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The Structure of Group C^*-algebras of the Generalized Dixmier Groups

By

Takahiro Sudo*

Abstract

In this paper we first analyze the algebraic structure of group C^*-algebras of the generalized Dixmier groups, and next consider that of group C^*-algebras of some Lie semi-direct products with multi-diagonal or diagonal actions. As an application, we estimate the stable rank and the connected stable rank of these C^*-algebras in terms of groups. Also, we show that some of these group C^*-algebras have no nontrivial projections.

§1. Introduction

Group C^*-algebras provide many important examples in some topics of the theory of C^*-algebras such as their representation theory, K-theory, extension theory, etc. (cf. [1], [2], [22]). The (algebraic) structure of group C^*-algebras in this paper means their composition series with well understood subquotients. The structure of group C^*-algebras for some connected Lie groups was examined by some mathematicians (cf. [5], [14], [18], [21] and [23]). In particular, the author [18] analyzed the structure of group C^*-algebras of the Lie semi-direct products $\mathbb{C}^n \rtimes_\alpha \mathbb{R}$ (we often omit the action's symbol α). However, the structure of group C^*-algebras for general Lie groups is still mysterious. On the other hand, the stable rank theory of C^*-algebras was initiated by M. A. Rieffel [12], who raised an interesting problem of determining the stable rank of group C^*-algebras of Lie groups in terms of groups. See [15], [18], [19] and [20] for some partial answers of this problem.

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*Department of Mathematical Sciences, Faculty of Science, University of the Ryukyus, Nishihara, Okinawa 903-0213, Japan.
e-mail: sudo@math.u-ryukyu.ac.jp

This paper is organized as follows. First of all, we consider the structure of group C^*-algebras of the generalized Dixmier groups. For analysis of their subquotients we use a Green's result [5, Corollary 15], a corollary of Green's imprimitivity theorem [6, Corollary 2.10], a Dixmier-Douady's result (cf. [4, Chapter 10]), and some techniques of Connes' foliation C^*-algebras ([2], [9]). These known results are used frequently in this paper. As a corollary, we estimate the stable rank and the connected stable rank of these group C^*-algebras. Moreover, it is shown that these group C^*-algebras have no nontrivial projections. We next investigate the case of Lie semi-direct products of $\mathbb{C}^n$ by connected Lie groups with multi-diagonal actions. Finally, we analyze the case of Lie semi-direct products of the product groups $\mathbb{R}^u \times \mathbb{C}^v$ by connected Lie groups with diagonal actions.

Notation. Let G be a Lie group, $C^*(G)$ its (full) group C^*-algebra (cf. [4, Part II]), and $\hat{G}_1$ the space of all 1-dimensional representations of G. Denote by $\mathfrak{A} \rtimes_\alpha G$ the C^*-crossed product of a C^*-algebra $\mathfrak{A}$ by G with an action α (we often omit the symbol α), (cf. [1]). Denote by $C_0(X)$ the C^*-algebra of all continuous complex-valued functions on a locally compact T_2-space X vanishing at infinity. Set $C_0(X) = C(X)$ when X is compact. We say that an action of G on X is wandering if any compact set of X is wandering under the action [5]. Let $\mathbb{K} = \mathbb{K}(H)$ be the C^*-algebra of all compact operators on a separable Hilbert space H.

Denote by $\mathrm{sr}(\mathfrak{A})$, $\mathrm{csr}(\mathfrak{A})$ the stable rank and the connected stable rank of a C^*-algebra $\mathfrak{A}$ respectively [12]. $\vee, \wedge$ respectively mean the maximum and the minimum.

Set $\dim_{\mathbb{C}}(X) = [\dim(X)/2] + 1$ where $\dim X$ is the covering dimension of a space X and $[x]$ means the greatest integer with $[x] \le x$. Let $\mathbb{R}_+$ be the space of all nonzero positive real numbers, and $\mathbb{T}^k$ the k-torus group (or space).

Basic formulas of stable ranks.

(F1): For an exact sequence $0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{I} \to 0$ of C^*-algebras,

$$\mathrm{sr}(\mathfrak{I}) \vee \mathrm{sr}(\mathfrak{A}/\mathfrak{I}) \le \mathrm{sr}(\mathfrak{A}) \le \mathrm{sr}(\mathfrak{I}) \vee \mathrm{sr}(\mathfrak{A}/\mathfrak{I}) \vee \mathrm{csr}(\mathfrak{A}/\mathfrak{I}), \quad \mathrm{csr}(\mathfrak{A}) \le \mathrm{csr}(\mathfrak{I}) \vee \mathrm{csr}(\mathfrak{A}/\mathfrak{I}).$$

(F2): For the C^*-tensor product $\mathfrak{A} \otimes \mathbb{K}$ for a C^*-algebra $\mathfrak{A}$,

$$\mathrm{sr}(\mathfrak{A} \otimes \mathbb{K}) = 2 \wedge \mathrm{sr}(\mathfrak{A}), \quad \mathrm{csr}(\mathfrak{A} \otimes \mathbb{K}) \le 2 \wedge \mathrm{csr}(\mathfrak{A}).$$

(F3): $\mathrm{sr}(C_0(X)) = \dim_{\mathbb{C}} X^+$, where X^+ means the one-point compactification of a locally compact T_2-space X, and

$\mathrm{csr}(C_0(\mathbb{R})) = 2$, $\mathrm{csr}(C_0(\mathbb{R}^2)) = 1$, and $\mathrm{csr}(C_0(\mathbb{R}^n)) = [(n+1)/2] + 1$, $n \geq 3$. See [10], [12] and [15] for (F1), (F2) and (F3).

§2. Group C^*-algebras of the Generalized Dixmier Groups

First of all, we review the structure of the generalized Heisenberg groups. Let H_{2n+1} be the real $(2n+1)$-dimensional generalized Heisenberg group of all the matrices:

$$g = (c, b, a) = \begin{pmatrix} 1 & a & c \\ 0_n^t & I_n & b^t \\ 0 & 0_n & 1 \end{pmatrix}$$

with $c \in \mathbb{R}$, $b = (b_1, \ldots, b_n), a = (a_1, \ldots, a_n), 0_n = (0, \ldots, 0) \in \mathbb{R}^n$, where I_n means the $n \times n$ identity matrix and $0_n^t, b^t$ respectively mean the transposes of $0_n, b$. The group H_{2n+1} is a simply connected nilpotent Lie group isomorphic to the semi-direct product $\mathbb{R}^{n+1} \rtimes_\alpha \mathbb{R}^n$ with the action α defined by $\alpha_a(c, b) = (c + \sum_{i=1}^n a_i b_i, b)$. It is obtained by definition of crossed products and the Fourier transform that

$$C^*(H_{2n+1}) \cong C^*(\mathbb{R}^{n+1}) \rtimes_\alpha \mathbb{R}^n \cong C_0(\mathbb{R}^{n+1}) \rtimes_{\hat{\alpha}} \mathbb{R}^n$$

where $\hat{\alpha}_a(l, m) = (l, (m_i + a_i l))$ for $l \in \mathbb{R}$, $m = (m_i) \in \mathbb{R}^n$. Since $\{0\} \times \mathbb{R}^n$ is fixed under $\hat{\alpha}$ and closed in $\mathbb{R}^{n+1}$, the following exact sequence is obtained:

$$0 \to C_0((\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n) \rtimes \mathbb{R}^n \to C_0(\mathbb{R}^{n+1}) \rtimes \mathbb{R}^n \to C_0(\mathbb{R}^{2n}) \to 0.$$

