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Group Actions on Spaces of Rational Functions

By

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Abstract

Let Hol_d be the space consisting of all holomorphic maps $f : S^2 \to S^2$ of degree d. The group $\mathrm{Hol}_1 = \mathrm{PSL}_2(\mathbb{C})$ acts on Hol_d freely by the post-composition and we shall study the orbit space $X_d = \mathrm{Hol}_1 \backslash \mathrm{Hol}_d$. As an application we shall determine the homotopy types of the universal covering spaces of Hol_d and X_d explicitly.

§1. Introduction

For each integer $d \geq 0$, let Hol_d denote the space consisting of all holomorphic maps f of degree d from the Riemann sphere $S^2 = \mathbb{C} \cup \infty$ to itself. The corresponding space of continuous maps $f : S^2 \to S^2$ is denoted by $\mathrm{Map}_d(S^2, S^2)$. Similarly, we denote by $\mathrm{Hol}_d^* \subset \mathrm{Hol}_d$ the subspace consisting of all maps $f \in \mathrm{Hol}_d$ which preserve the base-points, and the corresponding space of continuous maps is also denoted by $\mathrm{Map}_d^*(S^2, S^2) = \Omega_d^2 S^2$. The spaces Hol_d and Hol_d^* are of interest both from a classical and modern point of view ([1], [3], [5]), and they can be easily identified with the following spaces of rational functions.

$$\begin{cases} \mathrm{Hol}_d = \{p(z)/q(z) : \text{(i), (ii) are satisfied}\} \\ \mathrm{Hol}_d^* = \{p(z)/q(z) : \text{(i), (iii) are satisfied}\}, \end{cases}$$

where the conditions (i)–(iii) are given by the following:

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(i) $p(z), q(z) \in \mathbb{C}[z]$ are mutually coprime polynomials.

(ii) $\max\{\deg(p(z)), \deg(q(z))\} = d$.

(iii) $p(z), q(z) \in \mathbb{C}[z]$ are monic polynomials of degree d.

In particular, if $d = 1$, Hol_1 is the group of fractional linear transformations $\mathrm{PSL}_2(\mathbb{C})$ and Hol_1^* may be identified with the affine transformation group of $\mathbb{C}$. It is an elementary and fundamental fact that Hol_d and Hol_d^* are connected spaces. More generally, the following result is known.

Theorem 1.1 (S. Epshtein, G. Segal; [6], [17]). *Let $d \geq 1$ be an integer.*

(i) $\pi_1(\mathrm{Hol}_d^*) = \mathbb{Z}$ *and* $\pi_1(\mathrm{Hol}_d) = \mathbb{Z}/2d$.

(ii) *If $i_d : \mathrm{Hol}_d \to \mathrm{Map}_d(S^2, S^2)$ and $\tilde{i}_d : \mathrm{Hol}_d^* \to \Omega_d^2 S^2$ are inclusion maps, then i_d and $\tilde{i}_d$ are homotopy equivalences up to dimension d, where the map $f : X \to Y$ is said to be a homotopy equivalence up to dimension N if the induced homomorphism $f_* : \pi_k(X) \to \pi_k(Y)$ is bijective when $k < N$ and surjective when $k = N$.*

Since the topology of the space Hol_d^* is now well studied ([3], [7], [9], [10], [17]), we would like to study the topology of Hol_d. However, because Hol_d is non-simply connected, we shall mainly consider the homotopy types of the universal covering of Hol_d. We denote by $\widetilde{\mathrm{Hol}}_d$ and $\widetilde{\mathrm{Hol}}_d^*$ the universal coverings of Hol_d and Hol_d^*, respectively. If $d = 1$ or 2, the following result is known.

Theorem 1.2 (M. Guest et al.; [7]).

