

$II_{1,0}1: r_1 = 5, r_2 = r_3 = r_4 = 3, r_5 \geq 4$, $II_{1,0}2: r_1 = 7, r_2 = 2, r_3 = r_4 = r_5 = 3$.

$III_{1,0}1: r_1 = 8, r_2 = r_3 = 3, r_4 \geq 4$, $III_{1,0}2: r_1 = 10, r_2 = 2, r_3 = r_4 = 3$.

$IV_{1,0}1: r_1 = 5, r_2 = 7, r_3 = r_4 = 3$.

$V_{1,0}1: r_1 = 11, r_2 = 3, r_4 \geq 4$, $V_{1,0}2: r_1 = 13, r_2 = 2, r_3 = 3$.

$VI_{1,0}1: r_1 = 8, r_2 = 7, r_3 = 3$, $VI_{1,0}2: r_1 = 10, r_2 = 5, r_3 = 3$.

(4.5) *Proof of the Theorem* C *for the cases of* 4.4. Here we have $q_1 = 3, q_3 \geq q_2 \geq 5$, and $D^2 = 2 - \sum_\alpha ((i_\alpha)^2/r_\alpha)$. We can easily see $D^2 < 0$ for $\mathrm{II}_{1,0}2$, $\mathrm{III}_{1,0}2$, $\mathrm{IV}_{1,0}1$, $\mathrm{V}_{1,0}2$, $\mathrm{VI}_{1,0}1$, $\mathrm{VI}_{1,0}2$. Hence these cases do not happen.
($\mathrm{I}_{1,0}1$) Since $0 < D^2 = (1/6) - (1/r_6)$, we have $r_6 \geq 7$. If $r_6 \geq 8$, then $D^2 \geq 1/24$ and $q_1q_2q_3D^2 > 2$. Now assume $r_6 = 7$. Then $D^2 = 1/42$. A direct computation shows $(q_1, q_2, q_3) = (3, 6, 7)$. Hence $q_1q_2q_3D^2 = 3$.
($\mathrm{II}_{1,0}1$) Since $0 < D^2 = (1/5) - (1/r_5)$, we have $r_5 \geq 6$. Hence $D^2 \geq (1/30)$ and $q_1q_2q_3D^2 > 2$.
($\mathrm{III}_{1,0}1$) Since $0 < D^2 = (5/24) - (1/r_4)$, we have $r_4 \geq 5$. If $r_4 \geq 6$, then $D^2 \geq 1/24$. Hence $q_1q_2q_3D^2 > 2$. Now assume $r_4 = 5$. Then $D^2 = 1/120$. Here we can show $(q_1, q_2, q_3) = (3, 5, 16)$. Hence we obtain $q_1q_2q_3D^2 = 2$.
($\mathrm{V}_{1,0}1$) Since $0 < D^2 = (7/33) - (1/r_3)$, we have $r_3 \geq 5$. If $r_3 \geq 6$, then $D^2 > (1/30)$ and $q_1q_2q_3D^2 > 2$. Now assume $r_3 = 5$. Then $D^2 = 2/165$. Here we can show $(q_1, q_2, q_3) = (3, 5, 11)$. Hence we obtain $q_1q_2q_3D^2 = 2$.

Lemma 4.6. *Assume $b_3 = 0, b_4 = 1, b_5 \leq 1$. Then, for the cases* I–VI, *one of the followings occurs.*

$I_{0,1}1: r_1 = r_2 = 2, r_3 = 3, r_4 = 4 \leq r_5 \leq r_6$.

$II_{0,1}1: r_1 = 5, r_2 = 2, r_3 = r_4 = 4 \leq r_5$, $II_{0,1}2: r_1 = 7, r_2 = r_3 = 2, r_4 \geq 4, r_5 \geq 5$, $II_{0,1}3: r_1 = 9, r_2 = r_3 = 2, r_4 = 3, r_5 \geq 5$, $II_{0,1}4: r_1 \geq 11, r_2 = r_3 = 2, r_4 = 3, r_5 = 4$.

$III_{0,1}1: r_1 = 7, r_2 = r_3 = 4, r_4 \geq 4$, $III_{0,1}2: r_1 = 13, r_2 = r_3 = 2, r_4 = 3$, $III_{0,1}3: r_1 = 14, r_2 = r_3 = 2, r_4 = 3$, $III_{0,1}4: r_1 = 11, r_2 = r_3 = 2, r_4 \geq 4$.

$IV_{0,1}1: r_1 = 5, r_2 = 9, r_3 = 2, r_4 = 4$, $IV_{0,1}2: r_1 = 7, r_2 \geq 9, r_3 = r_4 = 2$.

$V_{0,1}1: r_1 = 15, r_2 = r_3 = 2$.

$VI_{0,1}1: r_1 = 7, r_2 = 9, r_3 = 4$, $VI_{0,1}2: r_1 \geq 10, r_2 = 5, r_3 = 2$.

(4.7) *Proof of the Theorem* C *for the cases of* 4.6. Here we have $q_1 = 4, q_2 \geq 5, q_3 \geq 6$, and $D^2 = 2 - \sum_\alpha ((i_\alpha)^2/r_\alpha)$. We can easily see $D^2 > (1/60)$ for $\mathrm{II}_{1,0}3$, $\mathrm{II}_{1,0}4$, $\mathrm{III}_{0,1}3$. Hence $q_1q_2q_3D^2 > 2$ holds for these cases. Further we can see $D^2 < 0$ for $\mathrm{III}_{0,1}2$ and $\mathrm{V}_{0,1}2$. Hence these cases do not happen.
($\mathrm{I}_{0,1}1$) We have $b_5 = 1 - (5 - r_5)_+ - (5 - r_6)_+$.

If $b_5 = 1$, then $5 \leq r_4 \leq r_5$. Hence $D^2 = (5/12) - (1/r_5) - (1/r_6) \geq 1/60$. Hence $q_1q_2q_3D^2 \geq 2$. Here the equality holds if and only if $r_5 = r_6 = 5$.

If $b_5 = 0$, then $r_5 = 4, r_6 \geq 5$. Since $0 < D^2 = (1/6) - (1/r_6)$, we have $r_6 \geq 7$. Hence $D^2 > (1/60)$, and $q_1q_2q_3D^2 > 2$.

($\mathrm{II}_{0,1}1$) Since $0 < D^2 = (1/5) - (1/r_5)$, we have $r_5 \geq 6$. Hence then $D^2 > (1/60)$ and $q_1q_2q_3D^2 > 2$.

($\mathrm{II}_{0,1}2$) We have $b_5 = 1 - (5 - r_4)_+ - (5 - r_5)_+$.

If $b_5 = 1$, then $5 \leq r_4 \leq r_5$. Hence $D^2 = (3/7) - (1/r_4) - (1/r_5) > 1/60$. Hence $q_1q_2q_3D^2 > 2$.

If $b_5 = 0$, then $r_4 = 4, r_5 \geq 5$. Since $0 < D^2 = (5/28) - (1/r_5)$, we have $r_5 \geq 6$. Here $D^2 \geq (1/84)$. Further we can see $b_6 = 1$ and $q_2 = 6, q_3 \geq 7$. Hence $q_1q_2q_3D^2 \geq 2$ follows. We can check the equality holds when $r_5 = 6$.