Moreover, $\hat{\alpha}$ on $(\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n$ is free and wandering. Green's result [5] implies that

$$\begin{aligned} C_0((\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n) \rtimes \mathbb{R}^n &\cong C_0(((\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n)/\mathbb{R}^n) \otimes \mathbb{K}(L^2(\mathbb{R}^n)) \\ &\cong C_0(\mathbb{R} \setminus \{0\}) \otimes \mathbb{K} \end{aligned}$$

where the orbit space $((\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n)/\mathbb{R}^n$ is homeomorphic to $\mathbb{R} \setminus \{0\}$.

We now give the following definition:

Definition. Denote by D_{6n+1} the real $(6n+1)$-dimensional generalized Dixmier group defined by the semi-direct product $\mathbb{C}^{2n} \rtimes_\beta H_{2n+1}$ with the action β as follows:

$$\beta_g(z, z') = ((e^{ia_i} z_i), (e^{ib_i} z_{n+i})), \quad z = (z_i)_{i=1}^n, z' = (z_{n+i})_{i=1}^n \in \mathbb{C}^n, g \in H_{2n+1},$$

$$\beta_g = \begin{pmatrix} e^{ia_1} & & 0 \\ & \ddots & \\ 0 & & e^{ia_n} \end{pmatrix} \oplus \begin{pmatrix} e^{ib_1} & & 0 \\ & \ddots & \\ 0 & & e^{ib_n} \end{pmatrix} \in GL_{2n}(\mathbb{C}).$$

The group D_{6n+1} is a simply connected solvable Lie group of non type I. When $n = 1$, D_7 is said to be the Dixmier group [3]. It is obtained by the Fourier transform that

$$C^*(D_{6n+1}) \cong C^*(\mathbb{C}^{2n}) \rtimes_\beta H_{2n+1} \cong C_0(\mathbb{C}^{2n}) \rtimes_{\hat{\beta}} H_{2n+1},$$

where $\hat{\beta}_g(w, w') = ((e^{-ia_i} w_i), (e^{-ib_i} w_{n+i}))$ for $w = (w_i), w' = (w_{n+i}) \in \mathbb{C}^n$. Since the origin $0_{2n} \in \mathbb{C}^{2n}$ is fixed under $\hat{\beta}$ and closed in $\mathbb{C}^{2n}$, we have that

$$0 \to C_0(\mathbb{C}^{2n} \setminus \{0_{2n}\}) \rtimes H_{2n+1} \to C_0(\mathbb{C}^{2n}) \rtimes H_{2n+1} \to C^*(H_{2n+1}) \to 0.$$

Moreover, since the subspace $\mathbb{C} \setminus \{0\}$ in each direct factor of $\mathbb{C}^{2n}$ is $\hat{\beta}$-invariant and closed in $\mathbb{C}^{2n} \setminus \{0_{2n}\}$, it is obtained that

$$0 \to C_0(X_1) \rtimes H_{2n+1} \to C_0(\mathbb{C}^{2n} \setminus \{0_{2n}\}) \rtimes H_{2n+1} \to \oplus^{2n} C_0(\mathbb{C} \setminus \{0\}) \rtimes H_{2n+1} \to 0$$

where X_1 means the complement of the disjoint union $\sqcup^{2n} \mathbb{C} \setminus \{0\}$ of all $\mathbb{C} \setminus \{0\}$ in $\mathbb{C}^{2n} \setminus \{0_{2n}\}$. Since the direct products of either $\mathbb{C} \setminus \{0\}$ or $\{0\}$ in direct factors of $\mathbb{C}^{2n}$, homeomorphic to $(\mathbb{C} \setminus \{0\})^k$ for $2 \le k \le 2n - 1$ are invariant under $\hat{\beta}$, the following exact sequences ($2 \le k \le 2n - 1$) are obtained inductively:

$$\begin{aligned} 0 \to C_0(X_k) \rtimes H_{2n+1} &\to C_0(X_{k-1}) \rtimes H_{2n+1} \\ &\to \oplus_{1 \le i_1 < \cdots < i_k \le 2n} C_0((\mathbb{C} \setminus \{0\})^k) \rtimes H_{2n+1} \to 0 \end{aligned}$$

with $X_{k-1} \setminus X_k = \sqcup^{\binom{2n}{k}} (\mathbb{C} \setminus \{0\})^k$ and $X_{2n-1} = (\mathbb{C} \setminus \{0\})^{2n}$, where $\oplus_{1 \le i_1 < \cdots < i_k \le 2n}$ means the combination $\binom{2n}{k}$-direct sum. Since $\hat{\beta}$ on $(\mathbb{C} \setminus \{0\})^k$ is the multi-rotation, $C_0((\mathbb{C} \setminus \{0\})^k) \rtimes H_{2n+1}$ is isomorphic to $C_0(\mathbb{R}_+^k) \otimes (C(\mathbb{T}^k) \rtimes H_{2n+1})$. Moreover, the action $\hat{\beta}$ on $\mathbb{T}^k$ is transitive. Thus Green's result [6] implies that

$$C(\mathbb{T}^k) \rtimes H_{2n+1} \cong C(H_{2n+1}/(H_{2n+1})_{1_k}) \rtimes H_{2n+1} \cong C^*((H_{2n+1})_{1_k}) \otimes \mathbb{K}(L^2(\mathbb{T}^k))$$

where $(H_{2n+1})_{1_k}$ is the stabilizer of $1_k \in \mathbb{T}^k$.