(i) *If $d = 1$, $\widetilde{\mathrm{Hol}_1^*}$ is contractible and there is a homotopy equivalence $\widetilde{\mathrm{Hol}}_1 \simeq S^3$.*

(ii) *If $d = 2$, there are homotopy equivalences*

$$\begin{cases} \widetilde{\mathrm{Hol}_2^*} \simeq S^2 \\ \widetilde{\mathrm{Hol}}_2 \simeq S^2 \times S^3. \end{cases}$$

We identify $\mathrm{Hol}_1 = \mathrm{PSL}_2(\mathbb{C})$ and define the left action of Hol_1 on Hol_d by post-composition:

$$A \cdot f(z) = A(f(z)) \quad \text{for} \quad (A, f(z)) \in \mathrm{Hol}_1 \times \mathrm{Hol}_d.$$

We also define the left action of Hol_1^* on Hol_d^* by post-composition in a similar way. Let X_d denote the orbit space $X_d = \mathrm{Hol}_1 \backslash \mathrm{Hol}_d$ and let $\mathcal{F}_{d,0}$ be the space consisting of all non-singular $(d \times d)$-Toeplitz matrices. We recall the following result.

Theorem 1.3 (R. J. Milgram; [15]). *If $d \geq 1$ is an integer, there is a homeomorphism $X_d \cong \mathcal{F}_{d,0}$.*

Finite Toeplitz matrices appear in many areas of mathematics ranging from applied mathematics and mathematical physics, through algebraic geometry (e.g. [2], [13]). So it seems interesting to study the topology of the space $\mathcal{F}_{d,0}$ and in this paper we shall study the topology of the orbit space X_d. Since X_d is not simply connected, we shall also study the universal covering space $\tilde{X}_d$ of X_d. The main purpose of this paper is to prove the following two results:

Theorem 1.4. *Let $d \geq 1$ be an integer and $\tilde{X}_d$ denote the universal covering of X_d. Then there is a homotopy equivalence $\widetilde{\mathrm{Hol}}_d \simeq S^3 \times \tilde{X}_d$.*

Theorem 1.5. *Let $d \geq 1$ be an integer. Then there is a homotopy equivalence $\tilde{X}_d \simeq \widetilde{\mathrm{Hol}}_d^*$.*

It follows from the above two results that we also have:

Corollary 1.1 ([20]). *Let $d \geq 1$ be an integer.*

(i) *There is a homotopy equivalence $\widetilde{\mathrm{Hol}}_d \simeq S^3 \times \widetilde{\mathrm{Hol}}_d^*$.*

(ii) *There is an isomorphism $\pi_k(\mathrm{Hol}_d) \cong \pi_k(\mathrm{Hol}_d^*) \oplus \pi_k(S^3)$ for any $k \geq 2$.*

(iii) *In particular, if $d > k \geq 2$, there is an isomorphism $\pi_k(\mathrm{Hol}_d) \cong \pi_{k+2}(S^2) \oplus \pi_k(S^3)$.*

Remark. The above corollary was first obtained by the analysis of the evaluation fibration in [20]. On the other hand, in this paper, we shall study the Hol_1-action on Hol_d and prove the above two theorems. Then Corollary 1.1 easily follows from them and so it seems easier and natural to study the topology of $\widetilde{\mathrm{Hol}}_d$ from the point of view of group actions.

The plan of this paper is as follows. In Section 2, we shall study the Hol_1 action on Hol_d and give the proof of Theorem 1.4. In Section 3, we shall prove Theorem 1.5.

§2. Group Actions and Their Orbit Spaces

First recall the following result.

Lemma 2.1 ([7]). *Let $d \geq 1$ be an integer.*

(i) *The group* Hol_1 *acts on* Hol_d *freely by post-composition. Similarly, the group* Hol_1^* *also acts on* Hol_d^* *freely by post-composition.*

(ii) *The natural inclusion* $j_d : \mathrm{Hol}_d^* \to \mathrm{Hol}_d$ *induces a homeomorphism* $p_d : \mathrm{Hol}_1^* \backslash \mathrm{Hol}_d^* \xrightarrow{\cong} \mathrm{Hol}_1 \backslash \mathrm{Hol}_d = X_d$ *such that the diagram*

$$\begin{array}{llcccc} (*)_d^* & & \mathrm{Hol}_1^* & \xrightarrow{s_1'} & \mathrm{Hol}_d^* & \xrightarrow{q_d'} & \mathrm{Hol}_1^* \backslash \mathrm{Hol}_d^* \\ & & \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_d} & & \downarrow{\scriptstyle p_d}\ \cong \\ (*)_d & & \mathrm{Hol}_1 & \xrightarrow{s_1} & \mathrm{Hol}_d & \xrightarrow{q_d} & \mathrm{Hol}_1 \backslash \mathrm{Hol}_d = X_d \end{array}$$

is commutative, where horizontal sequences $(*)_d^*$ *and* $(*)_d$ *are principal fibration sequences.*

