Publ. RIMS, Kyoto Univ.
39 (2002), 183–203

Hukuhara's Topological Degree for non Compact Valued Multifunctions

By

Francesco S. De Blasi* and Pando Gr. Georgiev**

Abstract

We present a direct construction of a topological degree for multivalued vector fields $I-F$ in a Banach space, where F takes closed, bounded, convex (or non convex) values and the set-valued range of F is precompact in the Pompeiu-Hausdorff metric. Some useful properties of our topological degree are established. Applications to fixed point theory including a Borsuk's type result are considered.

§1. Introduction

Let $\mathbb{E}$ be a real Banach space and let F be a multifunction defined on a non empty open bounded subset of $\mathbb{E}$, whose values are non empty subsets of $\mathbb{E}$. If I stands for the identity mapping in $\mathbb{E}$, the multifunction $I-F$ will be called a *multivalued vector field*.

A topological degree theory for multivalued vector fields $I-F$, when F is a Pompeiu-Hausdorff upper semicontinuous (h-u.s.c.) multifunction with non empty compact convex values, was developed by Hukuhara [16] in a classical

Communicated by Y. Takahashi. Received August 9, 1999. Revised November 14, 2000 and January 17, 2002.
2000 Mathematics Subject Classification(s): 47H10, 58C06.
Key words: topological degree, multivalued mappings, fixed points.
*Department of Mathematics, University of Roma II 'Tor Vergata', Via della Ricerca Scientifica, 00133 Roma, Italy.
e-mail: deblasi@mat.uniroma2.it
**Department of Mathematics and Informatics, Sofia University 'St. Kl. Ohridski', 5 James Bourchier Blvd., 1126 Sofia, Bulgaria.
Current address: Laboratory for Advanced Brain Signal Processing, Brain Science Institute, The Institute of Physical and Chemical Research (RIKEN), 2-1, Hirosawa, Wako-shi, Saitama 351-0198, Japan.
e-mail: georgiev@bsp.brain.riken.go.jp

paper published in 1967. Hukuhara's topological degree retains the fundamental properties of Leray-Schauder topological degree [20] and, like the latter, has several applications (see Hu and Papageorgiou [15], Lloyd [21], Ma [22]). In particular, it permits to give alternative proofs of fixed point theorems for multifunctions of the type of Kakutani [18] and Ky Fan [7]. Significant developments of Hukuhara's theory can be found among others, in Cellina and Lasota [3], Lasry and Robert [19], Ma [22], Fitzpatrick and Petryshyn [8], [24] and Borisovitch, Gelman, Myshkis and Obukhovskii [2].

The problem of extending the Hukuhara topological degree to multivalued vector fields $I - F$, in which F takes non empty closed bounded and convex values, was considered by De Blasi and Myjak [6]. However the notion of topological degree introduced in [6] was too weak, sufficient only to prove existence of almost fixed points. Recently the problem has been considered, in a general setting, by Dawidowicz [4] who has introduced a more appropriate notion of topological degree, which is useful also in fixed point theory. The approach of Dawidowicz is, to a certain extent, not elementary, since it relies on advanced techniques of homology theory along a line of research which goes back to the fundamental contributions of Granas [13], Çeba and Granas [9] and Górniewicz [10].

The aim of this paper is to present an elementary and direct construction of a topological degree for multivalued vector fields $I - F$, where F takes non empty closed bounded convex, or non convex, values. For a multifunction F the usual notion of compactness is too restrictive for our needs, and thus we will replace it by the h-compactness of F, a notion introduced in [6], which essentially requires that the set-valued range of F be precompact in the Pompeiu-Hausdorff metric h. In our approach a fundamental role is played by approximation techniques very much as in Hukuhara [16] and Cellina and Lasota [3]. Further developments in this spirit can be found in Górniewicz, Granas and Kryszewski [11], Górniewicz and Lassonde [12], Borisovitch, Gelman, Myshkis and Obukhovskii [2] as well as in the comprehensive monograph on multifunctions by Hu and Papageorgiou [15].

The paper consists of five sections. Section 2 contains notation and preliminaries. In Section 3 we define a topological degree for multivalued vector fields $I - F$, in which F is a regular multifunction (see Definition 3.2). Our degree reduces to that of Hukuhara, when F is a h-u.s.c. compact multifunction taking non empty compact convex values. A few useful properties of our topological degree are established in Section 4. Applications to fixed point theory including a Borsuk's type result are contained in Section 5.

§2. Notation and Terminology

Throughout this paper M denotes a metric space, $\mathbb{E}$ a real Banach space with norm $\|\cdot\|$, I is the identity map in $\mathbb{E}$. Furthermore, $2^{\mathbb{E}}$ will denote the family of all non empty subsets of $\mathbb{E}$, and

$$\begin{aligned}
\mathcal{H}(\mathbb{E}) &= \{X \in 2^{\mathbb{E}} : X \text{ is compact}\},\\
\mathcal{K}(\mathbb{E}) &= \{X \in 2^{\mathbb{E}} : X \text{ is compact convex}\},\\
\mathcal{C}(\mathbb{E}) &= \{X \in 2^{\mathbb{E}} : X \text{ is closed bounded convex}\},\\
\mathcal{B}(\mathbb{E}) &= \{X \in 2^{\mathbb{E}} : X \text{ is closed bounded}\}.
\end{aligned}$$

For $a \in \mathbb{E}$ and $X \in 2^{\mathbb{E}}$, we set $d(a, X) = \inf_{x \in X} \|a - x\|$. Each space $\mathcal{H}(\mathbb{E})$, $\mathcal{K}(\mathbb{E}), \mathcal{C}(\mathbb{E}), \mathcal{B}(\mathbb{E})$ is endowed with the Pompeiu-Hausdorff metric:

$$h(X, Y) = \max\{e(X, Y), e(Y, X)\},$$

where $e(X, Y) = \sup_{x \in X} d(x, Y)$ and $e(Y, X) = \sup_{y \in Y} d(y, X)$.

Remark 2.1. Under the Pompeiu-Hausdorff metric h each space $\mathcal{K}(\mathbb{E})$, $\mathcal{C}(\mathbb{E})$, $\mathcal{B}(\mathbb{E})$ is complete. Furthermore, $\mathcal{H}(\mathbb{E})$, $\mathcal{K}(\mathbb{E})$ and $\mathcal{C}(\mathbb{E})$ are closed subsets of $\mathcal{B}(\mathbb{E})$.

For $X \subset M$, by $\overline{X}$ or $cl_M X$ we mean the closure of X in M, and by ∂X the boundary of X. Moreover, $U_M(a, r)$ is the open ball in M with center a and radius r. In $\mathbb{E}$ instead of $U_{\mathbb{E}}(a, r)$ and $U_{\mathbb{E}}(0, 1)$ we write, for brevity, $U(a, r)$ and U.

The convex hull and closed convex hull of $X \subset \mathbb{E}$ are denoted by coX and $\overline{co}X$, respectively.

Let F be a map which associates with each $x \in M$ a non empty subset $F(x)$ of $\mathbb{E}$. When, for each $x \in M, F(x)$ is a subset of $\mathbb{E}$ in a family, say $\mathcal{F}(\mathbb{E})$, of subsets of $\mathbb{E}$, we write (by abuse of notation) $F : M \to \mathcal{F}(\mathbb{E})$, and we call F a *multifunction*, or $\mathcal{F}(\mathbb{E})$-*valued multifunction.*

The $\mathbb{E}$-*range* $\mathcal{R}_{\mathbb{E}}(F)$ of F and the $\mathcal{F}(\mathbb{E})$-*range* $\mathcal{R}_{\mathcal{F}(\mathbb{E})}(F)$ of F are defined by

$$\begin{aligned}
\mathcal{R}_{\mathbb{E}}(F) &= \{y \in E : \text{ there is } x \in M \text{ such that } y \in F(x)\},\\
\mathcal{R}_{\mathcal{F}(\mathbb{E})}(F) &= \{Y \in \mathcal{F}(\mathbb{E}) : \text{ there is } x \in M \text{ such that } Y = F(x)\}.
\end{aligned}$$

When $F : M \to \mathbb{E}$ is single-valued, the $\mathbb{E}$-range $\mathcal{R}_{\mathbb{E}}(F)$ of F is also denoted by $F(M)$.

Definition 2.1. A multifunction $F : M \to \mathcal{B}(\mathbb{E})$ is said to be Pompeiu-Hausdorff upper semicontinuous (resp. lower semicontinuous and continuous) if for every $x_0 \in M$ and $\varepsilon > 0$ there is $\delta > 0$ such that $x \in U_M(x_0, \delta)$ implies $e(F(x), F(x_0)) < \varepsilon$ (resp. $e(F(x_0), F(x)) < \varepsilon$ and $h(F(x), F(x_0)) < \varepsilon$).