Summing up the above argument, the following theorem is obtained:

Theorem 2.1. *The C^*-algebra $C^*(D_{6n+1})$ has a finite composition series $\{\mathfrak{I}_j\}_{j=1}^{2n+1}$ with each subquotient $\mathfrak{I}_{2n+1-k}/\mathfrak{I}_{2n-k}$ isomorphic to $C^*(H_{2n+1})$ for $k = 0$, and*

$$\oplus_{1 \le i_1 < \cdots < i_k \le 2n} C_0(\mathbb{R}_+^k) \otimes C^*((H_{2n+1})_{1_k}) \otimes \mathbb{K}(L^2(\mathbb{T}^k)) \quad \text{for } 1 \le k \le 2n.$$

We next analyze the structure of group C^*-algebras of the stabilizers $(H_{2n+1})_{1_k}$ in the following. Note that $D_{6n+1} = (\mathbb{C}^n \times \mathbb{C}^n) \rtimes_\beta H_{2n+1}$.

Case 1. First suppose that 1_k is contained in $\mathbb{C}^n \times \{0_n\}$. Then we may have that $(H_{2n+1})_{1_k}$ is isomorphic to $\mathbb{R}^{n+1} \rtimes_\alpha (\mathbb{Z}^k \times \mathbb{R}^{n-k})$. It is obtained by the Fourier transform that

$$C^*((H_{2n+1})_{1_k}) \cong C^*(\mathbb{R}^{n+1}) \rtimes_\alpha (\mathbb{Z}^k \times \mathbb{R}^{n-k}) \cong C_0(\mathbb{R}^{n+1}) \rtimes_{\hat{\alpha}} (\mathbb{Z}^k \times \mathbb{R}^{n-k}),$$

where $\hat{\alpha}_a(l,m) = (l,(m_i + a_i l))$ for $l \in \mathbb{R}, m = (m_i) \in \mathbb{R}^n, a \in \mathbb{Z}^k \times \mathbb{R}^{n-k}$. Since $\{0\} \times \mathbb{R}^n$ is fixed under $\hat{\alpha}$ and closed in $\mathbb{R}^{n+1}$, the following exact sequence is obtained:

$$\begin{aligned} 0 \to\ & C_0((\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n) \rtimes (\mathbb{Z}^k \times \mathbb{R}^{n-k}) \\ & \to C_0(\mathbb{R}^{n+1}) \rtimes_{\hat{\alpha}} (\mathbb{Z}^k \times \mathbb{R}^{n-k}) \to C_0(\mathbb{T}^k \times \mathbb{R}^{2n-k}) \to 0. \end{aligned}$$

Moreover, the action of $\mathbb{Z}^k \times \mathbb{R}^{n-k}$ on $(\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n$ is free and wandering, so that Green's result [6] implies that

$$\begin{aligned} & C_0((\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n) \rtimes (\mathbb{Z}^k \times \mathbb{R}^{n-k}) \\ & \quad \cong C_0(((\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n)/(\mathbb{Z}^k \times \mathbb{R}^{n-k})) \otimes \mathbb{K}(L^2(\mathbb{Z}^k \times \mathbb{R}^{n-k})). \end{aligned}$$

Furthermore, since the orbit of the point $(l,m) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n$ is parameterized with the point $(l,(m_i \bmod l)_{i=1}^k)$, the orbit space $((\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n)/(\mathbb{Z}^k \times \mathbb{R}^{n-k})$ has the fiber structure whose base space is $\mathbb{R} \setminus \{0\}$ and fibers are $\mathbb{T}^k$. This orbit space splits into the product space $(\mathbb{R} \setminus \{0\}) \times \mathbb{T}^k$ since any orbit in $(\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n$ has the same type.

Case 2. Next suppose that 1_k is contained in $\{0_n\} \times \mathbb{C}^n$. Then the stabilizer $(H_{2n+1})_{1_k}$ is isomorphic to $(\mathbb{R} \times (\mathbb{Z}^k \times \mathbb{R}^{n-k})) \rtimes_\alpha \mathbb{R}^n$. By the Fourier transform,

$$C^*((H_{2n+1})_{1_k}) \cong C^*(\mathbb{R} \times (\mathbb{Z}^k \times \mathbb{R}^{n-k})) \rtimes_\alpha \mathbb{R}^n \cong C_0(\mathbb{R} \times (\mathbb{T}^k \times \mathbb{R}^{n-k})) \rtimes_{\hat{\alpha}} \mathbb{R}^n,$$

where $\hat{\alpha}_a(l,m) = (l,(e^{i(m_i + a_i l)})_{i=1}^k, (m_i + a_i l)_{i=k+1}^n)$ for $m = ((e^{im_i})_{i=1}^k, (m_i)_{i=k+1}^n) \in \mathbb{T}^k \times \mathbb{R}^{n-k}$, $l \in \mathbb{R}, a \in \mathbb{R}^n$. Since $\{0\} \times \mathbb{T}^k \times \mathbb{R}^{n-k}$ is fixed under $\hat{\alpha}$ and closed in $\mathbb{R} \times \mathbb{T}^k \times \mathbb{R}^{n-k}$, the following exact sequence is obtained:

$$\begin{aligned} 0 \to\ & C_0((\mathbb{R} \setminus \{0\}) \times \mathbb{T}^k \times \mathbb{R}^{n-k}) \rtimes \mathbb{R}^n \\ & \to C_0(\mathbb{R} \times \mathbb{T}^k \times \mathbb{R}^{n-k}) \rtimes_{\hat{\alpha}} \mathbb{R}^n \to C_0(\mathbb{T}^k \times \mathbb{R}^{2n-k}) \to 0. \end{aligned}$$

Moreover, the above ideal is decomposed into $\oplus^2 C_0(\mathbb{R}_+ \times \mathbb{T}^k \times \mathbb{R}^{n-k}) \rtimes \mathbb{R}^n$ since two connected components of $(\mathbb{R} \setminus \{0\}) \times \mathbb{T}^k \times \mathbb{R}^{n-k}$ are $\hat{\alpha}$-invariant, and each direct factor is assumed to be the C^*-algebra of continuous fields over $\mathbb{R}_+$ with the fibers $C_0(\mathbb{T}^k \times \mathbb{R}^{n-k}) \rtimes \mathbb{R}^n$, and denoted by $C_0(\mathbb{R}_+, \cup_{\mathbb{R}_+} C_0(\mathbb{T}^k \times \mathbb{R}^{n-k}) \rtimes_{\hat{\alpha}}$