Proof. This follows from [[7]; (3.1), (3.2)]. □

Lemma 2.2. *For each integer $d \geq 1$, there is a commutative diagram*

$$\begin{array}{ccccc} \pi_1(\mathrm{Hol}_1) & \xrightarrow{(s_1)_*} & \pi_1(\mathrm{Hol}_d) & \xrightarrow{(q_d)_*} & \pi_1(X_d) \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\ \mathbb{Z}/2 & \xrightarrow{i} & \mathbb{Z}/2d & \xrightarrow{\rho} & \mathbb{Z}/d \end{array}$$

where three vertical maps are isomorphisms, and $\rho : \mathbb{Z}/2d \to \mathbb{Z}/d$ *and* $i : \mathbb{Z}/2 \to \mathbb{Z}/2d$ *denote the natural epimorphism and the natural inclusion homomorphism, respectively.*

Proof. We note ([12], (3.4)) that $\pi_1(\mathrm{Hol}_1^* \backslash \mathrm{Hol}_d^*) = \mathbb{Z}/d$. Furthermore, since $p_d : \mathrm{Hol}_1^* \backslash \mathrm{Hol}_d^* \xrightarrow{\cong} X_d$ is a homeomorphism, $\pi_1(X_d) = \mathbb{Z}/d$. We also remark that $\pi_1(\mathrm{Hol}_k) = \mathbb{Z}/2k$ by Theorem 1.1. Hence, if we consider the homotopy exact sequence of the fibration $(*)_d$, the assertion easily follows. □

Proposition 2.1. *For each integer $d \geq 1$, there is a fibration sequence (up to homotopy),*

$$\widetilde{\mathrm{Hol}}_1 \xrightarrow{\tilde{s_1}} \widetilde{\mathrm{Hol}}_d \xrightarrow{\tilde{q_d}} \tilde{X}_d. \tag{2.1.1}$$

Proof. We remark that $\pi_1(\mathrm{Hol}_d) = \mathbb{Z}/2d$ and $\pi_1(X_d) = \mathbb{Z}/d$. Let $\tilde{\iota}_d : \mathrm{Hol}_d \to K(\mathbb{Z}/2d, 1) = B(\mathbb{Z}/2d)$ and $\iota'_d : X_d \to K(\mathbb{Z}/d, 1) = B(\mathbb{Z}/d)$ denote the maps which represent the generators of $[\mathrm{Hol}_d, K(\mathbb{Z}/2d, 1)] \cong H^1(\mathrm{Hol}_d, \mathbb{Z}/2d) = \mathbb{Z}/2d$ and $[X_d, K(\mathbb{Z}/d, 1)] \cong H^1(X_d, \mathbb{Z}/d) = \mathbb{Z}/d$, respectively.

Consider the universal coverings $\pi_d : \widetilde{\mathrm{Hol}}_d \to \mathrm{Hol}_d$ and $\pi'_d : \tilde{X}_d \to X_d$. Then it follows from Lemma 2.2 that there is a homotopy commutative diagram

$$\begin{array}{ccccc}
 & & \widetilde{\mathrm{Hol}}_d & \xrightarrow{\tilde{q}_d} & \tilde{X}_d \\
 & & {\scriptstyle \pi_d}\downarrow & & {\scriptstyle \pi'_d}\downarrow \\
\mathrm{Hol}_1 & \xrightarrow{s_1} & \mathrm{Hol}_d & \xrightarrow{q_d} & X_d \\
{\scriptstyle \tilde{\iota}_1}\downarrow & & {\scriptstyle \tilde{\iota}_d}\downarrow & & {\scriptstyle \iota'_d}\downarrow \\
B(\mathbb{Z}/2) & \xrightarrow{Bi} & B(\mathbb{Z}/2d) & \xrightarrow{B\rho} & B(\mathbb{Z}/d)
\end{array}$$

where all vertical and horizontal sequences are fibration sequences.