($\mathrm{III}_{0,1}1$) Since $0 < D^2 = (3/14) - (1/r_4)$, we have $r_4 \geq 5$. If $r_4 \geq 6$, we have $D^2 > (1/60)$ and $q_1q_2q_3D^2 > 2$. Now assume $r_4 = 5$. Then we can see $(q_1, q_2, q_3) = (4, 5, 7)$. Hence $q_1q_2q_3D^2 = 2$ follows.

($\mathrm{III}_{0,1}4$) Since $0 < D^2 = (2/11) - (1/r_4)$, we have $r_4 \geq 6$. Here we have $D^2 \geq (1/66)$. We can see $b_5 = 0$ and $q_2 \geq 6$. Hence $q_1q_2q_3D^2 > 2$ follows.

($\mathrm{IV}_{0,1}1$) We can check $D^2 = 1/180$ and $(q_1, q_2, q_3) = (4, 5, 18)$. Hence $q_1q_2q_3D^2 = 2$ holds.

($\mathrm{IV}_{0,1}2$) Since $0 < D^2 = (3/7) - (4/r_2)$, we have $r_2 \geq 11$. Here $D^2 > (1/60)$ and $q_1q_2q_3D^2 > 2$.

($\mathrm{VI}_{0,1}1$) We can check $D^2 = 5/252$ and $(q_1, q_2, q_3) = (4, 5, 7)$. Hence $q_1q_2q_3D^2 > 2$ holds.

($\mathrm{VI}_{0,1}2$) Since $0 < D^2 = (7/10) - (9/r_1)$, we have $r_1 \geq 13$ and $D^2 \geq (1/130)$. If $r_1 \geq 14$, then $D^2 \geq (4/70) > (1/30)$. Hence $q_1q_2q_3D^2 > 2$. Now assume $r_1 = 13$. Here we can show $(q_1, q_2, q_3) = (4, 5, 13)$ and $q_1q_2q_3D^2 = 2$ follows.

Lemma 4.8. *Assume $b_3 = b_4 = 0, b_5 \leq 1, b_6 \leq 1$. Then, for the cases* I–VI, *one of the followings occurs.*

$I_{0,0}1 : r_1 = r_2 = 2, r_3 = r_4 = 3, 4 \leq r_5 \leq r_6$.

$II_{0,0}1 : r_1 = 5, r_2 = 2, r_3 = 3, 4 \leq r_4 \leq r_5$, $II_{0,0}2 : r_1 \geq 9, r_2 = r_3 = 2, r_4 = r_5 = 3$, $II_{0,0}3 : r_1 = 7, r_2 = r_3 = 2, r_4 = 3, r_5 \geq 4$.

$III_{0,0}1 : r_1 = 7, r_2 = 3, 4 \leq r_3 \leq r_4$, $III_{0,0}2 : r_1 = 8, r_2 = 2, 4 \leq r_3 \leq r_4$, $III_{0,0}3 : r_1 = 10, r_2 = r_3 = 2, r_4 \geq 4$, $III_{0,0}4 : r_1 = 11, r_2 = r_3 = 2, r_4 = 3$.

$IV_{0,0}1 : r_1 = r_2 = 5, 4 \leq r_3 \leq r_4$, $IV_{0,0}2 : r_1 = 5, r_2 = 7, r_3 = 2, r_4 \geq 4$, $IV_{0,0}3 : r_1 = 5, r_2 \geq 9, r_3 = 2, r_4 = 3$. $IV_{0,0}4 : r_1 = r_2 = 7, r_3 = r_4 = 2$.

$V_{0,0}$: There are no such cases.

$VI_{0,0}1 : r_1 = r_2 = 7, r_3 \geq 4$, $VI_{0,0}2 : r_1 = 7, r_2 \geq 9, r_3 = 3$ $VI_{0,0}3 : r_1 = 8, r_2 \geq 7, r_3 = 2$.

(4.9) *Proof of the Theorem* C *for the cases of* 4.8. Here we have $q_1 \geq 5, q_2 \geq 6, q_3 \geq 7$, and $D^2 = 2 - \sum_\alpha ((i_\alpha)^2/r_\alpha)$. We can easily see $D^2 > (1/105)$ for $\mathrm{V}_{0,0}1$. Hence $q_1q_2q_3D^2 > 2$ holds for these cases. Further we can see $D^2 < 0$ for $\mathrm{III}_{0,0}4$, $\mathrm{IV}_{0,0}4$. Hence these cases do not happen.

($\mathrm{I}_{0,0}1$) Since $b_5 = -(5-r_5)_+ - (5-r_6)_+ \leq 0$, we have $b_5 = 0$ and $r_5 \geq 5$. Here $b_6 = 2 - (6-r_5)_+ - (6-r_6)_+$. We have assumed that $b_6 \leq 1$. Hence $r_5 = 5$. Since $0 < D^2 = (2/15) - (1/r_6)$, we have $r_6 \geq 8$. Now $b_6 = 1$. We obtain $q_1 = 6, q_2 \geq 7$ and $D^2 \geq (1/120)$. Hence $q_1q_2q_3D^2 > 2$.

($\mathrm{II}_{0,0}1$) We have $b_5 = 1 - (5-r_4)_+ - (5-r_5)_+$.

If $b_5 = 1$, then $r_4 \geq 5$. Since we have assumed $1 \geq b_6 = 2-(6-r_4)_+ - (6-r_5)$, $r_4 = 5$. Since $0 < D^2 = (1/6) - (1/r_5)$, we have $r_5 \geq 7$ and $D^2 \geq (1/42)$. Hence $q_1q_2q_3D^2 > 2$.

Now assume $b_5 = 0$. We have $r_4 = 4, r_5 \geq 5$. Since $0 < D^2 = (7/60) - (1/r_5)$, we have $r_5 \geq 9$ and $D^2 \geq (1/180)$. On the other hand we have $b_6 = -(6-r_5)_+ = 0$. We may assume $b_7 \leq 1, b_8 \leq 1$ by Section 3. Hence $q_1q_2q_3D^2 > 2$.

($\mathrm{II}_{0,0}2$) Since $0 < D^2 = (1/3) - (4/r_1)$, we have $r_1 \geq 13$ and $D^2 \geq (1/39)$. Hence $q_1q_2q_3D^2 > 2$.

($\mathrm{II}_{0,0}3$) We have $b_5 = -1 - (5-r_5)_+ \leq -1$. Hence this case dose not happen.

($\mathrm{III}_{0,1}1$) We have $b_5 = 1 - (5-r_3)_+ - (5-r_4)_+$.

If $b_5 = 1$, then $r_3 \geq 5$. Since we have assumed $1 \geq b_6 = 2 - (6-r_3)_+ - (6-r_4)_+$, we obtain $r_3 = 5$. Since $0 < D^2 = (19/105) - (1/r_4)$, we have $r_4 \geq 6$ and $D^2 \geq (1/70)$. Therefore $q_1q_2q_3D^2 > 2$.

Now assume $b_5 = 0$. We have $r_3 = 4, r_4 \geq 5$. Now $b_6 = -(6-r_4)_+$, hence $b_6 = 0$ and $r_4 \geq 6$. Since $0 < D^2 = (11/84) - (1/r_4)$, we have $r_4 \geq 8$ and $D^2 \geq (1/168)$. We may assume $b_7 \leq 1, b_8 \leq 1$ by Section 3. Hence $q_1q_2q_3D^2 > 2$.

($\mathrm{III}_{0,0}2$) We have $b_5 = 1 - (5-r_3)_+ - (5-r_4)_+$.

