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Principal Bundle Maps via Rational Homotopy Theory

By

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Abstract

Let P be a finite complex on which S^1 acts freely. In this paper, we shall give a sufficient condition that the kernel of the natural map (forgetful map) $\mathrm{Aut}_{S^1}P \to \mathrm{Aut}P$ is a finite group.

§1. Introduction

Let

$$S^1 \to P \overset{\pi}{\to} B \tag{1.1}$$

be a principal S^1-bundle over the base space B. We denote by $autP$ (resp. $aut_{S^1}P$) the space of self homotoy equivalences of P (resp. the space of S^1-equivariant self homotopy equivalences of P). Let $\mathrm{Aut}P$ (resp. $\mathrm{Aut}_{S^1}P$) be the group of path connected components of $autP$ (resp. $aut_{S^1}P$). Then we have a natural homomorphism

$$\mathcal{F} : \mathrm{Aut}_{S^1}P \to \mathrm{Aut}P$$

obtained by forgetting S^1-action, which is called forgetful map. The kernel Ker $\mathcal{F}$ was discussed in [5, Problem 13] and [9]. There are the examples where Ker $\mathcal{F}$ are not zero but finite, countable and uncountable [8]. In this paper we assume that B is a connected, simply connected and finite complex. We study $\mathcal{F}$ from the view point of rational homotopy theory. We prove

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Theorem 1.1. *Suppose* $\pi_2(aut_{Id}B) \otimes Q = 0$. *Then* Ker $\mathcal{F}$ *is a finite group for any principal* S^1*-bundle over* B*, where* $aut_{Id}B$ *is the identity component of* $autB$.

In particular, by [7] we have

Corollary 1.2. *If the base* B *has the rational homotopy type of the homogeneous space* G/U *with* $rankG = rankU$*, then* Ker $\mathcal{F}$ *is a finite group for any principal* S^1*-bundle over* B.

The outline of the proof of Theorem 1.1 goes as follows:

We recall [Gottlieb] that the group $\mathrm{Aut}_{S^1}P$ is isomorphic to the subgroup Aut_kB of $\mathrm{Aut}B$, which preserves the homotopy class of the classifying map of P, $k : B \to BS^1$.

In rational homotopy theory, $\mathrm{Aut}P$ and Aut_kB correspond to $\mathrm{Aut}_Q\mathcal{M}(P)$ and $\mathrm{Aut}_Q^t\mathcal{M}(B)$ respectively, where $\mathcal{M}(X)$ is the minimal model of a space X and $\mathrm{Aut}_Q\mathcal{M}(X)$ denotes the group of D.G.A. homotopy classes of automorphisms of $\mathcal{M}(X)$. (See the detail in Section 2.)

Then we shall show that a certain homomorphism $[\pi_t] : \mathrm{Aut}_Q^t\mathcal{M}(B) \to \mathrm{Aut}_Q\mathcal{M}(P)$ corresponds to $\mathcal{F}$ (see Proposition 2.2) and we show that $\mathrm{Ker}[\pi_t] = Id_{\mathcal{M}(B)}$ is zero if the 0-dimensional homology of certain derivations $H_0(Der^t(\mathcal{M}(B),(t)))$ is trivial (see Proposition 2.3). We then prove that $H_0(Der^t(\mathcal{M}(B),(t))) = 0$ if $\pi_2(aut_{Id}B) \otimes Q = 0$. Finally by Sullivan's theorem ([6, p. 307 Theorem 10.2 (i)]) if $\mathrm{Ker}[\pi_t] = Id_{\mathcal{M}(B)}$ then Ker $\mathcal{F}$ is finite.

This paper is organized as follows: In Section 2 we study $\mathcal{F}$ by using automorphism group of minimal models when the bundle (1.1) is not trivial (Proposition 2.2) and then prove Proposition 2.3. In Section 3 we prove Theorem 1.1 and give some examples.