$\mathbb{R}^n$). The action $\hat{\alpha}$ on $\mathbb{T}^k \times \mathbb{R}^{n-k}$ is transitive. Thus, it is obtained by Green's result [4] that

$$\begin{aligned} C_0(\mathbb{T}^k \times \mathbb{R}^{n-k}) \rtimes_{\hat{\alpha}} \mathbb{R}^n &\cong C_0(\mathbb{R}^n/(\mathbb{R}^n)_{(l,m)}) \rtimes \mathbb{R}^n \\ &\cong C^*((\mathbb{R}^n)_{(l,m)}) \otimes \mathbb{K}(L^2(\mathbb{T}^k \times \mathbb{R}^{n-k})) \cong C(\mathbb{T}^k) \otimes \mathbb{K} \end{aligned}$$

where $(\mathbb{R}^n)_{(l,m)}$ is the stabilizer of (l,m), isomorphic to $\mathbb{Z}^k$. Since the cohomology group $H^3(\mathbb{R},\mathbb{Z})$ vanishes, it is obtained by [4] that $C_0(\mathbb{R}_+, \cup_{\mathbb{R}_+} C_0(\mathbb{T}^k \times \mathbb{R}^{n-k}) \rtimes \mathbb{R}^n) \cong C_0(\mathbb{R} \times \mathbb{T}^k) \otimes \mathbb{K}$.

Case 3. We consider the other cases such that 1_k is not contained in $\mathbb{C}^n \times \{0_n\}$ and $\{0_n\} \times \mathbb{C}^n$. We may assume that $(H_{2n+1})_{1_k} \cong (\mathbb{R} \times \mathbb{Z}^{k_1} \times \mathbb{R}^{n-k_1}) \rtimes (\mathbb{Z}^{k_2} \times \mathbb{R}^{n-k_2})$ for $k = k_1 + k_2$, where $1 \le k_1 = k_2 \le n$, or $1 \le k_1 < k_2 \le n$, or $n \ge k_1 > k_2 \ge 1$. In each case, it is obtained by the Fourier transform that

$$\begin{aligned} C^*((H_{2n+1})_{1_k}) &\cong C^*(\mathbb{R} \times \mathbb{Z}^{k_1} \times \mathbb{R}^{n-k_1}) \rtimes_{\alpha} (\mathbb{Z}^{k_2} \times \mathbb{R}^{n-k_2}) \\ &\cong C_0(\mathbb{R} \times \mathbb{T}^{k_1} \times \mathbb{R}^{n-k_1}) \rtimes_{\hat{\alpha}} (\mathbb{Z}^{k_2} \times \mathbb{R}^{n-k_2}), \end{aligned}$$

where the action $\hat{\alpha}$ is defined by $\hat{\alpha}_a(l,m) = (l, (e^{i(m_i + a_i l)})_{i=1}^{k_1}, (m_j + a_j l)_{j=k_1+1}^{n})$ for $(l,m) = (l, (e^{im_i})_{i=1}^{k_1}, (m_i)_{i=k_1+1}^{n}) \in \mathbb{R} \times \mathbb{T}^{k_1} \times \mathbb{R}^{n-k_1}$, $a \in \mathbb{Z}^{k_1} \times \mathbb{R}^{n-k_2}$. Since $\{0\} \times \mathbb{T}^{k_1} \times \mathbb{R}^{n-k_1}$ is fixed under $\hat{\alpha}$ and closed in $\mathbb{R} \times \mathbb{T}^{k_1} \times \mathbb{R}^{n-k_1}$, it follows that

$$\begin{aligned} 0 &\to C_0((\mathbb{R} \setminus \{0\}) \times \mathbb{T}^{k_1} \times \mathbb{R}^{n-k_1}) \rtimes (\mathbb{Z}^{k_2} \times \mathbb{R}^{n-k_2}) \\ &\to C_0(\mathbb{R} \times \mathbb{T}^{k_1} \times \mathbb{R}^{n-k_1}) \rtimes (\mathbb{Z}^{k_2} \times \mathbb{R}^{n-k_2}) \to C_0(\mathbb{T}^k \times \mathbb{R}^{2n-k}) \to 0. \end{aligned}$$

The above ideal is decomposed into $\oplus^2 C_0(\mathbb{R}_+ \times \mathbb{T}^{k_1} \times \mathbb{R}^{n-k_1}) \rtimes (\mathbb{Z}^{k_2} \times \mathbb{R}^{n-k_2})$ since two connected components of $(\mathbb{R} \setminus \{0\}) \times \mathbb{T}^{k_1} \times \mathbb{R}^{n-k_1}$ are $\hat{\alpha}$-invariant. Then each direct factor of the above decomposition is regarded as the C^*-algebra of continuous fields over $\mathbb{R}_+$ with the fibers $C_0(\mathbb{T}^{k_1} \times \mathbb{R}^{n-k_1}) \rtimes_{\theta} (\mathbb{Z}^{k_2} \times \mathbb{R}^{n-k_2})$, and denoted by

$$C_0(\mathbb{R}_+, \cup_{\theta \in \mathbb{R}_+} C_0(\mathbb{T}^{k_1} \times \mathbb{R}^{n-k_1}) \rtimes_{\theta} (\mathbb{Z}^{k_2} \times \mathbb{R}^{n-k_2})),$$

where the action θ corresponds to the restriction of $\hat{\alpha}$ to $\{\theta\} \times \mathbb{T}^{k_1} \times \mathbb{R}^{n-k_1}$. Since each direct factor of $\mathbb{Z}^{k_2} \times \mathbb{R}^{n-k_2}$ acts on one of direct factors of $\mathbb{T}^{k_1} \times \mathbb{R}^{n-k_1}$ componentwise, each fiber is isomorphic to one of the following tensor products:

$$\begin{cases} (\otimes^{k_1} C(\mathbb{T}) \times_{\theta} \mathbb{Z}) \otimes (\otimes^{n-k_1} C_0(\mathbb{R}) \rtimes \mathbb{R}) & k_1 = k_2 \\ (\otimes^{k_1} C(\mathbb{T}) \times_{\theta} \mathbb{Z}) \otimes (\otimes^{k_2 - k_1} C_0(\mathbb{R}) \rtimes \mathbb{Z}) \otimes (\otimes^{n-k_2} C_0(\mathbb{R}) \rtimes \mathbb{R}) & k_1 < k_2 \\ (\otimes^{k_2} C(\mathbb{T}) \times_{\theta} \mathbb{Z}) \otimes (\otimes^{k_1 - k_2} C(\mathbb{T}) \rtimes \mathbb{R}) \otimes (\otimes^{n-k_2} C_0(\mathbb{R}) \rtimes \mathbb{R}) & k_1 > k_2, \end{cases}$$

which is also proved by considering correspondence between generators of each fiber and those of tensor products. The above tensor factors have the following isomorphisms:

