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Backward Global Solutions Characterizing Annihilation Dynamics of Travelling Fronts

By

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Abstract

We consider a reaction-diffusion equation $u_t = u_{xx} + f(u)$, where f has exactly three zeros 0, α and 1 ($0 < \alpha < 1$), $f_u(0) < 0$, $f_u(1) < 0$ and $\int_0^1 f(u)du \geq 0$. Then, the equation has a travelling wave solution $u(x,t) = \phi(x - ct)$ with $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$. Known results suggest that for an initial state $u_0(x)$ with $\underline{\lim}_{x \to \pm\infty} u_0(x) > \alpha$ having two interfaces at a large distance, $u(x,t)$ approaches a pair of travelling wave solutions $\phi(x - p_1(t)) + \phi(-x + p_2(t))$ for a long time, and then the travelling fronts eventually disappear by colliding with each other. While our results establish this process, they show that *there is a* (*backward*) *global solution* $\psi(x,t)$ and that *the annihilation process is approximated by a solution* $\psi(x - x_0, t - t_0)$.

§1. Introduction

In this paper, we consider the scalar bistable reaction-diffusion equation

$$(1.1) \qquad \begin{cases} u_t = u_{xx} + f(u), & t > 0, \quad x \in \mathbf{R}, \\ u(0) = u_0 \in BU(\mathbf{R}), \end{cases}$$

where $BU(\mathbf{R})$ is the space of bounded uniformly continuous functions from $\mathbf{R}$ to $\mathbf{R}$ with the supremum norm, and the reaction term f satisfies the following conditions:

1 $f \in C^2(\mathbf{R})$,

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2 f has exactly three zeros 0, α and 1 $(0 < \alpha < 1)$,

3 $f_u(0) < 0,\ f_u(1) < 0$,

4 $\int_0^1 f(u)du \geq 0$.

It is known (e.g. [4, Section 4.4]) that the reaction-diffusion equation (1.1) has a unique (except for translation) travelling wave solution $u(x,t) = \phi(x-ct)$, where (ϕ, c) satisfies

$$\phi''(z) + c\phi'(z) + f(\phi(z)) = 0 \tag{1.2}$$

with $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$. Then $c \leq 0$ holds from $\int_0^1 f(u)du \geq 0$. We normalize the definition of ϕ by requiring $\phi(0) = 1/2$.

This solution is linearly stable except for neutral translational perturbations. Specifically, the following is known (e.g. [10, Section 5.4]).

Theorem A. (1) *The operator* $-(\partial^2/\partial z^2 + c(\partial/\partial z) + f_u(\phi(z))) : BU(\mathbf{R}) \to BU(\mathbf{R})$ *is a sectorial one with a simple eigenvalue* 0. *The remainder of the spectrum has real part greater than some positive constant.*

(2) *There exist* δ, C *and* $\gamma > 0$ *such that for any* $u_0 \in BU(\mathbf{R})$ *with* $\|u_0(x) - \phi(x)\|_{C^0} \leq \delta$, *there exists* $x_0 \in \mathbf{R}$ *satisfying*

$$\|u(x,t) - \phi(x - x_0 - ct)\|_{C^0} \leq Ce^{-\gamma t}\|u_0(x) - \phi(x)\|_{C^0}$$

for all $t \geq 0$.

Moreover, Fife and McLeod [6] showed the following theorem, which gives a global stability result for the travelling wave solution $\phi(x - ct)$.

Theorem B. *If* $\overline{\lim}_{x\to-\infty} u_0(x) < \alpha$ *and* $\underline{\lim}_{x\to+\infty} u_0(x) > \alpha$ *hold, then*

$$\inf_{x_0 \in \mathbf{R}} \|u(x,t) - \phi(x - x_0)\|_{C^0} \to 0 \quad as \quad t \to +\infty$$

holds.

Also, Fife and McLeod [6] showed the following, which means that the pair of the travelling wave solutions going to $x = \pm\infty$ has strong attractivity.

Theorem C. *Suppose that* $c < 0$, $\overline{\lim}_{x\to\pm\infty} u_0(x) < \alpha$, $u_0(x) \geq \eta$ $(|x| < L)$ *for some* $\eta > \alpha$ *and* $u_0(x) \geq \zeta$ $(|x| < \infty)$ *for some* $\zeta > -\infty$ *hold. If* L *is large enough depending on* η *and* ζ, *then* $u(x,t)$ *approaches* (*uniformly in* x *and exponentially in* t) *a pair of diverging travelling wave solutions*

$$\phi(x - x_1 - ct) + \phi(-x - x_2 - ct) - 1.$$

On the other hand, when $\underline{\lim}_{x\to\pm\infty} u_0(x) > \alpha$ holds, the following is known (e.g. [5]).

Proposition D. *If* $\underline{\lim}_{x\to\pm\infty} u_0(x) > \alpha$ *holds, then* $\lim_{t\to+\infty} \|u(x,t) - 1\|_{C^0} = 0$ *holds.*

For an initial state $u_0(x)$ with $\underline{\lim}_{x\to\pm\infty} u_0(x) > \alpha$ having two interfaces at a large distance, Theorems A, B and C suggest that $u(x,t)$ approaches a pair of travelling wave solutions

$$\phi(x - p_1(t)) + \phi(-x + p_2(t))$$

for a long time. Then, Proposition D suggests that the travelling fronts eventually disappear by colliding with each other. While our main results (Theorem 1.1 and Corollary 1.4) establish this process, they show that *there is a* (*backward*) *global solution* $\psi(x,t)$ and that *the annihilation process is approximated by a solution* $\psi(x - x_0, t - t_0)$.[1]

The following theorem shows that there is a (backward) global solution $\psi(x,t)$ such that it approaches a pair of travelling wave solutions as $t \to -\infty$ and is locally asymptotic stable uniformly in $t \in \mathbf{R}$.

Theorem 1.1. *There exists a solution* $\psi \in C(\mathbf{R}, BU(\mathbf{R}))$ *of* $u_t = u_{xx} + f(u)$ *satisfying* $\lim_{t\to+\infty} \|\psi(t) - 1\|_{C^0(\mathbf{R})} = 0$, $\psi(-x,t) = \psi(x,t)$ *and the following.*

(1) *There exists* $p \in C^1(\mathbf{R})$ *such that*

$$p(-\infty) = +\infty,\ \dot{p}(-\infty) = c$$

and

$$\lim_{t\to-\infty} \|\psi(x,t) - (\phi(x - p(t)) + \phi(-x - p(t)))\|_{C^0(\mathbf{R})} = 0$$

hold.

(2) *There exist* $\delta > 0$, $C > 0$ *and* $\gamma > 0$ *such that for any* $t_0 \in \mathbf{R}$ *and* $u_0 \in BU(\mathbf{R})$ *satisfying* $\|u_0 - \psi(t_0)\|_{C^0(\mathbf{R})} \le \delta$, *there exist* $x_0 \in \mathbf{R}$ *and* $t_0' \in \mathbf{R}$ *such that the solution* u *of* (1.1) *satisfies*

$$\|u(x,t) - \psi(x - x_0, t - t_0')\|_{C^0(\mathbf{R})} \le Ce^{-\gamma t}\|u_0(x) - \psi(x, t_0)\|_{C^0(\mathbf{R})}$$

for all $t \ge 0$.

[1] For mathematical studies on motion and collapse of fronts in (1.1) from other aspects, we can refer to, e.g., [1], [2], [3], [7], [8], [9], [11] and [12].

Theorem 1.1 leads to the following. This is a uniqueness result for the global solution $\psi(x,t)$.

Corollary 1.2. *For any $T \in [-\infty, +\infty)$ and solution $\bar{\psi} \in C((T,+\infty), BU(\mathbf{R}))$ of $u_t = u_{xx} + f(u)$, if there exist $\{p_n\}_{n=1}^{\infty}$, $\{q_n\}_{n=1}^{\infty} \subset \mathbf{R}$ and $\{T_n\}_{n=1}^{\infty} \subset (T,+\infty)$ such that*

$$\lim_{n\to\infty}(p_n - q_n) = +\infty$$

and

$$\lim_{n\to\infty} \|\bar{\psi}(x,T_n) - (\phi(x-p_n) + \phi(-x+q_n))\|_{C^0(\mathbf{R})} = 0 \tag{1.3}$$

hold, then $T = -\infty$ holds and there exist x_0 and $t_0 \in \mathbf{R}$ satisfying

$$\psi(x,t) = \bar{\psi}(x+x_0, t+t_0).$$

Proof. By Theorem 1.1 (1), there exists $\{t'_n\}_{n=1}^{\infty} \subset \mathbf{R}$ with $\lim_{n\to\infty} t'_n = -\infty$ such that

$$\lim_{n\to\infty} \left\| \psi(x,t'_n) - \left(\phi\left(x - \frac{p_n - q_n}{2}\right) + \phi\left(-x - \frac{p_n - q_n}{2}\right)\right)\right\|_{C^0(\mathbf{R})} = 0$$

holds. From this and (1.3),

$$\lim_{n\to\infty} \left\| \bar{\psi}\left(x + \frac{p_n + q_n}{2}, T_n\right) - \psi(x,t'_n)\right\|_{C^0(\mathbf{R})} = 0$$

holds. By Theorem 1.1 (2), if $n \in \{1,2,\cdots\}$ is sufficiently large, then there exist x_n and $t_n \in \mathbf{R}$ such that

$$\begin{aligned}&\|\bar{\psi}(x,t+T_n) - \psi(x-x_n, t+T_n-t_n)\|_{C^0(\mathbf{R})} \\ &\qquad \le Ce^{-\gamma t}\left\| \bar{\psi}\left(x + \frac{p_n + q_n}{2}, T_n\right) - \psi(x,t'_n)\right\|_{C^0(\mathbf{R})}\end{aligned}$$

holds for all $t \ge 0$. Therefore, we obtain

$$\lim_{n\to\infty} \sup_{t \ge T_n - t_n} \|\bar{\psi}(x+x_n, t+t_n) - \psi(x,t)\|_{C^0(\mathbf{R})} = 0. \tag{1.4}$$

Hence, from (1.3),

$$\lim_{n\to\infty} \|\psi(x,T_n - t_n) - (\phi(x-(p_n-x_n)) + \phi(-x+(q_n-x_n)))\|_{C^0(\mathbf{R})} = 0$$

holds. Because of this and $\lim_{n\to\infty}((p_n - x_n) - (q_n - x_n)) = +\infty$, we obtain $\lim_{n\to\infty}(T_n - t_n) = -\infty$ by Theorem 1.1 (1).

Now, we show that there exists $\bar{t}_0 \in \mathbf{R}$ such that $\lim_{n\to\infty} t_n = \bar{t}_0$ holds. Assume that there exist $\{N_n\}_{n=1}^{\infty}$ and $\{M_n\}_{n=1}^{\infty} \subset \{1, 2, \cdots\}$ such that $\lim_{n\to\infty} N_n = \lim_{n\to\infty} M_n = \infty$ and $\inf_{n=1,2,\cdots}(t_{N_n} - t_{M_n}) > 0$ hold. Then, by (1.4),

$$\lim_{n\to\infty} \|\psi(x,t) - \psi(x + x_{N_n} - x_{M_n}, t + t_{N_n} - t_{M_n})\|_{C^0(\mathbf{R})} = 0$$

holds for all $t \in \mathbf{R}$. This is contradiction with $\inf_{n=1,2,\cdots}(t_{N_n} - t_{M_n}) > 0$. Hence, $\lim_{n\to\infty} t_n = \bar{t}_0 \in \mathbf{R}$ holds.

Because of $\lim_{n\to\infty}(T_n - t_n) = -\infty$ and $\lim_{n\to\infty} t_n = \bar{t}_0 \in \mathbf{R}$, we obtain $T \le \lim_{n\to\infty} T_n = -\infty$. Also, by (1.4),

$$\lim_{(n,m)\to(\infty,\infty)} \|\psi(x, t - \bar{t}_0) - \psi(x + x_n - x_m, t - \bar{t}_0)\|_{C^0(\mathbf{R})} = 0$$

holds for all $t \in \mathbf{R}$. Hence, we have $\lim_{(n,m)\to(\infty,\infty)} |x_n - x_m| = 0$. There exists $\bar{x}_0 \in \mathbf{R}$ such that $\lim_{n\to\infty} x_n = \bar{x}_0$ holds. Therefore, by (1.4), we obtain $\bar{\psi}(x + \bar{x}_0, t + \bar{t}_0) = \psi(x,t)$. □

Definition 1. For $l > 0$, $\delta \in (0, \min\{\alpha, 1-\alpha\})$ and $L > 0$, a closed subset $\Xi_{l,\delta,L}$ of $BU(\mathbf{R})$ is defined by

$$\Xi_{l,\delta,L} = \{u \in BU(\mathbf{R}) |\, 0 \le u(x) \le \alpha - \delta \ (|x| < l - L),$$
$$0 \le u(x) \le 1 \ (l - L \le |x| \le l + L),\ \alpha + \delta \le u(x) \le 1 \ (l + L < |x|)\}.$$

For $\bar{l} > 0$, $\bar{\delta} \in (0, \min\{\alpha, 1-\alpha\})$ and $\bar{L} > 0$, a closed subset $\Pi_{\bar{l},\bar{\delta},\bar{L}}$ of $BU(\mathbf{R})$ is defined by

$$\Pi_{\bar{l},\bar{\delta},\bar{L}} = \bigcup_{l \ge \bar{l}} \Xi_{l,\bar{\delta},\bar{L}}.$$

Under the above definition, the following proposition holds. This is proved in Section 6.

Proposition 1.3. *For any $\bar{\delta}_0 \in (0, \min\{\alpha, 1-\alpha\})$, $\bar{L}_0 > 0$ and $\varepsilon > 0$, there exist $\bar{l}_0 > 0$, $L > 0$ and $T > 0$ such that for any $l \ge \bar{l}_0$ and $u_0 \in \Xi_{l,\bar{\delta}_0,\bar{L}_0}$, there exist x_1 and $x_2 \in [l - L, l + L]$ such that the solution u of* (1.1) *satisfies*

$$\|u(x,T) - (\phi(x - x_1 - cT) + \phi(-x - x_2 - cT))\|_{C^0(\mathbf{R})} < \varepsilon.$$

Theorem 1.1 and Proposition 1.3 lead to the following. This states that when an initial state has two interfaces at a large distance, the annihilation process of the interfaces is approximated by a solution $\psi(x - x_0, t - t_0)$.

Corollary 1.4. *For any* $\bar{\delta}_0 \in (0, \min\{\alpha, 1-\alpha\})$, $\bar{L}_0 > 0$, $T_0 \in \mathbf{R}$ *and* $\varepsilon > 0$, *there exists* $\bar{l}_0 > 0$ *such that for any* $u_0 \in \Pi_{\bar{l}_0, \bar{\delta}_0, \bar{L}_0}$, *there exist* $x_0 \in \mathbf{R}$ *and* $t_0 \geq -T_0$ *such that the solution* u *of* (1.1) *satisfies*

$$\sup_{t \geq T_0} \|u(x + x_0, t + t_0) - \psi(x, t)\|_{C^0(\mathbf{R})} < \varepsilon.$$

Proof. We first show that there exist $M > 0$ and $\varepsilon' \in (0, \varepsilon)$ such that for any p, q and $t \in \mathbf{R}$, if

$$p + q \geq M$$

and

$$\|\psi(x,t) - (\phi(x-p) + \phi(-x-q))\|_{C^0(\mathbf{R})} < \left(1 + \frac{1}{2C}\right)\varepsilon' \tag{1.5}$$

hold, then $t \leq T_0$ holds. Assume that there exist $\{p_n\}_{n=1}^{\infty}$, $\{q_n\}_{n=1}^{\infty} \subset \mathbf{R}$ and $\{t_n\}_{n=1}^{\infty} \subset (T_0, +\infty)$ such that

$$\lim_{n \to \infty} (p_n + q_n) = +\infty$$

and

$$\lim_{n \to \infty} \|\psi(x, t_n) - (\phi(x - p_n) + \phi(-x - q_n))\|_{C^0(\mathbf{R})} = 0$$

hold. Then, from Corollary 1.2, $T_0 = -\infty$ holds. This is contradiction for $T_0 \in \mathbf{R}$.

By Proposition 1.3, there exist L, T and $\bar{l}'_0 > 0$ such that for any $l \geq \bar{l}'_0$ and $u_0 \in \Xi_{l, \bar{\delta}_0, \bar{L}_0}$, there exist x_1 and $x_2 \geq l - (L - cT)$ such that

$$\|u(x,T) - (\phi(x - x_1) + \phi(-x - x_2))\|_{C^0(\mathbf{R})} < \min\left\{\frac{\varepsilon'}{2C}, \frac{\delta}{2}\right\} \tag{1.6}$$

holds. Now, we let $\bar{l}_0 > 0$ be sufficiently large. Then, because $(x_1 + x_2)/2 > 0$ is sufficiently large, by Theorem 1.1 (1), there exists $t'_0 \in \mathbf{R}$ such that

$$\left\|\psi(x, t'_0) - \left(\phi\left(x - \frac{x_1 + x_2}{2}\right) + \phi\left(-x - \frac{x_1 + x_2}{2}\right)\right)\right\|_{C^0(\mathbf{R})} < \min\left\{\frac{\varepsilon'}{2C}, \frac{\delta}{2}\right\}$$

holds. Therefore, we have

$$\left\|u\left(x + \frac{x_1 - x_2}{2}, T\right) - \psi(x, t'_0)\right\|_{C^0(\mathbf{R})} < \min\{\varepsilon'/C, \delta\}.$$

Hence, by Theorem 1.1 (2), there exist x_0 and $t_0 \in \mathbf{R}$ such that

$$\sup_{t \geq T} \|u(x,t) - \psi(x - x_0, t - t_0)\|_{C^0(\mathbf{R})} < \varepsilon' \tag{1.7}$$

holds. From (1.6) and (1.7), we have

$$\|\psi(x, T - t_0) - (\phi(x - (x_1 - x_0)) + \phi(-x - (x_2 + x_0)))\|_{C^0(\mathbf{R})} < \left(1 + \frac{1}{2C}\right)\varepsilon'.$$

Because $(x_1 - x_0) + (x_2 + x_0)$ is sufficiently large and (1.5) holds, $T - t_0 \leq T_0$ holds. Hence, from (1.7), $\sup_{t \geq T_0} \|u(x+x_0, t+t_0) - \psi(x,t)\|_{C^0(\mathbf{R})} < \varepsilon$ holds. □

We prove Theorem 1.1 by Sections 2, 3, 4 and 5. In order to do, we need to construct a *global* invariant manifold with asymptotic stability, where the word of *global* means that the invariant manifold includes a solution having two interfaces at any sufficiently large distance. In Section 2, we construct a semilinear parabolic system including a part of the reaction-diffusion equation. This part consists of solutions near pairs of the travelling wave solutions at a large distance. Further, such pairs are contained in a two-dimensional *linear* subspace of the system. Hence, we can construct a global invariant manifold near the subspace by a standard technique. While we do it in Section 5, we state the result in the end of Section 2. In Section 3, we show that there exists a solution on the invariant manifold of the system satisfying Theorem 1.1 (1) in the reaction-diffusion equation, i.e., it becomes the pair of the travelling wave solutions as $t \to -\infty$. This solution is denoted by $\psi(x,t)$. In Section 4, we show that the set of solutions $\psi(x - x_0, t - t_0)$ by translation of $\psi(x,t)$ corresponds to the invariant manifold. This argument is rather troublesome. Then, we prove Theorem 1.1 (2), i.e., we show that this set has asymptotic stability in the reaction-diffusion equation. This is also somewhat troublesome, as the topologies of the equation and the system are different. Finally, Proposition 1.3 is proved in Section 6.

§2. An Extended System and an Invariant Manifold

In this section, we define a semilinear parabolic system $(\mathrm{ES})_{l,r}$ for $l > 0$ and $r > 0$ (Definition 4). Then, we show that the system $(\mathrm{ES})_{l,r}$ includes a part of the reaction-diffusion equation (Proposition 2.2). This part consists of solutions near pairs of the travelling wave solutions at a large distance. Also, we state that there exists a two-dimensional invariant manifold S with asymptotic stability near such pairs in $(\mathrm{ES})_{l_0,r_0}$ for some $l_0 > 0$ and $r_0 > 0$ (Theorem 2.3).