Instead of Pompeiu-Hausdorff upper semicontinuous (resp. lower semicontinuous and continuous) we write, for convenience, h-u.s.c. (resp. h-l.s.c. and h-continuous).

Definition 2.2. A multifunction $F : M \to \mathcal{B}(\mathbb{E})$ is called h-compact, if the set $\mathcal{R}_{\mathcal{B}(\mathbb{E})}(F)$ is precompact in $\mathcal{B}(\mathbb{E})$. If $\mathcal{R}_{\mathbb{E}}(F)$ is precompact in $\mathbb{E}$, then F is called *compact.*

Remark 2.2. Since $\mathbb{E}$ and $\mathcal{B}(\mathbb{E})$ are complete metric spaces, a multifunction $F : M \to \mathcal{B}(\mathbb{E})$ is compact (resp. h-compact), if and only if $\overline{\mathcal{R}_{\mathbb{E}}(F)}$ is a compact subset of $\mathbb{E}$ (resp. $cl_{\mathcal{B}(\mathbb{E})}\mathcal{R}_{\mathcal{B}(\mathbb{E})}(F)$) is a compact subset of $\mathcal{B}(\mathbb{E})$).

The following elementary proposition will be useful in the sequel.

Proposition 2.1. *Let $F_i : M \to \mathcal{B}(\mathbb{E}), i = 1, 2,$ be h-u.s.c.* (*resp. h-compact*) *and let T be a non empty* (*resp. non empty compact*) *subset of* $\mathbf{R}$. *Then the multifunction $K : T \times T \times M \to \mathcal{B}(\mathbb{E})$ given by*

$$K(t_1, t_2, x) = \overline{t_1 F_1(x) + t_2 F_2(x)} \quad \textit{for every} \quad (t_1, t_2, x) \in T \times T \times M$$

is h-u.s.c. (*resp. h-compact*).

The class of h-compact multifunctions is strictly larger than the class of all compact multifunctions.

Proposition 2.2. *For any $F : M \to \mathcal{B}(\mathbb{E})$ we have that F is compact if and only if F is h-compact and takes non empty compact values.*

Proof. Set $A = \overline{\mathcal{R}_{\mathbb{E}}(F)}, \mathcal{A} = cl_{\mathcal{B}(\mathbb{E})}\mathcal{R}_{\mathcal{B}(\mathbb{E})}(F)$. By Remark 2.2 it suffices to show that the following are equivalent:

(j) A is a compact subset of $\mathbb{E}$,

(jj) $\mathcal{A}$ is a compact subset of $\mathcal{B}(\mathbb{E})$ and F is $\mathcal{H}(\mathbb{E})$-valued.

Assume (j). F is $\mathcal{H}(\mathbb{E})$-valued, since $F(x) \subset A$ for every $x \in M$. Set $\mathcal{H}_A = \{Y \in \mathcal{H}(\mathbb{E}) : Y \subset A\}$ and observe that $\mathcal{H}_A$, equipped with the metric h, is compact, because A is so. Furthermore $\mathcal{H}(\mathbb{E})$ is closed in $\mathcal{B}(\mathbb{E})$, by Remark 2.1, and thus $\mathcal{A} \subset \mathcal{H}_A$. Consequently $\mathcal{A}$ is compact, proving (jj).

Assume (jj). For every $Y \in \mathcal{A}$ we have $Y \subset A$, since A is closed in $\mathbb{E}$. Now let $\{y_n\} \subset A$. From the definition of A, there is a sequence $\{F(x_n)\} \subset \mathcal{A}$ such that

$$d(y_n, F(x_n)) \to 0 \qquad \text{as } n \to +\infty. \tag{2.1}$$

Since $\mathcal{A}$ is compact, there exists a subsequence, say $\{F(x_n)\}$, and a set $Y \in \mathcal{A}$ such that

$$h(F(x_n), Y) \to 0 \qquad \text{as } n \to +\infty. \tag{2.2}$$

From $d(y_n, Y) \le d(y_n, F(x_n)) + h(F(x_n), Y)$, in view of (2.1) and (2.2), it follows that

$$d(y_n, Y) \to 0 \qquad \text{as } n \to +\infty.$$

But Y is compact in $\mathbb{E}$, and hence a subsequence, say $\{y_n\}$, converges to some $y \in Y$. As $Y \subset A$, it follows that $y \in A$. Thus A is compact and (j) holds, completing the proof. □

Let $\{\varphi_n\}$ and $\{\psi_n\}$ be sequences of multifunctions, where

$$\varphi_n : M \to \mathcal{B}(\mathbb{E}) \text{ and } \psi_n : T \times M \to \mathcal{B}(\mathbb{E}), T \text{ a metric space.}$$

Definition 2.3. A sequence $\{\varphi_n\}$ (resp. $\{\psi_n\}$) is said to be e-convergent to $F : M \to \mathcal{B}(\mathbb{E})$ if for every $x \in M$ and $\varepsilon > 0$ there exist $\delta > 0$ and $n_0 \in \mathbf{N}$ such that

$$\sup_{x' \in U_M(x,\delta)} e(\varphi_n(x'), F(x)) < \varepsilon, \quad (\text{resp.} \sup_{(t',x') \in T \times U_M(x,\delta)} e(\psi_n(t', x'), F(x)) < \varepsilon)$$

for every $n \ge n_0$.

In the above definition φ_n and ψ_n can be single valued. For brevity we write $\varphi_n \xrightarrow{e} F$ to mean that $\{\varphi_n\}$ is e-convergent to F.

Definition 2.4. A sequence $\{f_n\}$, $f_n : M \to \mathbb{E}$, is called collectively compact, if the set $\bigcup_{n \in \mathbf{N}} f_n(M)$ is a precompact subset of $\mathbb{E}$.

Given $F : M \to 2^{\mathbb{E}}$, a function $f : M \to \mathbb{E}$ satisfying $f(x) \in F(x)$ for every $x \in M$ is called a *selection* of F. The notion of a *multivalued selection* of F is analogous.

§3. Topological Degree for Regular Multivalued Vector Fields

In this section we define a topological degree for multivalued vector fields $I - F$, when F is a $\mathcal{B}(\mathbb{E})$-valued regular multifunction (see Definition 3.2). Each $\mathcal{C}(\mathbb{E})$-valued h-u.s.c. and h-compact F, in particular each $\mathcal{K}(\mathbb{E})$-valued h-u.s.c. and compact F, is a regular multifunction. The topological degree we introduce reduces to that of Hukuhara [16], when F is h-u.s.c. compact and takes its values in $\mathcal{K}(\mathbb{E})$.

Definition 3.1. Let $\varphi : M \to \mathcal{C}(\mathbb{E})$ be h-u.s.c. A sequence $\{f_n\}$ of continuous functions $f_n : M \to \mathbb{E}$ is called an admissible approximating sequence for φ if (i) $\{f_n\}$ is collectively compact, and (ii) $\{f_n\}$ is e-convergent to φ. The family of all admissible approximating sequences for φ will be denoted by $\mathcal{A}_\varphi$.

Proposition 3.1. *Let $\varphi : M \to \mathcal{C}(\mathbb{E})$ be h-u.s.c. and h-compact. Then we have*:

(i) *φ admits a h-u.s.c. and compact multivalued selection $\omega : M \to \mathcal{K}(\mathbb{E})$,*

(ii) *for every $x \in M$, $\varphi(x) = \bigcup\{\omega(x) : \omega : M \to \mathcal{K}(\mathbb{E})$ is a h-u.s.c. compact selection of $\varphi\}$,*

(iii) *$\mathcal{A}_\varphi \neq \emptyset$.*

Proof.