If $b_5 = 1$, then $r_3 \geq 5$. Since we have assumed $1 \geq b_6 = 2 - (6-r_3)_+ - (6-r_4)_+$, we have $r_3 = 5$, and $q_1 = 5, q_2 \geq 6, q_3 \geq 7$. Since $0 < D^2 = (7/40) - (1/r_4)$, we have $r_4 \geq 6$. In the case $r_4 \geq 7$, $q_1q_2q_3D^2 > 2$ follows. Now assume $r_4 = 6$. Then we can check $(q_1, q_2, q_3) = (5, 6, 8)$ and $D^2 = 1/120$. Therefore $q_1q_2q_3D^2 = 2$ follows.

Now assume $b_5 = 0$. Then $r_3 = 4, r_4 \geq 5$. Since $0 < D^2 = (1/8) - (1/r_4)$, we have $r_4 \geq 9$ and $D^2 \geq (1/72)$. Hence $q_1q_2q_3D^2 > 2$.

($\mathrm{III}_{0,0}3$) We have $b_5 = -1 - (5-r_4)_+ \leq -1$. Hence this case does not happen.

($\mathrm{IV}_{0,0}1$) Since $1 \geq b_5 = 2 - (5-r_3)_+ - (5-r_4)_+$, we have $r_3 = 4$. Since $0 < D^2 = (3/20) - (1/r_4)$, we have $r_4 \geq 7$ and $D^2 \geq (1/140)$. Then we can see $b_5 = 1, b_6 = 0, b_7 = 1$ and $q_1 = 5, q_2 = 7, q_3 \geq 8$. Hence, $q_1q_2q_3D^2 > 2$ when

$r_4 \geq 8$. For $r_4 = 7$, we can see $q_3 = 8$ and $q_1q_2q_3D^2 = 2$.
($\mathrm{IV}_{0,0}2$) Since $0 < D^2 = (9/70) - (1/r_4)$, we have $r_4 \geq 8$ and $D^2 \geq (1/280)$. Here we see $b_5 = 0$, $b_6 = 0$, $b_7 = 1$, $b_8 = 1$, $b_9 = 1 - (9 - r_4)_+$. Hence $q_1 = 7, q_2 = 8, q_3 \geq 9$. If $r_4 \geq 9$, $D^2 \geq (11/630)$ and $q_1q_2q_3D^2 > 2$. Further, if $r_4 = 8$, then we can see $q_3 = 10$ and $q_1q_2q_3D^2 = 2$.
($\mathrm{IV}_{0,0}3$) Since $0 < D^2 = (11/30) - (4/r_2)$, we have $r_2 \geq 11$ and $D^2 \geq (1/330)$. If $r_2 \neq 11$, then $r_2 \geq 13$ and $D^2 > (1/30)$, hence $q_1q_2q_3D^2 > 2$. In the case $r_2 = 11$, we can see $(q_1, q_2, q_3) = (5, 6, 22)$ and $D^2 = 1/330$. Hence $q_1q_2q_3D^2 = 2$.
($\mathrm{VI}_{0,0}1$) Since $0 < D^2 = (1/7) - (1/r_3)$, we have $r_3 \geq 8$ and $D^2 > (1/105)$. Hence $q_1q_2q_3D^2 > 2$.
($\mathrm{VI}_{0,0}2$) Since $0 < D^2 = (8/21) - (4/r_2)$, we have $r_2 \geq 11$ and $D^2 \geq (1/105)$. Hence $q_1q_2q_3D^2 > 2$.
($\mathrm{VI}_{0,0}3$) Since $D^2 = (3/8) - (4/r_2)$, we have $r_2 \geq 11$ and $D^2 \geq (1/88)$. Hence $q_1q_2q_3D^2 > 2$.

(4.10) Now one can show $q_1q_2q_3D^2 > 2$ for the cases VII–XII by similar arguments. These are not difficult. Hence we left the details to the readers and we omit them.

§5. The Case of $b_1 = 0, b_2 = 1$ and $b_3 \leq 1$

(5.1) In this section, we assume that $\underline{b_1 = 0, b_2 = 1, b_3 \leq 1}$. By (3.4) we have $\sum_{\alpha=1}^{N} i_\alpha = 7$. Hence we have the following basic cases for $\mathcal{I} = (i_1, \ldots, i_N)$ and r_i's:

I. $N = 7$, $\mathcal{I} = (1,1,1,1,1,1,1)$ with $2 \leq r_1 \leq \cdots \leq r_7$.

II. $N = 6$, $\mathcal{I} = (2,1,1,1,1,1)$ with $r_1 \geq 5$, $(r_1, 2) = 1$, $2 \leq r_2 \leq \cdots \leq r_6$.

III. $N = 5$, $\mathcal{I} = (3,1,1,1,1)$ with $r_1 \geq 7$, $(r_1, 3) = 1$, $2 \leq r_2 \leq \cdots \leq r_5$.

IV. $N = 5$, $\mathcal{I} = (2,2,1,1,1)$ with $5 \leq r_1 \leq r_2$, $(r_1, 2) = (r_2, 2) = 1$, $2 \leq r_3 \leq r_4 \leq r_5$.

V. $N = 4$, $\mathcal{I} = (4,1,1,1)$ with $r_1 \geq 9$, $(r_1, 4) = 1$, $2 \leq r_2 \leq r_3 \leq r_4$.

VI. $N = 4$, $\mathcal{I} = (3,2,1,1)$ with $r_1 \geq 7$, $(r_1, 3) = 1$, $r_2 \geq 5$, $(r_2, 2) = 1$, $2 \leq r_3 \leq r_4$.

VII. $N = 4$, $\mathcal{I} = (2,2,2,1)$ with $5 \leq r_1 \leq r_2 \leq r_3$, $(r_i, 2) = 1$ for $i = 1,2,3$, $r_4 \geq 2$.

VIII. $N = 3$, $\mathcal{I} = (5,1,1)$ with $r_1 \geq 11$, $(r_1, 5) = 1$, $2 \leq r_2 \leq r_3$.

IX. $N = 3$, $\mathcal{I} = (4, 2, 1)$ with $r_1 \geq 9$, $(r_1, 4) = 1$, $r_2 \geq 5$, $(r_2, 2) = 1$, $r_3 \geq 2$.

X. $N = 3$, $\mathcal{I} = (3, 3, 1)$ with $7 \leq r_1 \leq r_2$, $(r_i, 3) = 1$ for $i = 1, 2$, $r_3 \geq 2$.

XI. $N = 3$, $\mathcal{I} = (3, 2, 2)$ with $r_1 \geq 7$, $(r_1, 3) = 1$, $5 \leq r_2 \leq r_3$, $(r_2, 2) = (r_3, 2) = 1$.

XII. $N = 2$, $\mathcal{I} = (6, 1)$ with $r_1 \geq 13$, $(r_1, 6) = 1$, $r_2 \geq 2$.

XIII. $N = 2$, $\mathcal{I} = (5, 2)$ with $r_1 \geq 11$, $(r_1, 5) = 1$, $r_2 \geq 5$, $(r_2, 2) = 1$.

XIV. $N = 2$, $\mathcal{I} = (4, 3)$ with $r_1 \geq 9$, $(r_1, 4) = 1$, $r_2 \geq 7$, $(r_2, 2) = 1$.

XV. $N = 1$, $\mathcal{I} = (7)$ with $r_1 \geq 15$, $(r_1, 7) = 1$.

In these cases, we can show $q_1q_2q_3D^2 \geq 2$ for I–VI and $q_1q_2q_3D^2 > 2$ for VII–XV. In the below we will discuss the cases I–VI according to the types of b_3. We have the relations (3.4.3): $b_3 = 5 - \sum_\alpha (3i_\alpha - r_\alpha)_+, b_4 = 12 - \sum_\alpha (4i_\alpha - r_\alpha)_+$. By using these we can show the following.