In virtue of (1.2) and Theorem A (1), the following definition is allowed.

Definition 2. A Banach space X and a sectorial operator A in X with $\mathrm{Re}\,\sigma(A) > 0$ are defined by

$$X = \left\{ u \in BU(\mathbf{R}) |\, \langle u(y), e^{cy}\phi'(y)\rangle := \int_{-\infty}^{+\infty} u(y) e^{cy}\phi'(y) dy = 0 \right\}^2,$$
$$\|(g,h)\|_X = \|(g,h)\|_{C^0} := \|g\|_{C^0} + \|h\|_{C^0}$$

and

$$\begin{aligned} D(A) &= X \cap \{u \in C^2(\mathbf{R}) |\, u, u_y, u_{yy} \in BU(\mathbf{R})\}^2, \\ A(g,h) &= -(g_{yy}(y) + c g_y(y) + f_u(\phi(y)) g(y), \\ &\qquad h_{yy}(y) + c h_y(y) + f_u(\phi(y)) h(y)), \end{aligned}$$

respectively. A Banach space X' and an open subset U of X' are defined by

$$X' = D(A^{3/4}),\ \|(g,h)\|_{X'} = \|A^{3/4}(g,h)\|_X$$

and

$$U = \{(g,h) \in X' |$$
$$\langle \phi'(y) + g_y(y), e^{cy}\phi'(y)\rangle \neq 0,\, \langle \phi'(y) + h_y(y), e^{cy}\phi'(y)\rangle \neq 0\},$$

respectively.

A Banach space Y and a sectorial operator B in Y with $\mathrm{Re}\,\sigma(B) > 0$ are defined by

$$Y = BU(\mathbf{R}),\ \|u\|_Y = \|u\|_{C^0}$$

and

$$\begin{aligned} D(B) &= \{u \in C^2(\mathbf{R}) |\, u, u_x, u_{xx} \in BU(\mathbf{R})\}, \\ Bu &= -(u_{xx} + f_u(1) u), \end{aligned}$$

respectively. Banach spaces Y' and Y'' are defined by

$$Y' = D(B^{3/4}),\ \|u\|_{Y'} = \|B^{3/4} u\|_Y$$

and

$$Y'' = D(B^{7/8}),\ \|u\|_{Y''} = \|B^{7/8} u\|_Y,$$

respectively.

Definition 3. For $l > 0$, a map $f_l \in C^1(X' \times \mathbf{R}^2, Y)$ is defined by

$$\begin{aligned}
f_l[g,h,p,q](y) &= f(\phi(y)+g(y)) - (f(\phi(y)) + f_u(\phi(y))g(y)) \\
&- \frac{2}{l}\frac{1}{1+e^{(y+p)/l}}(\phi'(y)+g_y(y)) + \frac{1}{l^2}\frac{1}{1+e^{(y+p)/l}}(\phi(y)+g(y)) \\
&+ \frac{1}{1+e^{-(y+p)/l}} f(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q)) \\
&- f\left(\frac{1}{1+e^{-(y+p)/l}}(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q))\right).
\end{aligned}$$

For $l > 0$, maps $F_l^1, F_l^2 \in C^1(U \times \mathbf{R}^2, Y)$ and $F_l \in C^1(U \times \mathbf{R}^2, X)$ are defined by

$$\begin{aligned}
F_l^1[g,h,p,q](y) &= f_l[g,h,p,q](y) - \frac{\langle f_l[g,h,p,q](y), e^{cy}\phi'(y)\rangle}{\langle \phi'(y)+g_y(y), e^{cy}\phi'(y)\rangle}(\phi'(y)+g_y(y)), \\
F_l^2[g,h,p,q] &= F_l^1[h,g,q,p]
\end{aligned}$$

and

$$F_l = (F_l^1, F_l^2),$$

respectively.

For $l > 0$, maps $G_l^1, G_l^2 \in C^1(U \times \mathbf{R}^2)$ and $G_l \in C^1(U \times \mathbf{R}^2, \mathbf{R}^2)$ are defined by

$$\begin{aligned}
G_l^1[g,h,p,q] &= -\frac{\langle f_l[g,h,p,q](y), e^{cy}\phi'(y)\rangle}{\langle \phi'(y)+g_y(y), e^{cy}\phi'(y)\rangle}, \\
G_l^2[g,h,p,q] &= G_l^1[h,g,q,p]
\end{aligned}$$

and

$$G_l = (G_l^1, G_l^2),$$

respectively.

The following lemma is easily seen.

Lemma 2.1. *There exists a cutoff function* $\chi(x) \in C^\infty(\mathbf{R})$ *satisfying* $\chi(x) = 0$ $(x < 1/2)$, $\chi(x) = 1$ $(x > 1)$ *and* $0 \le \chi'(x) \le 4$.

The following defines some semilinear parabolic systems.

Definition 4. For $l > 0$ and $r > 0$, semilinear parabolic systems $(\mathrm{ES})_l$, $(\mathrm{ES})_{l,r}$ in $U \times \mathbf{R}^2$ and (RD) in Y are defined by

$$(\mathrm{ES})_l \quad \begin{cases} \frac{d}{dt}(g,h) + A(g,h) = F_l[g,h,p,q], \\ \frac{d}{dt}(p,q) = (c,c) + G_l[g,h,p,q], \\ \qquad (g,h,p,q) \in U \times \mathbf{R}^2, \end{cases}$$

$$(\mathrm{ES})_{l,r} \quad \begin{cases} \frac{d}{dt}(g,h) + A(g,h) = \chi(\frac{p}{lr})\chi(\frac{q}{lr})F_l[g,h,p,q], \\ \frac{d}{dt}(p,q) = (c,c) + \chi(\frac{p}{lr})\chi(\frac{q}{lr})G_l[g,h,p,q], \\ \qquad (g,h,p,q) \in U \times \mathbf{R}^2 \end{cases}$$

and

$$(\mathrm{RD}) \quad \frac{d}{dt}u + Bu = f(u) - f_u(1)u, \qquad u \in Y,$$

respectively.

Definition 5. For $l > 0$, a closed subset Ω_l of $U \times \mathbf{R}^2$ is defined by

$$\Omega_l = \{(g,h,p,q) \in U \times \mathbf{R}^2 |$$
$$\frac{1}{1+e^{x/l}}(\phi(x-p) + g(x-p)) = \frac{1}{1+e^{-x/l}}(\phi(-x-q) + h(-x-q))\}.$$

For p and $q \in \mathbf{R}$, a map $\Theta_{p,q} \in C^1(U, Y')$ is defined by

$$\Theta_{p,q}[g,h](x) = \phi(x-p) + \phi(-x-q) + g(x-p) + h(-x-q).$$

The following states that the system $(\mathrm{ES})_l$ is an extension of the equation (RD) in a sense, i.e., the set Ω_l is positively invariant in $(\mathrm{ES})_l$ and a solution of $(\mathrm{ES})_l$ on Ω_l corresponds to one of (RD) by the transformation Θ.

Proposition 2.2. *Suppose that* $l > 0$, $T > 0$, $(g,h,p,q) \in C([0,T), U \times \mathbf{R}^2)$ *is a solution of* $(\mathrm{ES})_l$ *and* $(g,h,p,q)(0) \in \Omega_l$. *Then,* $(g,h,p,q) \in C([0,T), \Omega_l)$ *holds and* $\Theta_{p,q}[g,h] \in C([0,T), Y')$ *is a solution of* (RD).

Proof. Let $v \in C([0,T), Y)$ be defined by

$$v(x,t) = \frac{1}{1+e^{x/l}}(\phi(x-p(t)) + g(x-p(t),t))$$
$$- \frac{1}{1+e^{-x/l}}(\phi(-x-q(t)) + h(-x-q(t),t)).$$

Then,

$$(2.1) \qquad \frac{1}{1+e^{-x/l}}\Theta(x,t) + v(x,t) = \phi(x-p(t)) + g(x-p(t),t)$$

and

$$\frac{1}{1+e^{x/l}}\Theta(x,t) - v(x,t) = \phi(-x-q(t)) + h(-x-q(t),t) \tag{2.2}$$

hold with $\Theta(x,t) := \Theta_{p,q}[g,h](x,t)$. We have

$$\begin{aligned} &\frac{1}{1+e^{-x/l}}\Theta_x(x,t) + \frac{1}{l}\frac{1}{1+e^{-x/l}}\frac{1}{1+e^{x/l}}\Theta(x,t) + v_x(x,t) \\ &\qquad = \phi'(x-p(t)) + g_y(x-p(t),t), \end{aligned} \tag{2.3}$$

$$\begin{aligned} &\frac{1}{1+e^{x/l}}\Theta_x(x,t) - \frac{1}{l}\frac{1}{1+e^{-x/l}}\frac{1}{1+e^{x/l}}\Theta(x,t) - v_x(x,t) \\ &\qquad = -(\phi'(-x-q(t)) + h_y(-x-q(t),t)), \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\frac{1}{1+e^{-x/l}}\Theta_{xx}(x,t) + \frac{2}{l}\frac{1}{1+e^{-x/l}}\frac{1}{1+e^{x/l}}\Theta_x(x,t) \\ &-\frac{1}{l^2}\frac{1}{1+e^{-x/l}}\frac{1}{1+e^{x/l}}\left(\frac{1}{1+e^{-x/l}} - \frac{1}{1+e^{x/l}}\right)\Theta(x,t) + v_{xx}(x,t) \\ &\qquad = \phi''(x-p(t)) + g_{yy}(x-p(t),t) \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} &\frac{1}{1+e^{x/l}}\Theta_{xx}(x,t) - \frac{2}{l}\frac{1}{1+e^{-x/l}}\frac{1}{1+e^{x/l}}\Theta_x(x,t) \\ &+\frac{1}{l^2}\frac{1}{1+e^{-x/l}}\frac{1}{1+e^{x/l}}\left(\frac{1}{1+e^{-x/l}} - \frac{1}{1+e^{x/l}}\right)\Theta(x,t) - v_{xx}(x,t) \\ &\qquad = \phi''(-x-q(t)) + h_{yy}(-x-q(t),t). \end{aligned} \tag{2.6}$$

From (1.2) and (2.1), we get

$$\begin{aligned} &\frac{1}{1+e^{-x/l}}\Theta_t(x,t) + v_t(x,t) \\ &= -c\phi'(x-p(t)) + g_{yy}(x-p(t),t) \\ &\quad + f_u(\phi(x-p(t)))g(x-p(t),t) + f_l[g,h,p,q](x-p(t)) \\ &= \phi''(x-p(t)) + g_{yy}(x-p(t),t) \\ &\quad + f(\phi(x-p(t))) + f_u(\phi(x-p(t)))g(x-p(t),t) + f_l[g,h,p,q](x-p(t)) \\ &= \phi''(x-p(t)) + g_{yy}(x-p(t),t) \\ &\quad - \frac{2}{l}\frac{1}{1+e^{x/l}}(\phi'(x-p(t)) + g_y(x-p(t),t)) \\ &\quad + \frac{1}{l^2}\frac{1}{1+e^{x/l}}(\phi(x-p(t)) + g(x-p(t),t)) \\ &\quad + f(\phi(x-p(t)) + g(x-p(t),t)) \end{aligned}$$

$$+\frac{1}{1+e^{-x/l}}f(\phi(x-p(t))+\phi(-x-q(t))+g(x-p(t),t)+h(-x-q(t),t))$$
$$-f\left(\frac{1}{1+e^{-x/l}}(\phi(x-p(t))+\phi(-x-q(t))+g(x-p(t),t)+h(-x-q(t),t))\right).$$

Hence, from (2.1), (2.2), (2.3) and (2.5), we obtain

(2.7)
$$\begin{aligned}\frac{1}{1+e^{-x/l}}\Theta_t(x,t)+v_t(x,t)=&\frac{1}{1+e^{-x/l}}\Theta_{xx}(x,t)+v_{xx}(x,t)\\&-\frac{2}{l}\frac{1}{1+e^{x/l}}v_x(x,t)+\frac{1}{l^2}\frac{1}{1+e^{x/l}}v(x,t)\\&+f\left(\frac{1}{1+e^{-x/l}}\Theta(x,t)+v(x,t)\right)\\&-f\left(\frac{1}{1+e^{-x/l}}\Theta(x,t)\right)+\frac{1}{1+e^{-x/l}}f(\Theta(x,t)).\end{aligned}$$

From (1.2) and (2.2), we get

$$\begin{aligned}&\frac{1}{1+e^{x/l}}\Theta_t(x,t)-v_t(x,t)\\&=-c\phi'(-x-q(t))+h_{yy}(-x-q(t),t)+f_u(\phi(-x-q(t)))h(-x-q(t),t)\\&\quad+f_l[h,g,q,p](-x-q(t))\\&=\phi''(-x-q(t))+h_{yy}(-x-q(t),t)\\&\quad+f(\phi(-x-q(t)))+f_u(\phi(-x-q(t)))h(-x-q(t),t)+f_l[h,g,q,p](-x-q(t))\\&=\phi''(-x-q(t))+h_{yy}(-x-q(t),t)\\&\quad-\frac{2}{l}\frac{1}{1+e^{-x/l}}(\phi'(-x-q(t))+h_y(-x-q(t),t))\\&\quad+\frac{1}{l^2}\frac{1}{1+e^{-x/l}}(\phi(-x-q(t))+h(-x-q(t),t))\\&\quad+f(\phi(-x-q(t))+h(-x-q(t),t))\\&\quad+\frac{1}{1+e^{x/l}}f(\phi(x-p(t))+\phi(-x-q(t))+g(x-p(t),t)+h(-x-q(t),t))\\&\quad-f\left(\frac{1}{1+e^{x/l}}(\phi(x-p(t))+\phi(-x-q(t))+g(x-p(t),t)+h(-x-q(t),t))\right).\end{aligned}$$

Hence, from (2.1), (2.2), (2.4) and (2.6), we obtain

(2.8)
$$\begin{aligned}\frac{1}{1+e^{x/l}}\Theta_t(x,t)-v_t(x,t)=&\frac{1}{1+e^{x/l}}\Theta_{xx}(x,t)-v_{xx}(x,t)\\&-\frac{2}{l}\frac{1}{1+e^{-x/l}}v_x(x,t)-\frac{1}{l^2}\frac{1}{1+e^{-x/l}}v(x,t)\end{aligned}$$

$$+ f\left(\frac{1}{1+e^{x/l}}\Theta(x,t) - v(x,t)\right)$$
$$- f\left(\frac{1}{1+e^{x/l}}\Theta(x,t)\right) + \frac{1}{1+e^{x/l}} f(\Theta(x,t)).$$

From (2.7) and (2.8), on $t \in (0,T)$,

$$\begin{aligned}\Theta_t = \Theta_{xx} + f(\Theta) - \frac{2}{l}v_x - \frac{1}{l^2}\left(\frac{1}{1+e^{-x/l}} - \frac{1}{1+e^{x/l}}\right)v \\ + f\left(\frac{1}{1+e^{-x/l}}\Theta + v\right) - f\left(\frac{1}{1+e^{-x/l}}\Theta\right) \\ + f\left(\frac{1}{1+e^{x/l}}\Theta - v\right) - f\left(\frac{1}{1+e^{x/l}}\Theta\right)\end{aligned}$$

and

$$\begin{aligned}v_t = v_{xx} + \frac{2}{l}\left(\frac{1}{1+e^{-x/l}} - \frac{1}{1+e^{x/l}}\right)v_x \\ + \frac{1}{l^2}\left(\frac{1}{(1+e^{-x/l})^2} + \frac{1}{(1+e^{x/l})^2}\right)v \\ + \frac{1}{1+e^{x/l}}\left(f\left(\frac{1}{1+e^{-x/l}}\Theta(x,t) + v(x,t)\right) - f\left(\frac{1}{1+e^{-x/l}}\Theta(x,t)\right)\right) \\ - \frac{1}{1+e^{-x/l}}\left(f\left(\frac{1}{1+e^{x/l}}\Theta(x,t) - v(x,t)\right) - f\left(\frac{1}{1+e^{x/l}}\Theta(x,t)\right)\right)\end{aligned}$$

hold. Therefore, from $v(x,0)=0$, we obtain

$$v(x,t) = 0$$

on $t \in [0,T)$. Hence,

$$\Theta_t = \Theta_{xx} + f(\Theta)$$

holds on $t \in (0,T)$. □

We now state the main technical result of the paper. This gives a two-dimensional invariant manifold S with asymptotic stability of $(\mathrm{ES})_{l_0,r_0}$ for some l_0 and $r_0 > 0$. The proof of this is given in Section 5.

Theorem 2.3. *There exist $D_0 > 0$, $L_0 > 0$, $l_0 > 0$ and $r_0 > 0$ satisfying the following.*

There uniquely exists an invariant manifold

$$S = \left\{(g,h,p,q) \in U \times \mathbf{R}^2 \middle|\ (g,h) = \Sigma(p,q)\ \left(= (\Sigma^1(p,q), \Sigma^2(p,q))\right)\right\}$$

for $(\mathrm{ES})_{l_0,r_0}$ *with* $\sup_{(p,q)\in\mathbf{R}^2}\|\Sigma(p,q)\|_{X'}\le D_0$. *Further,*

$$\Sigma^2(p,q)=\Sigma^1(q,p),$$
$$\|\Sigma(p_1,q_1)-\Sigma(p_2,q_2)\|_{X'}\le\|(p_1,q_1)-(p_2,q_2)\|_{\mathbf{R}^2},$$
$$\|\tilde G(p_1,q_1)-\tilde G(p_2,q_2)\|_{\mathbf{R}^2}\le L_0\|(p_1,q_1)-(p_2,q_2)\|_{\mathbf{R}^2}$$

and

$$\lim_{(p,q)\to(+\infty,+\infty)}(\|\Sigma(p,q)\|_{X'}+\|\tilde F(p,q)\|_X+\|\tilde G(p,q)\|_{\mathbf{R}^2})=0$$

hold, where $\tilde F\in C(\mathbf{R}^2,X)$ *and* $\tilde G\in C(\mathbf{R}^2,\mathbf{R}^2)$ *be defined by* $\tilde F(p,q)=\chi(p/l_0r_0)\chi(q/l_0r_0)F_{l_0}[\Sigma(p,q),p,q]$ *and* $\tilde G(p,q)=\chi(p/l_0r_0)\chi(q/l_0r_0)G_{l_0}[\Sigma(p,q),p,q]$*, respectively. Also, there exist* $C_0>0$ *and* $\gamma_0>0$ *such that for any* $(g_0,h_0,p_0,q_0)\in X'\times\mathbf{R}^2$ *satisfying* $\|(g_0,h_0)\|_{X'}\le 2D_0$*, there exist solutions* $(g,h,p,q)\in C([0,+\infty),U\times\mathbf{R}^2)$ *and* $(\bar g,\bar h,\bar p,\bar q)\in C([0,+\infty),S)$ *of* $(\mathrm{ES})_{l_0,r_0}$ *such that*

$$(g,h,p,q)(0)=(g_0,h_0,p_0,q_0)$$

and for any $t\ge 0$*,*

$$\|(g,h,p,q)(t)-(\bar g,\bar h,\bar p,\bar q)(t)\|_{X'\times\mathbf{R}^2}\le C_0e^{-\gamma_0 t}\|(g_0,h_0)-\Sigma(p_0,q_0)\|_{X'}$$

hold.

Definition 6. A local invariant manifold $\tilde S$ for $(\mathrm{ES})_{l_0}$ is defined by

$$\tilde S=\{(g,h,p,q)\in S|\ p>l_0r_0,\ q>l_0r_0\}.$$

§3. Existence of Backward Global Solutions

In this section, we prove Theorem 1.1 (1). First, we show that there exists a function $\omega\in C^1(\{h\in BU(\mathbf{R})|\,\|h\|_{C^0}<\delta_0\})$ satisfying

$$\langle(\phi(x)+h(x))-\phi(x-\omega[h]),e^{cx}\phi'(x-\omega[h])\rangle$$
$$:=\int_{-\infty}^{+\infty}((\phi(x)+h(x))-\phi(x-\omega[h]))e^{cx}\phi'(x-\omega[h])dx=0,$$

i.e., the point $x=\omega[h]$ gives the position of the front of $\phi(x)+h(x)$.