(i) Let $J : \mathcal{C}(\mathbb{E}) \to \mathcal{C}(\mathbb{E})$ be the multifunction given by

$$J(X) = X \qquad \text{for every} \;\; X \in \mathcal{C}(\mathbb{E}).$$

Thus J assigns to each X in the metric space $(\mathcal{C}(\mathbb{E}), h)$ the non empty closed bounded and convex subset X of $\mathbb{E}$. J is h-continuous and hence lower semicontinuous in the sense of Michael [23], that is, for every open $V \subset \mathbb{E}$ the set $\{X \in \mathcal{C}(\mathbb{E}) : J(X) \cap V \neq \emptyset\}$ is open in $\mathcal{C}(\mathbb{E})$. By the selection theorem of Michael [23] (see also Hu and Papageorgiou [15], p. 92) there is a continuous selection $s : \mathcal{C}(\mathbb{E}) \to \mathbb{E}$ such that

$$s(X) \in X \qquad \text{for every} \;\; X \in \mathcal{C}(\mathbb{E}).$$

Now consider the Filippov regularization of $s \circ \varphi$ that is the multifunction $\omega_{s\circ\varphi} : M \to \mathcal{K}(\mathbb{E})$ defined by

$$\omega_{s\circ\varphi}(x) = \bigcap_{n\in\mathbf{N}} \overline{co}(s \circ \varphi)\left(U_M\left(x, \frac{1}{n}\right)\right) \qquad \text{for every} \;\; x \in M.$$

The set $(s \circ \varphi)(M)$ is precompact in $\mathbb{E}$, because the set $\mathcal{A} = cl_{\mathcal{C}(\mathbb{E})}(\mathcal{R}_{\mathcal{C}(\mathbb{E})} \times(\varphi))$ is compact in $\mathcal{C}(\mathbb{E})$ and s is continuous. Clearly $\omega_{s\circ\varphi}(x) \in \mathcal{K}(\mathbb{E})$ for every $x \in M$.

The multifunction $\omega_{s\circ\varphi}$ is h-u.s.c. Let $x_0 \in M$ and $\varepsilon > 0$ be given. The sequence of compact sets $\overline{co}(s \circ \varphi)(U_M(x_0, 1/n))$ is monotonic decreasing, by inclusion, and thus there exists $n_0 \in \mathbf{N}$ such that $\overline{co}(s \circ \varphi)(U_M(x_0, 1/n_0)) \subset \omega_{s\circ\varphi}(x_0) + \varepsilon U$. For every $x \in U_M(x_0, 1/n_0)$ we have

$$\begin{aligned}\omega_{s\circ\varphi}(x) &= \bigcap_{n\in\mathbf{N}} \overline{co}(s \circ \varphi)\left(U_M\left(x, \frac{1}{n}\right)\right)\\ &\subset \overline{co}(s \circ \varphi)\left(U_M\left(x_0, \frac{1}{n_0}\right)\right) \subset \omega_{s\circ\varphi}(x_0) + \varepsilon U,\end{aligned}$$

and thus $\omega_{s\circ\varphi}$ is h-u.s.c. Further $\omega_{s\circ\varphi}$ is compact, because, for every $x \in M$, we have $\omega_{s\circ\varphi}(x) \subset \overline{co}(\overline{(s \circ \varphi)(M)})$, where the latter set is compact, by Mazur's theorem.

It remains to show that $\omega_{s\circ\varphi}$ is a selection of φ. Let $x \in M$ and $\varepsilon > 0$ be arbitrary. Fix $\delta > 0$ so that $x' \in U_M(x, \delta)$ implies $\varphi(x') \subset \varphi(x) + \varepsilon U$. For every $x' \in U_M(x, \delta)$ we have $(s \circ \varphi)(x') \in \varphi(x') \subset \varphi(x) + (\varepsilon/2)U$, thus for all n large enough $\overline{co}(s\circ\varphi)(U_M(x, 1/n)) \subset \varphi(x) + \varepsilon U$ and so, a fortiori, $\omega s \circ \varphi(x) \subset \varphi(x) + \varepsilon U$. Therefore $\omega_{s\circ\varphi}(x) \subset \varphi(x)$. Letting $\omega = \omega_{s\circ\varphi}$, (i) is proved.

(ii) Let $x \in M$ and $y \in \varphi(x)$ be arbitrary, and let $\tilde{X} = \varphi(x)$. Let $\tilde{J} : \mathcal{C}(\mathbb{E}) \to \mathcal{C}(\mathbb{E})$ be given by $\tilde{J}(X) = X$ if $X \neq \tilde{X}$ and $\tilde{J}(X) = \{y\}$ if $X = \tilde{X}$. $\tilde{J}$ is h-l.s.c., and hence also lower semicontinuous in the sense of Michael. Therefore there is a continuous function $s : \mathcal{C}(\mathbb{E}) \to \mathbb{E}$ such that $s(X) \in \tilde{J}(X)$ for every $X \in \mathcal{C}(\mathbb{E})$. Since

$$y = (s \circ \varphi)(x) \in \bigcap_{n\in\mathbf{N}} \overline{co}(s \circ \varphi)\left(U_M\left(x, \frac{1}{n}\right)\right) = \omega_{s\circ\varphi}(x),$$

it follows that

$$\varphi(x) \subset \{\omega(x) : \omega : M \to \mathcal{K}(\mathbb{E}) \text{ is a } h\text{-u.s.c. compact selection of } \varphi\}.$$

The opposite inclusion is obvious, and thus (ii) is proved.

(iii) Let $\omega : M \to \mathcal{K}(\mathbb{E})$ be a h-u.s.c. compact selection of φ. By the Hukuhara approximation theorem [16] (see also [5] and [15], p. 119) there is a sequence $\{\varphi_n\}$ of h-continuous multifunctions $\varphi_n : M \to \mathcal{K}(\mathbb{E})$ such that (j) $\omega(x) \subset \varphi_{n+1}(x) \subset \varphi_n(x)$ for every $x \in M$ and every $n \in \mathbf{N}$, (jj) $\mathcal{R}_{\mathbb{E}}(\varphi_n) \subset$

$\overline{co}\mathcal{R}_{\mathbb{E}}(\omega)$ for every $n \in \mathbf{N}$, and (jjj) $h(\varphi_n(x), \omega(x)) \to 0$, as $n \to \infty$, for every $x \in M$.

For $n \in \mathbf{N}$, let $f_n : M \to \mathbb{E}$ be a continuous selection of φ_n. By (jj), $\{f_n\}$ is collectively compact. Let $x \in M$ and $\varepsilon > 0$ be arbitrary. By (jjj), there exists $n_0 \in \mathbf{N}$ such that $n \geq n_0$ implies $\varphi_n(x) \subset \omega(x) + \varepsilon U$. Since φ_{n_0} is h-continuous, there is $\delta > 0$ such that $\varphi_{n_0}(x') \subset \varphi_{n_0}(x) + \varepsilon U \subset \omega(x) + 2\varepsilon U$ for every $x' \in U_M(x, \delta)$. In view of (j) we have

$$f_n(x') \in \varphi_n(x') \subset \varphi_{n_0}(x') \subset \omega(x) + 2\varepsilon U \subset \varphi(x) + 2\varepsilon U$$

for every $x' \in U_M(x, \delta)$ and $n \geq n_0$, and thus $\{f_n\}$ is e-convergent to φ. Therefore $\{f_n\} \in \mathcal{A}_\varphi$, and (iii) is proved. This completes the proof. □

Given $\varphi : A \to 2^{\mathbb{E}}$, where $\emptyset \neq A \subset \mathbb{E}$, and $r : \mathbb{E} \to \mathbb{E}$, we denote by $\overline{r \circ \varphi}$ the multifunction defined on A with values in $2^{\mathbb{E}}$, defined by $\overline{(r \circ \varphi)}(x) = \overline{r(\varphi(x))}$, for every $x \in A$.

In the sequel D will always denote a non empty open bounded subset of $\mathbb{E}$, p a point of $\mathbb{E}$ and I the identity map in $\mathbb{E}$. Further we set

$$\mathcal{L}_{\mathbb{E}} = \{r : \mathbb{E} \to \mathbb{E} : r \text{ is Lipschitzean with constant } L_r \geq 0\},$$
$$\mathcal{H}(\overline{D}, \mathcal{C}(\mathbb{E})) = \{\varphi : \overline{D} \to \mathcal{C}(\mathbb{E}) : \varphi \text{ is } h\text{-u.s.c. and } \mathcal{A}_\varphi \neq \emptyset\}.$$

Remark 3.1. By Proposition 3.1, each h-u.s.c. and h-compact multifunction $\varphi : \overline{D} \to \mathcal{C}(\mathbb{E})$ is in $\mathcal{H}(\overline{D}, \mathcal{C}(\mathbb{E}))$.

Definition 3.2. A multifunction $F : \overline{D} \to \mathcal{B}(\mathbb{E})$ admits a regular representation if there exists a pair $(r, \varphi) \in \mathcal{L}_{\mathbb{E}} \times \mathcal{H}(\overline{D}, \mathcal{C}(\mathbb{E}))$ such that

$$F(x) = \overline{(r \circ \varphi)}(x)$$

for every $x \in \overline{D}$. In this case, $\overline{r \circ \varphi} : \overline{D} \to \mathcal{B}(\mathbb{E})$ is called a regular representation of F, and F is called a regular multifunction. If F is regular and admits a regular representation $\overline{r \circ \varphi}$, in which r has inverse r^{-1} Lipschitzean on $\mathbb{E}$, then F is called a strongly regular multifunction.

It is evident that a regular multifunction is not necessarily convex valued.