Lemma 5.2. *Assume $b_3 = 1$. Then, for the cases* I–VI, *one of the followings occurs.*

$I_1 1 : r_1 = r_2 = r_3 = r_4 = 2, 3 \leq r_5 \leq r_6 \leq r_7$.

$II_1 1 : r_1 = 5, r_2 = r_3 = r_4 = 2, 3 \leq r_5 \leq r_6$, $II_1 2 : r_1 \geq 7, r_2 = 2, r_3 = r_4 = r_5 = 2, r_6 \geq 3$.

$III_1 1 : r_1 = 7, r_2 = r_3 = 2, 3 \leq r_4 \leq r_5$, $III_1 2 : r_1 = 8, r_2 = r_3 = r_4 = 2, r_5 \geq 3$, $III_1 3 : r_1 \geq 10, r_2 = r_3 = r_4 = r_5 = 2$.

$IV_1 1 : r_1 = r_2 = 5, r_3 = r_4 = 2, r_5 \geq 3$, $IV_1 2 : r_1 = 5, r_2 \geq 7, r_3 = r_4 = r_5 = 2$.

$V_1 1 : r_1 = 9, r_2 = 2, r_3 \geq 3$, $V_1 2 : r_1 = 11, r_2 = r_3 = r_4 = 2$.

$VI_1 1 : r_1 = 7, r_2 = 5, r_3 = 2, r_4 \geq 3$, $VI_1 2 : r_1 = 7, r_2 \geq 7, r_3 = r_4 = 2$, $VI_1 3 : r_1 = 8, r_2 = 5, r_3 = r_4 = 2$.

(5.3) *Proof of the Theorem* C *for the cases of* 5.2. Here we have $q_1 = 2, q_2 = 3, q_3 \geq 4$, and $D^2 = 3 - \sum_\alpha ((i_\alpha)^2 / r_\alpha)$. We can easily see $D^2 > (1/12)$ for $\mathrm{II}_1 2$, $\mathrm{III}_1 3$, $\mathrm{IV}_1 2$, $\mathrm{VI}_1 2$. Hence $q_1q_2q_3D^2 > 2$ holds for these cases.

($\mathrm{I}_1 1$) Since $0 < D^2 = 1 - (1/r_5) - (1/r_6) - (1/r_7)$, we have $r_7 \geq 4$. Hence $b_4 = 4 - (4 - r_5)_+ - (4 - r_6)_+ - (4 - r_7)_+ \geq 2$, and $q_3 = 4$. Here $D^2 \geq (1/12)$. Hence $q_1q_2q_3D^2 \geq 2$. The equalities holds if and only if $r_5 = r_6 = 3$, and $r_7 = 4$.

($\mathrm{II}_1 1$) We have $b_4 = 3 - (4 - r_5)_+ - (4 - r_6)_+$. If $b_4 \geq 2$, then $r_6 \geq 4$. We have $D^2 = (7/10) - (1/r_5) - (1/r_6) \geq (7/60) > (1/12)$. Hence $q_1q_2q_3D^2 > 2$. Now assume $b_4 = 1$. Then $r_1 = 5$, $r_2 = r_3 = r_4 = 2$, $r_5 = r_6 = 3$, and $D^2 = 1/30$. A direct computation shows $q_3 = 10$. Hence $q_1q_2q_3D^2 = 2$.

(III$_1$) If $r_5 \geq 4$, then $D^2 = (5/7)-(1/r_4)-(1/r_5) > (1/12)$. Hence $q_1q_2q_3D^2 > 2$. If $r_4 = r_5 = 3$, then $D^2 = 1/21$. A direct computation shows $q_3 = 7$. Hence $q_1q_2q_3D^2 = 2$.
(III$_1$2) If $r_5 \geq 4$, then $D^2 = (3/8)-(1/r_5) > (1/12)$. Hence $q_1q_2q_3D^2 > 2$. If $r_5 = 3$, then $D^2 = 1/24$. A direct computation shows $q_3 = 8$. Hence $q_1q_2q_3D^2 = 2$.
(IV$_1$1) If $r_5 \geq 4$, then $D^2 = (2/5)-(1/r_5) \geq (3/20) > (1/12)$. Hence $q_1q_2q_3D^2 > 2$. If $r_5 = 3$, we have $D^2 = 1/15$. We can show $q_3 = 5$. Hence $q_1q_2q_3D^2 = 2$.
(V$_1$1) If $r_4 \geq 4$, we have $D^2 > (1/12)$. Hence $q_1q_2q_3D^2 > 2$. If $r_3 = r_4 = 3$, then $D^2 = 1/18$. A direct computation shows $q_3 = 7$. Hence $q_1q_2q_3D^2 > 2$.
(V$_1$2) We have $D^2 = 1/22$. A direct computation shows $q_3 = 8$. Hence $q_1q_2q_3D^2 > 2$.
(VI$_1$1) If $r_4 \geq 4$, then $D^2 = (29/70) - (1/r_4) > (1/12)$. Hence $q_1q_2q_3D^2 > 2$. If $r_4 = 3$, then $D^2 = 17/210$. We can show $q_3 = 5$. Hence $q_1q_2q_3D^2 > 2$.
(VI$_1$3) Here we have $D^2 = 3/40$. A direct computation shows $q_3 = 5$. Hence $q_1q_2q_3D^2 = 2 + (1/4) > 2$.

Lemma 5.4. *Assume $b_3 = 0$. Then, for the cases* I–VI*, one of the followings occurs.*

$I_01 : r_1 = r_2 = r_3 = r_4 = r_5 = 2, 3 \leq r_6 \leq r_7$.

$II_01 : r_1 = 5, r_2 = r_3 = r_4 = r_5 = 2, 3 \leq r_6$, $II_02 : r_1 \geq 9, r_2 = 2, r_3 = r_4 = r_5 = r_6 = 2$.

$III_01 : r_1 = 7, r_2 = r_3 = r_4 = 2, 3 \leq r_5$, $III_02 : r_1 = 8, r_2 = r_3 = r_4 = r_5 = 2$.

$IV_01 : r_1 = r_2 = 5, r_3 = r_4 = r_5 = 2$.

$V_01 : r_1 = 9, r_2 = r_3 = 2, r_4 \geq 3$.

$VI_01 : r_1 = 7, r_2 = 5, r_3 = r_4 = 2$.

(5.5) *Proof of the Theorem C for the cases of* 5.4. Here we have $q_1 = 2, q_3 \geq q_2 \geq 4$, and $D^2 = 3 - \sum_\alpha((i_\alpha)^2/r_\alpha)$. By this we can easily see $D^2 < 0$ for III$_0$2, IV$_0$1, VI$_0$1. Hence these cases do not happen.
(I$_0$1) We have $b_4 = 2 - (4 - r_6)_+ - (4 - r_7)_+$.

If $b_4 = 2$, then $r_6 \geq 4$ and $q_2 = 4$. Since $0 < D^2 = (1/2) - (1/r_6) - (1/r_7)$, we have $r_7 \geq 5$. Here we can see $b_5 \geq 1$ and $q_3 = 5$. Now $D^2 \geq (1/20)$. Hence $q_1q_2q_3D^2 \geq 2$. The equality holds if and only if $r_6 = 4, r_7 = 5$.