Lemma 3.1. *There exist* $\delta_0 > 0$ *and* $\omega \in C^1(\{h \in BU(\mathbf{R}) | \, \|h\|_{C^0} < \delta_0\})$ *such that*

$$\begin{gathered}\|\omega\|_{C^1} < +\infty, \\ \langle(\phi(y) + h(y)) - \phi(y - \omega[h]), e^{cy}\phi'(y - \omega[h])\rangle = 0, \\ \langle(\phi(y) + h(y)) - \phi(y - \omega[h]), e^{cy}\phi''(y - \omega[h])\rangle \\ \neq \langle\phi'(y - \omega[h]), e^{cy}\phi'(y - \omega[h])\rangle\end{gathered}$$

and

$$\langle h(y), e^{cy}\phi'(y)\rangle = 0 \implies \omega[h] = 0$$

hold.

Proof. By (1.2), there exist C and $\sigma > 0$ such that $(1 + e^{cz})(|\phi'(z)| + |\phi''(z)|) \leq Ce^{-\sigma|z|}$ holds for all $z \in \mathbf{R}$. Hence, we define $f \in C^1(BU(\mathbf{R}) \times \mathbf{R})$ by $f(h, x) = \int_{-\infty}^{+\infty} ((\phi(z) + h(z)) - \phi(z - x))\, e^{cz}\phi'(z - x)dz$.

Then, $f(h, 0) = 0$ holds when $\langle h(y), e^{cy}\phi'(y)\rangle = 0$ holds. Hence, because $f_x(h, x) = \langle\phi'(y - x), e^{cy}\phi'(y - x)\rangle - \langle(\phi(y) + h(y)) - \phi(y - x), e^{cy}\phi''(y - x)\rangle$ also holds, this lemma follows from the implicit function theorem. □

Definition 7. Open subsets V^1, V^2 of $\mathbf{R} \times Y''$ and V of $\mathbf{R}^2 \times Y''$ are defined by

$$\begin{aligned} V^1 &= \left\{(\tilde{p}, u) \in \mathbf{R} \times Y'' | \, \left\| \frac{1}{1 + e^{-x/l_0}} u(x) - \phi(x - \tilde{p}) \right\|_{C^0} < \delta_0 \right\}, \\ V^2 &= \left\{(\tilde{q}, u) \in \mathbf{R} \times Y'' | \, \left\| \frac{1}{1 + e^{x/l_0}} u(x) - \phi(-x - \tilde{q}) \right\|_{C^0} < \delta_0 \right\} \end{aligned}$$

and

$$V = \{(\tilde{p}, \tilde{q}, u) \in \mathbf{R}^2 \times Y'' | \, (\tilde{p}, u) \in V^1, (\tilde{q}, u) \in V^2\},$$

respectively.

Maps $P \in C^1(V^1)$ and $Q \in C^1(V^2)$ are defined by

$$P[\tilde{p}, u] = \tilde{p} + \omega\left[\frac{1}{1 + e^{-(y+\tilde{p})/l_0}} u(y + \tilde{p}) - \phi(y)\right]$$

and

$$Q[\tilde{q}, u] = \tilde{q} + \omega\left[\frac{1}{1 + e^{-(y+\tilde{q})/l_0}} u(-y - \tilde{q}) - \phi(y)\right],$$

respectively.

For p and $q \in \mathbf{R}$, maps Φ_p and $\Psi_q \in C^1(Y'', Y')$ are defined by

$$\Phi_p[u](y) = \frac{1}{1+e^{-(y+p)/l_0}} u(y+p) - \phi(y)$$

and

$$\Psi_q[u](y) = \frac{1}{1+e^{-(y+q)/l_0}} u(-y-q) - \phi(y),$$

respectively.

Roughly speaking, the points $x = P[\tilde{p}, u]$ and $-Q[\tilde{q}, u]$ represent the position of the fore and hind fronts of u, respectively. Also, the functions $\Phi_{P[\tilde{p},u]}[u](y)$ and $\Psi_{Q[\tilde{q},u]}[u](y)$ give the difference of the fore and hind ones from $\phi(y)$.

From this definition, the following lemma holds.

Lemma 3.2.

$$(\Phi_{P[\tilde{p},u]}[u], \Psi_{Q[\tilde{q},u]}[u], P[\tilde{p}, u], Q[\tilde{q}, u]) \in \Omega_{l_0}$$

and

$$\Theta_{P[\tilde{p},u],Q[\tilde{q},u]}[\Phi_{P[\tilde{p},u]}[u], \Psi_{Q[\tilde{q},u]}[u]] = u$$

hold for all $(\tilde{p}, \tilde{q}, u) \in V$.

Proof. Because $\omega[(1/(1+e^{-(y+\tilde{p})/l_0}))u(y+\tilde{p}) - \phi(y)] = P[\tilde{p}, u] - \tilde{p}$ holds, from Lemma 3.1,

$$\left\langle \frac{1}{1+e^{-(y+\tilde{p})/l_0}} u(y+\tilde{p}) - \phi(y - (P[\tilde{p}, u] - \tilde{p})), e^{cy}\phi'(y - (P[\tilde{p}, u] - \tilde{p})) \right\rangle = 0$$

and

$$\left\langle \frac{1}{1+e^{-(y+\tilde{p})/l_0}} u(y+\tilde{p}) - \phi(y - (P[\tilde{p}, u] - \tilde{p})), e^{cy}\phi''(y - (P[\tilde{p}, u] - \tilde{p})) \right\rangle \neq \langle \phi'(y - (P[\tilde{p}, u] - \tilde{p})), e^{cy}\phi'(y - (P[\tilde{p}, u] - \tilde{p})) \rangle$$

hold. Hence,

$$\langle \Phi_{P[\tilde{p},u]}[u](y), e^{cy}\phi'(y) \rangle = 0$$

and

$$\langle \Phi_{P[\tilde{p},u]}[u](y), e^{cy}\phi''(y)\rangle \neq \langle \phi'(y), e^{cy}\phi'(y)\rangle$$

hold. We have $\langle \phi'(y) + (\Phi_{P[\tilde{p},u]}[u])_y(y), e^{cy}\phi'(y)\rangle \neq 0$. Because we also have $\langle \phi'(y)+(\Psi_{Q[\tilde{q},u]}[u])_y(y), e^{cy}\phi'(y)\rangle \neq 0$, $(\Phi_{P[\tilde{p},u]}[u], \Psi_{Q[\tilde{q},u]}[u]) \in U$ holds. Then, we can immediately get this lemma. □

From Proposition D, the following lemma immediately follows.

Lemma 3.3. *There exist $N > 0$ and $\delta > 0$ such that for any $u_0 \in Y$ satisfying $\inf_{x_0 \geq N} \|u_0(x) - (\phi(x - x_0) + \phi(-x - x_0))\|_Y \leq \delta$, the solution $u \in C([0, +\infty), Y)$ of* (RD) *with $u(0) = u_0$ satisfies $\lim_{t\to+\infty} \|u(t) - 1\|_Y = 0$.*

The following shows that there is a solution $(\Sigma(p, p), p, p)$ of $(\mathrm{ES})_{l_0}$ on the local invariant manifold $\tilde{S}$ with $(\Sigma(p, p), p, p) \in \Omega_{l_0}$ and $p(-\infty) = +\infty$.

Proposition 3.4. *There exist $N_0 > l_0r_0$ and $p \in C^1((-\infty, 0])$ such that $(\Sigma(p, p), p, p) \in C((-\infty, 0], \tilde{S})$ is a solution of $(\mathrm{ES})_{l_0}$ with $p(0) = N_0$. Further, $p(-\infty) = +\infty$ and $(\Sigma(p(t), p(t)), p(t), p(t)) \in \Omega_{l_0}$ hold for all $t \leq 0$.*

Proof. We remember $\lim_{p'\to+\infty} \|\Sigma(p', p')\|_{X'} = 0$ from Theorem 2.3. Let $N_0 > l_0r_0$ be a constant such that $\phi(-p') \leq 1/8$ and $\|\Sigma(p', p')\|_{C^0} \leq 1/8$ hold for all $p' \geq N_0$. Then, from $\Theta_{p',p'}[\Sigma(p', p')](0) \leq 1/2$,

$$\|1 - \Theta_{p',p'}[\Sigma(p', p')]\|_{C^0} \geq 1/2 \tag{3.1}$$

holds for all $p' \geq N_0$.

Let $N > 0$ be a constant satisfying Lemma 3.3 and $(x_0, x_0, \phi(x - x_0) + \phi(-x - x_0)) \in V$ for $x_0 \geq N$. For $x_0 \geq N$, by $u_{x_0,0}$, $p_{x_0,0}$ and $g_{x_0,0}$, we denote $\phi(x-x_0)+\phi(-x-x_0)$, $P[x_0, u_{x_0,0}]$ $(= Q[x_0, u_{x_0,0}])$ and $\Phi_{p_{x_0,0}}[u_{x_0,0}]$ $(= \Psi_{p_{x_0,0}}[u_{x_0,0}])$, respectively. Then, $\lim_{x_0\to+\infty}(|p_{x_0,0}-x_0|+\|g_{x_0,0}\|_{C^2}) = 0$ holds. Hence, by Theorem 2.3, if x_0 is sufficiently large, then there exist solutions $(g_{x_0}, g_{x_0}, p_{x_0}, p_{x_0})$ and $(\Sigma(\bar{p}_{x_0}, \bar{p}_{x_0}), \bar{p}_{x_0}, \bar{p}_{x_0}) \in C([0, +\infty), U \times \mathbf{R}^2)$ of $(\mathrm{ES})_{l_0,r_0}$ such that

$$(g_{x_0}, g_{x_0}, p_{x_0}, p_{x_0})(0) = (g_{x_0,0}, g_{x_0,0}, p_{x_0,0}, p_{x_0,0})$$

and

$$\lim_{t\to+\infty} \|(g_{x_0}, g_{x_0}, p_{x_0}, p_{x_0})(t) - (\Sigma(\bar{p}_{x_0}, \bar{p}_{x_0}), \bar{p}_{x_0}, \bar{p}_{x_0})(t)\|_{X'\times\mathbf{R}^2} = 0 \tag{3.2}$$

hold. Because of $\lim_{x_0\to+\infty}(\|(g_{x_0,0},g_{x_0,0})\|_{X'}+\|\Sigma(p_{x_0,0},p_{x_0,0})\|_{X'})=0$, we also have

$$\lim_{x_0\to+\infty}\sup_{t\geq 0}\|(g_{x_0},g_{x_0},p_{x_0},p_{x_0})(t)-(\Sigma(\bar{p}_{x_0},\bar{p}_{x_0}),\bar{p}_{x_0},\bar{p}_{x_0})(t)\|_{X'\times\mathbf{R}^2}=0. \tag{3.3}$$

Fix any constant $\delta\geq 0$. By contradiction, we now show that if x_0 is sufficiently large, then there exists $T_{\delta,x_0}>0$ satisfying $p_{x_0}(T_{\delta,x_0})=N_0+\delta$ and $p_{x_0}(t)>N_0+\delta$ for $t\in[0,T_{\delta,x_0})$. Assume that x_0 is sufficiently large and $p_{x_0}(t)>N_0+\delta$ holds for all $t\in[0,+\infty)$. Then, from $N_0>l_0r_0$, $(g_{x_0},g_{x_0},p_{x_0},p_{x_0})$ is a solutions of $(\mathrm{ES})_{l_0}$ on $t\in[0,+\infty)$. Hence, because $(g_{x_0,0},g_{x_0,0},p_{x_0,0},p_{x_0,0})\in\Omega_{l_0}$ holds from Lemma 3.2, by Proposition 2.2, $(g_{x_0},g_{x_0},p_{x_0},p_{x_0})(t)\in\Omega_{l_0}$ holds and $\Theta_{(p_{x_0},p_{x_0})(t)}[(g_{x_0},g_{x_0})(t)]$ is a solution of (RD) on $t\in[0,+\infty)$. Because $\Theta_{(p_{x_0,0},p_{x_0,0})}[(g_{x_0,0},g_{x_0,0})]=u_{x_0,0}$ also holds from Lemma 3.2, by Lemma 3.3, $\lim_{t\to+\infty}\|\Theta_{(p_{x_0},p_{x_0})(t)}[(g_{x_0},g_{x_0})(t)]-1\|_{C^0}=0$ holds. This is contradiction by (3.1) and (3.2). Therefore, if x_0 is sufficiently large, then there exists $T_{\delta,x_0}>0$ satisfying $p_{x_0}(T_{\delta,x_0})=N_0+\delta$ and $p_{x_0}(t)>N_0+\delta$ for $t\in[0,T_{\delta,x_0})$.

Because of $(g_{x_0,0},g_{x_0,0},p_{x_0,0},p_{x_0,0})\in\Omega_{l_0}$ and $p_{x_0}(t)>l_0r_0$ for $t\in[0,T_{\delta,x_0}]$, by Proposition 2.2, if x_0 is sufficiently large, then $(g_{x_0},g_{x_0},p_{x_0},p_{x_0})(T_{\delta,x_0})\in\Omega_{l_0}$ holds. Also, by (3.3) and $p_{x_0}(T_{\delta,x_0})=N_0+\delta$, we get $\lim_{x_0\to+\infty}\|(g_{x_0},g_{x_0},p_{x_0},p_{x_0})(T_{\delta,x_0})-(\Sigma(N_0+\delta,N_0+\delta),N_0+\delta,N_0+\delta)\|_{X'\times\mathbf{R}^2}=0$. Therefore, $(\Sigma(N_0+\delta,N_0+\delta),N_0+\delta,N_0+\delta)\in\Omega_{l_0}$ holds for all $\delta\geq 0$.

We conclude this proof by showing that if $\delta>0$ is sufficiently large, then there exist $T_\delta>0$ and a solution $(\Sigma(p_\delta,p_\delta),p_\delta,p_\delta)\in C([0,+\infty),S)$ of $(\mathrm{ES})_{l_0,r_0}$ with $p_\delta(0)=N_0+\delta$ and $p_\delta(T_\delta)=N_0$. Assume that $\delta>0$ is sufficiently large and the solution $(\Sigma(p_\delta,p_\delta),p_\delta,p_\delta)\in C([0,+\infty),S)$ of $(\mathrm{ES})_{l_0,r_0}$ with $p_\delta(0)=N_0+\delta$ satisfies $p_\delta(t)>N_0$ on $t\in[0,+\infty)$. Then, $(\Sigma(p_\delta,p_\delta),p_\delta,p_\delta)$ is a solutions of $(\mathrm{ES})_{l_0}$ on $t\in[0,+\infty)$. Hence, by Proposition 2.2 and $(\Sigma(p_\delta(0),p_\delta(0)),p_\delta(0),p_\delta(0))\in\Omega_{l_0}$, $\Theta_{(p_\delta(t),p_\delta(t))}[\Sigma(p_\delta(t),p_\delta(t))]$ is a solution of (RD) on $t\in[0,+\infty)$. By Lemma 3.3 and $\lim_{\delta\to+\infty}\|\Sigma(p_\delta(0),p_\delta(0))\|_{X'}=0$, $\lim_{t\to+\infty}\|\Theta_{(p_\delta(t),p_\delta(t))}[\Sigma(p_\delta(t),p_\delta(t))]-1\|_{C^0}=0$ holds. This is contradiction with (3.1) and $p_\delta(t)>N_0$ on $t\in[0,+\infty)$. □

Definition 8. For $p\geq N_0$, $\tilde{\Theta}_p\in Y'$ is defined by $\tilde{\Theta}_p=\Theta_{p,p}[\Sigma(p,p)]$.

We define the (backward) global solution ψ by the following definition, which is allowed in virtue of Propositions 2.2 and 3.4.

Definition 9. A solution $\psi\in C(\mathbf{R},Y)$ of (RD) is defined such that $\psi(t)=\tilde{\Theta}_{p(t)}$ holds on $t\leq 0$ with $p(0)=N_0$.

Now, we prove Theorem 1.1 (1).

Proof of Theorem 1.1 (1). Because of $\Sigma^2(p,q) = \Sigma^1(q,p)$ from Theorem 2.3, we have $\psi(-x,t) = \psi(x,t)$. From Proposition 3.4, $p(-\infty) = +\infty$ holds. Hence, because of $\lim_{p\to+\infty}(\|\Sigma(p,p)\|_{X'} + \|\tilde{G}(p,p)\|_{\mathbf{R}^2}) = 0$ from Theorem 2.3, we obtain $\lim_{t\to-\infty} \|\psi(x,t)-(\phi(x-p(t))+\phi(-x-p(t)))\|_Y = 0$ and $\dot{p}(-\infty) = c$. By Lemma 3.3, $\lim_{t\to+\infty} \|\psi(t) - 1\|_Y = 0$ also holds. □

§4. Uniformly Asymptotic Stability

In this section, we prove Theorem 1.1 (2) using the asymptotic stability of the invariant manifold S. Lemma 4.4 below becomes the key in this proof, but it is rather technical. We also need additional argument for the difference between the topologies of $X' \times \mathbf{R}^2$ and Y.

Proposition 4.1. *Let* $T > 0$ *and* $(\tilde{p}, \tilde{q}, u) \in C([0,T), V)$ *be given. Suppose that* u *is a solution of* (RD)*. Then,*

$$(\Phi_{P[\tilde{p},u]}[u], \Psi_{Q[\tilde{q},u]}[u], P[\tilde{p},u], Q[\tilde{q},u]) \in C([0,T), \Omega_{l_0})$$

is a solution of $(\mathrm{ES})_{l_0}$ *and*

$$\Theta_{P[\tilde{p},u],Q[\tilde{q},u]}[\Phi_{P[\tilde{p},u]}[u], \Psi_{Q[\tilde{q},u]}[u]] = u$$

holds on $t \in [0,T)$.

Proof. From Lemma 3.2, $(\Phi_{P[\tilde{p},u]}[u], \Psi_{Q[\tilde{q},u]}[u], P[\tilde{p},u], Q[\tilde{q},u]) \in \Omega_{l_0}$ and $\Theta_{P[\tilde{p},u],Q[\tilde{q},u]}[\Phi_{P[\tilde{p},u]}[u], \Psi_{Q[\tilde{q},u]}[u]] = u$ hold on $t \in [0,T)$. We have $P[\tilde{p},u] \in C([0,T))$. Hence, because $B^{3/4}((1/(1+e^{-x/l_0}))u(x,t)) \in C([0,T), BU(\mathbf{R}))$ holds, we have $B^{3/4}(\Phi_{P[\tilde{p},u]}[u]) \in C([0,T), BU(\mathbf{R}))$. Because we also have $Q[\tilde{q},u] \in C([0,T))$ and $B^{3/4}(\Psi_{Q[\tilde{q},u]}[u]) \in C([0,T), BU(\mathbf{R}))$, we get

$$(\Phi_{P[\tilde{p},u]}[u], \Psi_{Q[\tilde{q},u]}[u], P[\tilde{p},u], Q[\tilde{q},u]) \in C([0,T), \Omega_{l_0}).$$

We denote $(1/(1+e^{-x/l_0}))u(x,t)$, $\Phi_{P[\tilde{p}(t),u(t)]}[u(t)](y)$ and $P[\tilde{p}(t),u(t)]$ by $v(x,t)$, $g(y,t)$ and $p(t)$, respectively. Then, $\omega[v(y+\tilde{p}(t),t) - \phi(y)] = p(t) - \tilde{p}(t)$ holds. From Lemma 3.1, we have

$$\langle v(x,t) - \phi(x-p(t)), e^{cx}\phi'(x-p(t))\rangle = 0$$

and

$$\langle v(x,t) - \phi(x-p(t)), e^{cx}\phi''(x-p(t))\rangle$$

$$\neq \langle \phi'(x-p(t)), e^{cx}\phi'(x-p(t))\rangle .$$

Hence, by the implicit function theorem, $p \in C^1((0,T),\mathbf{R})$ holds. Hence,

$$g_t(y,t) = v_t(y+p(t),t) + \dot{p}(t)v_x(y+p(t),t) \tag{4.1}$$

holds for all $t \in (0,T)$. From $\langle g(y,t), e^{cy}\phi'(y)\rangle = 0$ and $\langle g(y,t), e^{cy}\phi''(y)\rangle \neq \langle \phi'(y), e^{cy}\phi'(y)\rangle$,

$$\langle g_t(y,t), e^{cy}\phi'(y)\rangle = 0 \tag{4.2}$$

and

$$\langle \phi'(y) + g_y(y,t), e^{cy}\phi'(y)\rangle \neq 0 \tag{4.3}$$

hold for all $t \in (0,T)$.