For any regular multifunction $F : \overline{D} \to \mathcal{B}(\mathbb{E})$ we put

$$Repr(F) = \{(r, \varphi) \in \mathcal{L}_{\mathbb{E}} \times \mathcal{H}(\overline{D}, \mathcal{C}(\mathbb{E})) : \overline{r \circ \varphi} \text{ is a regular representation of } F\}.$$

Remark 3.2. Each h-u.s.c. and h-compact multifunction $F : \overline{D} \to \mathcal{C}(\mathbb{E})$ is strongly regular and admits infinitely many regular representations $\overline{r_\lambda \circ \varphi_\lambda}$, where $r_\lambda = \lambda I$ and $\varphi_\lambda = \lambda^{-1} F, \lambda \neq 0$.

Remark 3.3. Each regular multifunction $F : \overline{D} \to \mathcal{B}(\mathbb{E})$ is h-u.s.c.

We shall consider the following classes of multivalued vector fields:

$$\mathcal{V}_{comp}(\overline{D}, \mathcal{K}(\mathbb{E})) = \{G : \overline{D} \to \mathcal{K}(\mathbb{E}) : G = I - F \text{ and } F \text{ is } h\text{-u.s.c. and compact}\};$$
$$\mathcal{V}_{h-comp}(\overline{D}, \mathcal{C}(\mathbb{E})) = \{G : \overline{D} \to \mathcal{C}(\mathbb{E}) : G = I - F \text{ and } F \text{ is } h\text{-u.s.c. and } h\text{-compact}\};$$
$$\mathcal{V}_{r-mult}(\overline{D}, \mathcal{B}(\mathbb{E})) = \{G : \overline{D} \to \mathcal{B}(\mathbb{E}) : G = I - F \text{ and } F \text{ is a regular multifunction}\}.$$

Any G which is in $\mathcal{V}_{comp}(\overline{D}, \mathcal{K}(\mathbb{E}))$, (resp. in $\mathcal{V}_{h-comp}(\overline{D}, \mathcal{C}(\mathbb{E}))$, $\mathcal{V}_{r-mult} \times (\overline{D}, \mathcal{B}(\mathbb{E}))$) is called a *compact* (resp. *h-compact, regular*) *multivalued vector field.*

Remark 3.4. We have

$$\mathcal{V}_{comp}(\overline{D}, \mathcal{K}(\mathbb{E})) \subset \mathcal{V}_{h-comp}(\overline{D}, \mathcal{C}(\mathbb{E})) \subset \mathcal{V}_{r-mult}(\overline{D}, \mathcal{B}(\mathbb{E})),$$

where each inclusion is strict.

Proposition 3.2. *Let $I - F \in \mathcal{V}_{r-mult}(\overline{D}, \mathcal{B}(\mathbb{E}))$ and let $\overline{r \circ \varphi}$ be a regular representation of F for some $(r, \varphi) \in Repr(F)$. We have*

(i) *For every $\{f_n\}, \{\tilde{f}_m\} \in \mathcal{A}_\varphi$ the sequence $\{K_{n,m}\}$ of continuous functions $K_{n,m} : [0,1] \times \overline{D} \to \mathbb{E}$ given by*

$$K_{n,m}(t,x) = r((1-t)f_n(x) + t\tilde{f}_m(x)) \quad \text{for every} \quad (t,x) \in [0,1] \times \overline{D}$$

is collectively compact, and e-convergent to F as $n, m \to +\infty$.

(ii) *Let $\{f_n\}, \{\tilde{f}_n\} \in \mathcal{A}_\varphi$. If for some point $p \in \mathbb{E}$ there exist sequences $\{f_{n_k}\}$, $\{\tilde{f}_{n_k}\}$, and $\{(t_k, x_k)\} \subset [0,1] \times \partial D$, such that*

$$p = x_k - r((1-t_k)f_{n_k}(x_k) + t_k\tilde{f}_{n_k}(x_k)) \quad \text{for every} \quad k \in \mathbf{N}, \tag{3.1}$$

then there is a subsequence of $\{(t_k, x_k)\}$ which converges to a point $(t, x) \in [0,1] \times \partial D$, where x satisfies $p \in x - F(x)$.

Proof.

(i) $\{K_{n,m}\}$ is collectively compact, for $\{f_n\}$ and $\{\tilde{f}_m\}$ are so, and r is continuous. To show that $\{K_{n,m}\}$ is e-convergent to F, take arbitrary $x \in \overline{D}$ and $\varepsilon > 0$. Since $\{f_n\}$ and $\{\tilde{f}_n\}$ are e-convergent to φ, there exist $\delta > 0$ and $n_0 \in \mathbf{N}$ such that

$$f_n(x'), \tilde{f}_n(x') \in \varphi(x) + \varepsilon U \quad \text{for every} \quad x' \in U_{\overline{D}}(x, \delta), n \geq n_0. \tag{3.2}$$

Hence, if L_r is a Lipschitz constant for r, we have

$$K_{n,m}(t, x') \in r(\varphi(x) + \varepsilon U) \subset F(x) + \varepsilon L_r U$$

for every $(t, x') \in [0, 1] \times U_{\overline{D}}(x, \delta)$ and $n, m \geq n_0$, and thus $\{K_{n,m}\}$ is e-convergent to F as $n, m \to +\infty$.

(ii) Since $\{K_{n,m}\}$ is collectively compact, from (3.1) it follows that $\{(t_k, x_k)\}$ contains a subsequence, say $\{(t_k, x_k)\}$, which converges to a point $(t, x) \in [0, 1] \times \partial D$. Given $\varepsilon > 0$, let $\delta > 0$ and $n_0 \in \mathbf{N}$ be such that (3.2) holds. Take $k_0 \geq n_0$ so that $x_k \in U_{\overline{D}}(x, \delta)$ for all $k \geq k_0$. In view of (3.2), for every $k \geq k_0$ we have $f_{n_k}(x_k), \tilde{f}_{n_k}(x_k) \in \varphi(x) + \varepsilon U$, and hence

$$r((1 - t_k) f_{n_k}(x_k) + t_k \tilde{f}_{n_k}(x_k)) \in r(\varphi(x) + \varepsilon U) \subset F(x) + \varepsilon L_r U. \tag{3.3}$$

From (3.1) and (3.3) we have $p \in x_k - F(x) + \varepsilon L_r U$ for every $k \geq k_0$. Letting $k \to +\infty$ it follows that $p \in x - F(x)$, completing the proof. □

Given a continuous and compact function $f : \overline{D} \to E$ and a point $p \notin \cup_{x \in \partial D} (I - f)(x)$ we shall denote by $\deg(I - f, D, p)$ the Leray-Schauder topological degree of the vector field $I - f$ at p relative to D (see [20] and also [17], [21], [26]).

Proposition 3.3. *Let $I - F \in \mathcal{V}_{r-mult}(\overline{D}, \mathcal{B}(\mathbb{E}))$ and let $p \notin \bigcup_{x \in \partial D} (I - F)(x)$. Let $\overline{r \circ \varphi}$ be a regular representation of F for some $(r, \varphi) \in Repr(F)$. Then we have*

(i) *for every $\{f_n\} \in \mathcal{A}_\varphi$ there exists $n_0 \in \mathbf{N}$ such that*

$$\deg(I - r \circ f_n, D, p) = \deg(I - r \circ f_m, D, p) \quad \textit{for every} \quad n, m \geq n_0, \tag{3.4}$$

(ii) *for every $\{f_n\}, \{\tilde{f}_n\} \in \mathcal{A}_\varphi$ there exists $n_0 \in \mathbf{N}$ such that*

$$\deg(I - r \circ f_n, D, p) = \deg(I - r \circ \tilde{f}_n, D, p) \quad \textit{for every} \quad n \geq n_0. \tag{3.5}$$

Proof.

(i) For $n, m \in \mathbf{N}$, define $K_{n,m} : [0, 1] \times \overline{D} \to \mathbb{E}$ by

$$K_{n,m}(t, x) = r((1 - t) f_n(x) + t f_m(x)) \quad \text{for every} \quad (t, x) \in [0, 1] \times \overline{D}.$$

$K_{n,m}$ is continuous and satisfies

(3.6) $K_{n,m}(0,x) = (r \circ f_n)(x), \quad K_{n,m}(1,x) = (r \circ f_m)(x) \quad$ for every $\; x \in \overline{D}$.

Further, by Proposition 3.2 (i), $\{K_{n,m}\}$ is collectively compact and e-convergent to F as $n, m \to +\infty$.