§2. Bundle Maps and Automorphism Groups of Minimal Models

Let $k : B \to BS^1$ be the classifying map of (1.1). There is a Serre fibration ([1]):

$$map(B, S^1) \to aut_{S^1}P \xrightarrow{\psi} aut_kB,$$

where $map(B, S^1)$ denotes the space of maps $B \to S^1$ and aut_kB is the subspace of $autB$ consisting of all $f \in autB$ satisfying the condition $k \circ f \simeq k$. Since B is simply conected, $map(B, S^1)$ is connected and ψ induces the isomorphism

$$\pi_0(\psi) : \mathrm{Aut}_{S^1}P \cong \mathrm{Aut}_kB,$$

where $\mathrm{Aut}_k B$ denotes the group of path components of $aut_k B$. For $f \in aut_k B$ there is an element $\overline{f} \in aut_{S^1} P$ such that $\psi(\overline{f}) = f$. The diagram

$$\begin{array}{ccc} P & \xrightarrow{\overline{f}} & P \\ \pi\downarrow & & \downarrow\pi \\ B & \xrightarrow{f} & B \end{array} \tag{2.1}$$

is commutative. We define $T : \mathrm{Aut}_k B \to \mathrm{Aut} P$ by $T([f]) = [\overline{f}]$, which is well defined and

$$\mathcal{F} = T \circ \pi_0(\psi).$$

We study T by automorphism groups of Sullivan minimal models ([6]). From now on, we assume that the bundle (1.1) is not trivial. Let $\mathcal{M}(P) = (\wedge V, d)$ be the minimal model of P.

Since there is a fibration $P \to B \to BS^1$, it follows from the non-triviality of (1.1) that the minimal model of B has the form ([3]; p. 206, Theorem 4.6)

$$\mathcal{M}(B) = (Q[t] \otimes \wedge V, D) \tag{2.2}$$

which satisfies the following conditions:

(i) $\deg t = 2$, $D(t) = 0$.

(ii) $(Q[t], 0)$ is the minimal model of BS^1 such that

$$D(1 \otimes v) = 1 \otimes dv + D_t v. \quad (*)$$

(iii) $D_t v$ is a decomposable element contained in the ideal $(t \otimes 1)$.

Let $\pi^* : \mathcal{M}(B) \to \mathcal{M}(P)$ be the D.G.A. (differential graded algebra) map induced by the projection π. Then

$$\begin{aligned} \pi^*(1 \otimes v) &= v \qquad \text{for } v \in \wedge V \\ \pi^*(t \otimes 1) &= 0. \end{aligned}$$

Let $\mathrm{Aut}\mathcal{M}(B)$ be the group of D.G.A. automorphisms of $\mathcal{M}(B)$ and $\mathrm{Aut}^t\mathcal{M}(B)$ be the subgroup of $\mathrm{Aut}\mathcal{M}(B)$ which fixes the element $t \otimes 1$.

Now we quote some results on nilpotent derivations and unipotent automorphism group ([6]). Let $(\mathcal{M}(Y), d_Y)$ be a minimal model and $Der_i\mathcal{M}(Y)$ be the set of Q-derivations of $\mathcal{M}(Y)$ decreasing the degree by i. The boundary operator

$$\delta_Y : Der_i\mathcal{M}(Y) \to Der_{i-1}\mathcal{M}(Y)$$

is defined by

$$\delta_Y \phi = \phi \circ d_Y + (-1)^{i+1} d_Y \circ \phi, \qquad \phi \in Der_i \mathcal{M}(Y).$$

Then ${\delta_Y}^2 = 0$.

If $i > 0$, any element ϕ of $Der_i\mathcal{M}(Y)$ is nilpotent. Hence, for each element v of $\mathcal{M}(Y)$, there is a positive integer m such that $\phi^m(v) = 0$. If an element ϕ of $Der_0\mathcal{M}(Y)$ is nilpotent, exp $\phi = id + \phi + \phi^2/2 + \cdots + \phi^n/n! + \cdots$ is well defined. An element $f \in \mathrm{Aut}\mathcal{M}(Y)$ is D.G.A. homotopic to identity if and only if f can be written as

$$f = \exp\, \delta_Y \phi,$$

where ϕ is a nilpotent derivation of degree one [6, Propositions 6.3 and 6.5]. Then

$$\mathrm{Aut}_Q\mathcal{M}(Y) = \mathrm{Aut}\mathcal{M}(Y)/\exp(\delta_Y Der_1\mathcal{M}(Y))$$

represents the group of D.G.A. homotopy classes of D.G.A. autmorphisms.