We also denote $(1/(1+e^{x/l_0}))u(x,t)$ by $w(x,t)$. We get

$$\begin{aligned}
v_t(x,t) &= \frac{1}{1+e^{-x/l_0}}(u_{xx}(x,t) + f(u(x,t))) \\
&= \frac{1}{1+e^{-x/l_0}}\Big((1+e^{-x/l_0})v(x,t)\Big)_{xx} + \frac{1}{1+e^{-x/l_0}} f(v(x,t)+w(x,t)) \\
&= v_{xx}(x,t) + f(v(x,t)) - \frac{2}{l_0}\frac{1}{1+e^{x/l_0}}v_x(x,t) + \frac{1}{l_0^2}\frac{1}{1+e^{x/l_0}}v(x,t) \\
&\quad + \frac{1}{1+e^{-x/l_0}} f(v(x,t)+w(x,t)) - f\left(\frac{1}{1+e^{-x/l_0}}(v(x,t)+w(x,t))\right).
\end{aligned}$$

Further, we denote $\Psi_{Q[\tilde{q},u]}[u]$ and $Q[\tilde{q},u]$ by h and q, respectively. Then, we have

$$\begin{aligned}
v_t(y+p(t),t) = {} & \phi''(y) + g_{yy}(y,t) + f(\phi(y)+g(y,t)) \\
& - \frac{2}{l_0}\frac{1}{1+e^{(y+p(t))/l_0}}(\phi'(y) + g_y(y,t)) \\
& + \frac{1}{l_0^2}\frac{1}{1+e^{(y+p(t))/l_0}}(\phi(y)+g(y,t)) \\
& + \frac{1}{1+e^{-(y+p(t))/l_0}} f(\phi(y) + \phi(-y-p(t)-q(t)) \\
& + g(y,t) + h(-y-p(t)-q(t),t)) \\
& - f\left(\frac{1}{1+e^{-(y+p(t))/l_0}}(\phi(y) + \phi(-y-p(t)-q(t))\right. \\
& \left. + g(y,t) + h(-y-p(t)-q(t),t))\right)
\end{aligned}$$

and

$$v_x(y+p(t),t) = \phi'(y) + g_y(y,t).$$

Hence, from (1.2) and (4.1), we obtain

$$\begin{aligned} g_t(y,t) = {} & g_{yy}(y,t) + cg_y(y,t) + f_u(\phi(y))g(y,t) \\ & + (\dot{p}(t) - c)(\phi'(y) + g_y(y,t)) \\ & + f_{l_0}[(g,h,p,q)(t)](y). \end{aligned}$$

From (4.2) and (4.3), we have $\dot{p}(t) = c - (\langle f_{l_0}[(g,h,p,q)(t)](y), e^{cy}\phi'(y)\rangle)/(\langle \phi'(y) + g_y(y,t), e^{cy}\phi'(y)\rangle)$. Therefore, we have $g_t(y,t) = g_{yy}(y,t) + cg_y(y,t) + f_u(\phi(y))g(y,t) + F_l^1[(g,h,p,q)(t)](y)$ and $\dot{p}(t) = c + G_l^1[(g,h,p,q)(t)]$. Because we also have $h_t(y,t) = h_{yy}(y,t) + ch_y(y,t) + f_u(\phi(y))h(y,t) + F_l^2[(g,h,p,q)(t)](y)$ and $\dot{q}(t) = c + G_l^2[(g,h,p,q)(t)]$, $(\Phi_{P[\tilde{p},u]}[u], \Psi_{Q[\tilde{q},u]}[u], P[\tilde{p},u], Q[\tilde{q},u])$ is a solution of $(\mathrm{ES})_{l_0}$ on $t \in [0,T)$. □

Lemma 4.2. *For any $\varepsilon > 0$, there exists $M > N_0$ such that*

$$\begin{gathered} (p + x_0, p - x_0, \tilde{\Theta}_p(x - x_0)) \in V, \\ \|(P[p + x_0, \tilde{\Theta}_p(x - x_0)], Q[p - x_0, \tilde{\Theta}_p(x - x_0)]) \\ -(p + x_0, p - x_0)\|_{\mathbf{R}^2} < \varepsilon \end{gathered}$$

and

$$\begin{gathered} (\Phi_{P[p+x_0,\tilde{\Theta}_p(x-x_0)]}[\tilde{\Theta}_p(x - x_0)], \Psi_{Q[p-x_0,\tilde{\Theta}_p(x-x_0)]}[\tilde{\Theta}_p(x - x_0)], \\ P[p + x_0, \tilde{\Theta}_p(x - x_0)], Q[p - x_0, \tilde{\Theta}_p(x - x_0)]) \in \tilde{S} \end{gathered}$$

hold for all $x_0 \in \mathbf{R}$ and $p \geq M + |x_0|$.

Proof. Let $\varepsilon > 0$ be given. Then, there exists $\delta > 0$ such that for any $\tilde{p}$, $\tilde{q} \in \mathbf{R}$ and $u \in Y''$ satisfying $\|(1/(1+e^{-x/l_0}))u(x) - \phi(x - \tilde{p})\|_{C^0} < \delta$ and $\|(1/(1+e^{x/l_0}))u(x) - \phi(-x - \tilde{q})\|_{C^0} < \delta$,

$$(\tilde{p}, \tilde{q}, u) \in V$$

and

$$\|(P[\tilde{p},u], Q[\tilde{q},u]) - (\tilde{p},\tilde{q})\|_{\mathbf{R}^2} < \min\{\varepsilon, 1\}$$

hold. Suppose that $M > N_0$ is sufficiently large. Then, because $\lim_{p\to+\infty} \|\Sigma(p,p)\|_{X'} = 0$ holds from Theorem 2.3, we have $\|(1/(1+e^{-x/l_0}))\tilde{\Theta}_p(x-x_0) -$

$\phi(x-(p+x_0))\|_{C^0} < \delta$ and $\|(1/(1+e^{x/l_0}))\tilde{\Theta}_p(x-x_0)-\phi(-x-(p-x_0))\|_{C^0} < \delta$. Therefore, we obtain

$$(p+x_0, p-x_0, \tilde{\Theta}_p(x-x_0)) \in V,$$
$$\|(P[p+x_0, \tilde{\Theta}_p(x-x_0)], Q[p-x_0, \tilde{\Theta}_p(x-x_0)])$$
$$-(p+x_0, p-x_0)\|_{\mathbf{R}^2} < \varepsilon,$$
$$P[p+x_0, \tilde{\Theta}_p(x-x_0)] > l_0 r_0$$

and

$$Q[p-x_0, \tilde{\Theta}_p(x-x_0)] > l_0 r_0.$$

There exist $t_0 > 0$ and $\bar{p} \in C^1((-\infty, t_0], [p-1, +\infty))$ with $\bar{p}(0) = p$ and $\bar{p}(t_0) = p-1$ such that $\tilde{\Theta}_{\bar{p}} \in C((-\infty, t_0], Y')$ is a solution of (RD). Then, $\tilde{\Theta}_{\bar{p}}(x-x_0) \in C((-\infty, t_0], Y')$ is also a solution of (RD). Because $\bar{p} \geq (M-1)+|x_0|$ holds, we obtain

$$(\bar{p}+x_0, \bar{p}-x_0, \tilde{\Theta}_{\bar{p}}(x-x_0)) \in V,$$
$$P[\bar{p}+x_0, \tilde{\Theta}_{\bar{p}}(x-x_0)] > l_0 r_0$$

and

$$Q[\bar{p}-x_0, \tilde{\Theta}_{\bar{p}}(x-x_0)] > l_0 r_0$$

for $t \leq t_0$. Therefore, by Proposition 4.1,

$$(\Phi_{P[\bar{p}+x_0, \tilde{\Theta}_{\bar{p}}(x-x_0)]}[\tilde{\Theta}_{\bar{p}}(x-x_0)], \Psi_{Q[\bar{p}-x_0, \tilde{\Theta}_{\bar{p}}(x-x_0)]}[\tilde{\Theta}_{\bar{p}}(x-x_0)],$$
$$P[\bar{p}+x_0, \tilde{\Theta}_{\bar{p}}(x-x_0)], Q[\bar{p}-x_0, \tilde{\Theta}_{\bar{p}}(x-x_0)])$$

is a solution of $(\mathrm{ES})_{l_0, r_0}$ on $t \in (-\infty, 0]$.

Also, because of $\lim_{p\to+\infty}(\|\Sigma(p,p)\|_{X'} + \|\tilde{F}(p,p)\|_X) = 0$ from Theorem 2.3, $\lim_{p\to+\infty}(\|\Sigma^1(p,p)\|_{Y''} + \|\Sigma^2(p,p)\|_{Y''}) = 0$ holds. Hence, because $M > N_0$ is sufficiently large, $\|\Sigma^1(\bar{p},\bar{p})\|_{Y''}$ and $\|\Sigma^2(\bar{p},\bar{p})\|_{Y''}$ are sufficiently small. Hence, because $|P[\bar{p}+x_0, \tilde{\Theta}_{\bar{p}}(x-x_0)] - (\bar{p}+x_0)|$ and $|Q[\bar{p}-x_0, \tilde{\Theta}_{\bar{p}}(x-x_0)] - (\bar{p}-x_0)|$ are also sufficiently small, we have

$$\|(\Phi_{P[\bar{p}+x_0, \tilde{\Theta}_{\bar{p}}(x-x_0)]}[\tilde{\Theta}_{\bar{p}}(x-x_0)], \Psi_{Q[\bar{p}-x_0, \tilde{\Theta}_{\bar{p}}(x-x_0)]}[\tilde{\Theta}_{\bar{p}}(x-x_0)])\|_{X'}$$
$$\leq 2D_0.$$

Therefore, by Theorem 2.3, for any $T > 0$, there exists $(p_T, q_T) \in \mathbf{R}^2$ satisfying

$$\|(\Phi_{P[p+x_0, \tilde{\Theta}_p(x-x_0)]}[\tilde{\Theta}_p(x-x_0)], \Psi_{Q[p-x_0, \tilde{\Theta}_p(x-x_0)]}[\tilde{\Theta}_p(x-x_0)])$$

$$-\Sigma(p_T, q_T)\|_{X'}$$
$$+\|(P[p+x_0, \tilde{\Theta}_p(x-x_0)], Q[p-x_0, \tilde{\Theta}_p(x-x_0)]) - (p_T, q_T)\|_{\mathbf{R}^2}$$
$$\leq 6C_0 D_0 e^{-\gamma_0 T}.$$

Hence, from

$$\lim_{T\to+\infty}(p_T, q_T) = (P[p+x_0, \tilde{\Theta}_p(x-x_0)], Q[p-x_0, \tilde{\Theta}_p(x-x_0)])$$

and

$$\lim_{T\to+\infty} \Sigma(p_T, q_T)$$
$$= (\Phi_{P[p+x_0,\tilde{\Theta}_p(x-x_0)]}[\tilde{\Theta}_p(x-x_0)], \Psi_{Q[p-x_0,\tilde{\Theta}_p(x-x_0)]}[\tilde{\Theta}_p(x-x_0)]),$$

we get

$$\Sigma(P[p+x_0, \tilde{\Theta}_p(x-x_0)], Q[p-x_0, \tilde{\Theta}_p(x-x_0)])$$
$$= (\Phi_{P[p+x_0,\tilde{\Theta}_p(x-x_0)]}[\tilde{\Theta}_p(x-x_0)], \Psi_{Q[p-x_0,\tilde{\Theta}_p(x-x_0)]}[\tilde{\Theta}_p(x-x_0)]).$$

□

Lemma 4.3. *For any $\varepsilon > 0$, there exists $M > 0$ such that for any p_0 and $q_0 \geq M$, there exist p_0' and $x_0 \in \mathbf{R}$ such that*

$$\left|p_0' - \frac{p_0+q_0}{2}\right| \leq \varepsilon, \left|x_0 - \frac{p_0-q_0}{2}\right| \leq \varepsilon,$$
$$(p_0'+x_0, p_0'-x_0, \tilde{\Theta}_{p_0'}(x-x_0)) \in V$$

and

$$(p_0, q_0) = (P[p_0'+x_0, \tilde{\Theta}_{p_0'}(x-x_0)], Q[p_0'-x_0, \tilde{\Theta}_{p_0'}(x-x_0)])$$

hold.

Proof. Fix any $\varepsilon > 0$. Then, let $M > 0$ be sufficiently large.

If $(p_0', x_0) \in [(p_0+q_0)/2-\varepsilon, (p_0+q_0)/2+\varepsilon] \times [(p_0-q_0)/2-\varepsilon, (p_0-q_0)/2+\varepsilon]$ holds, then $p_0' - |x_0| \geq ((p_0+q_0)/2-\varepsilon) - ((|p_0-q_0|)/2+\varepsilon) \geq M - 2\varepsilon$ holds. Hence, by Lemma 4.2, $|P[p_0'+x_0, \tilde{\Theta}_{p_0'}(x-x_0)] - (p_0'+x_0)| < \varepsilon$ and $|Q[p_0'-x_0, \tilde{\Theta}_{p_0'}(x-x_0)] - (p_0'-x_0)| < \varepsilon$ hold. We have

$$\begin{pmatrix} p_1' \\ x_1 \end{pmatrix} := \begin{pmatrix} \frac{p_0+q_0}{2} \\ \frac{p_0-q_0}{2} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} P[p_0'+x_0, \tilde{\Theta}_{p_0'}(x-x_0)] - (p_0'+x_0) \\ Q[p_0'-x_0, \tilde{\Theta}_{p_0'}(x-x_0)] - (p_0'-x_0) \end{pmatrix}$$

$$\in \left[\frac{p_0+q_0}{2}-\varepsilon, \frac{p_0+q_0}{2}+\varepsilon\right]\times\left[\frac{p_0-q_0}{2}-\varepsilon, \frac{p_0-q_0}{2}+\varepsilon\right].$$

Therefore, by Brouwer's fixed point theorem, there exists $(p_0', x_0) \in [(p_0+q_0)/2-\varepsilon, (p_0+q_0)/2+\varepsilon] \times [(p_0-q_0)/2-\varepsilon, (p_0-q_0)/2+\varepsilon]$ satisfying

$$\begin{pmatrix} p_0' \\ x_0 \end{pmatrix} = \begin{pmatrix} \frac{p_0+q_0}{2} \\ \frac{p_0-q_0}{2} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} P[p_0'+x_0, \tilde{\Theta}_{p_0'}(x-x_0)] - (p_0'+x_0) \\ Q[p_0'-x_0, \tilde{\Theta}_{p_0'}(x-x_0)] - (p_0'-x_0) \end{pmatrix}.$$

Hence, there exists $(p_0', x_0) \in [(p_0+q_0)/2-\varepsilon, (p_0+q_0)/2+\varepsilon] \times [(p_0-q_0)/2-\varepsilon, (p_0-q_0)/2+\varepsilon]$ satisfying

$$(p_0, q_0) = (P[p_0'+x_0, \tilde{\Theta}_{p_0'}(x-x_0)], Q[p_0'-x_0, \tilde{\Theta}_{p_0'}(x-x_0)]).$$

□

From Proposition 4.1, Lemmas 4.2 and 4.3, we prove the following lemma. This lemma is the key one in this section, and it shows that if $N > 0$ is sufficiently large, then for any $\bar{p}_0$ and $\bar{q}_0 \geq N+1/2$ with $|\bar{p}_0 - \bar{q}_0| \leq 1$, there is a solution $(\Sigma(\bar{p},\bar{q}), \bar{p}, \bar{q})$ of $(\mathrm{ES})_{l_0}$ on the local invariant manifold $\tilde{S}$ with initial data $(\bar{p}(0), \bar{q}(0)) = (\bar{p}_0, \bar{q}_0)$ such that $\tilde{\Theta}_{\bar{p}''}(x-x_0) = \Theta_{\bar{p},\bar{q}}[\Sigma(\bar{p},\bar{q})](x)$ holds on $t \in (-\infty, T)$ with $\bar{p}''(T'') = N$ and $T > T'' > 0$.

Lemma 4.4. *There exists $M > N_0$ such that for any $N > M$, $\bar{p}_0 \geq N+1/2$ and $\bar{q}_0 \geq N+1/2$ with $|\bar{p}_0 - \bar{q}_0| \leq 1$, there exist $x_0 \in (-1,1)$, $T'' > 0$, $T > T''$, solutions $\tilde{\Theta}_{\bar{p}''} \in C((-\infty,T), Y'')$ of* (RD) *with $\bar{p}''(T'') = N$ and $(\Sigma(\bar{p},\bar{q}), \bar{p}, \bar{q}) \in C((-\infty,T), \tilde{S})$ of* $(\mathrm{ES})_{l_0}$ *with $(\bar{p}(0), \bar{q}(0)) = (\bar{p}_0, \bar{q}_0)$ such that*

$$\bar{p} > l_0 r_0 + 1, \quad \bar{q} > l_0 r_0 + 1$$

and

$$\tilde{\Theta}_{\bar{p}''}(x-x_0) = \Theta_{\bar{p},\bar{q}}[\Sigma(\bar{p},\bar{q})](x)$$

hold on $t \in (-\infty, T)$.

Proof. Let $M > N_0$ be sufficiently large. By Lemma 4.3, there exist $\bar{p}_0'' > N$ and $x_0 \in (-1,1)$ satisfying

$$(\bar{p}_0, \bar{q}_0) = (P[\bar{p}_0''+x_0, \tilde{\Theta}_{\bar{p}_0''}(x-x_0)], Q[\bar{p}_0''-x_0, \tilde{\Theta}_{\bar{p}_0''}(x-x_0)]).$$

Also, there exist $T'' > 0$, $T > T''$ and a solution $\tilde{\Theta}_{\bar{p}''} \in C((-\infty,T], Y'')$ of (RD) with $\bar{p}''(0) = \bar{p}_0''$, $\bar{p}''(T'') = N$ and $\bar{p}''(T) = M$. From Lemma 4.2,

$$(\bar{p}''+x_0, \bar{p}''-x_0, \tilde{\Theta}_{\bar{p}''}(x-x_0)) \in V,$$

$$P[\bar{p}'' + x_0, \tilde{\Theta}_{\bar{p}''}(x - x_0)] > l_0 r_0 + 1,$$
$$Q[\bar{p}'' - x_0, \tilde{\Theta}_{\bar{p}''}(x - x_0)] > l_0 r_0 + 1$$

and

$$(\Phi_{P[\bar{p}''+x_0,\tilde{\Theta}_{\bar{p}''}(x-x_0)]}[\tilde{\Theta}_{\bar{p}''}(x - x_0)], \Psi_{Q[\bar{p}''-x_0,\tilde{\Theta}_{\bar{p}''}(x-x_0)]}[\tilde{\Theta}_{\bar{p}''}(x - x_0)])$$
$$= \Sigma(P[\bar{p}'' + x_0, \tilde{\Theta}_{\bar{p}''}(x - x_0)], Q[\bar{p}'' - x_0, \tilde{\Theta}_{\bar{p}''}(x - x_0)])$$

hold for all $t < T$. We denote $(P[\bar{p}'' + x_0, \tilde{\Theta}_{\bar{p}''}(x - x_0)], Q[\bar{p}'' - x_0, \tilde{\Theta}_{\bar{p}''}(x - x_0)]) \in C((-\infty, T), \mathbf{R}^2)$ by $(\bar{p}, \bar{q})$. Then, from Proposition 4.1, $(\Sigma(\bar{p}, \bar{q}), \bar{p}, \bar{q}) \in C((-\infty, T), \tilde{S})$ is a solution of $(\mathrm{ES})_{l_0}$ and

$$\Theta_{\bar{p},\bar{q}}[\Sigma(\bar{p}, \bar{q})](x) = \tilde{\Theta}_{\bar{p}''}(x - x_0)$$

holds for all $t < T$. □

Lemma 4.5. *There exists $C > 0$ such that $\|\Phi_{p_1}[\tilde{\Theta}_p] - \Phi_{p_2}[\tilde{\Theta}_p]\|_{Y'} \leq C|p_1 - p_2|$ and $\|\Psi_{q_1}[\tilde{\Theta}_p] - \Psi_{q_2}[\tilde{\Theta}_p]\|_{Y'} \leq C|q_1 - q_2|$ hold for all $p_1, p_2, q_1, q_2 > 0$ and $p \geq N_0$.*

Proof. The solution $\psi \in C(\mathbf{R}, BU(\mathbf{R}))$ of (RD) satisfies $\sup_{t\in\mathbf{R}} \|\psi\|_{C^0} < +\infty$ and $\sup_{t\in\mathbf{R}} \|f(\psi)\|_{C^0} < +\infty$. Hence, by (1.4) of [6], we have $\sup_{t\in\mathbf{R}} \|\psi\|_{C^1} < +\infty$. Also, we have $\sup_{t\in\mathbf{R}} \|f(\psi)\|_{C^1} < +\infty$. Hence, by (1.5) of [6], $\sup_{t\in\mathbf{R}} \|\psi\|_{C^2} < +\infty$ holds. Also, $\sup_{t\in\mathbf{R}} \|f(\psi)\|_{C^2} < +\infty$ holds. Hence, because $\sup_{t\in\mathbf{R}} \|B\psi\|_Y < +\infty$ and $\sup_{t\in\mathbf{R}} \|B(f(\psi))\|_Y < +\infty$ hold, we obtain $\sup_{t\in\mathbf{R}} \|B^{1+3/4}\psi\|_Y < +\infty$. Hence, $\sup_{t\in\mathbf{R}} \|\psi\|_{C^3} < +\infty$ holds. There exists $C' > 0$ such that $\|\Phi_{p_1}[\psi(t)] - \Phi_{p_2}[\psi(t)]\|_{C^2} \leq C'|p_1 - p_2|$ and $\|\Psi_{q_1}[\psi(t)] - \Psi_{q_2}[\psi(t)]\|_{C^2} \leq C'|q_1 - q_2|$ hold for all $p_1, p_2, q_1, q_2 > 0$ and $t \in \mathbf{R}$. □

We also need the following lemma in the proof of Proposition 4.8 below, as the topologies of $X' \times \mathbf{R}^2$ and Y are different.