There is $n_0 \in \mathbf{N}$ such that

$$p \notin \bigcup_{n,m \geq n_0} \bigcup_{(t,x) \in [0,1] \times \partial D} (x - K_{n,m}(t,x)). \tag{3.7}$$

In the contrary case, there are sequences $\{f_{n_k}\}$, $\{\tilde{f}_{m_k}\}$, and $\{(t_k, x_k)\} \subset [0,1] \times \partial D$, such that

$$p = x_k - r((1-t_k) f_{n_k}(x_k) + t_k f_{m_k}(x_k)) \quad \text{for every} \quad k \in \mathbf{N}.$$

Proposition 3.2 (ii) implies that a subsequence of $\{(t_k, x_k)\}$, say $\{(t_k, x_k)\}$, converges to a point $(t, x) \in [0,1] \times \partial D$, where x satisfies $p \in x - F(x)$, a contradiction. Hence, for some $n_0 \in \mathbf{N}$, (3.7) holds.

Now, for each $n, m \geq n_0, K_{n,m}$ is continuous compact and satisfies (3.7). By the homotopy property of the Leray-Schauder topological degree, in view of (3.6), (3.4) follows, and so (i) is proved.

(ii) The proof of (3.5) is similar, if one defines $K_n : [0,1] \times \overline{D} \to \mathbb{E}$ by

$$K_n(t,x) = r((1-t) f_n(x) + t \tilde{f}_n(x)) \quad \text{for every} \quad (t,x) \in [0,1] \times \overline{D}.$$

This completes the proof. □

Definition 3.3. Let $I - F \in \mathcal{V}_{r-mult}(\overline{D}, \mathcal{B}(\mathbb{E}))$ and let $p \notin \bigcup_{x \in \partial D} (I - F)(x)$. Let $\overline{r \circ \varphi}$ be a regular representation of F for some $(r, \varphi) \in Repr(F)$. For arbitrary $\{f_n\} \in \mathcal{A}_\varphi$ the topological degree $d(I - \overline{r \circ \varphi}, D, p)$ of $I - \overline{r \circ \varphi}$ at p relative to D is defined by

$$d(I - \overline{r \circ \varphi}, D, p) = \lim_{n \to \infty} \deg(I - r \circ f_n, D, p). \tag{3.8}$$

The topological degree $\mathrm{Deg}(I - F, D, p)$ of $I - F$ at p relative to D is defined by

$$\mathrm{Deg}(I - F, D, p) = \{d(I - \overline{r \circ \varphi}, D, p) : (r, \varphi) \in Repr(F)\}.$$

Remark 3.5. The limit (3.8) exists and is finite by Proposition 3.3 (i), and it is independent of $\{f_n\} \in \mathcal{A}_\varphi$ by Proposition 3.3 (ii).

Proposition 3.4. *Let* $I - F \in \mathcal{V}_{r-mult}(\overline{D}, \mathcal{B}(\mathbb{E}))$ *and let* $p \notin \bigcup_{x \in \partial D} (I - F)(x)$. *If, in addition,* F *is a strongly regular multifunction, then* $\mathrm{Deg}(I - F, D, p)$ *is singleton.*

Proof. Let $\overline{r \circ \varphi}$ be an arbitrary regular representation of F for some $(r, \varphi) \in Repr(F)$. By hypothesis, there exists a regular representation $\overline{\tilde{r} \circ \tilde{\varphi}}$ of F for some $(\tilde{r}, \tilde{\varphi}) \in Repr(F)$, where $\tilde{r}$ has inverse $\tilde{r}^{-1}$, Lipschitzean on $\mathbb{E}$. Let $\{f_n\} \in \mathcal{A}_{\varphi}$ and $\{\tilde{f}_n\} \in \mathcal{A}_{\tilde{\varphi}}$ be arbitrary. For $n \in \mathbf{N}$ define $g_n : \overline{D} \to \mathbb{E}$ by $g_n(x) = (\tilde{r}^{-1} \circ r \circ f_n)(x), x \in \overline{D}$.

We have $\{g_n\} \in \mathcal{A}_{\tilde{\varphi}}$. Clearly, g_n is continuous, and the sequence $\{g_n\}$ is collectively compact, for $\{f_n\}$ is so and $\tilde{r}^{-1} \circ r$ is continuous. Moreover, $\{g_n\}$ is e-convergent to $\tilde{\varphi}$. Let $x \in \overline{D}$ and $\varepsilon > 0$ be given. Since $f_n \stackrel{e}{\longrightarrow} \varphi$, there are $\delta > 0$ and $n_0 \in \mathbf{N}$ such that

$$f_n(x') \in \varphi(x) + \varepsilon U \quad \text{for every} \quad x' \in U_{\overline{D}}(x, \delta), n \geq n_0. \tag{3.9}$$

Let L_r and $L_{\tilde{r}^{-1}}$ be Lipschitz constants for r and $\tilde{r}^{-1}$, respectively. In view of (3.9), for every $x' \in U_{\overline{D}}(x, \delta)$ and $n \geq n_0$ we have

$$g_n(x') \in \tilde{r}^{-1}(r(\varphi(x) + \varepsilon U)) \subset \tilde{r}^{-1}(F(x) + \varepsilon L_r U) \subset \tilde{\varphi}(x) + \varepsilon L_r L_{\tilde{r}^{-1}} U,$$

and so $\{g_n\}$ is e-convergent to $\tilde{\varphi}$. Hence $\{g_n\} \in \mathcal{A}_{\tilde{\varphi}}$.

For $n \in \mathbf{N}$ define $K_n : [0,1] \times \overline{D} \to \mathbb{E}$ by

$$K_n(t, x) = \tilde{r}((1-t)g_n(x) + t\tilde{f}_n(x)) \quad \text{for every} \quad (t, x) \in [0,1] \times \overline{D}. \tag{3.10}$$

Since $\{g_n\}, \{\tilde{f}_n\} \in \mathcal{A}_{\tilde{\varphi}}$, by Proposition 3.2 (i) $(n = m)$, the sequence $\{K_n\}$ is collectively compact and e-convergent to F. Further, by virtue of Proposition 3.2 (ii), one can show that there exists $n_0 \in \mathbf{N}$ such that

$$p \notin \bigcup_{n \geq n_0} \bigcup_{(t,x) \in [0,1] \times \overline{D}} (x - K_n(t, x)). \tag{3.11}$$

The K_n are continuous compact and satisfy (3.10) and (3.11), and thus by the homotopy property of the Leray-Schauder topological degree we have

$$\deg(I - r \circ f_n, D, p) = \deg(I - \tilde{r} \circ \tilde{f}_n, D, p) \quad \text{for every} \quad n \geq n_0.$$

But $\{f_n\} \in \mathcal{A}_{\varphi}$ and $\{\tilde{f}_n\} \in \mathcal{A}_{\tilde{\varphi}}$, and hence by Definition 3.3 and Remark 3.5 it follows

$$d(I - \overline{r \circ \varphi}, D, p) = d(I - \overline{\tilde{r} \circ \tilde{\varphi}}, D, p).$$

As $(r, \varphi) \in Repr(F)$ is arbitrary, the topological degree $\mathrm{Deg}(I - F, D, p)$ is singleton. This completes the proof. □

Remark 3.6. Since each h-u.s.c. and h-compact multifunction $F : \overline{D} \to \mathcal{C}(\mathbb{E})$ is strongly regular, the conclusion of Proposition 3.4 remains valid if $I - F \in \mathcal{V}_{h-comp}(\overline{D}, \mathcal{C}(\mathbb{E}))$ and, in particular, if $I - F \in \mathcal{V}_{comp}(\overline{D}, \mathcal{K}(\mathbb{E}))$.

Proposition 3.5. *Let* $I - F \in \mathcal{V}_{comp}(\overline{D}, \mathcal{K}(\mathbb{E}))$ *and let* $p \notin \bigcup_{x \in \partial D}(I - F)(x)$. *Then* $\mathrm{Deg}(I - F, D, p)$ *coincides with the Hukuhara topological degree of* $I - F$ *at* p *relative to* D.

Proof. Clearly $\mathrm{Deg}(I - F, D, p)$ is singleton, by Remark 3.6, and $(I, F) \in Repr(F)$, by Remark 3.2. Let $\{f_n\}$ be constructed as in the proof of Proposition 3.1 (iii) (with $\omega = \varphi = F$), and thus $\{f_n\} \in \mathcal{A}_F$. By Definition 3.3 we have

$$\mathrm{Deg}(I - F, D, p) = d(I - F, D, p) = \lim_{n \to \infty} \deg(I - f_n, D, p),$$

from which the result follows, as the limit is the Hukuhara topological degree of $I - F$ at p relative to D. □

Remark 3.7. If $F : \overline{D} \to E$ is continuous and compact, the Hukuhara topological degree reduces to that of Leray and Schauder.