Because $\mathrm{Hol}_1 \simeq SO_3$ and $\tilde{\iota}_1$ induces an isomorphism on π_1, the homotopy fibre of $\tilde{\iota}_1$ is S^3 (up to homotopy). Hence it follows from [[4], (2.1)] that we obtain the homotopy commutative diagram

$$(2.1.2) \qquad \begin{array}{ccccc}
S^3 & \xrightarrow{\tilde{s}_1} & \widetilde{\mathrm{Hol}}_d & \xrightarrow{\tilde{q}_d} & \tilde{X}_d \\
{\scriptstyle q}\downarrow & & {\scriptstyle \pi_d}\downarrow & & {\scriptstyle \pi'_d}\downarrow \\
\mathrm{Hol}_1 & \xrightarrow{s_1} & \mathrm{Hol}_d & \xrightarrow{q_d} & X_d \\
{\scriptstyle \tilde{\iota}_1}\downarrow & & {\scriptstyle \tilde{\iota}_d}\downarrow & & {\scriptstyle \iota'_d}\downarrow \\
B(\mathbb{Z}/2) & \xrightarrow{Bi} & B(\mathbb{Z}/2d) & \xrightarrow{B\rho} & B(\mathbb{Z}/d)
\end{array}$$

where all vertical and horizontal sequences are fibration sequences and $q : S^3 \to SO_3 \simeq \mathrm{Hol}_1$ denotes the universal covering projection. So we obtain the desired fibration sequence (2.1.1). □

Definition 2.1. Let $ev_d : \mathrm{Hol}_d \to S^2$ denote the evaluation map defined by $ev_d(f) = f(\infty)$ for $f \in \mathrm{Hol}_d$, where we identify $S^2 = \mathbb{C} \cup \infty$.

There is a fibration sequence $\mathrm{Hol}^*_d \xrightarrow{j_d} \mathrm{Hol}_d \xrightarrow{ev_d} S^2$.

Lemma 2.3. *The following two diagrams are homotopy commutative*:

$$\text{(i)} \quad \begin{array}{ccc}
\mathrm{Hol}_1 & \xrightarrow{s_1} & \mathrm{Hol}_d \\
{\scriptstyle ev_1}\downarrow & & {\scriptstyle ev_d}\downarrow \\
S^2 & \xrightarrow{=} & S^2
\end{array} \qquad \text{(ii)} \quad \begin{array}{ccc}
\mathrm{Hol}_d & \xrightarrow{ev_d} & S^2 \\
{\scriptstyle \tilde{\iota}_d}\downarrow & & {\scriptstyle \iota}\downarrow \\
B(\mathbb{Z}/2d) & \xrightarrow{B\rho'} & BS^1 = K(\mathbb{Z}, 2)
\end{array}$$

where $\rho' : \mathbb{Z}/2d \to S^1$ *and* $\iota : S^2 \to BS^1 = K(\mathbb{Z},2)$ *denote the natural inclusion homomorphism and the map which represents the generator of* $\pi_2(K(\mathbb{Z},2)) = \mathbb{Z}$, *respectively.*

Proof. (i) Without loss of generalities, we may suupose that the map $s_1 : \mathrm{Hol}_1 \to \mathrm{Hol}_d$ is given by $s_1((az+b)/(cz+d)) = (az^d+b)/(cz^d+d)$. Then it is easy to see that the diagram (i) is commutative.

(ii) It suffices to show that two induced homomorphisms

$$\begin{cases} \alpha_1 : \mathbb{Z} = H^2(K(\mathbb{Z},2),\mathbb{Z}) \xrightarrow{\iota^*} H^2(S^2,\mathbb{Z}) \xrightarrow{ev_d^*} H^2(\mathrm{Hol}_d,\mathbb{Z}) \\ \alpha_2 : \mathbb{Z} = H^2(K(\mathbb{Z},2),\mathbb{Z}) \xrightarrow{(B\rho')^*} H^2(B(\mathbb{Z}/2d),\mathbb{Z}) \xrightarrow{\tilde{\iota}_d^*} H^2(\mathrm{Hol}_d,\mathbb{Z}) \end{cases}$$

coincide. Remark that ι^* is an isomorphism and that $(B\rho')^*$ can be identified with the natural projection homomorphism $\pi' : \mathbb{Z} \to \mathbb{Z}/2d$. Next, consider the Serre spectral sequence of the evaluation fibration: $\mathrm{Hol}_d^* \xrightarrow{j_d} \mathrm{Hol}_d \xrightarrow{ev_d} S^2$,

$$E_2^{p,q} = H^p(S^2,\mathbb{Z}) \otimes H^q(\mathrm{Hol}_d^*,\mathbb{Z}) \Rightarrow H^{p+q}(\mathrm{Hol}_d,\mathbb{Z}).$$