$$C_0(\mathbb{R}) \rtimes \mathbb{R} \cong \mathbb{K}, \quad C_0(\mathbb{R}) \rtimes \mathbb{Z} \cong C(\mathbb{T}) \otimes \mathbb{K}, \quad C(\mathbb{T}) \rtimes \mathbb{R} \cong C(\mathbb{T}) \otimes \mathbb{K}$$

since each action is the shift, and $C(\mathbb{T}) \rtimes_\theta \mathbb{Z} \equiv \mathfrak{A}_\theta$ is the irrational or rational rotation algebra. Thus, each fiber is isomorphic to one of the following:

$$\begin{cases} \otimes^n \mathfrak{A}_\theta \text{ for } k = 2n, \text{ and } (\otimes^{k_1}\mathfrak{A}_\theta) \otimes \mathbb{K} & \text{for } k = 2k_1 \leq 2n-2, \\ (\otimes^{k_1}\mathfrak{A}_\theta) \otimes C(\mathbb{T}^{k_2-k_1}) \otimes \mathbb{K} & \text{for } k_1 < k_2, \\ (\otimes^{k_2}\mathfrak{A}_\theta) \otimes C(\mathbb{T}^{k_1-k_2}) \otimes \mathbb{K} & \text{for } k_1 > k_2. \end{cases}$$

Summing up the above argument, the following theorem is deduced:

Theorem 2.2. *The group C^*-algebras $C^*((H_{2n+1})_{1_k})$ of the stabilizers $(H_{2n+1})_{1_k}$ have the following decompositions*:

$$0 \to \mathfrak{L}_k \to C^*((H_{2n+1})_{1_k}) \to C(\mathbb{T}^k) \otimes C_0(\mathbb{R}^{2n-k}) \to 0$$

for $0 \leq k \leq 2n$ and $\mathfrak{L}_k$ is isomorphic to

$$\begin{cases} C_0(\mathbb{R}\setminus\{0\}) \otimes \mathbb{K} \quad \textit{for } k = 0, \textit{ and } C_0((\mathbb{R}\setminus\{0\}) \times \mathbb{T}) \otimes \mathbb{K} \quad \textit{for } k = 1, \\ C_0((\mathbb{R}\setminus\{0\}) \times \mathbb{T}^k) \otimes \mathbb{K} \quad \textit{or} \\ C_0(\mathbb{R}\setminus\{0\}, \cup_{\theta\in\mathbb{R}\setminus\{0\}}((\otimes^{s_1}\mathfrak{A}_\theta) \otimes C(\mathbb{T}^{s_2}) \otimes \mathbb{K})) \quad \textit{for } 2 \leq k \leq n, \\ C_0(\mathbb{R}\setminus\{0\}, \cup_{\theta\in\mathbb{R}\setminus\{0\}}((\otimes^{s_1}\mathfrak{A}_\theta) \otimes C(\mathbb{T}^{s_2}) \otimes \mathbb{K})) \quad \textit{for } n+1 \leq k \leq 2n-1, \\ C_0(\mathbb{R}\setminus\{0\}, \cup_{\theta\in\mathbb{R}\setminus\{0\}} \otimes^n \mathfrak{A}_\theta) \quad \textit{for } k = 2n \end{cases}$$

with $s_1 \geq 1$, $s_2 \geq 0$, $2s_1 + s_2 = k$.

Remark. Let Γ be the discrete central subgroup of both H_{2n+1} and D_{6n+1} defined by

$$\Gamma = \left\{ \begin{pmatrix} 1 & 0_n & 2\pi k \\ 0_n^t & I_n & 0_n^t \\ 0 & 0_n & 1 \end{pmatrix} : k \in \mathbb{Z} \right\}.$$

Then $D_{6n+1}/\Gamma \cong \mathbb{C}^{2n} \rtimes (H_{2n+1}/\Gamma)$. If H_{2n+1} is replaced by H_{2n+1}/Γ in the above theorem, then $C^*((H_{2n+1}/\Gamma)_{1_k}) \cong (\oplus_{\mathbb{Z}\setminus\{0\}} C(\mathbb{T}^k) \otimes \mathbb{K}) \oplus C_0(\mathbb{T}^k \times \mathbb{R}^{2n-k})$ for $0 \leq k \leq 2n$. It follows that $C^*(D_{6n+1}/\Gamma)$ is of type I while $C^*(D_{6n+1})$ is of non type I (cf. [3]).

Taking a refinement of the composition series of Theorems 2.1 and 2.2, we obtain

Theorem 2.3. *There exists a finite composition series* $\{\mathfrak{K}_j\}_{j=1}^K$ *of* $C^*(D_{6n+1})$ *with its subquotients* $\mathfrak{K}_j/\mathfrak{K}_{j-1}$ *given by* $C_0(\mathbb{R}^{2n})$ *for* $j = K$*, and*

$$\begin{cases} C_0(\mathbb{R}) \otimes \mathbb{K}, \quad or \quad C_0(\mathbb{T}^k \times \mathbb{R}^{2n}) \otimes \mathbb{K}, \quad or \quad C_0(\mathbb{T}^k \times \mathbb{R}^{k+1}) \otimes \mathbb{K}, \quad or \\ C_0(\mathbb{R}_+^k) \otimes \mathbb{K} \otimes C_0(\mathbb{R} \setminus \{0\}, \cup_{\theta \in \mathbb{R}\setminus\{0\}}((\otimes^{s_1}\mathfrak{A}_\theta) \otimes C(\mathbb{T}^{s_2}) \otimes \mathbb{K})) \end{cases}$$

for $1 \le j \le K-1$ *with* $1 \le k \le 2n$*,* $s_1 \ge 1$*,* $s_2 \ge 0$*,* $2s_1 + s_2 = k$*.*

Remark. The C^*-tensor product $(\otimes^{s_1}\mathfrak{A}_\theta) \otimes C(\mathbb{T}^{s_2})$ is isomorphic to the crossed product $C(\mathbb{T}^{s_1} \times \mathbb{T}^{s_2}) \rtimes \mathbb{Z}^{s_1}$ which is a special case of noncommutative tori. We see that $C^*(D_{6n+1})$ has $\mathbb{K}$ and $\mathbb{K} \otimes (\otimes^{s_1}\mathfrak{A}_\theta)$ for θ irrational as simple subquotients (cf. [6], [11]).