§4. Properties of the Topological Degree

In this section we establish a few properties of the topological degree $\mathrm{Deg}(I - F, D, p)$ that are useful in fixed point theory.

Proposition 4.1 (invariance under homotopy)**.** *Let* $I - F_1, I - F_2 \in \mathcal{V}_{r-mult}(\overline{D}, \mathcal{B}(\mathbb{E}))$ *and suppose that the multifunction* $H : [0,1] \times \overline{D} \to \mathcal{B}(\mathbb{E})$ *given by*

$$H(t, x) = \overline{(1 - t)F_1(x) + tF_2(x)} \quad \textit{for every} \quad (t, x) \in [0,1] \times \overline{D}$$

satisfies $p \notin \bigcup_{(t,x) \in [0,1] \times \partial D} (x - H(t, x))$. *Then we have*

$$\mathrm{Deg}(I - F_1, D, p) = \mathrm{Deg}(I - F_2, D, p),$$

where both sides are singletons.

Proof. For $i = 1, 2$, let $\overline{r_i \circ \varphi_i}$ be a regular representation of F_i for some $(r_i, \varphi_i) \in Repr(F_i)$. Let $\{f_n^i\} \in \mathcal{A}_{\varphi_i}, i = 1, 2$. For $n \in \mathbf{N}$ define $K_n : [0,1] \times \overline{D} \to \mathbb{E}$ by

$$K_n(t, x) = (1 - t)(r_1 \circ f_n^1)(x) + t(r_2 \circ f_n^2)(x) \quad \text{for every} \quad (t, x) \in [0,1] \times \overline{D}.$$

Each K_n is continuous and the sequence $\{K_n\}$ is collectively compact, for $\{f_n^1\}$ and $\{f_n^2\}$ are so. There is $n_0 \in \mathbf{N}$ such that

$$p \notin \bigcup_{n \geq n_0} \bigcup_{(t,x) \in [0,1] \times \partial D} (x - K_n(t, x)). \tag{4.1}$$

In the contrary case there are sequences $\{f^1_{n_k}\}$, $\{f^2_{n_k}\}$, and $\{(t_k, x_k)\} \subset [0,1] \times \partial D$, such that

$$p = x_k - ((1-t_k)(r_1 \circ f^1_{n_k})(x_k) + t_k(r_2 \circ f^2_{n_k})(x_k)) \quad \text{for every} \quad k \in \mathbf{N}. \tag{4.2}$$

Since $\{K_n\}$ is collectively compact, $\{(t_k, x_k)\}$ contains a subsequence, say $\{(t_k, x_k)\}$, which converges to a point $(t,x) \in [0,1] \times \partial D$. It will be shown that

$$p \in x - H(t,x). \tag{4.3}$$

Let $\varepsilon > 0$. For $i = 1,2, \{f^i_n\}$ is e-convergent to φ_i, and thus there are $\delta > 0$ and $n_0 \in \mathbf{N}$ such that

$$f^1_n(x') \in \varphi_1(x) + \varepsilon U, \quad f^2_n(x') \in \varphi_2(x) + \varepsilon U \quad \text{for every} \quad x' \in U_{\overline{D}}(x,\delta), n \geq n_0.$$

Now fix $k_0 \geq n_0$ so that $x_k \in U_{\overline{D}}(x,\delta)$ for all $k \geq k_0$. Hence

$$f^1_{n_k}(x_k) \in \varphi_1(x) + \varepsilon U, \quad f^2_{n_k}(x_k) \in \varphi_2(x) + \varepsilon U \quad \text{for every} \quad k \geq k_0. \tag{4.4}$$

From (4.2), in view of (4.4), for every $k \geq k_0$ we have

$$\begin{aligned} p &\in x_k - ((1-t_k)r_1(\varphi_1(x) + \varepsilon U)) + t_k r_2(\varphi_2(x) + \varepsilon U)) \\ &\subset x_k - ((1-t_k)(F_1(x) + \varepsilon L_{r_1} U) + t_k(F_2(x) + \varepsilon L_{r_2} U)) \\ &\subset x_k - H(t_k, x) + \varepsilon(L_{r_1} + L_{r_2})U, \end{aligned}$$

where L_{r_i} is a Lipschitz constant for $r_i, i = 1,2$. Letting $k \to +\infty$, (4.3) follows, a contradiction. Thus, for some $n_0 \in \mathbf{N}$, (4.1) holds.

The K_n are continuous compact and satisfy (4.1), and thus by the homotopy property of the Leray-Schauder topological degree we have

$$\deg(I - r_1 \circ f^1_n, D, p) = \deg(I - r_2 \circ f^2_n, D, p) \quad \text{for every} \quad n \geq n_0.$$

As $\{f^1_n\} \in \mathcal{A}_{\varphi_1}$ and $\{f^2_n\} \in \mathcal{A}_{\varphi_2}$, letting $n \to +\infty$ it follows that $d(I - \overline{r_1 \circ \varphi_1}, D, p) = d(I - \overline{r_2 \circ \varphi_2}, D, p)$. But $(r_1, \varphi_1) \in Repr(F_1)$ and $(r_2, \varphi_1) \in Repr(F_2)$ are arbitrary, and hence

$$\operatorname{Deg}(I - F_1, D, p) = \operatorname{Deg}(I - F_2, D, p),$$

where, clearly, both sides are singletons. This completes the proof. □

Proposition 4.2 (inclusion solving property). *Let $I - F \in \mathcal{V}_{r-mult}(\overline{D}, \mathcal{B}(\mathbb{E}))$, let $p \notin \bigcup_{x \in \partial D}(I - F)(x)$, and suppose that* $\operatorname{Deg}(I - F, D, p) \neq \{0\}$. *Then there exists an $x \in D$ such that*

$$p \in x - F(x). \tag{4.5}$$

Proof. Since $\mathrm{Deg}(I-F,D,p)\neq\{0\}$, there is a regular representation $\overline{r\circ\varphi}$ of F for some $(r,\varphi)\in Repr(F)$, such that $d(I-\overline{r\circ\varphi},D,p)\neq 0$. Let $\{f_n\}\in\mathcal{A}_\varphi$. By Definition 3.3 and Proposition 3.3 (i), there exists $m_0\in\mathbf{N}$ such that $\deg(I-r\circ f_n,D,p)\neq 0$ for every $n\geq m_0$. By a property of the Leray-Schauder topological degree, for every $n\geq m_0$ there exists $x_n\in D$ such that

$$p=x_n-(r\circ f_n)(x_n). \tag{4.6}$$

As $\{f_n\}$ is collectively compact, $\{x_n\}$ contains a subsequence, say $\{x_n\}$, which converges to a point $x\in\overline{D}$. Let $\varepsilon>0$ be arbitrary. Since $\{f_n\}$ is e-convergent to φ, there are $\delta>0$ and $m_1\geq m_0$ such that

$$f_n(x')\in\varphi(x)+\varepsilon U \quad\text{for every}\quad x'\in U_{\overline{D}}(x,\delta),n\geq m_1. \tag{4.7}$$

Now take $n_0\geq m_1$ so that $x_n\in U_{\overline{D}}(x,\delta)$ for all $n\geq n_0$. From (4.6), in view of (4.7), for every $n\geq n_0$ we have

$$p=x_n-r(f_n(x_n))\in x_n-r(\varphi(x)+\varepsilon U)\subset x_n-F(x)+\varepsilon L_r U,$$

where L_r is a Lipschitz constant for r. Letting $n\to+\infty$, (4.5) follows and, clearly, $x\in D$. This completes the proof. □

Proposition 4.3 (normalization). *If* $p\in D$ *then* $\mathrm{Deg}(I,D,p)=1$.

Proof. The Hukuhara topological degree has this property, and thus the statement follows from Proposition 3.5. □

Proposition 4.4 (continuity in p). *Let* $I-F\in\mathcal{V}_{r-mult}(\overline{D},\mathcal{B}(\mathbb{E}))$ *and let* $p,q\in A$, *where* A *is an open component of* $\mathbb{E}\setminus\cup_{x\in\partial D}(I-F)(x)$. *Then we have*

$$\mathrm{Deg}(I-F,D,p)=\mathrm{Deg}(I-F,D,q). \tag{4.8}$$

Proof. Let $(r,\varphi)\in Repr(F)$ be arbitrary, and let $\{f_n\}\in\mathcal{A}_\varphi$. Let $\gamma:[0,1]\to\mathbb{E}$ be a continuous path contained in A, joining p and q. For $\varepsilon>0$ put $C_\varepsilon=\bigcup_{t\in[0,1]}U(\gamma(t),\varepsilon)$. C_ε is open and connected, and $C_\varepsilon\subset A$, provided that ε is small enough.