Lemma 4.6. *There exist $M > 0$, $\delta > 0$ and $C > 0$ such that for any $p \geq M$ and $u \in Y''$ with $\|u - \tilde{\Theta}_p\|_{Y''} \leq \delta$,*

$$(p, p, u) \in V$$

and

$$\|(\Phi_{P[p,u]}[u], \Psi_{Q[p,u]}[u]) - \Sigma(P[p, u], Q[p, u])\|_{X'} \leq C\|u - \tilde{\Theta}_p\|_{Y''}$$

hold.

Proof. Suppose that $M > 0$ and $\delta > 0$ are sufficiently large and small, respectively. Because of $\lim_{p\to+\infty} \|\Sigma(p,p)\|_{X'} = 0$ from Theorem 2.3, $\|(1/(1+e^{-x/l_0}))u(x) - \phi(x-p)\|_{C^0}$ and $\|(1/(1+e^{x/l_0}))u(x) - \phi(-x-p)\|_{C^0}$ are sufficiently small. Hence, $(p,p,u) \in V$ holds.

Let $C > 0$ be sufficiently large. Because of $(\Phi_{P[p,\tilde{\Theta}_p]}[\tilde{\Theta}_p], \Psi_{Q[p,\tilde{\Theta}_p]}[\tilde{\Theta}_p]) = \Sigma(P[p,\tilde{\Theta}_p], Q[p,\tilde{\Theta}_p])$ from Lemma 4.2, we have

$$\begin{aligned}
&\|(\Phi_{P[p,u]}[u], \Psi_{Q[p,u]}[u]) - \Sigma(P[p,u], Q[p,u])\|_{X'} \\
&\quad \le \|(\Phi_{P[p,u]}[u], \Psi_{Q[p,u]}[u]) - (\Phi_{P[p,\tilde{\Theta}_p]}[\tilde{\Theta}_p], \Psi_{Q[p,\tilde{\Theta}_p]}[\tilde{\Theta}_p])\|_{X'} \\
&\qquad + \|\Sigma(P[p,\tilde{\Theta}_p], Q[p,\tilde{\Theta}_p]) - \Sigma(P[p,u], Q[p,u])\|_{X'} \\
&\quad \le C^{1/4}\|(\Phi_{P[p,u]}[u], \Psi_{Q[p,u]}[u]) - (\Phi_{P[p,\tilde{\Theta}_p]}[\tilde{\Theta}_p], \Psi_{Q[p,\tilde{\Theta}_p]}[\tilde{\Theta}_p])\|_{Y'^2} \\
&\qquad + C^{1/4}\|(P[p,\tilde{\Theta}_p], Q[p,\tilde{\Theta}_p]) - (P[p,u], Q[p,u])\|_{\mathbf{R}^2} \\
&\quad \le C^{1/4}\|(\Phi_{P[p,u]}[u], \Psi_{Q[p,u]}[u]) - (\Phi_{P[p,u]}[\tilde{\Theta}_p], \Psi_{Q[p,u]}[\tilde{\Theta}_p])\|_{Y'^2} \\
&\qquad + C^{1/4}\|(\Phi_{P[p,u]}[\tilde{\Theta}_p], \Psi_{Q[p,u]}[\tilde{\Theta}_p]) - (\Phi_{P[p,\tilde{\Theta}_p]}[\tilde{\Theta}_p], \Psi_{Q[p,\tilde{\Theta}_p]}[\tilde{\Theta}_p])\|_{Y'^2} \\
&\qquad + C^{1/4}\|(P[p,\tilde{\Theta}_p], Q[p,\tilde{\Theta}_p]) - (P[p,u], Q[p,u])\|_{\mathbf{R}^2}.
\end{aligned}$$

From this and Lemma 4.5,

$$\begin{aligned}
&\|(\Phi_{P[p,u]}[u], \Psi_{Q[p,u]}[u]) - \Sigma(P[p,u], Q[p,u])\|_{X'} \\
&\quad \le C^{1/4}\|(\Phi_{P[p,u]}[u], \Psi_{Q[p,u]}[u]) - (\Phi_{P[p,u]}[\tilde{\Theta}_p], \Psi_{Q[p,u]}[\tilde{\Theta}_p])\|_{Y'^2} \\
&\qquad + C^{1/2}\|(P[p,\tilde{\Theta}_p], Q[p,\tilde{\Theta}_p]) - (P[p,u], Q[p,u])\|_{\mathbf{R}^2} \\
&\quad \le C\|u - \tilde{\Theta}_p\|_{Y''}
\end{aligned}$$

holds. □

Lemma 4.7. *There exists $M > N_0$ such that $(p,p,\tilde{\Theta}_p) \in V$ and $P[p,\tilde{\Theta}_p] = Q[p,\tilde{\Theta}_p] = p$ hold for all $p \ge M$.*

Proof. Let $M > N_0$ be sufficiently large. Then, because of $(\Sigma(p,p), p, p) \in \Omega_{l_0}$ from Proposition 3.4,

$$\frac{1}{1+e^{-x/l_0}}\tilde{\Theta}_p(x) - \phi(x-p) = (\Sigma^1(p,p))(x-p)$$

and

$$\frac{1}{1+e^{x/l_0}}\tilde{\Theta}_p(x) - \phi(-x-p) = (\Sigma^2(p,p))(-x-p)$$

hold. Hence, because of $\lim_{p\to+\infty} \|\Sigma(p,p)\|_{X'} = 0$ from Theorem 2.3, we have $(p,p,\tilde{\Theta}_p) \in V$. From Lemma 3.1, we also have $P[p,\tilde{\Theta}_p] = Q[p,\tilde{\Theta}_p] = p$. □

Under the above preparations, we obtain the following.

Proposition 4.8. *There exist $N_0' > N_0$, $\delta_0' > 0$ and $C_0' > 0$ such that for any $p_0 \geq N_0' + 2$ and $u_0 \in Y$ with $\|u_0 - \tilde{\Theta}_{p_0}\|_Y \leq \delta_0'$, there exist $x_0 \in \mathbf{R}$, $T > 0$ and a solution $\tilde{\Theta}_{\bar{p}} \in C([0,T], Y)$ of* (RD) *with $\bar{p}(T) = N_0'$ such that the solution $u \in C([0,+\infty), Y)$ of* (RD) *with $u(0) = u_0$ satisfies*

$$\|u(x,t) - \tilde{\Theta}_{\bar{p}(t)}(x - x_0)\|_Y \leq C_0' e^{-\gamma_0 t} \|u_0(x) - \tilde{\Theta}_{p_0}(x)\|_Y$$

for all $t \in [0,T]$.

Proof.
Step 1. In this step, we show the following.

There exist $N_0' > N_0$, $\delta_0'' > 0$ and $C_0'' > 0$ such that for any $p_0'' \geq N_0' + 1$ and $u_0'' \in Y''$ with $\|u_0'' - \tilde{\Theta}_{p_0''}\|_{Y''} \leq \delta_0''$, there exist $x_0 \in (-1,1)$, $T'' > 0$ and a solution $\tilde{\Theta}_{\bar{p}''} \in C((-\infty, T''], Y'')$ of (RD) *with $\bar{p}''(T'') = N_0'$ such that the solution $u'' \in C([0,+\infty), Y'')$ of* (RD) *with $u''(0) = u_0''$ satisfies*

$$\|u''(x,t) - \tilde{\Theta}_{\bar{p}''(t)}(x - x_0)\|_Y \leq C_0'' e^{-\gamma_0 t} \|u_0''(x) - \tilde{\Theta}_{p_0''}(x)\|_{Y''}$$

for all $t \in [0, T'']$. Further, there exists a solution $(\Sigma(\bar{p},\bar{q}), \bar{p}, \bar{q}) \in C((-\infty, 0], \tilde{S})$ of $(\mathrm{ES})_{l_0}$ *satisfying $\|(\bar{p}(0), \bar{q}(0)) - (p_0'', p_0'')\|_{\mathbf{R}^2} \leq C_0'' \|u_0'' - \tilde{\Theta}_{p_0''}\|_{Y''}$ and $\tilde{\Theta}_{\bar{p}''}(x - x_0) = (\Theta_{\bar{p},\bar{q}}[\Sigma(\bar{p},\bar{q})])(x)$ on $t \in (-\infty, 0]$.*

Let $K > 0$ be sufficiently large, and fixed. Then, let $N_0' > N_0$ and $\delta_0'' > 0$ be sufficiently large and small, respectively. Because $\lim_{(p,q)\to(+\infty,+\infty)} \|\Sigma(p, q)\|_{X'} = 0$ holds from Theorem 2.3, by Lemma 4.6, $\|(\Phi_{P[p_0'',u_0'']}[u_0''], \Psi_{Q[p_0'',u_0'']}[u_0''])\|_{X'} \leq 2D_0$ holds. We denote $(\Phi_{P[p_0'',u_0'']}[u_0''], \Psi_{Q[p_0'',u_0'']}[u_0''], P[p_0'', u_0''], Q[p_0'', u_0''])$ by (g_0, h_0, p_0, q_0). By Theorem 2.3, there exist solutions $(g,h,p,q) \in C([0, +\infty), U \times \mathbf{R}^2)$ and $(\Sigma(\bar{p},\bar{q}), \bar{p}, \bar{q}) \in C(\mathbf{R}, S)$ of $(\mathrm{ES})_{l_0,r_0}$ such that

$$(g,h,p,q)(0) = (g_0, h_0, p_0, q_0)$$

and for any $t \geq 0$,

$$\|(g,h,p,q)(t) - (\Sigma(\bar{p},\bar{q}), \bar{p}, \bar{q})(t)\|_{X' \times \mathbf{R}^2} \leq C_0 e^{-\gamma_0 t} \|(g_0, h_0) - \Sigma(p_0, q_0)\|_{X'}$$

hold. By Lemma 4.6,

$$\|(g,h,p,q)(t) - (\Sigma(\bar{p},\bar{q}), \bar{p}, \bar{q})(t)\|_{X' \times \mathbf{R}^2} \leq K e^{-\gamma_0 t} \|u_0'' - \tilde{\Theta}_{p_0''}\|_{Y''} \tag{4.4}$$

holds for all $t \geq 0$.

From Lemma 4.7, we have

$$\begin{aligned}
&\|(p_0'' - \bar{p}(0), p_0'' - \bar{q}(0))\|_{\mathbf{R}^2} \\
&\quad = \|(P[p_0'', \tilde{\Theta}_{p_0''}] - \bar{p}(0), Q[p_0'', \tilde{\Theta}_{p_0''}] - \bar{q}(0))\|_{\mathbf{R}^2} \\
&\quad \leq \|(P[p_0'', \tilde{\Theta}_{p_0''}] - p_0, Q[p_0'', \tilde{\Theta}_{p_0''}] - q_0)\|_{\mathbf{R}^2} + \|(p_0 - \bar{p}(0), q_0 - \bar{q}(0))\|_{\mathbf{R}^2} \\
&\quad = \|(P[p_0'', \tilde{\Theta}_{p_0''}] - P[p_0'', u_0''], Q[p_0'', \tilde{\Theta}_{p_0''}] - Q[p_0'', u_0''])\|_{\mathbf{R}^2} \\
&\qquad + \|(p_0 - \bar{p}(0), q_0 - \bar{q}(0))\|_{\mathbf{R}^2}.
\end{aligned}$$

Hence, by (4.4), we obtain

$$\|(p_0'' - \bar{p}(0), p_0'' - \bar{q}(0))\|_{\mathbf{R}^2} \leq 2K \|u_0'' - \tilde{\Theta}_{p_0''}\|_{Y''}. \tag{4.5}$$

Because $\|(p_0'' - \bar{p}(0), p_0'' - \bar{q}(0))\|_{\mathbf{R}^2} \leq 2K\delta_0''$ and $p_0'' \geq N_0' + 1$ hold, we have $\bar{p}(0) \geq N_0' + 1/2$, $\bar{q}(0) \geq N_0' + 1/2$ and $|\bar{p}(0) - \bar{q}(0)| \leq 1$. Therefore, by Lemma 4.4, there exist $x_0 \in (-1, 1)$, $T'' > 0$, $T > T''$ and a solution $\tilde{\Theta}_{\bar{p}''} \in C((-\infty, T), Y'')$ of (RD) with $\bar{p}''(T'') = N_0'$ such that

$$\bar{p} > l_0 r_0 + 1,\ \bar{q} > l_0 r_0 + 1$$

and

$$\Theta_{\bar{p},\bar{q}}[\Sigma(\bar{p}, \bar{q})](x) = \tilde{\Theta}_{\bar{p}''}(x - x_0) \tag{4.6}$$

hold on $t \in (-\infty, T)$. Hence, from (4.4), $p(t) > l_0 r_0$ and $q(t) > l_0 r_0$ hold for all $t \in [0, T)$. Therefore, (g, h, p, q) is a solution of $(\mathrm{ES})_{l_0}$ on $t \in [0, T)$. Hence, because $(g_0, h_0, p_0, q_0) \in \Omega_{l_0}$ and $\Theta_{p_0,q_0}[g_0, h_0] = u_0''$ also hold from Lemma 3.2, by Proposition 2.2,

$$\Theta_{p,q}[g, h] = u'' \tag{4.7}$$

holds on $t \in [0, T)$. From (4.4), (4.5), (4.6) and (4.7), there exists $C_0'' > 0$ such that

$$\|(\bar{p}(0), \bar{q}(0)) - (p_0'', p_0'')\|_{\mathbf{R}^2} \leq C_0'' \|u_0'' - \tilde{\Theta}_{p_0''}\|_{Y''}$$

and for any $t \in [0, T)$,

$$\|u''(x, t) - \tilde{\Theta}_{\bar{p}''(t)}(x - x_0)\|_{C^0} \leq C_0'' e^{-\gamma_0 t} \|u_0''(x) - \tilde{\Theta}_{p_0''}(x)\|_{Y''}$$

hold.

Step 2. In this step, we prove Proposition 4.8.

We denote a positive constant $(\sup_{p\geq N_0'+1}\left|c+G_l^1[\Sigma(p,p),p,p]\right|)^{-1}$ by t_0. Let $K>0$ be sufficiently large, and fixed. Let $\delta_0'>0$ be sufficiently small. Then, there exists a solution $\tilde{\Theta}_p\in C((-\infty,t_0],Y'')$ of (RD) with $p(0)=p_0$ such that

$$p(t_0)\geq N_0'+1,$$
$$\|u(t_0)-\tilde{\Theta}_{p(t_0)}\|_{Y''}\leq K\|u_0-\tilde{\Theta}_{p_0}\|_Y$$

and

$$\sup_{t\in[0,t_0]}\|u(t)-\tilde{\Theta}_{p(t)}\|_Y\leq K\|u_0-\tilde{\Theta}_{p_0}\|_Y \tag{4.8}$$

hold. Hence, by Step 1, there exist $x_0\in(-1,1)$, $T>t_0$ and a solution $\tilde{\Theta}_{\bar{p}}\in C((-\infty,T],Y)$ of (RD) with $\bar{p}(T)=N_0'$ such that

$$\|u(x,t)-\tilde{\Theta}_{\bar{p}(t)}(x-x_0)\|_Y\leq C_0''Ke^{-\gamma_0(t-t_0)}\|u_0(x)-\tilde{\Theta}_{p_0}(x)\|_Y \tag{4.9}$$

holds for all $t\in[t_0,T]$. Further, there exists a solution $(\Sigma(\hat{p},\hat{q}),\hat{p},\hat{q})\in C([0,t_0],\tilde{S})$ of $(\mathrm{ES})_{l_0}$ satisfying $\|(p(t_0),p(t_0))-(\hat{p}(t_0),\hat{q}(t_0))\|_{\mathbf{R}^2}\leq C_0''K\|u_0-\tilde{\Theta}_{p_0}\|_Y$ and $\tilde{\Theta}_{\bar{p}(t)}(x-x_0)=(\Theta_{\hat{p}(t),\hat{q}(t)}[\Sigma(\hat{p}(t),\hat{q}(t))])(x)$ for $t\in[0,t_0]$. Hence, we obtain

$$\begin{aligned}\sup_{t\in[0,t_0]}&\|\tilde{\Theta}_{p(t)}(x)-\tilde{\Theta}_{\bar{p}(t)}(x-x_0)\|_Y\\ &=\sup_{t\in[0,t_0]}\|\Theta_{p(t),p(t)}[\Sigma(p(t),p(t))]-\Theta_{\hat{p}(t),\hat{q}(t)}[\Sigma(\hat{p}(t),\hat{q}(t))]\|_{C^0}\\ &\leq K\sup_{t\in[0,t_0]}\|(p(t),p(t))-(\hat{p}(t),\hat{q}(t))\|_{\mathbf{R}^2}\\ &\leq K^2\|(p(t_0),p(t_0))-(\hat{p}(t_0),\hat{q}(t_0))\|_{\mathbf{R}^2}\\ &\leq C_0''K^3\|u_0(x)-\tilde{\Theta}_{p_0}(x)\|_Y.\end{aligned} \tag{4.10}$$

From (4.8), (4.9) and (4.10) with $C_0'=(1+C_0'')(K+K^3)e^{\gamma_0t_0}$,

$$\|u(x,t)-\tilde{\Theta}_{\bar{p}(t)}(x-x_0)\|_Y\leq C_0'e^{-\gamma_0t}\|u_0(x)-\tilde{\Theta}_{p_0}(x)\|_Y$$

holds for all $t\in[0,T]$. □

Now, we prove Theorem 1.1 (2) from Proposition 4.8.