Applying (F1), (F2), (F3) to the composition series of Theorem 2.3, it follows that

Corollary 2.4. *For the group* C^**-algebra* $C^*(D_{6n+1})$*, it holds that*

$$\mathrm{sr}(C^*(D_{6n+1})) = n+1 = \dim_{\mathbb{C}}(D_{6n+1})_1^\wedge, \quad and \quad 2 \le \mathrm{csr}(C^*(D_{6n+1})) \le n+1.$$

Proof. Note that Theorem 2.3 implies that the space $(D_{6n+1})_1^\wedge$ of all 1-dimensional representations of D_{6n+1} is homeomorphic to $\mathbb{R}^{2n}$. By Theorem 2.3 and (F2), it is obtained that $\mathrm{sr}(\mathfrak{K}_j/\mathfrak{K}_{j-1}) \le 2$ and $\mathrm{csr}(\mathfrak{K}_j/\mathfrak{K}_{j-1}) \le 2$ for $1 \le j \le K-1$. Inductively applying (F1) to the composition series of Theorem 2.3, $\mathrm{sr}(\mathfrak{K}_j) \le 2$ and $\mathrm{csr}(\mathfrak{K}_j) \le 2$ for $1 \le j \le K-1$. Therefore, it is obtained by (F1) and (F3) that

$$\begin{aligned} \mathrm{sr}(C_0(\mathbb{R}^{2n})) = n+1 \le \mathrm{sr}(C^*(D_{6n+1})) &\le 2 \vee \mathrm{sr}(C_0(\mathbb{R}^{2n})) \\ &\quad \vee \mathrm{csr}(C_0(\mathbb{R}^{2n})) = n+1, \\ \mathrm{csr}(C^*(D_{6n+1})) &\le 2 \vee \mathrm{csr}(C_0(\mathbb{R}^{2n})) = n+1. \end{aligned}$$

On the other hand, note that D_{6n+1} is isomorphic to $((\mathbb{R}^{4n}) \rtimes \mathbb{R}^{n+1}) \rtimes \mathbb{R}^n)$, where $(z, z', g) \mapsto ((z, z'), (c, b), a)$. Thus, $C^*(D_{6n+1}) \cong (C_0(\mathbb{R}^{4n}) \rtimes \mathbb{R}^{n+1}) \rtimes \mathbb{R}^n$. By using Connes' Thom isomorphism for K-groups of C^*-algebras (cf. [2], [22]), $K_1(C^*(D_{6n+1})) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$. By Hassan's result [7], $\mathrm{csr}(C^*(D_{6n+1})) \ge 2$. $\square$

Remark. For D_{6n+1}/Γ with Γ in the remark of Theorem 2.2, it is obtained that

$$\mathrm{sr}(C^*(D_{6n+1}/\Gamma)) = n+1 = \dim_{\mathbb{C}}(D_{6n+1}/\Gamma)_1^\wedge, \quad \mathrm{csr}(C^*(D_{6n+1}/\Gamma)) \le n+1.$$

It follows from the composition series of Theorem 2.3 that

Corollary 2.5. *The group C^*-algebra $C^*(D_{6n+1})$ has no nontrivial projections.*

Proof. Notice that if a nontrivial projection exists in a C^*-algebra, its image in any quotient is a nontrivial projection or zero. On the other hand, each subquotients of $C^*(D_{6n+1})$ has no nontrivial projections since each subquotient $\mathfrak{K}_j/\mathfrak{K}_{j-1}$ $(1 \le j \le K)$ has a commutative C^*-algebra on a noncompact connected space as a tensor factor. □

Remark. There exist nontrivial projections in $\mathbb{K}$ and $\mathbb{K} \otimes (\otimes^{s_1}\mathfrak{A}_\theta)$ for θ irrational.

§3. The Lie Semi-direct Products of $\mathbb{C}^n$ by Connected Lie Groups with Multi-diagonal Actions

Let G be a connected Lie group defined by the semi-direct product $\mathbb{C}^n \rtimes_\alpha N$ with N a connected Lie group. The action α is also a Lie group homomorphism from N to $GL_n(\mathbb{C})$. Denote by $\mathrm{d}\alpha$ the differential of α from the Lie algebra $\mathfrak{N}$ of N to the Lie algebra $M_n(\mathbb{C})$ of all $n \times n$ matrices over $\mathbb{C}$. Moreover, suppose that the action α is induced from the following commutative diagram:

$$\begin{array}{ccccc} N & \longrightarrow & N/[N,N] & \xrightarrow{\alpha} & GL_n(\mathbb{C}) \\ {\scriptstyle\exp}\uparrow & & {\scriptstyle\exp}\uparrow & & {\scriptstyle\exp}\uparrow \\ \mathfrak{N} & \longrightarrow & \mathfrak{N}/[\mathfrak{N},\mathfrak{N}] & \xrightarrow{\mathrm{d}\alpha} & M_n(\mathbb{C}) \end{array}$$

where exp means the exponential map, and $[N,N]$, $[\mathfrak{N},\mathfrak{N}]$ mean the commutators of N and $\mathfrak{N}$ respectively. Then $N/[N,N] \cong \mathbb{R}^{l-m} \times \mathbb{T}^m$ and $\mathfrak{N}/[\mathfrak{N},\mathfrak{N}] \cong \mathbb{R}^l$ for some $l \ge 0$ and $0 \le m \le l$. First suppose that α is a complex 1-dimensional, multi-diagonal action of the form:

$$\alpha_t = \begin{pmatrix} e^{\lambda_1 t_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t_n} \end{pmatrix} \in GL_n(\mathbb{C})$$

with $t = ((t_i)_{i=1}^{n-m}, (e^{it_j})_{j=n-m+1}^{n}) \in \mathbb{R}^{n-m} \times \mathbb{T}^m$, $\lambda_i \in \mathbb{C}$ $(1 \le i \le n-m)$, $\lambda_j \in i\mathbb{R}$ $(n-m+1 \le j \le n)$. We may assume that $\lambda_k = 0$ for $1 \le k \le n_0$, $\lambda_k \notin i\mathbb{R}$ for $n_0+1 \le k \le n_1$, and $\lambda_k \in i(\mathbb{R} \setminus \{0\})$ for $n_1+1 \le k \le n$ with $n_0, n_1 \ge 0$. Note that if the action α of N is diagonal on $\mathbb{C}^n$, it reduces to that of $N/[N,N]$ automatically.