If $b_4 = 1$, then $r_6 = 3, r_7 \geq 4$. Since $0 < D^2 = (1/6) - (1/r_7)$, we have $r_7 \geq 7$. Here we can see $b_5 = 0, b_6 = 2, b_7 = 1$ and $q_2 = 6, q_3 = 7$. Now $D^2 \geq (1/42)$. Hence $q_1q_2q_3D^2 \geq 2$. The equality holds if and only if $r_7 = 7$.
(II$_0$1) Since $0 < D^2 = (1/5) - (1/r_6)$, we have $r_6 \geq 6$. Here we can show $b_4 = 1, b_5 = 1, b_6 = 2$, and $q_2 = 5, q_3 = 6$. Now we have $D^2 \geq (1/30)$. Hence $q_1q_2q_3D^2 \geq 2$. The equality holds if and only if $r_6 = 6$.

($\mathrm{II}_0 2$) Here we have $D^2 = (1/2) - (4/r_1) \geq (1/18)$. Further we can see $b_4 = 2$. Hence $q_2 = 4, q_3 \geq 5$ and $q_1 q_2 q_3 D^2 \geq 40(1/18) > 2$.
($\mathrm{III}_0 1$) Since $0 < D^2 = (3/14) - (1/r_5)$, we have $r_5 \geq 5$. Here we can show $b_4 = b_5 = 1$, and $q_2 = 5, q_3 \geq 6$.

If $r_5 \geq 6$, then $D^2 \geq (3/14) - (1/6) = (1/21)$. Hence $q_1 q_2 q_3 D^2 \geq 2 \cdot 5 \cdot 6(1/21) > 2$.

If $r_5 = 5$, we have $D^2 = (1/70)$. A direct computation shows $q_3 = 14$. Hence $q_1 q_2 q_3 D^2 = 2$.
($\mathrm{V}_0 1$) Since $0 < D^2 = (2/9) - (1/r_4)$, we have $r_4 \geq 5$. Here we can show $b_4 = b_5 = 1$. Hence $q_2 = 5, q_3 \geq 6$.

If $r_4 \geq 6$, then $D^2 > (1/30)$. Hence $q_1 q_2 q_3 D^2 > 2$.

If $r_4 = 5$, then $D^2 = (1/45)$. A direct computation shows $q_3 = 9$. Hence $q_1 q_2 q_3 D^2 = 2$.

(5.6) Now one can show $q_1 q_2 q_3 D^2 > 2$ for the cases VII–XV by similar arguments. These are not difficult. Hence we left the details to the readers and we omit them.

§6. In the Case with $b_1 = b_2 = 1$

In this section, we assume that $\underline{b_1 = 1, b_2 = 1}$. We have $q_1 = 1, q_2 \geq 3$, $\sum_{\alpha=1}^{N} i_\alpha = 3$, and $D^2 = 1 - \sum_{\alpha=1}((i_\alpha)^2/r_\alpha)$. We have the relations (3.4.3): $b_3 = 2 - \sum_\alpha (3i_\alpha - r_\alpha)_+$, $b_4 = 4 - \sum_\alpha (4i_\alpha - r_\alpha)_+$, $b_5 = 7 - \sum_\alpha (5i_\alpha - r_\alpha)_+ - \sum_\alpha (5i_\alpha - 2r_\alpha)_+, \dots$. In the below we separate the arguments into cases depending on the type $(i_1, \dots, i_N)$.

Case 6.1. $N = 3, (i_1, i_2, i_3) = (1, 1, 1)$, with $2 \leq r_1 \leq r_2 \leq r_3$.

We have $b_3 = 2 - \sum_{\alpha=1}^{3} (3 - r_\alpha)_+$.
(6.1.1) If $b_3 = 2$, then $r_1 \geq 3$, and we have $q_2 = 3, q_3 \geq 4$. We have $b_4 = 4 - \sum_{\alpha=1}^{3} (4 - r_\alpha)_+$.
(6.1.1.1) If $b_4 \geq 3$, then $r_2 \geq 4$, $q_3 = 4$. We have $D^2 = 1 - (1/r_1) - (1/r_2) - (1/r_3) \geq (1/6)$. Hence $q_1 q_2 q_3 D^2 \geq 2$. The equality holds if and only if $r_1 = 3, r_2 = r_3 = 4$.
(6.1.1.2) If $b_4 = 2$, then $r_1 = r_2 = 3, r_3 \geq 4$, $q_3 \geq 5$.
(6.1.1.2.a) If $r_3 \geq 6$, then $D^2 = (1/3) - (1/r_3) \geq (1/6) > (2/15)$. Hence $q_1 q_2 q_3 D^2 \geq 30 D^2 > 2$.
(6.1.1.2.b) If $r_3 = 5$, then $D^2 = 2/15$. A direct computation shows $q_3 = 5$. Hence $q_1 q_2 q_3 D^2 = 2$.
(6.1.1.2.c) If $r_3 = 4$, then $D^2 = 1/12$. A direct computation shows $q_3 = 8$. Hence $q_1 q_2 q_3 D^2 = 2$.

(6.1.2) Now we assume $b_3 = 1$. Then $r_1 = 2, r_2 \geq 3$ and we have $b_4 = 2 - (4 - r_2)_+ - (4 - r_3)_+$.
(6.1.2.1) If $b_4 = 2$, then $r_2 \geq 4$, $q_2 = 4$. We have $b_5 = 3 - (5 - r_2)_+ - (5 - r_3)_+$.
(6.1.2.1.1) If $b_5 = 3$, then $r_2 \geq 5$ and $q_3 = 5$. We have $D^2 = (1/2) - (1/r_2) - (1/r_3) \geq (1/10)$ hence $q_1 q_2 q_3 D^2 \geq 2$. Here the equality holds if and only if $r_2 = r_3 = 5$.
(6.1.2.1.2) If $b_5 = 2$, then $r_2 = 4, r_3 \geq 5$ and $q_3 \geq 6$.
(6.1.2.1.2.a) If $r_3 \geq 7$, then $D^2 \geq (3/28) > (1/12)$. Hence $q_1 q_2 q_3 D^2 > 2$.
(6.1.2.1.2.b) If $r_3 = 6$, then $D^2 = 1/12$. A direct computation shows $q_3 = 6$. Hence $q_1 q_2 q_3 D^2 = 2$.
(6.1.2.1.2.c) If $r_3 = 5$, then $D^2 = 1/20$. A direct computation shows $q_3 = 10$. Hence $q_1 q_2 q_3 D^2 = 2$.
(6.1.2.2) If $b_4 = 1$, then $r_1 = 2, r_2 = 3, r_3 \geq 4$. Since $0 < D^2 = (1/6) - (1/r_3)$, $r_3 \geq 7$ holds. Here we can prove $b_5 = 1, b_6 = 2, b_7 = 2$, hence $q_2 = 6, q_3 \geq 8$ hold.
(6.1.2.2.a) If $r_3 \geq 9$, then $D^2 \geq (1/18) > (1/24)$ and $q_1 q_2 q_3 D^2 > 2$.
(6.1.2.2.b) If $r_3 = 8$, then $D^2 = 1/24$. A direct computation shows $q_3 = 8$, hence $q_1 q_2 q_3 D^2 = 2$.
(6.1.2.2.c) If $r_3 = 7$, then $D^2 = 1/42$. A direct computation shows $q_3 = 14$, hence $q_1 q_2 q_3 D^2 = 2$.