For an element $f \in \mathrm{Aut}^t\mathcal{M}(B)$, we have a D.G.A. endomorphism $\pi_t(f)$ of $\mathcal{M}(P)$ defined by

$$\pi_t(f)(v) = \pi^*(f(1 \otimes v))$$

such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{M}(P) & \xrightarrow{\pi_t(f)} & \mathcal{M}(P) \\ {\scriptstyle \pi^*}\uparrow & & \uparrow{\scriptstyle \pi^*} \\ \mathcal{M}(B) & \xrightarrow{f} & \mathcal{M}(B). \end{array} \tag{2.3}$$

If $g : \mathcal{M}(P) \to \mathcal{M}(P)$ is a D.G.A. map satisfying the condition $\pi^* \circ f = g \circ \pi^*$, $g = \pi_t(f)$. Hence $\pi_t(f) \circ \pi_t(g) = \pi_t(f \circ g)$ and $\pi_t(Id_{\mathcal{M}(B)}) = Id_{\mathcal{M}(P)}$. Thus $\pi_t(f) \in \mathrm{Aut}\mathcal{M}(P)$.

Lemma 2.1. *If $f, g \in \mathrm{Aut}^t\mathcal{M}(B)$ are D.G.A. homotopic, then so are $\pi_t(f)$ and $\pi_t(g)$.*

Proof. Since $g^{-1} \circ f$ is D.G.A. homotopic to the identity, there exists $\phi \in Der_1\mathcal{M}(B)$ such that $g^{-1} \circ f = \exp\, \delta_B\phi$. Then $\pi_t(g)^{-1} \circ \pi_t(f)(v) = \pi_t(g^{-1} \circ f)(v) = \pi^*(\exp\, \delta_B\phi(1 \otimes v)) = \exp(\delta_P\overline{\phi}(v))$ where $\overline{\phi}$, the derivation of $\mathcal{M}(P)$, is defined by $\overline{\phi}(v) = \pi^*(\phi(1 \otimes v))$ for $v \in \mathcal{M}(P)$. □

We denote by $\mathrm{Aut}^t_Q\mathcal{M}(X)$ the group of D.G.A. homotopy classes of Aut^t $\mathcal{M}(X)$. Then we have

Proposition 2.2. *The following diagram is commutative.*

$$\begin{array}{ccc} \mathrm{Aut}P & \longrightarrow & \mathrm{Aut}_Q\mathcal{M}(P) \\ {\scriptstyle T}\uparrow & & \uparrow{\scriptstyle [\pi_t]} \\ \mathrm{Aut}_k B & \longrightarrow & \mathrm{Aut}_Q{}^t\mathcal{M}(B), \end{array}$$

where horizontal maps correspond to the induced maps on the minimal models and $[\pi_t]$ is the induced map from π_t on D.G.A. homotopy classes.

Proof. Let $f \in aut_k B$ and $f^* \in \mathrm{Aut}^t\mathcal{M}(B)$ the induced map on minimal model. Then $\pi_t(f^*)(v) = \pi^* f^*(1 \otimes v) = T(f)^*(v)$ by (2.1). Taking homotopy class, we have the assertion. □

Let $f \in \mathrm{Aut}^t\mathcal{M}(B)$ be an element such that $[f] \in \mathrm{Ker}[\pi_t]$. Then $\pi_t(f)$ is D.G.A. homotopic to identity and it follows from [6, Proposition 6.5] that we may write

$$\pi_t(f) = \exp\, \delta_P \psi$$

for some $\psi \in Der_1\mathcal{M}(P)$. Define $\tilde{\psi} \in Der_1\mathcal{M}(B)$ by

$$\begin{aligned} &\tilde{\psi}(1 \otimes v) = 1 \otimes \psi(v) \qquad \text{for } v \in \wedge V \\ &\tilde{\psi}(t \otimes 1) = 0. \end{aligned}$$