Proof of Theorem 1.1 (2). Because of $\lim_{t\to+\infty}\|\psi(t)-1\|_Y=0$ and $\mathrm{Re}\,\sigma(B)>0$, there exist T_1, δ_1, C_1 and $\gamma_1>0$ such that for any $t_0\geq T_1$ and $u_0\in Y$ with $\|u_0-\psi(t_0)\|_Y\leq\delta_1$, the solution $u\in C([0,+\infty),Y)$ of (RD) with $u(0)=u_0$ satisfies

$$\|u(t)-\psi(t+t_0)\|_Y\leq C_1e^{-\gamma_1t}\|u_0-\psi(t_0)\|_Y \tag{4.11}$$

for all $t \geq 0$. We define $T_2 < 0$ such that $\psi(T_2) = \tilde{\Theta}_{N_0'+2}$ holds. Then, there exist δ_2 and $C_2 > 0$ such that for any $t_0 \in [T_2, T_1]$ and $u_0 \in Y$ with $\|u_0 - \psi(t_0)\|_Y \leq \delta_2$, the solution $u \in C([0, +\infty), Y)$ of (RD) with $u(0) = u_0$ satisfies

$$\|u(t) - \psi(t + t_0)\|_Y \leq C_2 \|u_0 - \psi(t_0)\|_Y \tag{4.12}$$

for all $t \in [0, T_1 - t_0]$. From (4.11) and (4.12), for any $t_0 \geq T_2$ and $u_0 \in Y$ with $\|u_0 - \psi(t_0)\|_Y \leq \min\{\delta_1/C_2, \delta_2\}$, the solution $u \in C([0, +\infty), Y)$ of (RD) with $u(0) = u_0$ satisfies

$$\|u(t) - \psi(t + t_0)\|_Y \leq C_1 C_2 e^{\gamma_1(T_1 - T_2)} e^{-\gamma_1 t} \|u_0 - \psi(t_0)\|_Y$$

for all $t \geq 0$. Therefore, by Proposition 4.8 with $\delta = \min\{(\delta_1/C_2 C_0'), (\delta_2/C_0'), \delta_0'\}$, $C = C_1 C_2 C_0' e^{\gamma_1(T_1 - T_2)}$ and $\gamma = \min\{\gamma_1, \gamma_0\}$, for any $t_0 \in \mathbf{R}$ and $u_0 \in Y$ with $\|u_0 - \psi(t_0)\|_Y \leq \delta$, there exist x_0 and $t_0' \in \mathbf{R}$ such that the solution $u \in C([0, +\infty), Y)$ of (RD) with $u(0) = u_0$ satisfies

$$\|u(x, t) - \psi(x - x_0, t - t_0')\|_Y \leq C e^{-\gamma t} \|u_0(x) - \psi(x, t_0)\|_Y$$

for all $t \geq 0$. □

§5. Proof of Theorem 2.3

In this section, we construct the invariant manifold S with asymptotic stability by standard technique, i.e., we prove Theorem 2.3 according to Section 6.1 of [10].

Lemma 5.1. *There exist $C_1 > 0$ and a monotone increasing function $\epsilon_1 : (0, +\infty) \to (0, +\infty)$ with $\lim_{s \to +0} \epsilon_1(s) = +0$ such that for any $l \geq 1$, $p > 0$, $q > 0$ and $(g, h) \in X'$ satisfying $\|(g, h)\|_{X'} \leq 1$,*

$$\|f_l[g, h, p, q]\|_Y \leq \epsilon_1(\max\{l/p, l/q\}) + \epsilon_1(1/l)\|(g, h)\|_{X'} + C_1 \|(g, h)\|_{X'}^2$$

holds.

Proof. Because $f(a+b) - (f(a) + f_u(a)b) = \left(\int_0^1 f_{uu}(a + \sigma b)(1 - \sigma) d\sigma\right) b^2$ holds, we have

(5.1)

$$\|f(\phi(y) + g(y)) - (f(\phi(y)) + f_u(\phi(y))g(y))\|_{C^0} \leq \left(\sup_{u \in [-1, 2]} |f_{uu}(u)|\right) \|g\|_{C^0}^2.$$

When $y \leq -(1/2)p$ holds,

$$\left| -\frac{2}{l}\frac{1}{1+e^{(y+p)/l}}\phi'(y) + \frac{1}{l^2}\frac{1}{1+e^{(y+p)/l}}\phi(y) \right| \leq 2 \max_{x\leq -\frac{1}{2}p} |\phi'(x)| + \max_{x\leq -\frac{1}{2}p} |\phi(x)|$$

holds. When $-(1/2)p \leq y$ holds,

$$\left| -\frac{2}{l}\frac{1}{1+e^{(y+p)/l}}\phi'(y) + \frac{1}{l^2}\frac{1}{1+e^{(y+p)/l}}\phi(y) \right| \leq 3\|\phi\|_{C^1} e^{-\frac{1}{2}\frac{p}{l}}$$

holds. Therefore, we have

$$\left\| -\frac{2}{l}\frac{1}{1+e^{(y+p)/l}}\phi'(y) + \frac{1}{l^2}\frac{1}{1+e^{(y+p)/l}}\phi(y) \right\|_{C^0} \leq 2 \max_{x\leq -\frac{1}{2}p} |\phi'(x)| + \max_{x\leq -\frac{1}{2}p} |\phi(x)| + 3\|\phi\|_{C^1} e^{-\frac{1}{2}\frac{p}{l}}. \tag{5.2}$$

Also, we get

$$\left\| -\frac{2}{l}\frac{1}{1+e^{(y+p)/l}}g_y(y) + \frac{1}{l^2}\frac{1}{1+e^{(y+p)/l}}g(y) \right\|_{C^0} \leq \frac{3}{l}\|g\|_{C^1}. \tag{5.3}$$

We have $f(\lambda a) = (\int_0^1 f_u(\sigma\lambda a)d\sigma)\lambda a$. Hence, when $y \leq -p-(1/2)q$ holds,

$$\begin{aligned} &\left| \frac{1}{1+e^{-(y+p)/l}} f(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q)) \right. \\ &\left. -f\left(\frac{1}{1+e^{-(y+p)/l}}(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q))\right) \right| \\ &\leq \left(\sup_{u\in[-2,4]} |f(u)| + 4 \sup_{u\in[-2,4]} |f_u(u)| \right) e^{-\frac{1}{2}\frac{q}{l}} \end{aligned}$$

holds. Because $\lambda f(a) = f_u(0)\lambda a + (\int_0^1 f_{uu}(\sigma a)(1-\sigma)d\sigma)\lambda a^2$ and $f(\lambda a) = f_u(0)\lambda a + (\int_0^1 f_{uu}(\sigma\lambda a)(1-\sigma)d\sigma)\lambda^2 a^2$ hold, we have $\lambda f(a) - f(\lambda a) = (\int_0^1 f_{uu}(\sigma a)(1-\sigma)d\sigma)\lambda a^2 - (\int_0^1 f_{uu}(\sigma\lambda a)(1-\sigma)d\sigma)\lambda^2 a^2$. Hence, when $-p-(1/2)q \leq y \leq -(1/2)p$ holds,

$$\begin{aligned} &\left| \frac{1}{1+e^{-(y+p)/l}} f(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q)) \right. \\ &\left. -f\left(\frac{1}{1+e^{-(y+p)/l}}(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q))\right) \right| \\ &\leq 2\left(\sup_{u\in[-2,4]} |f_{uu}(u)| \right) \left(\max_{x\leq -\frac{1}{2}p} |\phi(x)| + \max_{x\leq -\frac{1}{2}q} |\phi(x)| + \|g\|_{C^0} + \|h\|_{C^0} \right)^2 \end{aligned}$$

holds. Because of $f(\lambda a)-f(a)=(\int_0^1 f_u(a+\sigma(\lambda-1)a)d\sigma)\,(\lambda-1)\,a$, we have $\lambda f(a)-f(\lambda a)=(\lambda-1)(f(a)-a(\int_0^1 f_u(a+\sigma(\lambda-1)a)d\sigma))$. Hence, when $-(1/2)p\le y$ holds,

$$\Bigg|\frac{1}{1+e^{-(y+p)/l}}f(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q))$$
$$-f\left(\frac{1}{1+e^{-(y+p)/l}}(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q))\right)\Bigg|$$
$$\le\left(\sup_{u\in[-2,4]}|f(u)|+4\sup_{u\in[-2,4]}|f_u(u)|\right)e^{-\frac{1}{2}\frac{p}{l}}$$

holds. Therefore, we obtain

$$\Bigg\|\frac{1}{1+e^{-(y+p)/l}}f(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q))\tag{5.4}$$
$$-f\left(\frac{1}{1+e^{-(y+p)/l}}(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q))\right)\Bigg\|_{C^0}$$
$$\le\left(\sup_{u\in[-2,4]}|f(u)|+4\sup_{u\in[-2,4]}|f_u(u)|\right)\left(e^{-\frac{1}{2}\frac{p}{l}}+e^{-\frac{1}{2}\frac{q}{l}}\right)$$
$$+2\left(\sup_{u\in[-2,4]}|f_{uu}(u)|\right)\left(\max_{x\le-\frac{1}{2}p}|\phi(x)|+\max_{x\le-\frac{1}{2}q}|\phi(x)|+\|g\|_{C^0}+\|h\|_{C^0}\right)^2.$$

From (5.1), (5.2), (5.3) and (5.4), we obtain

$$\|f_l[g,h,p,q]\|_{C^0}$$
$$\le\left(\sup_{u\in[-1,2]}|f_{uu}(u)|\right)\|g\|_{C^0}^2$$
$$+2\max_{x\le-\frac{1}{2}p}|\phi'(x)|+\max_{x\le-\frac{1}{2}p}|\phi(x)|+3\|\phi\|_{C^1}e^{-\frac{1}{2}\frac{p}{l}}+\frac{3}{l}\|g\|_{C^1}$$
$$+\left(\sup_{u\in[-2,4]}|f(u)|+4\sup_{u\in[-2,4]}|f_u(u)|\right)\left(e^{-\frac{1}{2}\frac{p}{l}}+e^{-\frac{1}{2}\frac{q}{l}}\right)$$
$$+2\left(\sup_{u\in[-2,4]}|f_{uu}(u)|\right)\left(\max_{x\le-\frac{1}{2}p}|\phi(x)|+\max_{x\le-\frac{1}{2}q}|\phi(x)|+\|g\|_{C^0}+\|h\|_{C^0}\right)^2$$
$$\le 2\max_{x\le-\frac{1}{2}p}|\phi'(x)|+\max_{x\le-\frac{1}{2}p}|\phi(x)|$$
$$+2\left(\sup_{u\in[-2,4]}|f_{uu}(u)|\right)(2\|\phi\|_{C^0}+4)\left(\max_{x\le-\frac{1}{2}p}|\phi(x)|+\max_{x\le-\frac{1}{2}q}|\phi(x)|\right)$$

$$+ 3\|\phi\|_{C^1} e^{-\frac{1}{2}\frac{p}{l}} + \left(\sup_{u\in[-2,4]} |f(u)| + 4 \sup_{u\in[-2,4]} |f_u(u)| \right) \left(e^{-\frac{1}{2}\frac{p}{l}} + e^{-\frac{1}{2}\frac{q}{l}} \right)$$
$$+ \frac{3}{l}\|g\|_{C^1} + 3 \left(\sup_{u\in[-2,4]} |f_{uu}(u)| \right) (\|g\|_{C^0} + \|h\|_{C^0})^2.$$

□

Lemma 5.2. *There exists a monotone increasing function* $\epsilon_2 : (0, +\infty) \to (0, +\infty)$ *with* $\lim_{s\to+0} \epsilon_2(s) = +0$ *such that for any* $l \geq 1$, $p > 0$, $q > 0$ *and* $(g, h) \in X'$ *satisfying* $\|(g, h)\|_{X'} \leq 1$,

$$\|(\partial f_l)[g, h, p, q]\|_{L(X'\times\mathbf{R}^2,Y)} \leq \epsilon_2(\max\{l/p, l/q, 1/l, \|(g, h)\|_{X'}\})$$

holds, where ∂f_l *denotes the Frechet derivative of* f_l.

Proof. Because of $f_u(\lambda a) - f_u(a) = (\int_0^1 f_{uu}(a + \sigma(\lambda - 1)a)d\sigma)(\lambda - 1)a$, when $-(1/2)p \leq y$ holds,

$$\Bigg| f_u(\phi(y) + \phi(-y - p - q) + g(y) + h(-y - p - q))$$
$$-f_u \left(\frac{1}{1 + e^{-(y+p)/l}} (\phi(y) + \phi(-y - p - q) + g(y) + h(-y - p - q)) \right) \Bigg|$$
$$\leq 4 \left(\sup_{u\in[-2,4]} |f_{uu}(u)| \right) e^{-\frac{1}{2}\frac{p}{l}}$$

holds, and when $-p - (1/2)q \leq y \leq -(1/2)p$ holds,

$$\Bigg| f_u(\phi(y) + \phi(-y - p - q) + g(y) + h(-y - p - q))$$
$$-f_u \left(\frac{1}{1 + e^{-(y+p)/l}} (\phi(y) + \phi(-y - p - q) + g(y) + h(-y - p - q)) \right) \Bigg|$$
$$\leq \left(\sup_{u\in[-2,4]} |f_{uu}(u)| \right) \left(\max_{x\leq -\frac{1}{2}p} |\phi(x)| + \max_{x\leq -\frac{1}{2}q} |\phi(x)| + \|g\|_{C^0} + \|h\|_{C^0} \right)$$

holds. Also, when $y \leq -p - (1/2)q$ holds,

$$\frac{1}{1 + e^{-(y+p)/l}} \Bigg| f_u(\phi(y) + \phi(-y - p - q) + g(y) + h(-y - p - q))$$

$$-f_u\left(\frac{1}{1+e^{-(y+p)/l}}(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q))\right)\Bigg|$$
$$\leq 2\left(\sup_{u\in[-2,4]}|f_u(u)|\right)e^{-\frac{1}{2}\frac{q}{l}}$$

holds. Therefore, we obtain

$$(5.5)\qquad \Bigg\|\frac{1}{1+e^{-(y+p)/l}}\Bigg(f_u(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q))$$
$$-f_u\left(\frac{1}{1+e^{-(y+p)/l}}(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q))\right)\Bigg)\Bigg\|_{C^0}$$
$$\leq 2\left(\sup_{u\in[-2,4]}|f_u(u)|\right)e^{-\frac{1}{2}\frac{q}{l}}$$
$$+\left(\sup_{u\in[-2,4]}|f_{uu}(u)|\right)\left(4e^{-\frac{1}{2}\frac{p}{l}}+\max_{x\leq-\frac{1}{2}p}|\phi(x)|+\max_{x\leq-\frac{1}{2}q}|\phi(x)|+\|g\|_{C^0}+\|h\|_{C^0}\right).$$

On the other hand, straightforward calculation shows

$$(((\partial f_l)[g,h,p,q])(\Delta g,\Delta h,\Delta p,\Delta q))(y)$$
$$=(f_u(\phi(y)+g(y))-f_u(\phi(y)))(\Delta g)(y)$$
$$-\frac{2}{l}\frac{1}{1+e^{(y+p)/l}}(\Delta g)_y(y)+\frac{1}{l^2}\frac{1}{1+e^{(y+p)/l}}(\Delta g)(y)$$
$$+\frac{1}{1+e^{-(y+p)/l}}\Bigg(f_u(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q))$$
$$-f_u\left(\frac{1}{1+e^{-(y+p)/l}}(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q))\right)\Bigg)$$
$$\times((\Delta g)(y)+(\Delta h)(-y-p-q))$$
$$+\frac{1}{1+e^{-(y+p)/l}}\frac{1}{1+e^{(y+p)/l}}\left(\frac{2}{l^2}(\phi'(y)+g_y(y))-\frac{1}{l^3}(\phi(y)+g(y))\right)\Delta p$$
$$+\frac{1}{l}\frac{1}{1+e^{-(y+p)/l}}\frac{1}{1+e^{(y+p)/l}}$$
$$\times f(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q))\Delta p$$
$$-\frac{1}{l}\frac{1}{1+e^{-(y+p)/l}}\frac{1}{1+e^{(y+p)/l}}$$
$$\times f_u\left(\frac{1}{1+e^{-(y+p)/l}}(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q))\right)$$
$$\times(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q))\Delta p$$
$$-\frac{1}{1+e^{-(y+p)/l}}\Bigg(f_u(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q))$$

$$- f_u\left(\frac{1}{1+e^{-(y+p)/l}}(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q))\right)\Bigg)$$
$$\times(\phi'(-y-p-q)+h_y(-y-p-q))(\Delta p+\Delta q).$$

From this and (5.5), Lemma 5.2 can be easily seen. □

From Lemmas 5.1 and 5.2, the following lemma immediately follows.

Lemma 5.3. *There exist $c_3 > 0$, $C_3 > 0$ and a monotone increasing function $\epsilon_3 : (0, +\infty) \to (0, +\infty)$ with $\lim_{s\to+0} \epsilon_3(s) = +0$ satisfying the following.*

(1) *For any $l \geq 1$, $r \geq 1$ and $(g, h, p, q) \in X' \times \mathbf{R}^2$ with $\|(g, h)\|_{X'} \leq c_3$,*

$$\|F_{l,r}[g, h, p, q]\|_X \leq \epsilon_3(1/r) + \epsilon_3(1/l)\|(g, h)\|_{X'} + C_3\|(g, h)\|^2_{X'}$$

and

$$\|(\partial F_{l,r})[g, h, p, q]\|_{L(X'\times\mathbf{R}^2,X)} + \|(\partial G_{l,r})[g, h, p, q]\|_{L(X'\times\mathbf{R}^2,\mathbf{R}^2)}$$
$$\leq \epsilon_3(\max\{1/r, 1/l, \|(g, h)\|_{X'}\})$$

hold, where $F_{l,r}[g, h, p, q]$ and $G_{l,r}[g, h, p, q]$ denote $\chi(p/lr)\chi(q/lr)F_l[g, h, p, q]$ and $\chi(p/lr)\chi(q/lr)G_l[g, h, p, q]$, respectively.

(2) *For any $l \geq 1$, $p > 0$, $q > 0$ and $(g, h) \in X'$ with $\|(g, h)\|_{X'} \leq c_3$,*

$$\|F_l[g, h, p, q]\|_X + \|G_l[g, h, p, q]\|_{\mathbf{R}^2}$$
$$\leq \epsilon_3(\max\{l/p, l/q, \|(g, h)\|_{X'}\})$$

holds.

We obtain the following by Lemma 5.3 (1).

Proposition 5.4. *For any $\Delta \in (0, 1]$, there exists $l_1 \geq 1$ such that for any $d > 0$, there exists $r_1 \geq 1$ such that for any $l \geq l_1$ and $r \geq r_1$, there exists an invariant manifold*

$$S = \{(g, h, p, q) \in U \times \mathbf{R}^2 | \, (g, h) = \sigma(p, q)\}$$

for $(\mathrm{ES})_{l,r}$ *satisfying $\|\sigma(p, q)\|_{X'} \leq d$ and $\|\sigma(p_1, q_1) - \sigma(p_2, q_2)\|_{X'} \leq \Delta\|(p_1, q_1) - (p_2, q_2)\|_{\mathbf{R}^2}$.*

Proof. Let M and $\beta > 0$ be constants such that $\|e^{-At}\|_{L(X,X)} \le Me^{-\beta t}$ and $\|A^{3/4}e^{-At}\|_{L(X,X)} \le Mt^{-3/4}e^{-\beta t}$ hold for all $t > 0$.

Let $\Delta \in (0,1]$ be given, and let $l_1 \ge 1$ be a constant such that

$$\epsilon_3(1/l_1) \le \frac{\beta}{2(1+\Delta)},$$
$$\epsilon_3(1/l_1)M\int_0^\infty u^{-3/4}e^{-\frac{\beta}{2}u}du \le \frac{\Delta}{1+\Delta}$$

and

$$\epsilon_3(1/l_1)M\int_0^\infty u^{-3/4}e^{-\beta u}du \le \frac{1}{8}$$

hold.

Let $d > 0$ be given, let $D > 0$ be a constant defined by

$$D = \min\left\{d, \frac{c_3}{2}, \frac{1}{2l_1}, \frac{1}{32C_3M\int_0^\infty u^{-3/4}e^{-\beta u}du}\right\},$$

and let $r_1 \ge l_1$ be a constant such that

$$\epsilon_3(1/r_1)M\int_0^\infty u^{-3/4}e^{-\beta u}du \le \frac{D}{8}$$

holds.

Then, by Lemma 5.3 (1), the hypotheses of Theorem 6.1.2 of [10] for $(\mathrm{ES})_{l,r}$ are satisfied under the notations $Y = \mathbf{R}^2$, $\alpha = 3/4$, $U = \{(g,h) \in X' \,|\, \|(g,h)\|_{X'} < 2D\}$, $\lambda = \epsilon_3(1/l_1)$, $N = D/(2M\int_0^\infty u^{-3/4}e^{-\beta u}du)$, $M_1 = 1$, $M_2 = \beta/(2(1+\Delta))$ and $\mu = \beta/(2(1+\Delta))$. Hence, this proposition follows from Theorem 6.1.2 of [10]. □

We also obtain the following by Lemma 5.3 (1).