Case 6.2. $N = 2, (i_1, i_2) = (2, 1)$, with $r_1 \geq 5$, $(r_1, 2) = 1$, $2 \leq r_2$.

We have $b_3 = 2 - (6 - r_1)_+ - (3 - r_2)_+$.
(6.2.1) If $b_3 = 2$, then $r_1 \geq 7$, $r_2 \geq 3$ and $q_2 = 3$. We have $b_4 = 4 - (8 - r_1)_+ - (4 - r_2)_+$.
(6.2.1.1) If $b_4 \geq 3$, then $q_3 = 4$ and either $r_1 \geq 9, r_2 \geq 3$ or $r_1 \geq 7, r_2 \geq 4$ hold. In both cases, we can easily show $D^2 > (1/6)$, hence $q_1 q_2 q_3 D^2 > 2$.
(6.2.1.2) If $b_4 = 2$, then $r_1 = 7, r_2 = 3$ and $D^2 = 2/21$. A direct computation shows $q_3 = 7$, hence $q_1 q_2 q_3 D^2 = 2$.
(6.2.2) Now we assume $b_3 = 1$. Then we have two cases: (6.2.2.a) $r_1 = 5, r_2 \geq 3$ or (6.2.2.b) $r_1 \geq 7, r_2 = 2$, occur.
(6.2.2.a) Since $0 < D^2 = (1/5) - (1/r_2)$, we have $r_2 \geq 6$. Here we can prove $b_4 = 1, b_5 = 2, b_6 = 2$, hence $q_2 = 5, q_3 \geq 7$.
(6.2.2.a.a) If $r_2 \geq 8$, then $D^2 \geq (3/40) > (2/35)$ and $q_1 q_2 q_3 D^2 > 2$.
(6.2.2.a.b) If $r_2 = 7$, then $D^2 = 2/35$. A direct computation shows $q_3 = 7$. Hence $q_1 q_2 q_3 D^2 = 2$.
(6.2.2.a.c) If $r_2 = 6$, then $D^2 = 1/30$. A direct computation shows $q_3 = 12$. Hence $q_1 q_2 q_3 D^2 = 2$.
(6.2.2.b) Since $0 < D^2 = (1/2) - (4/r_1)$, we have $r_1 \geq 9$.

(6.2.2.b.a) If $r_1 \geq 11$, then $D^2 \geq (3/22) > (1/8)$. Hence $q_1q_2q_3D^2 \geq 1 \cdot 4 \cdot 4D^2 > 2$.
(6.2.2.b.b) If $r_1 = 9$, then $D^2 = 1/18$. A direct computation shows $q_2 = 4, q_3 = 9$. Hence $q_1q_2q_3D^2 = 2$.

Case 6.3. $N = 1, i_1 = 3$, with $r_1 \geq 7, (r_1, 3) = 1$.

Since $0 < D^2 = 1 - (9/r_1)$, we have $r_1 \geq 10$. Here we have $b_3 = 2$, hence $q_2 = 3, q_3 \geq 4$.
(6.3.a) If $r_1 \geq 11$, then $D^2 \geq (2/11) > (1/6)$ and $q_1q_2q_3D^2 \geq 1 \cdot 3 \cdot 4D^2 > 2$.
(6.3.b) If $r_1 = 10$, then $D^2 = (1/10)$. A direct computation shows $q_3 = 7$. Hence $q_1q_2q_3D^2 = 2 + (1/10) > 2$.

This completes the proof of Theorem C.

§7. Simple K3 Singularities of Multiplicity Two

(7.1) Now the proof of Theorem B is given as follows: Here we can show the equality $D^2 = \lim_{\lambda \to 1}(1-\lambda)^3 P(G, \lambda)$ (2.1). Since $e(m, A) = 2$, we obtain the equality $2 = \deg x_1 \cdot \deg x_2 \cdot \deg x_3$. $\lim_{\lambda \to 1}(1-\lambda)^3 P(G, \lambda) = e(m, A)$ by Theorem A (i). Therefore $2 = e(m, A) = e(G_+, G)$ and there exists parameter system of $m \subset A$ whose initial form gives a parameter system of $G_+ \subset G$. In particular, G is a hypersurface and there is a system y_1, y_2, y_3, y_4 which generates $m \cdot A$ and initial form of them generates G_+. This completes the proof of Theorem B.

Now, for this special non-rational singularity, a conjecture of M. Reid (4.2) of [14] about the existence of such a good coordinates is proved in the following form.

Corollary 7.2. *Let (V, p) be a simple $K3$ singularity of multiplicity two and G the associated graded ring of the canonical filtration. Then in the coordinate which induced from the homogeneous minimal generator of G_+, there is a 3-dimensional compact face Γ_0 of the Newton boundary of the defining equation of V where $(1,1,1,1) \in \mathbf{R}^4$ is contained in the relative interior of Γ_0.*

Proof. Let $V = \{(y_1, y_2, y_3, y_4) \in \mathbf{C}^4 \mid f(y_1, y_2, y_3, y_4) = 0\}$ be the representation of (V, p) by the new coordinate (y_1, y_2, y_3, y_4) in the arguments of (7.1). Let $f = \sum_{k \geq 0} f_k$ be the weighted Taylor expansion with respect to the $wt(y_i) = q_i$, for $i = 1, 2, 3, 4$, and f_h the initial form. The proof of Theorem B implies that $G = R(E, D) \cong \mathbf{C}[y_1, y_2, y_3, y_4]/f_h$. Therefore $f_h \in \mathbf{C}[y_1, y_2, y_3, y_4]$ is a quasi-homogeneous polynomial with $h = \sum_{1 \leq i \leq 4} q_i$

and $\{f_h = 0\} - \{o\}$ has only rational singularities (see [19, (1.5)]). By Theorem 5.6 of [19], the Newton boundary $\Gamma(f_h) \subset \mathbf{R}^4$ is 3-dimensional and $(1, 1, 1, 1)$ is contained in the relative interior of $\Gamma(f_h)$. Now the assertion on the Newton boundary of f follows as $\Gamma_0 = \Gamma(f_h)$. $\square$

(7.3) **A new proof of classification of the lists of weights for simple K3 double points.** As noted in Introduction, there are two basic works [2] and [23] on classification of hypersurface simple K3 singularities. In both studies, the list of famous 95 weights is created from the studies on the weight systems on the coordinates. In [2], Fletcher gives the list as that of certain quasi-smooth weight hypersurfaces which give normal K3 surfaces, and in [23], Yonemura gives the same list from the points of classification of special convex polytopes which corresponds to the initial form of non-degenerate functions. Now, for the cases of multiplicity two, we can give the list as a corollary of the proof of Theorem C. We obtain the list of 48 cases with the datum D^2, q_1, q_2, q_3 with the baskets of singularities where the relation $q_1q_2q_3D^2 = 2$ holds. Set q_4 as $q_4 = q_1 + q_2 + q_3$ and set h as $h = 2q_4$. Then the set of these $(q_1, q_2, q_3, q_4; h)$ gives the list for weights of simple K3 singularities of multiplicity two.