Now consider the element

$$\tilde{f} = (\exp\, \delta_B \tilde{\psi})^{-1} \circ f \in \mathrm{Aut}^t\mathcal{M}(B) \qquad (**),$$

where $\delta_B\tilde{\psi} = \tilde{\psi}D + D\tilde{\psi}$. Since $D(1 \otimes v) - 1 \otimes dv$ is contained in the ideal $(t \otimes 1) = (t)$ by $(*)$, $\delta_B\tilde{\psi}(1 \otimes v) - 1 \otimes \delta_P\psi(v)$ is also contained in the ideal (t). Then we have:

$$\begin{aligned} \tilde{f}(1 \otimes v) &= (\exp\, \delta_B\tilde{\psi})^{-1} \circ f(1 \otimes v) \\ &= (\exp\, \delta_B\tilde{\psi})^{-1} \left(1 \otimes \pi_t(f)(v) + \sum_{j \geq 1} t^j \otimes w_j \right) \qquad (w_j \in \wedge V) \\ &= (\exp\, \delta_B\tilde{\psi})^{-1} \left(1 \otimes \exp\, \delta_P\psi(v) + \sum_{j \geq 1} t^j \otimes w_j \right) \\ &= 1 \otimes (\exp\, \delta_p\psi)^{-1} \circ \exp\, \delta_p\psi(v) + \sum_{j \geq 1} t^j \otimes (\exp\, \delta_P\psi)^{-1}(w_j) \\ &= 1 \otimes v + \sum_{j \geq 1} t^j \otimes (\exp\, \delta_P\psi)^{-1}(w_j). \end{aligned}$$

So we take

$$\tilde{f} - Id_{\mathcal{M}(B)} = X$$

and

$$\sigma = \log(Id_{\mathcal{M}(B)} + X) = X - X^2/2 + \cdots .$$

Then $\sigma \in Der_0^t(\mathcal{M}(B),(t))$ and $\delta_B \sigma = 0$ since X commutes with D. Then we can write $\tilde{f} = \exp\, \sigma$, and by $(**)$ f and $\tilde{f}$ are D.G.A. homotopic. Here $Der_0^t(\mathcal{M}(B),(t))$ denotes the set of degree zero $Q[t]$-derivations of $\mathcal{M}(B)$ with value in the ideal (t). Note that $Der_0^t(\mathcal{M}(B),(t))$ forms a Lie algebra by $[\sigma,\tau] = \sigma\circ\tau - \tau\circ\sigma$ for $\sigma,\tau \in Der_0^t(\mathcal{M}(B),(t))$ and that any element of it is nilpotent.

Conversely, if we can write $f = \exp\,\tau$ for $\tau \in Der_0^t(\mathcal{M}(B),(t))$ with $\delta_B\tau = 0$, then $\pi_t(f)(v) = \pi^*(\exp\,\tau(1\otimes v)) = v$.

Thus we see that

$$[\exp] : Z_0(\mathcal{M}(B),(t)) \to \mathrm{Ker}[\pi_t]$$

is surjective map, where [exp] is the D.G.A. homotopy class of the composition of the exponential map and $Z_0(\mathcal{M}(B),(t)) = \{Der_0^t(\mathcal{M}(B),(t)); \delta_B\phi = 0\}$.

Proposition 2.3. *If* $H_0(Der^t(\mathcal{M}(B),(t))) = 0$, *then* $\mathrm{Ker}[\pi_t] = \{Id_{\mathcal{M}(B)}\}$, *where* $H_0(Der^t(\mathcal{M}(B),(t))) = Z_0(\mathcal{M}(B),(t))/\delta_B Der_1\mathcal{M}(B) \cap Z_0(\mathcal{M}(B),(t))$.

Proof. We take $\bar{H}_0 = Z_0(\mathcal{M}(B),(t))/\sim$, where $\sim$ is defined as follows: For $\sigma,\tau \in Z_0(\mathcal{M}(B),(t))$, $\sigma\sim\tau$ if $\exp\,\sigma$ is D.G.A. homotopic to $\exp\,\tau$, that is, $\exp\,\sigma\circ\exp(-\tau) \sim Id_{\mathcal{M}(B)}$. By the Baker-Campbell-Hausdorff formula, it is equivalent to

$$\sigma - \tau + \frac{1}{2}[\sigma,\tau] - \frac{1}{12}[\sigma,[\sigma,\tau]] + \cdots \ \in \delta_B Der_1(\mathcal{M}(B)).$$