Since $H^1(\mathrm{Hol}_1^*,\mathbb{Z}) = \mathbb{Z}$, $H^2(\mathrm{Hol}_d^*,\mathbb{Z})$ is a torsion group and $\pi_1(\mathrm{Hol}_d) = \mathbb{Z}/2d$, the differential $d_2 : \mathbb{Z} = E_2^{0,1} \to E_2^{2,0} = \mathbb{Z}$ is identified with the $2d$-times multiplication. Hence $E_3^{2,0} = E_\infty^{2,0} = H^2(\mathrm{Hol}_d,\mathbb{Z}) = \mathbb{Z}/2d$ and we obtain that $ev_d^* = \pi' : \mathbb{Z} = H^2(S^2,\mathbb{Z}) \to H^2(\mathrm{Hol}_d,\mathbb{Z}) = \mathbb{Z}/2d$. Therefore, $\alpha_1 = \pi' : \mathbb{Z} \to \mathbb{Z}/2d$. Similarly, if we compute the Serre spectral sequence of the fibration sequence $\widetilde{\mathrm{Hol}}_d \xrightarrow{\pi_d} \mathrm{Hol}_d \xrightarrow{\tilde{\iota}_d} B(\mathbb{Z}/2d)$, we can easily see that $\tilde{\iota}_d^*$ is an isomorphsim. Hence α_2 can be also identified with π'. So $\alpha_1 = \alpha_2$ and the diagram (ii) is homotopy commutative. □

Theorem 2.1 (Theorem 1.4). *For each integer* $d \geq 1$, *there is a homotopy equivalence* $\phi_d : \widetilde{\mathrm{Hol}}_d \xrightarrow{\simeq} S^3 \times \tilde{X}_d$.

Proof. We remark that there is a fibration sequence $S^3 \xrightarrow{\eta_2} S^2 \xrightarrow{\iota} BS^1$, where $\eta_2 \in \pi_3(S^2) = \mathbb{Z} \cdot \eta_2$ denotes the Hopf map. Then it follows from Lemma 2.3 that there is a homotopy commutative diagram

$$\begin{array}{ccccc} \widetilde{\mathrm{Hol}}_d & \xrightarrow{\pi_d} & \mathrm{Hol}_d & \xrightarrow{\tilde{\iota}_d} & B(\mathbb{Z}/2d) \\ & & {\scriptstyle ev_d}\downarrow & & {\scriptstyle B\rho'}\downarrow \\ S^3 & \xrightarrow{\eta_2} & S^2 & \xrightarrow{\iota} & BS^1 \end{array}$$

where two horizontal sequences are fibration sequences.

Since $\iota \circ ev_d \circ \pi_d$ is null-homotopic, there is a map $\theta_d : \widetilde{\mathrm{Hol}}_d \to S^3$ such that $\eta_2 \circ \theta_d = ev_d \circ \pi_d$ (up to homotopy).

Then by using the diagram (2.1.2) and Lemma 2.3, we have

$$\eta_2 \circ \theta_d \circ \tilde{s}_1 = ev_d \circ \pi_d \circ \tilde{s}_1 = ev_d \circ s_1 \circ q = ev_1 \circ q.$$

On the other hand, because $\mathrm{Hol}_1^* \simeq S^1$ and $q : S^3 \to SO_3 \simeq \mathrm{Hol}_1$ is a universal covering projection, $(ev_1)_* : \mathbb{Z} \cdot q = \pi_3(\mathrm{Hol}_1) \xrightarrow{\cong} \pi_3(S^2) = \mathbb{Z} \cdot \eta_2$ is an isomorphism. Hence, $ev_1 \circ q = \pm\eta_2$ and we also obtain

$$\eta_2 \circ \theta_d \circ \tilde{s}_1 = \pm\eta_2 = \pm\eta_2 \circ \iota_3, \tag{2.1.3}$$

where we denote by $\iota_n \in \pi_n(S^n) = \mathbb{Z} \cdot \iota_n$ the homotopy class of identity map of S^n.