There exist $\varepsilon>0$ and $n_0\in\mathbf{N}$ such that

$$C_\varepsilon\cap\left(\bigcup_{n\geq n_0}\bigcup_{x\in\partial D}(I-r\circ f_n)(x)\right)=\emptyset. \tag{4.9}$$

In the contrary case, there exist sequences $\{f_{n_k}\}$, and $\{(t_k, x_k)\} \subset [0,1] \times \partial D$, such that

$$\gamma(t_k) \in x_k - (r \circ f_{n_k})(x_k) + \frac{1}{k}U \quad \text{for every} \quad k \in \mathbf{N}. \tag{4.10}$$

As $\{f_n\}$ is collectively compact and $\gamma([0,1])$ is compact, there exist subsequences, say $\{x_k\}$ and $\{\gamma(t_k)\}$, converging respectively to $x \in \partial D$ and $y \in \gamma([0,1])$. But $\{f_n\}$ is e-convergent to φ, and thus given $\varepsilon > 0$ there exist $\delta > 0$ and $m_0 \in \mathbf{N}$ such that $f_n(x') \in \varphi(x) + \varepsilon U$ for every $x' \in U_{\overline{D}}(x, \delta)$ and $n \geq m_0$. Fix a $k_0 \geq m_0$ so that $x_k \in U_{\overline{D}}(x, \delta)$ for all $k \geq k_0$. Hence

$$f_{n_k}(x_k) \in \varphi(x) + \varepsilon U \quad \text{for every} \quad k \geq k_0. \tag{4.11}$$

From (4.10) and (4.11) we obtain

$$\gamma(t_k) \in x_k - r(\varphi(x) + \varepsilon U) + \frac{1}{k}U \subset x_k - F(x) + \left(\varepsilon L_r + \frac{1}{k}\right)U \quad \text{for every} \quad k \geq k_0,$$

where L_r is a Lipschitz constant for r. Letting $k \to +\infty$ it follows that $y \in x - F(x)$, a contradiction, as $x \in \partial D$ and $y \in A$. Therefore for some $\varepsilon > 0$ and $n_0 \in \mathbf{N}$, (4.9) holds.

Now (4.9) implies

$$C_\varepsilon \subset \mathbb{E} \setminus \bigcup_{x \in \partial D} (I - r \circ f_n)(x) \quad \text{for every} \quad n \geq n_0.$$

Since p and q are in C_ε, by a property of the Leray-Schauder topological degree one has $\deg(I - r \circ f_n, D, p) = \deg(I - r \circ f_n, D, q)$ for every $n \geq n_0$. Hence, by Definition 3.3, $d(I - \overline{r \circ \varphi}, D, p) = d(I - \overline{r \circ \varphi}, D, q)$. As $(r, \varphi) \in \mathit{Repr}(F)$ is arbitrary, (4.8) follows, completing the proof. □

§5. Applications to Fixed Point Theory

In this section we use our topological degree to obtain simple proofs of fixed point theorems for regular multifunctions. A result of Borsuk's type for $\mathcal{C}(\mathbb{E})$-valued multifunctions will be considered as well.

For any $F : A \to 2^{\mathbb{E}}$, where $\emptyset \neq A \subset \mathbb{E}$, a point $x \in A$ such that $x \in F(x)$ is called a *fixed point* of F. In the sequel 0 denotes the zero of the space $\mathbb{E}$.

Proposition 5.1. *Let C be a convex open bounded subset of $\mathbb{E}$ and let $0 \in C$. Let $F : \overline{C} \to \mathcal{B}(\mathbb{E})$ be a regular multifunction satisfying $F(x) \subset \overline{C}$ for every $x \in \overline{C}$. Then F has a fixed point.*

Proof. Without loss of generality we suppose that

$$0 \notin \bigcup_{x \in \partial C} (I - F)(x). \tag{5.1}$$

Now define $H : [0,1] \times \overline{C} \to \mathcal{B}(\mathbb{E})$ by $H(t,x) = tF(x)$ for every $(t,x) \in [0,1] \times \overline{C}$, and observe that

$$0 \notin \bigcup_{(t,x) \in [0,1] \times \partial C} (x - H(t,x)). \tag{5.2}$$

In fact, if (5.2) fails, for some $x \in \partial C$ and $t \in (0,1)$ we have $x \in tF(x)$, which is impossible because $F(x) \subset \overline{C}$ and 0 is an interior point of $\overline{C}$. Clearly I and $I - F$ are both in $\mathcal{V}_{r-mult}(\overline{C}, \mathcal{B}(\mathbb{E}))$. By Proposition 4.1, taking into account (5.2) and Proposition 4.3, one has $\mathrm{Deg}(I - F, C, 0) = 1$. Hence, by Proposition 4.2, F has a fixed point, completing the proof. □

Proposition 5.2. *Let C be as in Proposition* 5.1. *Let $F : \overline{C} \to \mathcal{B}(\mathbb{E})$ be a regular multifunction which satisfies the condition*

$$\inf_{y \in F(x)} [\|y - x\|^2 - \|y\|^2 + \|x\|^2] \geq 0 \quad \text{for every} \quad x \in \partial C. \tag{5.3}$$

Then F has a fixed point.

Proof. Without loss of generality we suppose that (5.1) is satisfied. With H as in the proof of Proposition 5.1, (5.2) holds. In the contrary case, for some $x \in \partial C$ and $t \in (0,1)$ we have $x \in tF(x)$. Now (5.3) implies

$$\left\| \frac{x}{t} - x \right\|^2 \geq \left\| \frac{x}{t} \right\|^2 - \|x\|^2,$$

and therefore $(1/t - 1)^2 \geq 1/t^2 - 1$, for $x \neq 0$. As $t \in (0,1)$, a contradiction follows, and thus (5.2) holds. The conclusion is as for Proposition 5.1. □

Proposition 5.3. *Let C be as in Proposition* 5.1. *Let $F : \overline{C} \to \mathcal{B}(\mathbb{E})$ be a regular multifunction which satisfies the Leray-Schauder condition*

$$x \in tF(x) \quad \text{for some} \quad x \in \partial C \text{ and } t > 0 \text{ implies} \quad t = 1. \tag{5.4}$$

Then F has a fixed point.

Proof. Without loss of generality we suppose that (5.1) is satisfied. With H as in the proof of Proposition 5.1, (5.2) holds. Otherwise, for some $x \in \partial C$ and $t \in (0,1]$ we have $x \in tF(x)$ and, by (5.4), $x \in F(x)$, a contradiction to (5.1). The conclusion follows as for Proposition 5.1. □

Proposition 5.4. *Let C be a non empty open bounded subset of $\mathbb{E}$. Let $F : \overline{C} \to \mathcal{B}(\mathbb{E})$ be a regular multifunction which satisfies the Rothe condition*:

$$\text{(5.5)} \qquad \text{there exists } u \in C \text{ such that } t(x-u) \notin F(x) - u \text{ for every } x \in \partial C \text{ and } t > 1.$$

Then F has a fixed point.

Proof. Without loss of generality we suppose that (5.1) holds. Clearly $I - u$ and $I - F$ are in $\mathcal{V}_{r-appr}(\overline{C}, \mathcal{B}(\mathbb{E}))$. Let $H : [0,1] \times \overline{C} \to \mathcal{B}(\mathbb{E})$ be given by $H(t,x) = (1-t)u + tF(x)$ for every $(t,x) \in [0,1] \times \overline{C}$. H satisfies (5.2). In the contrary case, for some $x \in \partial C$ and $t \in [0,1]$ we have $x \in (1-t)u + tF(x)$, and so $x - u \in t(F(x) - u)$. But $t \in (0,1)$, as $u \notin \partial C$ and $x \notin F(x)$, whence $(1/t)(x-u) \in F(x) - u$, a contradiction to (5.5). Therefore (5.2) holds. By Proposition 4.1, in view of (5.2) and Proposition 3.5, we have $\text{Deg}(I - F, C, 0) = \text{Deg}(I - u, C, 0) = 1$. By Proposition 4.3, F has a fixed point, completing the proof. □

The fixed point results of Propositions 5.1–5.4 are variants of theorems established by Kakutani [18] and Ky Fan [7], Altman [1], Leray and Schauder [20] and Rothe [25].

We conclude with a Borsuk's type result for $\mathcal{C}(\mathbb{E})$-valued multifunctions. Under different assumptions, multivalued versions of Borsuk's theorems have been previously obtained by Granas [14], Ma [22] and Lasry and Robert [19].