Let $p : Z_0(\mathcal{M}(B),(t)) \to \bar{H}_0$ be the natural map. If $\sigma - \tau = \delta_B\psi$ for some $\psi \in Der_1(\mathcal{M}(B))$, we have

$$[\sigma,\tau] = [\sigma, \sigma - \delta_B\psi] = -[\sigma,\delta_B\psi] = -\delta_B[\sigma,\psi].$$

Similarly we see that each term of the Baker-Campbell-Hausdorff formula is δ_B-exact. Hence $\sigma\sim\tau$. Thus p induces a surjective map $\overline{p} : H_0(Der^t(\mathcal{M}(B),(t))) \to \bar{H}_0$. If $H_0(Der^t(\mathcal{M}(B),(t))) = 0$, then the set $\bar{H}_0$ consists of one element (represented by δ_B-exact element). Since [exp] induces the bijective correspondence $\bar{H}_0 \to \mathrm{Ker}[\pi_t]$, we have $\mathrm{Ker}[\pi_t] = \{Id_{\mathcal{M}(B)}\}$. □

§3. The Proof of Theorem 1.1 and Examples

Proposition 3.1. *If $H_2(Der\mathcal{M}(B)) = 0$, then*

$$[\pi_t] : \mathrm{Aut}^t_Q\mathcal{M}(B) \to \mathrm{Aut}_Q\mathcal{M}(P)$$

is monomorphic.

Proof. Consider the homomorphism

$$t_* : H_2(Der^t\mathcal{M}(B)) \to H_0(Der^t(\mathcal{M}(B), (t)))$$

induced from the multiplication by t with its value. Clearly it is epimorphic. Since

$$H_2(Der^t\mathcal{M}(B)) \subset H_2(Der\mathcal{M}(B)),$$

$H_0(Der^t(\mathcal{M}(B), (t))) = 0$. Then by Proposition 2.3, $[f] = Id_{\mathcal{M}(B)}$. □

Proof of Theorem 1.1. It follows from ([6, p. 313-314]) that $H_2(Der\mathcal{M}(B)) \cong \pi_2(aut_{Id}(B)) \otimes Q = 0$. Hence $[\pi_t]$ is monomorphic by Proposition 3.1. By [6, Theorem 10.2], the kernel of the horizontal maps of the diagram of Proposition 2.2 is finite group. Hence the assertion easily follows when the bundle (1.1) is non-trivial.

Next we consider the case the bundle (1.1) is trivial. Note that (2.2) is not minimal and $\pi_t(f)$ is not well-defined in (2.3) in this case. Then $P \simeq S^1 \times B$ and $\mathrm{Aut}_{S^1}(S^1 \times B) \cong \mathrm{Aut}B$. Then T is monomorphic. □

Remark. If P is 2-connected, then $\mathcal{M}^2(B)$ is the vector space spanned by t. Then

$$H_0(Der^t(\mathcal{M}(B), (t))) \subset H_0(Der(\mathcal{M}(B), \mathcal{M}^+(B) \cdot \mathcal{M}^+(B))),$$

where $Der(\mathcal{M}(B), \mathcal{M}^+(B) \cdot \mathcal{M}^+(B))$ is the Lie algebra of Q-derivations of $\mathcal{M}(B)$ whose values are decomposable elements. This implies that any element of $\mathrm{Ker}[\pi_t]$ induces identity on the rational homotopy group. In particular, if $\mathrm{Aut}_\sharp\mathcal{M}(B) = \{Id_{\mathcal{M}(B)}\}$, Ker $\mathcal{F}$ is a finite group. Here $\mathrm{Aut}_\sharp\mathcal{M}(B)$ denotes the group of D.G.A. homotopy classes of D.G.A. automorphisms of $\mathcal{M}(B)$ which induce identity on the rational homotopy group.