The data for the cases with $q_1q_2q_3D^2 = 2$

reference	D^2	q_1, q_2, q_3	$(r_\alpha, i_\alpha){:}\,cl(D) = i_\alpha \in \mathbf{Z}/r_\alpha\mathbf{Z}$	Example of E
(3.5)	$\frac{1}{6}$	$2, 2, 3$	$(2,1) \times 7, (3,1)$	$X_{14} \subset P(2.2.3.7)$
(3.6)	2	$1, 1, 1$	D is Cartier	$X_6 \subset P(1.1.1.3)$
(3.7)	1	$1, 1, 2$	$(2,1) \times 2$	$X_8 \subset P(1.1.2.4)$
(3.7)	$\frac{2}{3}$	$1, 1, 3$	$(3,1)$	$X_{10} \subset P(1.1.3.5)$
(3.7)	$\frac{1}{2}$	$1, 1, 4$	$(2,1)$	$X_{12} \subset P(1.1.4.6)$
(3.9)	$\frac{1}{2}$	$1, 2, 2$	$(2,1) \times 5$	$X_{10} \subset P(1.2.2.5)$
(3.10.1.1)	$\frac{1}{3}$	$1, 2, 3$	$(2,1) \times 2, (3,1) \times 2$	$X_{12} \subset P(1.2.3.6)$
(3.10.1.2.1)	$\frac{1}{4}$	$1, 2, 4$	$(2,1) \times 3, (4,1)$	$X_{14} \subset P(1.2.4.7)$
(3.10.1.2.2)	$\frac{1}{6}$	$1, 2, 6$	$(2,1) \times 3, (3,1)$	$X_{18} \subset P(1.2.6.9)$
(3.10.2.2)	$\frac{1}{5}$	$1, 2, 5$	$(5,2), (2,1) \times 2$	$X_{16} \subset P(1.2.5.8)$
(4.7.$\mathrm{I}_{0,1}$1)	$\frac{1}{60}$	$4, 5, 6$	$(2,1) \times 2, (3,1), (4,1), (5,1) \times 2$	$X_{30} \subset P(4.5.6.15)$
(4.3.$\mathrm{II}_{1,1}$1)	$\frac{1}{30}$	$3, 4, 5$	$(5,2), (3,1) \times 2, (4,1) \times 2$	$X_{24} \subset P(3.4.5.12)$
(4.3.$\mathrm{II}_{1,1}$2)	$\frac{1}{84}$	$3, 4, 14$	$(7,2), (2,1), (3,1) \times 2, (4,1)$	$X_{42} \subset P(3.4.14.21)$
(4.7.$\mathrm{II}_{0,1}$2)	$\frac{1}{84}$	$4, 6, 7$	$(7,2), (2,1) \times 2, (4,1), (6,1)$	$X_{34} \subset P(4.6.7.17)$
(4.5.$\mathrm{III}_{0,1}$1)	$\frac{1}{120}$	$3, 5, 16$	$(8,3), (3,1) \times 2, (5,1)$	$X_{48} \subset P(3.5.16.24)$
(4.3.$\mathrm{III}_{1,1}$2)	$\frac{1}{60}$	$3, 4, 10$	$(10,3), (2,1), (3,1), (4,1)$	$X_{34} \subset P(3.4.10.17)$
(4.3.$\mathrm{III}_{1,1}$3)	$\frac{1}{66}$	$3, 4, 11$	$(11,3), (2,1), (3,1) \times 2$	$X_{36} \subset P(3.4.11.18)$

(continued)

(continued)

reference	D^2	q_1, q_2, q_3	$(r_\alpha, i_\alpha) : cl(D) = i_\alpha \in \mathbf{Z}/r_\alpha\mathbf{Z}$	Example of E
$(4.7.\mathrm{III}_{0,1}1)$	$\frac{1}{70}$	$4, 5, 7$	$(7,3), (4,1) \times 2, (5,1)$	$X_{32} \subset P(4.5.7.16)$
$(4.9.\mathrm{III}_{0,0}2)$	$\frac{1}{120}$	$5, 6, 8$	$(8,3), (2,1), (5,1), (6,1)$	$X_{38} \subset P(5.6.8.19)$
$(4.3.\mathrm{IV}_{1,1}3)$	$\frac{1}{42}$	$3, 4, 7$	$(7,2) \times 2, (2,1), (3,1)$	$X_{28} \subset P(3.4.7.14)$
$(4.9.\mathrm{IV}_{0,0}1)$	$\frac{1}{140}$	$5, 7, 8$	$(5,2) \times 2, (4,1), (7,1)$	$X_{40} \subset P(5.7.8.20)$
$(4.7.\mathrm{IV}_{0,1}1)$	$\frac{1}{180}$	$4, 5, 18$	$(5,2), (9,2), (2,1), (4,1)$	$X_{54} \subset P(4.5.18.27)$
$(4.9.\mathrm{IV}_{0,0}2)$	$\frac{1}{280}$	$7, 8, 10$	$(5,2), (7,2), (2,1), (8,1)$	$X_{50} \subset P(7.8.10.25)$
$(4.9.\mathrm{IV}_{0,0}3)$	$\frac{1}{330}$	$5, 6, 22$	$(5,2), (11,2), (2,1), (3,1)$	$X_{66} \subset P(5.6.22.33)$
$(4.5.\mathrm{V}_{1,1}1)$	$\frac{2}{165}$	$3, 5, 11$	$(11,4), (3,1), (5,1)$	$X_{38} \subset P(3.5.11.19)$
$(4.7.\mathrm{VI}_{0,1}2)$	$\frac{1}{130}$	$4, 5, 13$	$(13,3), (5,2), (2,1)$	$X_{44} \subset P(4.5.13.22)$
$(5.3.\mathrm{I}_1 1)$	$\frac{1}{12}$	$2, 3, 4$	$(2,1) \times 4, (3,1) \times 2, (4,1)$	$X_{18} \subset P(2.3.4.9)$
$(5.5.\mathrm{I}_0 1)$	$\frac{1}{20}$	$2, 4, 5$	$(2,1) \times 5, (4,1), (5,1)$	$X_{22} \subset P(2.4.5.11)$
$(5.5.\mathrm{I}_0 1)$	$\frac{1}{42}$	$2, 6, 7$	$(2,1) \times 5, (3,1), (7,1)$	$X_{30} \subset P(2.6.7.15)$
$(5.3.\mathrm{II}_1 1)$	$\frac{1}{30}$	$2, 3, 10$	$(5,2), (2,1) \times 3, (3,1) \times 2$	$X_{30} \subset P(2.3.10.15)$
$(5.5.\mathrm{II}_0 1)$	$\frac{1}{30}$	$2, 5, 6$	$(5,2), (2,1) \times 4, (6,1)$	$X_{26} \subset P(2.5.6.13)$
$(5.3.\mathrm{III}_1 1)$	$\frac{1}{21}$	$2, 3, 7$	$(7,3), (2,1) \times 2, (3,1) \times 2$	$X_{24} \subset P(2.3.7.12)$
$(5.3.\mathrm{III}_1 2)$	$\frac{1}{24}$	$2, 3, 8$	$(8,3), (2,1) \times 3, (3,1)$	$X_{26} \subset P(2.3.8.13)$
$(5.5.\mathrm{III}_0 1)$	$\frac{1}{70}$	$2, 5, 14$	$(7,3), (2,1) \times 3, (5,1)$	$X_{42} \subset P(2.5.14.21)$
$(5.3.\mathrm{IV}_1 1)$	$\frac{1}{15}$	$2, 3, 5$	$(5,2) \times 2, (2,1) \times 2, (3,1)$	$X_{20} \subset P(2.3.5.10)$
$(5.5.\mathrm{V}_0 1)$	$\frac{1}{45}$	$2, 5, 9$	$(9,4), (2,1) \times 2, (5,1)$	$X_{32} \subset P(2.5.9.16)$
$(6.1.1.1)$	$\frac{1}{6}$	$1, 3, 4$	$(3,1), (4,1) \times 2$	$X_{16} \subset P(1.3.4.8)$
$(6.1.1.2.\mathrm{b})$	$\frac{2}{15}$	$1, 3, 5$	$(3,1) \times 2, (5,1)$	$X_{18} \subset P(1.3.5.9)$
$(6.1.1.2.\mathrm{c})$	$\frac{1}{12}$	$1, 3, 8$	$(3,1) \times 2, (4,1)$	$X_{24} \subset P(1.3.8.12)$
$(6.1.2.1.1)$	$\frac{1}{10}$	$1, 4, 5$	$(2,1), (5,1) \times 2$	$X_{20} \subset P(1.4.5.10)$
$(6.1.2.1.2.\mathrm{b})$	$\frac{1}{12}$	$1, 4, 6$	$(2,1), (4,1), (6,1)$	$X_{22} \subset P(1.4.6.11)$
$(6.1.2.1.2.\mathrm{c})$	$\frac{1}{20}$	$1, 4, 10$	$(2,1), (4,1), (5,1)$	$X_{30} \subset P(1.4.10.15)$
$(6.1.2.2.\mathrm{b})$	$\frac{1}{24}$	$1, 6, 8$	$(2,1), (3,1), (8,1)$	$X_{30} \subset P(1.6.8.15)$
$(6.1.2.2.\mathrm{c})$	$\frac{1}{42}$	$1, 6, 14$	$(2,1), (3,1), (7,1)$	$X_{42} \subset P(1.6.14.21)$
$(6.2.1.2)$	$\frac{2}{21}$	$1, 3, 7$	$(7,2), (3,1)$	$X_{22} \subset P(1.3.7.11)$
$(6.2.2.\mathrm{a.b})$	$\frac{2}{35}$	$1, 5, 7$	$(5,2), (7,1)$	$X_{26} \subset P(1.5.7.13)$
$(6.2.2.\mathrm{a.c})$	$\frac{1}{30}$	$1, 5, 12$	$(5,2), (6,1)$	$X_{36} \subset P(1.5.12.18)$
$(6.2.2.\mathrm{b.b})$	$\frac{1}{18}$	$1, 4, 9$	$(9,2), (2,1)$	$X_{28} \subset P(1.4.9.14)$