Proposition 5.5. *Let C be an open bounded symmetric subset of $\mathbb{E}$, and let $0 \in C$. Let $F : \overline{C} \to \mathcal{C}(\mathbb{E})$ be a h-u.s.c. and h-compact multifunction satisfying $0 \notin \bigcup_{x \in \partial C} (I - F)(x)$. If, in addition, F is odd on ∂C, i.e. $F(x) = -F(-x)$ for every $x \in \partial C$, then*

$$\text{(5.6)} \qquad \text{Deg}(I - F, C, 0) = 1 \pmod{2}.$$

Moreover, F has a fixed point.

Proof. By Proposition 3.1, $\mathcal{A}_F$ is non empty. Let $\{f_n\} \in \mathcal{A}_F$. For $n \in \mathbf{N}$ define $g_n : \overline{C} \to \mathbb{E}$ and $K_n : [0,1] \times \overline{C} \to \mathbb{E}$ by

$$g_n(x) = \frac{f_n(x) - f_n(-x)}{2} \quad \text{for every} \quad x \in \overline{C},$$

and

$$K_n(t,x) = (1-t)f_n(x) + tg_n(x) \quad \text{for every} \quad (t,x) \in [0,1] \times \overline{C}.$$

Clearly g_n is odd on $\overline{C}$.

There is $n_0 \in \mathbf{N}$ such that

$$0 \notin \bigcup_{n \geq n_0} \bigcup_{(t,x) \in [0,1] \times \partial C} (x - K_n(t,x)). \tag{5.7}$$

In the contrary case there are sequences $\{f_{n_k}\}$, and $\{(t_k, x_k)\} \subset [0,1] \times \partial C$, such that

$$x_k = (1 - t_k) f_{n_k}(x_k) + t_k \frac{f_{n_k}(x_k) - f_{n_k}(-x_k)}{2} \quad \text{for every} \quad k \in \mathbf{N}. \tag{5.8}$$

Since $\{f_n\}$ is collectively compact, $\{(t_k, x_k)\}$ contains a subsequence, say $\{(t_k, x_k)\}$, which converges to a point $(t,x) \in [0,1] \times \partial C$.

On the other hand $\{f_n\}$ is e-convergent to F and thus, given $\varepsilon > 0$, there exist $\delta > 0$ and $n_0 \in \mathbf{N}$ such that

$$f_n(x') \in F(x) + \varepsilon U \quad \text{for every} \quad x' \in U_{\overline{C}}(x, \delta), n \geq n_0,$$
$$f_n(x') \in F(-x) + \varepsilon U \quad \text{for every} \quad x' \in U_{\overline{C}}(-x, \delta), n \geq n_0.$$

Fix $k_0 \geq n_0$ so that $x_k \in U_{\overline{C}}(x, \delta)$ and $-x_k \in U_{\overline{C}}(-x, \delta)$ for all $k \geq k_0$. Hence

$$f_{n_k}(x_k) \in F(x) + \varepsilon U \quad f_{n_k}(-x_k) \in F(-x) + \varepsilon U \quad \text{for every} \quad k \geq k_0. \tag{5.9}$$

From (5.8), in view of (5.9) and the assumption that F is odd on ∂C, one has

$$x_k \in (1 - t_k)(F(x) + \varepsilon U) + t_k \frac{(F(x) + \varepsilon U) - (F(-x) + \varepsilon U)}{2} = F(x) + \varepsilon U.$$

Letting $k \to +\infty$, it follows that $x \in F(x)$, a contradiction. Therefore, for some $n_0 \in \mathbf{N}$, (5.7) holds.

In view of (5.7), the homotopy property of the Leray-Schauder topological degree implies

$$\deg(I - f_n, C, 0) = \deg(I - g_n, C, 0) \quad \text{for every} \quad n \geq n_0.$$

The right hand side is 1 (mod 2), for g_n is odd on $\overline{C}$, while when $n \to +\infty$ the left hand side tends to $\mathrm{Deg}(I - F, C, 0)$, a singleton set by Remark 3.6. Consequently (5.6) holds and, by Proposition 4.2, F has a fixed point. This completes the proof. □

Acknowledgement

This work was supported by the project *'Geometrical functional analysis in Banach spaces: variational principles and global approximation'* between

Italy and Bulgaria. The second named author thanks for the hospitality of the University Roma II, where a part of this work was done during his stay as a Visiting Professor in July 1998 and, with a fellowship of the above mentioned project, in July 1999.

References

[1] Altman, M., A fixed point theorem in Banach spaces, *Bull. Acad. Polon. Sci. Cl* III, **5** (1957), 89-92.

[2] Borisovitch, Y. G., Gelman, B. D., Myshkis, A. D. and Obukhovskii, V. V., Topological methods in the fixed point theory of multivalued mappings, *Russian Math. Surveys*, **35** (1980), 65-143.

[3] Cellina, A. and Lasota, A., A new approach to the definition of topological degree of multivalued mappings, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.*, **47** (1969), 434-440.

[4] Dawidowicz, A., Méthodes homologiques dans la théorie des applications et des champs de vecteurs sphériques dans les espaces de Banach, *Dissertationes Math.* (*Rozprawy Mat.*), **326** (1993), 1-50.

[5] De Blasi, F. S., Characterizations of certain classes of semicontinuous multifunctions by continuous approximations, *J. Math. Anal. Appl.*, **106** (1985), 1-18.

[6] De Blasi, F. S. and Myjak, J., A remark on the definition of topological degree for set-valued mappings, *J. Math. Anal. Appl.*, **92** (1983), 445-551.

[7] Fan, Ky Fixed points and minimax theorems in locally convex spaces, *Proc. Nat. Acad. Sc. U.S.*, **38** (1952), 121-126.

[8] Fitzpatrick, P. M. and Petryshyn, W. V., Fixed point theorems and fixed point index for multivalued mappings in cones, *J. London Math. Soc.* (2), **12** (1975), 75-85.

[9] Çeba, K. and Granas, A., Infinite dimensional cohomology theory, *J. Math. Pures Appl.*, **5** (1973), 147-270.

[10] Górniewicz, L., Homological methods in fixed point theory of multi-valued maps, *Dissertationes Math.* (*Rozprawy Mat.*), **129** (1976), 1-71.

[11] Górniewicz, L., Granas, A. and Kryszewski, W., On the homotopy method in the fixed point index theory of multi-valued mappings of compact absolute neighborhood retracts, *J. Math. Anal. Appl.*, **161** (1991), 457-473.

[12] Górniewicz, L. and Lassonde, M., Approximation and fixed points for compositions of R_δ-maps, *Topology Appl.*, **55** (1994), 239-250.

[13] Granas, A., Sur la notion de degré topologique pour une certaine classe de transformations multivalentes dans les espaces de Banach, *Bull. Acad. Polon. Sci., Série Sc. Math. Astronom. Phys.*, **7** (1959), 191-194.

[14] ———, Theorem on antipodes and theorems on fixed points for a certain class of multivalued mappings in Banach spaces, *Bull. Acad. Polon. Sci., Série Sc. Math. Astronom. Phys.*, **7** (1959), 271-275.

[15] Hu, S. and Papageorgiou, N. S., *Handbook of multivalued analysis*, Vol. I, Kluwer, Dordrecht, 1997.

[16] Hukuhara, M., Sur l'application semi-continue dont la valeur est un compact convexe, *Funkcial. Ekvac.*, **10** (1967), 43-66.

[17] Istrăţescu, V. I., *Fixed point theory*, Reidel, Dordrecht, 1981.

[18] Kakutani, S., A generalization of Brouwer's fixed point theorem, *Duke Math. J.*, **8** (1941), 457-459.

[19] Lasry, J. M. and Robert, R., *Analyse nonlineaire multivoque*, Cahier de Mathématiques de la décision, No 7611, CEREMADE, Université de Paris IX, Dauphine, 1976.

[20] Leray, J. and Schauder, J., Topologie et équations fonctionnelles, *Ann. Sci. École Norm. Sup. Sér.* 3, **51** (1934), 45-78.

[21] Lloyd, N. G., *Degree theory*, Cambridge University Press, Cambridge, 1978.

[22] Ma, T. W., Topological degree for set-valued compact vector fields in locally convex spaces, *Dissertationes Math.* (*Rozprawy Mat.*), **92** (1972), 1-43.

[23] Michael, E., Continuous selections I, *Ann. of Math.*, **63** (1956), 361-382.

[24] Petryshyn, W. V. and Fitzpatrick, P. M., A degree theory, fixed point theorems and mappings theorems for multivalued noncompact mappings, *Trans. Amer. Math. Soc.*, **194** (1974), 1-25.

[25] Rothe, E., Zur Theorie der topologischen Ordnung und der Vectorfelder in Banachshen Räumen, *Compositio Math.*, **5** (1937), 177-197.

[26] Schwartz, J. T., *Nonlinear functional analysis*, Gordon and Breach, New York, 1969.