Under the above situation, the following theorem is obtained:

Theorem 3.1. *Let G be a Lie semi-direct product of $\mathbb{C}^n$ by a connected Lie group N with a complex 1-dimensional, multi-diagonal action. Then $C^*(G)$ has a finite composition series $\{\mathfrak{I}_j\}_{j=1}^{n+1}$ with $\mathfrak{I}_{n-n_0-k+1}/\mathfrak{I}_{n-n_0-k}$ isomorphic to $C_0(\mathbb{C}^{n_0}) \otimes C^*(N)$ for $k = 0$, and*

$$\oplus_{n_0+1\le i_1<\cdots<i_k\le n} \begin{cases} C_0(\mathbb{C}^{n_0} \times \mathbb{T}^k) \otimes \mathbb{K} \\ C_0(\mathbb{C}^{n_0} \times \mathbb{T}^{k_1} \times \mathbb{R}^{k_2}) \otimes C^*(N_{1_k}) \otimes \mathbb{K} \end{cases} \quad \text{for } 1 \le k \le n - n_0$$

with $0 \le n_0 \le n$ and $k_2 \ge 1$, $k = k_1 + k_2$, where $\mathbb{C}^{n_0}$ is the fixed point subspace of $\mathbb{C}^n$ under the action of N, and the first alternative corresponds to that the action of N on invariant subspaces $(\mathbb{C} \setminus \{0\})^k$ of $\mathbb{C}^n$ is free and wandering.

Proof. Since the action of N on $\mathbb{C}^{n_0}$ is trivial, it follows that

$$C^*(G) \cong C^*(\mathbb{C}^n) \rtimes_\alpha N \cong C_0(\mathbb{C}^n) \rtimes_{\hat{\alpha}} N \cong C_0(\mathbb{C}^{n_0}) \otimes (C_0(\mathbb{C}^{n-n_0}) \rtimes N)$$

where $\hat{\alpha}_t(z_i) = (e^{\bar{\lambda}_i t_i} z_i)$ for $(z_i) \in \mathbb{C}^n$. By the same argument before Theorem 2.1, we obtain a finite composition series $\{\mathfrak{I}_j\}_{j=1}^{n-n_0+1}$ of $C_0(\mathbb{C}^{n-n_0}) \rtimes N$ with subquotients $\mathfrak{I}_{n-n_0-k+1}/\mathfrak{I}_{n-n_0-k}$ isomorphic to $C^*(N)$ for $k = 0$, and

$$\oplus_{n_0+1\le i_1<\cdots<i_k\le n} C_0((\mathbb{C} \setminus \{0\})^k) \rtimes_{\hat{\alpha}} N \quad \text{for } 1 \le k \le n - n_0.$$

For $0 \le k_1 \le n_1$ and $0 \le k_2 = l_1 + l_2 \le n - n_0 - n_1$ with $k = k_1 + k_2$, the action of N on each direct factor of $(\mathbb{C} \setminus \{0\})^{k_1}$ is free and wandering, and that on $(\mathbb{C} \setminus \{0\})^{l_1}$ is the multi-rotation by $\mathbb{R}^{l_1}$, and that on $(\mathbb{C} \setminus \{0\})^{l_2}$ is the multi-rotation by $\mathbb{T}^{l_2}$. If $k_2 = 0$, it is obtained by Green's result [5] that

$$C_0((\mathbb{C} \setminus \{0\})^k) \rtimes_{\hat{\alpha}} N \cong C_0((\mathbb{C} \setminus \{0\})^k/N) \otimes \mathbb{K} \cong C(\mathbb{T}^k) \otimes \mathbb{K},$$

where the orbit space $(\mathbb{C} \setminus \{0\})^k/N$ is homeomorphic to $\mathbb{T}^k$. Next suppose $k_2 \ge 1$. Note that for the restriction of the action of N to $(\mathbb{C} \setminus \{0\})^{k_1}$, the crossed product of $C_0((\mathbb{C} \setminus \{0\})^{k_1})$ by N has the same structure for whether $\lambda_{i_j} \notin i\mathbb{R}$ $(1 \le j \le k_1)$ are real or not. Thus we may assume that all λ_{i_j} $(1 \le j \le k_1)$ are real. Then the action of N on the circle direction of each direct factor of $(\mathbb{C} \setminus \{0\})^{k_1}$ is trivial, and that on the radius direction of each direct factor of $(\mathbb{C} \setminus \{0\})^{k_2}$ is also trivial. Hence it follows that

$$C_0((\mathbb{C} \setminus \{0\})^k) \rtimes_{\hat{\alpha}} N \cong C_0(\mathbb{T}^{k_1} \times \mathbb{R}_+^{k_2}) \otimes (C_0(\mathbb{R}_+^{k_1} \times \mathbb{T}^{k_2}) \rtimes N).$$

Since the action of N on $\mathbb{R}_+^{k_1} \times \mathbb{T}^{k_2}$ is transitive, it follows by Green's theorem [4] that

$$C_0(\mathbb{R}_+^{k_1} \times \mathbb{T}^{k_2}) \rtimes N \cong C_0(N/N_{1_k}) \rtimes N \cong C^*(N_{1_k}) \otimes \mathbb{K}$$

where N_{1_k} is the stabilizer of $1_k \in \mathbb{R}_+^{k_1} \times \mathbb{T}^{k_2}$. □

Remark. If $N = [N, N]$ in the above setting, then $C^*(G) \cong C_0(\mathbb{C}^n) \otimes C^*(N)$. Even if N is nilpotent, the structure of $C^*(N_{1_k})$ is still mysterious.

In the above setting, if G is a Lie semi-direct product of $\mathbb{R}^n$ by a connected Lie group N with a real 1-dimensional, multi-diagonal action ($\lambda_i \in \mathbb{R}$), it is obtained that

Theorem 3.2. *If G is a Lie semi-direct product of $\mathbb{R}^n$ by a connected Lie group N with a real 1-dimensional, multi-diagonal action ($\lambda_i \in \mathbb{R}$), then $C^*(G)$ has a finite composition series $\{\mathfrak{I}_j\}_{j=1}^n$ with $\mathfrak{I}_{n-n_0-k+1}/\mathfrak{I}_{n-n_0-k}$ isomorphic to $C_0(\mathbb{R}^{n_0}) \otimes C^*(N)$ for $k = 0$, and*

$$\oplus_{n_0+1 \le i_1 < \cdots < i_k \le n} C_0(\mathbb{R}^{n_0}) \otimes (\oplus^{2^k} \mathbb{K}) \quad \textit{for } 1 \le k \le n - n_0.$$

Proof. Since the action of $\mathbb{R}$ on $\mathbb{R}$ is trivial or the translation, and that of $\mathbb{T}$ on $\mathbb{R}$ is trivial, the action of N on each direct factor of $(\mathbb{R} \setminus \{0\})^k$ is free and wandering. Thus Green's result [5] implies that $C_0((\mathbb{R} \setminus \{0\})^k) \rtimes N \cong C_0((\mathbb{R} \setminus \{0\})^k / N) \otimes \mathbb{K} \cong \oplus^{2^k} \mathbb{K}$. $\square$