Here $X_h \subset P(q_1, q_2, q_3, q_4)$ denotes a weight hypersurface defined by a quasi-homegeneous polynomial $f_h \in \mathbf{C}[y_1, y_2, y_3, y_4]$ of type $(q_1, q_2, q_3, q_4; h)$ as in (7.2).

Here one can find many examples of simple K3 singularity for each weight in the lists of Fletcher [2]-Yonemura [23]. By the existence of such weighted hypersurfaces, we can show the following where the ring G is not necessarily a hypersurface, but the Poincare series for G is very simple.

Corollary 7.4. *Let $G = R(E, D)$ be a normal graded ring with E a normal K3 surface and D an integral Weil divisor. Suppose $q_1 q_2 q_3 D^2 = 2$*

holds. Then the Poincare series $P(G,\lambda)$ *is given by the following way*:

$$P(G,\lambda)=\frac{1-\lambda^{2(q_1+q_2+q_3)}}{(1-\lambda^{q_1})(1-\lambda^{q_2})(1-\lambda^{q_3})(1-\lambda^{q_1+q_2+q_3})}.$$

Proof. By the list (7.3), the basket of singularities is determined by q_1, q_2, q_3 and D^2. Hence $P(G,\lambda)$ is determined. Let $q_4 = q_1 + q_2 + q_3$, and $h = 2q_4$. Then for each data, we can find a quasi-homogeneous polynomial of type $(q_1, q_2, q_3, q_4; h)$ with isolated singularity by Fletcher [2] and Yonemura [23]. Hence, for each case, the Poincare series $P(G,\lambda)$ is given as in the assertion by (2.2). $\square$

The following is a corollary of S. Ishii's theory and our main theorem. The following theorem say "*the weight type of the simple* K3 *double point stays the same under arbitrary one-parameter* (FG)*-deformation*".

Theorem 7.5. *Let* $\pi : \mathcal{V} \to T \subset \mathbf{C}$ *be a one-parameter family of simple* K3 *double points such that the resolution* $\psi : \tilde{\mathcal{V}} \to \mathcal{V}$ *has the relative canonical model, i.e.,* π *is an* (FG)*-deformation after Ishii* [8]*. Assume that non-rational singularity of* V_t *appears along a section* $P : T \to \mathcal{V}$*. Let* $0 \in T$*, then in a neighborhood of* $P(0)$*, there is a good coordinate* (x_1, x_2, x_3, x_4, t) *such that* π *is written as* $\mathcal{V} = \{f(x_1, x_2, x_3, x_4, t) = 0\} \ni (x_1, x_2, x_3, x_4, t) \to t \in T$ *and the initial compact face* $\Gamma_0(t)$ *of* $\Gamma(f_t)$ *with respect to* (x_1, x_2, x_3, x_4) *such that* $\Gamma_0(t) \ni (1,1,1,1)$ *and belongs to the hypersurface which is independent of* $t \in T$*.*

Proof. In [8, Corollary 1.11, Theorem 2], S. Ishii showed that the π admits the simultaneous canonical model and $\gamma_m(\mathcal{V}_t, P(t))$ is constant. Let $F : (\mathcal{Y}, \mathcal{E}) \to (\mathcal{V}, P(T))$ be the simultaneous canonical model. Ishii's arguments show that $\mathcal{E} \to T$ is a family of normal K3 surfaces and $\mathcal{F}^k = F_*(O_{\mathcal{Y}}(-k\mathcal{E})) \subset O_{\mathcal{V}}$ for $k \geq 1$ define the canonical filtration for each $t \in T$, and $\mathcal{F}^k/\mathcal{F}^{k+1}$ are locally free O_T-modules. Hence there are $x_i \in \mathcal{F}^{q_i} - \mathcal{F}^{q_i+1}$, $i = 1,2,3,4$ such that x_1, x_2, x_3 defines a homogeneous parameter of $gr_{\mathcal{F}} O_{\mathcal{V}} \otimes \mathbf{C} = R(\mathcal{E}_0, D)$ with $q_1 q_2 q_3 D^2 = 2$ and (x_1, x_2, x_3, x_4) gives a homogeneous generator of the maximal ideal of $gr_{\mathcal{F}} O_{\mathcal{V}} \otimes \mathbf{C}$. This properties are preserved for $t \in T$ in a neighborhood of 0. Such a parameter of $gr_{\mathcal{F}} O_{\mathcal{V}} \otimes \mathbf{C}_t$ determines the good coordinates in the sense of (7.2). Hence in the coordinate (x_1, x_2, x_3, x_4, t), we obtain the desired properties. $\square$

Remark 7.6. (i) For deformations of simple K3 singularities of multiplicity three, the situation is not so simple as in (7.5), even we assume γ_m

is constant [9]. (ii) For some special simple K3 singularities, in some case for more general deformations, "*the constant-ness of weight of simple K3 singularity*" are studied by S. Ishii [8, Example 2.7], Y. Kaneko and M. Furuya [3], independently. In their studies, they assumed non-degenerate conditions of Newton boundary or quasi-homogeneous isolated-ness for the singularities.

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