In the following three examples the bundles are not trivial, so we can use the result due to S. Halperin [4, Proposition 4.2]:

Proposition 3.2 ([4]). *If the minimal models of P and B are given as in (2.2) and $\dim_Q H^*(\mathcal{M}(B)) < \infty$, then P has the same rational homotopy type as a total space of a principal S^1-bundle over B.*

Example 3.1. Let

$$\mathcal{M}(P) = (\wedge(x_3, y_3, z_5), d)$$

with $d(z) = xy$, $d(x) = d(y) = 0$, $\deg *_i = i$ and

$$\mathcal{M}(B) = (\wedge(t, x_3, y_3, z_5), D)$$

with $D(t) = 0$, $D(x) = D(y) = 0$, $D(z) = xy + t^3$. Then $\dim H^*(B; Q) < \infty$ and we can have $\pi_2(aut_{Id}B) \otimes Q \cong H_2(Der\mathcal{M}(B)) = 0$ by straigtfoward calculations.

Example 3.2. In Theorem 1.1, the condition $\pi_2(aut_{Id}B) \otimes Q = 0$ is not necessary for Ker $\mathcal{F}$ being finite. In fact, let

$$\mathcal{M}(P) = (\wedge(x_3, y_3, z_5, w_9), d)$$

with $d(z) = xy$ and $d(x) = d(y) = d(w) = 0$ and

$$\mathcal{M}(B) = (\wedge(t, x_3, y_3, z_5, w_9), D)$$

with $D(z) = xy + t^3$, $D(t) = D(x) = D(y) = D(w) = 0$. Then $\dim H^*(B; Q) < \infty$ and $H_2(Der\mathcal{M}(B))$ is the two dimensional vector space spanned by (w, xt^2) and (w, yt^2), where (u, v) denotes the Q-derivation which sends u to v and the other generators to zero. On the other hand, we have $H_0(Der^t(\mathcal{M}(B), (t))) = 0$. Hence by Proposition 2.3 Ker $\mathcal{F}$ is finite for P.

Example 3.3. There are two principal S^1-bundles P_1, P_2 over the same base B such that Ker $\mathcal{F}_{P_1}$ is not finite but Ker $\mathcal{F}_{P_2}$ is finite. Let

$$\mathcal{M}(P_1) = (\wedge(s_2, x_3, v_3, u_3, z_7, w_7), d_1)$$

with $d_1(s) = d_1(x) = d_1(v) = d_1(z) = d_1(w) = 0$, $d_1(u) = s^2$ and

$$\mathcal{M}(P_2) = (\wedge(t_2, x_3, v_3, u_3, z_7, w_7), d_2)$$

with $d_2(t) = d_2(x) = d_2(v) = d_2(u) = d_2(w) = 0$, $d_2(z) = t^4$. Let

$$\mathcal{M}(B) = (Q[t] \otimes \mathcal{M}(P_1), D) = (Q[s] \otimes \mathcal{M}(P_2), D)$$

with $D(s) = D(t) = D(x) = D(w) = 0$ and $D(z) = t^4$, $D(v) = st$, $D(u) = s^2$. Then $\dim H^*(B; Q) < \infty$ and $H_0(Der^t(\mathcal{M}(B), (t)))$ is one dimensional vector space spanned by (w, t^2x). But $H_0(Der^s(\mathcal{M}(B), (s))) = 0$.

Finally we note that the forgetful map can be defined via fiber homotopy equivalences.

Let P be a space on which S^1 acts (not necessarily free). We consider the fibration

$$P \to ES^1 \times_{S^1} P \to BS^1,$$

where S^1 acts on $ES^1 \times P$ by the usual manner. Let $\mathcal{L}(ES^1 \times_{S^1} P)$ be the group of homotopy classes of fiber homotopy equivalence of $ES^1 \times_{S^1} P$. For each $[f] \in \mathrm{Aut}_{S^1}P$, we denote $[f_1] \in \mathcal{L}(ES^1 \times_{S^1} P)$ be the induced map from $id \times f : ES^1 \times P \to ES^1 \times P$. Hence we have a homomorphism $\psi : \mathrm{Aut}_{S^1}P \to \mathcal{L}(ES^1 \times_{S^1} P)$ by $\psi([f]) = [f_1]$. If we recall the natural map ([2]) $R : \mathcal{L}(ES^1 \times_{S^1} P) \to \mathrm{Aut}P$, then we have

$$\mathcal{F} = R \circ \psi.$$

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