Proposition 5.5. *There exist $D_1 > 0$, $\delta_1 > 0$, $K_1 > 0$, $\gamma_1 > 0$, $L_1 \ge 1$ and $R_1 \ge 1$ such that for any $l \ge L_1$ and $r \ge R_1$, there exists an invariant manifold*

$$S = \{(g,h,p,q) \in U \times \mathbf{R}^2 \,|\, (g,h) = \sigma(p,q)\}$$

for $(\mathrm{ES})_{l,r}$ satisfying $\|\sigma(p,q)\|_{X'} \le D_1$, $\|\sigma(p_1,q_1) - \sigma(p_2,q_2)\|_{X'} \le \|(p_1,q_1) - (p_2,q_2)\|_{\mathbf{R}^2}$ and the following. For any $(g_0,h_0,p_0,q_0) \in X' \times \mathbf{R}^2$ with $\|(g_0,h_0)\|_{X'} \le \delta_1$, there exist solutions $(g,h,p,q) \in C([0,+\infty), U \times \mathbf{R}^2)$ and $(\bar{g},\bar{h},\bar{p},\bar{q}) \in C([0,+\infty), S)$ of $(\mathrm{ES})_{l,r}$ such that

$$(g,h,p,q)(0) = (g_0,h_0,p_0,q_0)$$

and for any $t \geq 0$,

$$\|(g,h,p,q)(t) - (\bar{g},\bar{h},\bar{p},\bar{q})(t)\|_{X' \times \mathbf{R}^2} \leq K_1 e^{-\gamma_1 t}\|(g_0,h_0) - \sigma(p_0,q_0)\|_{X'}$$

hold.

Proof. Let $K > 0$ be a constant such that Theorem 6.1.4 of [10] holds for $\alpha = 3/4$, and let M and $\beta > 0$ be constants such that $\|e^{-At}\|_{L(X,X)} \leq Me^{-\beta t}$ and $\|A^{3/4}e^{-At}\|_{L(X,X)} \leq Mt^{-3/4}e^{-\beta t}$ hold for all $t > 0$.

Let $L_1 \geq 1$ be a constant such that

$$\epsilon_3(1/L_1) \leq \frac{\beta}{4},$$
$$\epsilon_3(1/L_1)M\int_0^\infty u^{-3/4}e^{-\frac{\beta}{2}u}du \leq \frac{1}{4}$$

and

$$\epsilon_3(1/L_1)(2+KM)M\int_0^\infty u^{-3/4}e^{-\beta u}du \leq \frac{1}{4}$$

hold, let $D_1 > 0$ be a constant defined by

$$D_1 = \min\left\{\frac{c_3}{2}, \frac{1}{2L_1}, \frac{1}{8C_3(2+KM)^2 M\int_0^\infty u^{-3/4}e^{-\beta u}du}\right\},$$

and let $R_1 \geq L_1$ be a constant such that

$$\epsilon_3(1/R_1)M\int_0^\infty u^{-3/4}e^{-\beta u}du \leq \frac{D_1}{8}$$

holds.

Then, by Lemma 5.3 (1), the hypotheses of Theorem 6.1.4 of [10] for $(\mathrm{ES})_{l,r}$ are satisfied under the notations $Y = \mathbf{R}^2$, $\alpha = 3/4$, $U = \{(g,h) \in X' \mid \|(g,h)\|_{X'} < (2+KM)D_1\}$, $\lambda = \epsilon_3(1/L_1)$, $N = D_1/(2M\int_0^\infty u^{-3/4}e^{-\beta u}du)$, $M_1 = 1$, $M_2 = \beta/4$, $\mu = \beta/4$, $\Delta = 1$ and $D = D_1$. Hence, there exists an invariant manifold

$$S = \{(g,h,p,q) \in U \times \mathbf{R}^2 \mid (g,h) = \sigma(p,q)\}$$

for $(\mathrm{ES})_{l,r}$ with $\|\sigma(p,q)\|_{X'} \leq D_1$ and $\|\sigma(p_1,q_1) - \sigma(p_2,q_2)\|_{X'} \leq \|(p_1,q_1) - (p_2,q_2)\|_{\mathbf{R}^2}$ satisfying the following. If $(g,h,p,q)(t)$ is a solution of $(\mathrm{ES})_{l,r}$ on $0 \leq t < T$ with $\|(g,h)(t)\|_{X'} < (2+KM)D_1$, then

$$\|(g,h)(t) - \sigma((p,q)(t))\|_{X'} \leq KMe^{-\gamma_1 t}\|(g,h)(0) - \sigma((p,q)(0))\|_{X'}$$

holds on $0 \leq t < T$ with $\gamma_1 = (1 - 8(\epsilon_3(1/L_1)M \int_0^\infty u^{-3/4} e^{-(\beta/2)u} du)^4)\beta > (1/2)\beta$. Therefore, according to Proof of Corollary 6.1.5 of [10] with $\delta_1 = D_1/2KM$ and $K_1 = (1 + \beta/(2\gamma_1 - \beta)KM)KM$, we can see that for any $(g_0, h_0, p_0, q_0) \in X' \times \mathbf{R}^2$ with $\|(g_0, h_0)\|_{X'} \leq \delta_1$, there exist solutions $(g, h, p, q) \in C([0, +\infty), U \times \mathbf{R}^2)$ and $(\bar{g}, \bar{h}, \bar{p}, \bar{q}) \in C([0, +\infty), S)$ of $(\mathrm{ES})_{l,r}$ such that

$$(g, h, p, q)(0) = (g_0, h_0, p_0, q_0)$$

and for any $t \geq 0$,

$$\|(g, h, p, q)(t) - (\bar{g}, \bar{h}, \bar{p}, \bar{q})(t)\|_{X' \times \mathbf{R}^2} \leq K_1 e^{-\gamma_1 t} \|(g_0, h_0) - \sigma(p_0, q_0)\|_{X'}$$

hold. □

Now, we prove Theorem 2.3 from the above preparations.

Proof of Theorem 2.3. Let D_1, δ_1, K_1, $\gamma_1 > 0$, L_1 and $R_1 \geq 1$ be constants satisfying Proposition 5.5, and let l_1 and $r_1 \geq 1$ be constants such that Proposition 5.4 holds for $\Delta = 1$ and $d = \delta_1/2$.

We define positive constants D_0, l_0 and r_0 by $D_0 = d$, $l_0 = \max\{l_1, L_1\}$ and $r_0 = \max\{r_1, R_1\}$, respectively. Then, by Proposition 5.4, there exists an invariant manifold

$$S = \{(g, h, p, q) \in U \times \mathbf{R}^2 | \, (g, h) = \Sigma(p, q)\}$$

for $(\mathrm{ES})_{l_0, r_0}$ with $\|\Sigma(p, q)\|_{X'} \leq D_0$.

Also, by Proposition 5.5, there exists an invariant manifold

$$S' = \{(g, h, p, q) \in U \times \mathbf{R}^2 | \, (g, h) = \sigma(p, q)\}$$

for $(\mathrm{ES})_{l_0, r_0}$ with $\|\sigma(p, q)\|_{X'} \leq D_1$ and $\|\sigma(p_1, q_1) - \sigma(p_2, q_2)\|_{X'} \leq \|(p_1, q_1) - (p_2, q_2)\|_{\mathbf{R}^2}$ satisfying the following. For any $(g_0, h_0, p_0, q_0) \in X' \times \mathbf{R}^2$ with $\|(g_0, h_0)\|_{X'} \leq 2D_0$, there exist solutions $(g, h, p, q) \in C([0, +\infty), U \times \mathbf{R}^2)$ and $(\bar{g}, \bar{h}, \bar{p}, \bar{q}) \in C([0, +\infty), S')$ of $(\mathrm{ES})_{l_0, r_0}$ such that

$$(g, h, p, q)(0) = (g_0, h_0, p_0, q_0)$$

and for any $t \geq 0$,

$$\|(g, h, p, q)(t) - (\bar{g}, \bar{h}, \bar{p}, \bar{q})(t)\|_{X' \times \mathbf{R}^2} \leq K_1 e^{-\gamma_1 t} \|(g_0, h_0) - \sigma(p_0, q_0)\|_{X'}$$

hold. Fix any point $(p_0, q_0) \in \mathbf{R}^2$, and let (p, q) denote the solution on S with $(p, q)(0) = (p_0, q_0)$. Then, because of $\|\Sigma(p, q)\|_{X'} \leq D_0$, for any $t < 0$, there exists $(\bar{p}_0, \bar{q}_0) \in \mathbf{R}^2$ such that

$$\|(\Sigma(p_0, q_0), p_0, q_0) - (\sigma(\bar{p}_0, \bar{q}_0), \bar{p}_0, \bar{q}_0)\|_{X' \times \mathbf{R}^2} \leq (D_0 + D_1) K_1 e^{\gamma_1 t}$$

holds. Hence, because $\lim_{t\to-\infty}(\bar{p}_0,\bar{q}_0) = (p_0,q_0)$ and $\lim_{t\to-\infty}\sigma(\bar{p}_0,\bar{q}_0) = \Sigma(p_0,q_0)$ hold, we have $\Sigma(p_0,q_0)=\sigma(p_0,q_0)$. That is $S=S'$. Therefore, there uniquely exists an invariant manifold

$$S=\{(g,h,p,q)\in U\times\mathbf{R}^2 \,|\, (g,h)=\Sigma(p,q)\,(=(\Sigma^1(p,q),\Sigma^2(p,q)))\}$$

for $(\mathrm{ES})_{l_0,r_0}$ with $\|\Sigma(p,q)\|_{X'}\le D_0$. From symmetry of $(\mathrm{ES})_{l_0,r_0}$ and uniqueness of S, $\Sigma^2(p,q)=\Sigma^1(q,p)$ holds.

Fix any $\varepsilon>0$. By Proposition 5.4, there exist $r_\varepsilon\ge r_0$ and an invariant manifold

$$S_\varepsilon=\{(g,h,p,q)\in U\times\mathbf{R}^2 \,|\, (g,h)=\Sigma_\varepsilon(p,q)\}$$

for $(\mathrm{ES})_{l_0,r_\varepsilon}$ with $\|\Sigma_\varepsilon(p,q)\|_{X'}\le\min\{\varepsilon/2,D_0\}$. Let p_0 and $q_0>l_0r_\varepsilon$ be sufficiently large. Then, there exists $(\bar{p}_0,\bar{q}_0)\in\mathbf{R}^2$ such that

$$\begin{aligned}&\|(\Sigma_\varepsilon(p_0,q_0),p_0,q_0)-(\Sigma(\bar{p}_0,\bar{q}_0),\bar{p}_0,\bar{q}_0)\|_{X'\times\mathbf{R}^2}\\ &\qquad\le(D_0+D_1)K_1e^{-\gamma_1\frac{\min\{p_0,q_0\}-l_0r_\varepsilon}{2\sup_{p>l_0r_\varepsilon,q>l_0r_\varepsilon}\|(c,c)+G_{l_0}[\Sigma_\varepsilon(p,q),p,q]\|_{\mathbf{R}^2}}}\end{aligned}$$

holds. Hence, because p_0 and q_0 are sufficiently large, $\|(\Sigma_\varepsilon(p_0,q_0),p_0,q_0)-(\Sigma(\bar{p}_0,\bar{q}_0),\bar{p}_0,\bar{q}_0)\|_{X'\times\mathbf{R}^2}\le\varepsilon/8$ holds. We have $\|\Sigma(p_0,q_0)\|_{X'}\le\|\Sigma_\varepsilon(p_0,q_0)\|_{X'}$ $+\|\Sigma(\bar{p}_0,\bar{q}_0)-\Sigma_\varepsilon(p_0,q_0)\|_{X'}+\|\Sigma(p_0,q_0)-\Sigma(\bar{p}_0,\bar{q}_0)\|_{X'}\le\varepsilon/2+\varepsilon/8+\varepsilon/8<\varepsilon$. Therefore, we obtain $\lim_{(p,q)\to(+\infty,+\infty)}\|\Sigma(p,q)\|_{X'}=0$. Hence, by Lemma 5.3 (2), $\lim_{(p,q)\to(+\infty,+\infty)}(\|\tilde{F}(p,q)\|_X+\|\tilde{G}(p,q)\|_{\mathbf{R}^2})=0$ also holds. □

§6. Characterizing Annihilation Dynamics

In this section, we give the proof of Proposition 1.3. This section is independent of Sections 2, 3, 4 and 5.

Lemma 6.1. *For any $\delta_0\in(0,\alpha)$, there exist P_1, D_1, k_1 and $\lambda_1>0$ such that for any $\overline{p}(t)$ and $\overline{\sigma}(t)$ satisfying*

$$\begin{cases}\dot{\overline{p}}=c-D_1(e^{-\lambda_1\overline{p}}+\overline{\sigma}), & \overline{p}\ge P_1,\\ \dot{\overline{\sigma}}=-k_1\overline{\sigma}, & 0\le\overline{\sigma}\le\alpha-\delta_0,\end{cases}\tag{6.1}$$

the function $\overline{u}(x,t)$ defined by

$$\overline{u}(x,t)=\phi(|x|-\overline{p}(t))+e^{-\lambda_1(|x|+\overline{p}(t))}+\overline{\sigma}(t)\tag{6.2}$$

is a super-solution of $u_t=u_{xx}+f(u)$ in $\mathbf{R}$.

Proof.

Step 1. In this step, we show the following.

There exist δ_1 *and* $k_1 > 0$ *such that*

$$f(a) - f(a+b) \geq k_1 b \tag{6.3}$$

holds for all $a \in [0, \delta_1] \cup [1-\delta_1, 1]$ *and* $b \in [0, (\alpha - \delta_0) + \delta_1]$.

Take a constant $\delta > 0$ such that $\inf_{u \in [0,\delta] \cup [1-\delta,1]}(-f_u(u)) > 0$ holds. Then, let $\delta_1 \in (0, \min\{\delta, \delta_0/4\}]$ be a constant satisfying $\inf_{u \in [0,\delta_1]} f(u) > \sup_{u \in [\delta, \alpha-\delta_0/2]} f(u)$.

Because of $\inf_{u \in [0,\delta]}(-f_u(u)) > 0$, $f(a) - f(a+b) > 0$ holds when $0 \leq a < a + b \leq \delta$ holds. Because of $\inf_{u \in [0,\delta_1]} f(u) > \sup_{u \in [\delta, \alpha-\delta_0/2]} f(u)$, $f(a) - f(a+b) > 0$ holds when $0 \leq a \leq \delta_1$ and $\delta \leq a + b$ hold. Because of $\inf_{u \in [1-\delta,1]}(-f_u(u)) > 0$, $f(a) - f(a+b) > 0$ holds when $1 - \delta_1 \leq a < a+b \leq 1$ holds. Because of $f(a) \geq 0$ and $f(a+b) < 0$, $f(a) - f(a+b) > 0$ holds when $1 - \delta_1 \leq a \leq 1$ and $1 < a + b$ hold.

Therefore, $(f(a) - f(a+b))/b > 0$ holds when $b > 0$ holds. Because $-f_u(a) > 0$ also holds, we have

$$k_1 := \inf_{a \in [0,\delta_1] \cup [1-\delta_1,1],\ b \in (0,(\alpha-\delta_0)+\delta_1]} \frac{f(a) - f(a+b)}{b} > 0.$$

Then, (6.3) holds.

Step 2. In this step, we prove Lemma 6.1.

Let $\lambda_1 > 0$ be a constant satisfying

$$\lambda_1(\lambda_1 + c) \leq k_1 \tag{6.4}$$

and

$$\lim_{p \to +\infty} \phi'(-p) e^{\lambda_1 p} = 0.$$

Let $P_1 > 0$ be a constant such that

$$\phi'(-p) \leq \lambda_1 e^{-\lambda_1 p} \tag{6.5}$$

and

$$e^{-\lambda_1 p} \leq \delta_1 \tag{6.6}$$

hold for all $p \geq P_1$. Let $L_1 > 0$ be a constant satisfying

$$\phi(z) \in [0, \delta_1] \cup [1 - \delta_1, 1] \tag{6.7}$$

for $|z| \geq L_1$, and let $D_1 > 0$ be defined by

$$D_1 = \frac{\sup_{0 \leq u \leq 1+\alpha-\delta_0+\delta_1} f_u(u) + k_1}{\inf_{|z| \leq L_1} \phi'(z)}. \tag{6.8}$$

Then, on $x > 0$, because

$$\begin{aligned}-\overline{u}_{xx} + \overline{u}_t - f(\overline{u}) &= -\phi'' - \dot{\overline{p}}\phi' - f(\overline{u}) - \lambda_1(\lambda_1 + \dot{\overline{p}})e^{-\lambda_1(x+\overline{p})} + \dot{\overline{\sigma}} \\ &= (c - \dot{\overline{p}})\phi' + f(\phi) - f(\overline{u}) - \lambda_1(\lambda_1 + \dot{\overline{p}})e^{-\lambda_1(x+\overline{p})} + \dot{\overline{\sigma}}\end{aligned}$$

holds from (6.2), by (6.1) and (6.4), we get

$$-\overline{u}_{xx} + \overline{u}_t - f(\overline{u}) \geq f(\phi) - f(\overline{u}) - k_1(e^{-\lambda_1(x+\overline{p})} + \overline{\sigma}).$$

Hence, by (6.1), (6.2), (6.3), (6.6) and (6.7),

$$-\overline{u}_{xx} + \overline{u}_t - f(\overline{u}) \geq 0 \tag{6.9}$$

holds in $x \in \{x \in \mathbf{R} | \, x > 0, |x - \overline{p}| \geq L_1\}$.

On $x > 0$, because

$$-\overline{u}_{xx} + \overline{u}_t - f(\overline{u}) = (c - \dot{\overline{p}})\phi' + f(\phi) - f(\overline{u}) - \lambda_1(\lambda_1 + \dot{\overline{p}})e^{-\lambda_1(x+\overline{p})} + \dot{\overline{\sigma}}$$

holds, by (6.1) and (6.4), we also get

$$-\overline{u}_{xx} + \overline{u}_t - f(\overline{u}) \geq D_1\phi'(e^{-\lambda_1\overline{p}} + \overline{\sigma}) + f(\phi) - f(\overline{u}) - k_1(e^{-\lambda_1(x+\overline{p})} + \overline{\sigma}).$$

Hence, because

$$\begin{aligned}-\overline{u}_{xx} + \overline{u}_t - f(\overline{u}) &\geq D_1\phi'(e^{-\lambda_1\overline{p}} + \overline{\sigma}) \\ &\quad - \left(\sup_{0 \leq u \leq 1+\alpha-\delta_0+\delta_1} f_u(u) + k_1\right)(e^{-\lambda_1(x+\overline{p})} + \overline{\sigma})\end{aligned}$$

holds from (6.1), (6.2) and (6.6), by (6.8),

$$-\overline{u}_{xx} + \overline{u}_t - f(\overline{u}) \geq 0 \tag{6.10}$$

holds in $x \in \{x \in \mathbf{R} | \, x > 0, |x - \overline{p}| \leq L_1\}$.

Also, we have

$$\lim_{x \downarrow +0} \frac{\overline{u}(x,t) - \overline{u}(0,t)}{x} = \phi'(-\overline{p}) - \lambda_1 e^{-\lambda_1\overline{p}} \leq 0$$

by (6.1) and (6.5). Hence, by (6.9) and (6.10), $\overline{u}(x,t)$ is a super-solution of $u_t = u_{xx} + f(u)$ in $\mathbf{R}$. □

Lemma 6.2. *For any $\delta_0 \in (0, 1-\alpha)$, there exist D_2 and $k_2 > 0$ such that for any $\underline{p}(t)$ and $\underline{\sigma}(t)$ satisfying*

$$\begin{cases} \dot{\underline{p}} = c + D_2 \underline{\sigma}, \\ \dot{\underline{\sigma}} = -k_2 \underline{\sigma}, \quad 0 \le \underline{\sigma} \le (1-\alpha) - \delta_0, \end{cases} \tag{6.11}$$

the function $\underline{u}(x,t)$ defined by

$$\underline{u}(x,t) = \phi(|x| - \underline{p}(t)) - \underline{\sigma}(t) \tag{6.12}$$

is a sub-solution of $u_t = u_{xx} + f(u)$ in $\mathbf{R}$.

Proof.

Step 1. In this step, we show the following.