Now we recall the isomorphism $(\eta_2)_* : \pi_3(S^3) = \mathbb{Z} \cdot \iota_3 \xrightarrow{\cong} \pi_3(S^2) = \mathbb{Z} \cdot \eta_2$. Then it follows from (2.1.3) that we have the equality

$$\theta_d \circ \tilde{s}_1 = \pm\iota_3. \tag{2.1.4}$$

Consider the fibration sequence (2.1.1): $\widetilde{\mathrm{Hol}}_1 \xrightarrow{\tilde{s}_1} \widetilde{\mathrm{Hol}}_d \xrightarrow{\tilde{q}_d} \tilde{X}_d$.

Define the map $\phi_d : \widetilde{\mathrm{Hol}}_d \to S^3 \times \tilde{X}_d$ by $\phi_d = (\theta_d, \tilde{q}_d)$. Then using (2.1.4) and the homotopy exact sequence induced from (2.1.1), we can easily see that $(\phi_d)_* : \pi_k(\widetilde{\mathrm{Hol}}_d) \xrightarrow{\cong} \pi_k(S^3 \times \tilde{X}_d)$ is an isomorphism for any $k \geq 0$. Hence, ϕ_d is a homotopy equivalence. □

§3. The Universal Covering Space $\tilde{X}_d$

Since $p_d : \mathrm{Hol}_1^* \backslash \mathrm{Hol}_d^* \xrightarrow{\cong} \mathrm{Hol}_1 \backslash \mathrm{Hol}_d = X_d$ is a homeomorphism, without loss of generalities we may assume $X_d = \mathrm{Hol}_1^* \backslash \mathrm{Hol}_d^*$ and that there is a fibration sequence

$$\mathrm{Hol}_1^* \xrightarrow{s_1'} \mathrm{Hol}_d^* \xrightarrow{q_d'} X_d.$$

Lemma 3.1. *For each integer $d \geq 1$, there is a commutative diagram*

$$\begin{array}{ccccc}
\pi_1(\mathrm{Hol}_1^*) & \xrightarrow{(s_1')_*} & \pi_1(\mathrm{Hol}_d^*) & \xrightarrow{(q_d')_*} & \pi_1(X_d) \\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
\mathbb{Z} & \xrightarrow{\mu_d} & \mathbb{Z} & \xrightarrow{\pi} & \mathbb{Z}/d
\end{array}$$

where $\pi : \mathbb{Z} \to \mathbb{Z}/d$ denotes the natural epimorphism and $\mu_d : \mathbb{Z} \to \mathbb{Z}$ denotes the multiplication map given by $\mu_d(x) = dx$ for $x \in \mathbb{Z}$.

Proof. Since the proof is analogous to that of Lemma 2.2, we omit the detail. □

Theorem 3.1 (Theorem 1.5). *For each integer $d \geq 1$, there is a homotopy equivalence $\tilde{q}'_d : \widetilde{\mathrm{Hol}}^*_d \xrightarrow{\simeq} \tilde{X}_d$.*

Proof. It follows from Lemma 3.1, [[4], (2.1)] and the analogous method given in the proof of Proposition 2.1 that we can easily obtain the homotopy commutative diagram

$$\begin{array}{ccccc}
F & \longrightarrow & \widetilde{\mathrm{Hol}}^*_d & \xrightarrow{\tilde{q}'_d} & \tilde{X}_d \\
\downarrow & & \pi^*_d \downarrow & & \pi''_d \downarrow \\
\mathrm{Hol}^*_1 & \xrightarrow{s'_1} & \mathrm{Hol}^*_d & \xrightarrow{q'_d} & X_d \\
\tilde{\iota}'_1 \downarrow \simeq & & \tilde{\iota}'_d \downarrow & & \iota'_d \downarrow \\
B\mathbb{Z} & \xrightarrow{B\mu_d} & B\mathbb{Z} & \xrightarrow{B\pi} & B(\mathbb{Z}/d)
\end{array}$$

where all horizontal and vertical sequences are fibrations.

Since $\tilde{\iota}'_1$ is a homotopy equivalence ([7]), the homotopy fibre F of $\tilde{q}'_d$ is contractible. Hence the map $\tilde{q}'_d : \widetilde{\mathrm{Hol}}^*_d \xrightarrow{\simeq} \tilde{X}_d$ is a homotopy equivalence. □

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