There exist δ_2 and $k_2 > 0$ such that

$$f(a-b) - f(a) \ge k_2 b \tag{6.13}$$

holds for all $a \in [0, \delta_2] \cup [1-\delta_2, 1]$ and $b \in [0, (1-\alpha) - \delta_0]$.

Take a constant $\delta > 0$ such that $\inf_{u \in [0,\delta] \cup [1-\delta, 1]} (-f_u(u)) > 0$ holds. Then, let $\delta_2 \in (0, \min\{\delta, \delta_0/2\}]$ be a constant satisfying $\sup_{u \in [1-\delta_2, 1]} f(u) < \inf_{u \in [\alpha + \delta_0/2, 1-\delta]} f(u)$.

Because of $\inf_{u \in [1-\delta, 1]} (-f_u(u)) > 0$, $f(a-b) - f(a) > 0$ holds when $1 - \delta \le a - b < a \le 1$ holds. Because of $\sup_{u \in [1-\delta_2, 1]} f(u) < \inf_{u \in [\alpha+\delta_0/2, 1-\delta]} f(u)$, $f(a-b) - f(a) > 0$ holds when $1 - \delta_2 \le a \le 1$ and $a - b \le 1 - \delta$ hold. Because of $\inf_{u \in [0,\delta]} (-f_u(u)) > 0$, $f(a-b) - f(a) > 0$ holds when $0 \le a - b < a \le \delta$ holds. Because of $f(a) \le 0$ and $f(a-b) > 0$, $f(a-b) - f(a) > 0$ holds when $0 \le a \le \delta_2$ and $a - b < 0$ hold.

Therefore, $(f(a-b) - f(a))/b > 0$ holds when $b > 0$ holds. Because $-f_u(a) > 0$ also holds, we have

$$k_2 := \inf_{a \in [0,\delta_2] \cup [1-\delta_2, 1],\ b \in (0, (1-\alpha) - \delta_0]} \frac{f(a-b) - f(a)}{b} > 0.$$

Then, (6.13) holds.

Step 2. In this step, we prove Lemma 6.2.

Let $L_2 > 0$ be a constant satisfying

$$\phi(z) \in [0, \delta_2] \cup [1 - \delta_2, 1] \tag{6.14}$$

for $|z| \ge L_2$, and let $D_2 > 0$ be defined by

$$D_2 = \frac{\sup_{-(1-\alpha)+\delta_0 \le u \le 1} f_u(u) + k_2}{\inf_{|z| \le L_2} \phi'(z)}. \tag{6.15}$$

Then, on $x > 0$, because

$$\begin{aligned}\underline{u}_{xx} - \underline{u}_t + f(\underline{u}) &= \phi'' + \dot{\underline{p}}\phi' + f(\underline{u}) + \dot{\underline{\sigma}} \\ &= (\dot{\underline{p}} - c)\phi' + f(\underline{u}) - f(\phi) + \dot{\underline{\sigma}}\end{aligned}$$

holds from (6.12), by (6.11), we get

$$\underline{u}_{xx} - \underline{u}_t + f(\underline{u}) \geq f(\underline{u}) - f(\phi) - k_2\underline{\sigma}.$$

Hence, by (6.11), (6.12), (6.13) and (6.14),

$$\underline{u}_{xx} - \underline{u}_t + f(\underline{u}) \geq 0 \tag{6.16}$$

holds in $x \in \{x \in \mathbf{R} \,|\, x > 0, |x - \underline{p}| \geq L_2\}$.

On $x > 0$, because

$$\underline{u}_{xx} - \underline{u}_t + f(\underline{u}) = (\dot{\underline{p}} - c)\phi' + f(\underline{u}) - f(\phi) + \dot{\underline{\sigma}}$$

holds, by (6.11), we get

$$\underline{u}_{xx} - \underline{u}_t + f(\underline{u}) \geq D_2\phi'\underline{\sigma} + f(\underline{u}) - f(\phi) - k_2\underline{\sigma}.$$

Hence, because we have

$$\underline{u}_{xx} - \underline{u}_t + f(\underline{u}) \geq D_2\phi'\underline{\sigma} - \left(\sup_{-(1-\alpha)+\delta_0 \leq u \leq 1} f_u(u) + k_2 \right) \underline{\sigma}$$

from (6.11) and (6.12), by (6.15),

$$\underline{u}_{xx} - \underline{u}_t + f(\underline{u}) \geq 0 \tag{6.17}$$

holds in $x \in \{x \in \mathbf{R} \,|\, x > 0, |x - \underline{p}| \leq L_2\}$.

Also, we have

$$\lim_{x \downarrow +0} \frac{\underline{u}(x,t) - \underline{u}(0,t)}{x} = \phi'(-\underline{p}) \geq 0.$$

Hence, by (6.16) and (6.17), $\underline{u}(x,t)$ is a sub-solution. □

Definition 10. For $L > 0$, a closed subset Λ_L of $BU(\mathbf{R})$ is defined by

$$\Lambda_L = \{u \in BU(\mathbf{R}) \,|\, \phi(x - L) \leq u(x) \leq \phi(x + L)\}.$$

Under the above definition, the following holds from Lemmas 6.1 and 6.2.

Lemma 6.3. *Let $\bar{\delta}_0 \in (0, \min\{\alpha, 1-\alpha\})$ and $\bar{L}_0 > 0$ be given. Then, there exist $L > 0$ satisfying the following.*

For any $\varepsilon' > 0$, there exists $\bar{l}_0 > 0$ such that for any $l \geq \bar{l}_0$ and $u_0 \in \Xi_{l,\bar{\delta}_0,\bar{L}_0}$, the solution $u \in C([0,+\infty), BU(\mathbf{R}))$ of $u_t = u_{xx} + f(u)$ with $u(0) = u_0$ satisfies

$$\inf_{v_0 \in \Lambda_L} \left\| \chi(x+1)u(x, \bar{l}_0^{1/2}) - v_0(x - (l + c\bar{l}_0^{1/2})) \right\|_{C^0(\mathbf{R})} < \varepsilon'$$

and

$$\sup_{(x,t)\in[-1,1]\times[\bar{l}_0^{1/2}, 2\bar{l}_0^{1/2}]} (|u(x,t)| + |u_x(x,t)|) < \varepsilon',$$

where χ is a cutoff function given in Lemma 2.1.

Proof. We take constants $y_1 > y_2$ such that $\phi(y_1) = 1 - \alpha + \bar{\delta}_0$ and $\phi(y_2) = 1 - \alpha - \bar{\delta}_0$ hold. Let $L > 0$ be defined by $L = \max\{\bar{L}_0 + y_1 + D_1/k_1(\alpha - \bar{\delta}_0) + 1, \bar{L}_0 - y_2 + D_2/k_2(1 - \alpha - \bar{\delta}_0)\}$.

Then,

$$\begin{aligned} \phi(|x| - (l + \bar{L}_0 - y_2)) - (1 - \alpha - \bar{\delta}_0) &\leq u_0(x) \\ &\leq \phi(|x| - (l - \bar{L}_0 - y_1)) + (\alpha - \bar{\delta}_0) \end{aligned} \tag{6.18}$$

holds for all $u_0 \in \Xi_{l,\bar{\delta}_0,\bar{L}_0}$. Let $(\overline{p}, \overline{\sigma})$ and $(\underline{p}, \underline{\sigma}) \in C^1([0, 2\bar{l}_0^{1/2}], \mathbf{R}^2)$ be solutions of

$$\begin{cases} \dot{\overline{p}} = c - D_1(e^{-\lambda_1 \overline{p}} + \overline{\sigma}), & \overline{p}(0) = l - \bar{L}_0 - y_1, \\ \dot{\overline{\sigma}} = -k_1 \overline{\sigma}, & \overline{\sigma}(0) = \alpha - \bar{\delta}_0 \end{cases}$$

and

$$\begin{cases} \dot{\underline{p}} = c + D_2 \underline{\sigma}, & \underline{p}(0) = l + \bar{L}_0 - y_2, \\ \dot{\underline{\sigma}} = -k_2 \underline{\sigma}, & \underline{\sigma}(0) = 1 - \alpha - \bar{\delta}_0, \end{cases}$$

respectively. If $\bar{l}_0 > 0$ is sufficiently large, then

$$\overline{p}(t) \geq l/2,$$
$$\overline{p}(t) \geq l + ct - \left(\bar{L}_0 + y_1 + \frac{D_1}{k_1}(\alpha - \bar{\delta}_0) + 1 \right)$$

and

$$\underline{p}(t) \leq l + ct + \left(\bar{L}_0 - y_2 + \frac{D_2}{k_2}(1 - \alpha - \bar{\delta}_0) \right)$$

hold on $t \in [0, 2\bar{l}_0^{1/2}]$. Hence, by Lemmas 6.1, 6.2 and (6.18), when $\bar{l}_0 > 0$ is sufficiently large,

$$\phi(|x| - (l + ct + L)) - (1 - \alpha - \bar{\delta}_0)e^{-k_2 t} \leq u(x,t) \\ \leq \phi(|x| - (l + ct - L)) + e^{-\frac{\lambda_1}{2}l} + (\alpha - \bar{\delta}_0)e^{-k_1 t}$$

holds on $t \in [0, 2\bar{l}_0^{1/2}]$ for all $u_0 \in \Xi_{l,\bar{\delta}_0,\bar{L}_0}$. Therefore, for any $\varepsilon > 0$, if $\bar{l}_0 > 0$ is sufficiently large, then

$$\phi(|x| - (l + c\bar{l}_0^{1/2} + L)) - \varepsilon \leq u(x, \bar{l}_0^{1/2}) \\ \leq \phi(|x| - (l + c\bar{l}_0^{1/2} - L)) + \varepsilon$$

and

$$\sup_{(x,t)\in[-2,2]\times[\bar{l}_0^{1/2}-1, 2\bar{l}_0^{1/2}]} |u(x,t)| < \varepsilon$$

hold for all $u_0 \in \Xi_{l,\bar{\delta}_0,\bar{L}_0}$. Hence, by (1.4) of [6], there exists $\bar{l}_0 > 0$ such that

$$\inf_{v_0 \in \Lambda_L} \left\| \chi(x+1)u(x, \bar{l}_0^{1/2}) - v_0(x - (l + c\bar{l}_0^{1/2})) \right\|_{C^0(\mathbf{R})} < \varepsilon'$$

and

$$\sup_{(x,t)\in[-1,1]\times[\bar{l}_0^{1/2}, 2\bar{l}_0^{1/2}]} (|u(x,t)| + |u_x(x,t)|) < \varepsilon'$$

hold for all $u_0 \in \Xi_{l,\bar{\delta}_0,\bar{L}_0}$. □

We obtain the following by Theorems A and B.

Lemma 6.4. *For any L and $\varepsilon > 0$, there exist T' and $\delta > 0$ such that for any $g \in C([0,T'], BU(\mathbf{R}))$ with $\sup_{t\in[0,T']} \|g(t)\|_{C^0} < \delta$ and solution $v \in C([0,T'], BU(\mathbf{R}))$ of $v_t = v_{xx} + f(v) + g(t)$ with $\inf_{u_0\in\Lambda_L} \|v(0) - u_0\|_{C^0} < \delta$ and $0 \leq v(x,t) \leq 1$ in $\mathbf{R} \times [0,T']$,*

$$\inf_{x_0\in[-L,L]} \|v(x,T') - \phi(x - x_0 - cT')\|_{C^0(\mathbf{R})} < \varepsilon/2$$

holds.

Proof.
Step 1. In this step, we show the following.

For any $L > 0$ and $\varepsilon > 0$, there exists $T' > 0$ satisfying the following. For any $u_0 \in \Lambda_L$, the solution $u \in C([0, +\infty), BU(\mathbf{R}))$ of $u_t = u_{xx} + f(u)$ with $u(0) = u_0$ satisfies

$$\inf_{x_0 \in [-L,L]} \|u(x, T') - \phi(x - x_0 - cT')\|_{C^0(\mathbf{R})} < \varepsilon/4.$$

Let $\delta > 0$ be sufficiently small. By Theorem B, for any $u_0 \in \Lambda_{L-c}$, there exists $\bar{T}_{u_0} > 0$ such that

$$\inf_{x_0 \in \mathbf{R}} \|u(x, \bar{T}_{u_0}) - \phi(x - x_0)\|_{C^0(\mathbf{R})} < \delta/2$$

holds. Hence, for any $v_0 \in \Lambda_{L-c}$, there exists a open neighborhood W_{v_0} of v_0 in $BU(\mathbf{R})$ such that

$$\inf_{x_0 \in \mathbf{R}} \|u(x, \bar{T}_{v_0}) - \phi(x - x_0)\|_{C^0(\mathbf{R})} < \delta$$

holds for all $u_0 \in W_{v_0}$. Also, we let Λ_L^1 denote the set

$$\{u(x, 1) \in BU(\mathbf{R}) |\, u \text{ is a solution of } u_t = u_{xx} + f(u) \text{ with } u_0(x) \in \Lambda_L\}.$$

Then, because $\Lambda_L^1 \subset \Lambda_{L-c}$ and $\sup_{v \in \Lambda_L^1} \|v\|_{C^1} < \infty$ hold, Λ_L^1 is relatively compact in $BU(\mathbf{R})$. There exist $v_1, v_2, \ldots$ and $v_n \in \Lambda_{L-c}$ such that $\Lambda_L^1 \subset \bigcup_{i=1}^n W_{v_i}$ holds. Therefore, for any $u_0 \in \Lambda_L$, there exists $i \in \{1, 2, \ldots, n\}$ such that

$$\inf_{x_0 \in \mathbf{R}} \|u(x, 1 + \bar{T}_{v_i}) - \phi(x - x_0)\|_{C^0(\mathbf{R})} < \delta$$

holds. Hence, because $\delta > 0$ is sufficiently small, by Theorem A (2) with $T' = 1 + \max_{i=1,\ldots,n} \bar{T}_{v_i} > 0$,

$$\inf_{x_0 \in \mathbf{R}} \|u(x, T') - \phi(x - x_0)\|_{C^0(\mathbf{R})} < \varepsilon/4$$

holds for all $u_0 \in \Lambda_L$. Because $\phi(x - L - cT') \leq u(x, T') \leq \phi(x + L - cT')$ also holds for all $u_0 \in \Lambda_L$, we have

$$\inf_{x_0 \in [-L,L]} \|u(x, T') - \phi(x - x_0 - cT')\|_{C^0(\mathbf{R})} < \varepsilon/4.$$

Step 2. In this step, we prove Lemma 6.4.

Let $T' > 0$ be a constant such that Step 1 holds, and fixed. Then, let $\delta > 0$ be sufficiently small.

Let u be a solution of $u_t = u_{xx} + f(u)$ with $u(0) = u_0 \in \Lambda_L$ satisfying $\|v(0) - u_0\|_{C^0} < \delta$. Then, because $(v-u)_t - (v-u)_{xx} = f(v) - f(u) + g(t)$ holds,

$$\|(v-u)(t)\|_{C^0} \le (1+T')\delta + \left(\sup_{0\le u\le 1} |f_u(u)|\right) \int_0^t \|(v-u)(s)\|_{C^0} ds$$

holds on $t \in [0, T']$. Because $\delta > 0$ is sufficiently small, by Gronwall's inequality, we obtain $\|(v-u)(T')\|_{C^0} < \varepsilon/4$. Hence, by Step 1,

$$\inf_{x_0 \in [-L,L]} \|v(x,T') - \phi(x - x_0 - cT')\|_{C^0(\mathbf{R})} < \varepsilon/2$$

holds. □

Now, we prove Proposition 1.3 from Lemmas 6.3 and 6.4.

Proof of Proposition 1.3. Let $L > 0$ be a constant such that Lemma 6.3 holds for $\bar{\delta}_0$ and $\bar{L}_0$, and let T' and $\delta > 0$ be constants such that Lemma 6.4 holds for L and ε. Then, let $\bar{l}_0 > 0$ be sufficiently large.

Fix any $l \ge \bar{l}_0$ and $u_0 \in \Xi_{l,\bar{\delta}_0,\bar{L}_0}$. We define $v \in C([\bar{l}_0^{1/2}, \bar{l}_0^{1/2} + T'], BU(\mathbf{R}))$ and $g \in C([\bar{l}_0^{1/2}, \bar{l}_0^{1/2} + T'], BU(\mathbf{R}))$ by

$$v(x,t) = \chi(x+1)u(x,t)$$

and

$$\begin{aligned} g(x,t) = &\chi(x+1)u_{xx}(x,t) - (\chi(x+1)u(x,t))_{xx} \\ &+ \chi(x+1)f(u(x,t)) - f(\chi(x+1)u(x,t)), \end{aligned}$$

respectively. Then, $v_t = v_{xx} + f(v) + g(t)$ holds on $t \in [\bar{l}_0^{1/2}, \bar{l}_0^{1/2} + T']$. Further, by Lemma 6.3, we have $\inf_{v_0 \in \Lambda_L} \left\| v(x + (l + c\bar{l}_0^{1/2}), \bar{l}_0^{1/2}) - v_0(x) \right\|_{C^0(\mathbf{R})} < \delta$ and $\sup_{t \in [\bar{l}_0^{1/2}, \bar{l}_0^{1/2} + T']} \|g(x,t)\|_{C^0(\mathbf{R})} < \delta$. Therefore, by Lemma 6.4 with $T = \bar{l}_0^{1/2} + T'$, there exists $x_1 \in [l - L, l + L]$ such that

$$\|\chi(x+1)u(x,T) - \phi(x - x_1 - cT)\|_{C^0(\mathbf{R})} < \varepsilon/2$$

holds. Also, there exists $x_2 \in [l - L, l + L]$ such that

$$\|\chi(x+1)u(-x,T) - \phi(x - x_2 - cT)\|_{C^0(\mathbf{R})} < \varepsilon/2$$

holds. Because $\bar{l}_0 > 0$ is sufficiently large, $x_1 + cT > 0$ and $x_2 + cT > 0$ are also sufficiently large. Therefore,

$$\|u(x,T) - (\phi(x - x_1 - cT) + \phi(-x - x_2 - cT))\|_{C^0(\mathbf{R})} < \varepsilon$$

holds. □

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References

[1] Carr, J. and Pego, R. L., Metastable patterns in solutions of $u_t = \varepsilon^2 u_{xx} - f(u)$, *Comm. Pure. Appl. Math.*, **42** (1989), 523-576.

[2] Carr, J. and Pego, R., Invariant manifolds for metastable patterns in $u_t = \varepsilon^2 u_{xx} - f(u)$, *Proc. Roy. Soc. Edinburgh A*, **116** (1990), 133-160.

[3] Eckmann, J.-P. and Rougemont, J., Coarsening by Ginzburg-Landau dynamics, *Comm. Math. Phys.*, **199** (1998), 441-470.

[4] Fife, P. C., *Mathematical Aspects of Reacting and Diffusing Systems*, Springer-Verlag, 1979.

[5] ———, Long time behavior of solutions of bistable nonlinear diffusion equations, *Arch. Ration. Mech. Anal.*, **70** (1979), 31-46.

[6] Fife, P. C. and McLeod, J. B., The approach of solutions of nonlinear diffusion equations to travelling front solutions, *Arch. Ration. Mech. Anal.*, **65** (1977), 335-361.

[7] Fusco, G., A geometric approach to the dynamics of $u_t = \varepsilon^2 u_{xx} + f(u)$ for small ε, In: K. Kirchgässner, (ed.), *Problems Involving Change of Type*, Springer-Verlag, pp. 53-73, 1990.

[8] Fusco, G. and Hale, J. K., Slow-motion manifolds, dormant instability, and singular perturbations, *J. Dynam. Diff. Equations*, **1** (1989), 75-94.

[9] Fusco, G., Hale, J. K. and Xun, J., Travelling waves as limits of solutions on bounded domains, *SIAM J. Math. Anal.*, **27** (1996), 1544-1558.

[10] Henry, D., *Geometric Theory of Semilinear Parabolic Equations*, Springer-Verlag, 1981.

[11] Morita, Y. and Mimoto, Y., Collision and collapse of layers in a 1D scalar reaction-diffusion equation, *Physica D*, **140** (2000), 151-170.

[12] Rougemont, J., Dynamics of kinks in the Ginzburg-Landau equation: approach to a metastable shape and collapse of embedded pairs of kinks, *Nonlinearity*, **12** (1999), 539-554.