As a special case of Theorem 3.1, let $N = H_{2n+1}$ and $G = \mathbb{C}^{2n} \rtimes_\beta H_{2n+1}$. We assume that the action β on $\mathbb{C}^{2n}$ is the diagonal sum:

$$\beta_g = \begin{pmatrix} e^{\lambda_1 b_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n b_n} \end{pmatrix} \oplus \begin{pmatrix} e^{\mu_1 a_1} & & 0 \\ & \ddots & \\ 0 & & e^{\mu_n a_n} \end{pmatrix} \in GL_{2n}(\mathbb{C})$$

with $g = (c, b, a) \in H_{2n+1}$, $\lambda_i, \mu_i \in \mathbb{C}$ $(1 \le i \le n)$. Then it follows that

Proposition 3.3. *If the action of H_{2n+1} on $\mathbb{C}^{2n}$ is given as above, then group C^*-algebras of the stabilizers $(H_{2n+1})_{1_k}$ $(0 \le k \le 2n)$ are isomorphic to the C^*-algebras of continuous fields over $\mathbb{R}$ with the following fibers:*

$$\begin{cases} C_0(\mathbb{T}^{p+2q} \times \mathbb{R}^{2n-k}) & \theta = 0 \\ C(\mathbb{T}^p) \otimes (\otimes^q \mathfrak{A}_\theta) \otimes \mathbb{K} & \theta \ne 0 \end{cases} \quad \textit{for } 0 \le p + 2q \le k,\ \theta \in \mathbb{R}.$$

Proof. For $1_k \in (\mathbb{R}^{m_1} \times \mathbb{T}^{m_2}) \times (\mathbb{R}^{l_1} \times \mathbb{T}^{l_2}) \subset \mathbb{C}^n \times \mathbb{C}^n$ and $k = m_1 + m_2 + l_1 + l_2$, and $m_i, l_i \ge 0$ $(i = 1, 2)$, we may assume that

$$(H_{2n+1})_{1_k} \cong (\mathbb{R} \times \{0_{m_1}\} \times \mathbb{Z}^{m_2} \times \mathbb{R}^{n-m_1-m_2}) \rtimes_\alpha (\{0_{l_1}\} \times \mathbb{Z}^{l_2} \times \mathbb{R}^{n-l_1-l_2}).$$

By the Fourier transform,

$$C^*((H_{2n+1})_{1_k}) \cong C_0(\mathbb{R} \times \mathbb{T}^{m_2} \times \mathbb{R}^{n-m_1-m_2}) \rtimes_{\hat{\alpha}} (\mathbb{Z}^{l_2} \times \mathbb{R}^{n-l_1-l_2}).$$

Since the subspace $\{0\} \times \mathbb{T}^{m_2} \times \mathbb{R}^{n-m_1-m_2}$ is fixed under $\hat{\alpha}$, it is obtained that

$$\begin{aligned}0 &\to C_0((\mathbb{R} \setminus \{0\}) \times \mathbb{T}^{m_2} \times \mathbb{R}^{n-m_1-m_2}) \rtimes_{\hat{\alpha}} (\mathbb{Z}^{l_2} \times \mathbb{R}^{n-l_1-l_2}) \\ &\to C_0(\mathbb{R} \times \mathbb{T}^{m_2} \times \mathbb{R}^{n-m_1-m_2}) \rtimes_{\hat{\alpha}} (\mathbb{Z}^{l_2} \times \mathbb{R}^{n-l_1-l_2}) \\ &\to C_0(\mathbb{T}^{m_2+l_2} \times \mathbb{R}^{2n-k}) \to 0.\end{aligned}$$

By the similar reasons as before Theorem 2.2, the ideal in the above exact sequence is isomorphic to the 2-direct sum $\oplus^2 C_0(\mathbb{R}_+, \cup_{\theta\in\mathbb{R}_+}(C(\mathbb{T}^p) \otimes (\otimes^q \mathfrak{A}_\theta) \otimes \mathbb{K}))$, where $q \geq 0$ is the cardinal number of the intersection $\{m_1+1, \ldots, m_1+m_2\} \cap \{l_1+1, \ldots, l_1+l_2\}$, and $p = m_2 + l_2 - 2q \geq 0$. □

Combining Proposition 3.3 with Theorem 3.1, it is obtained that

Theorem 3.4. *Let G be a Lie semi-direct product of $\mathbb{C}^{2n}$ by H_{2n+1} with a complex 1-dimensional, multi-diagonal action. Then $C^*(G)$ has a finite composition series $\{\mathfrak{I}_j\}_{j=1}^K$ with $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ isomorphic to $C_0(\mathbb{C}^{n_0} \times \mathbb{R}^{2n})$ for $j = K$, and $C_0(\mathbb{C}^{n_0} \times (\mathbb{R} \setminus \{0\})) \otimes \mathbb{K}$ for $j = K-1$, and*

$$\begin{cases} C_0(\mathbb{C}^{n_0} \times \mathbb{T}^k) \otimes \mathbb{K}, & or \\ C_0(\mathbb{C}^{n_0} \times \mathbb{T}^{k_1+m_2+l_2} \times \mathbb{R}^{2n-k_1}) \otimes \mathbb{K}, & or \\ C_0(\mathbb{C}^{n_0} \times \mathbb{T}^{k_1} \times \mathbb{R}^{k_2}) \otimes \mathbb{K} \otimes C_0(\mathbb{R}_+, \cup_{\theta\in\mathbb{R}_+}(C(\mathbb{T}^p) \otimes (\otimes^q \mathfrak{A}_\theta) \otimes \mathbb{K})) \end{cases}$$

for $1 \leq k \leq n - n_0$ with $0 \leq n_0 \leq n$ and $k_2 \geq 1$, $k = k_1 + k_2$.

Remark. In the above statement, $\otimes^q \mathfrak{A}_\theta$ is regarded as a noncommutative torus of the form $C(\mathbb{T}^q) \rtimes_\Theta \mathbb{Z}^q$ where Θ is the multi-rotation by the same angle θ (cf. [13]).

As a corollary, it follows from the same argument of Corollary 2.4 that

Corollary 3.5. *Under the same situation with Theorem 3.4, it is obtained that*

$$\mathrm{sr}(C^*(G)) = n_0 + n + 1 = \dim_{\mathbb{C}} \hat{G}_1, \quad \text{and} \quad \mathrm{csr}(C^*(G)) \leq n_0 + n + 1.$$

To compute the stable rank and the connected stable rank of $C^*(G)$ in Theorem 3.1, we need to compute the stable ranks of $C^*(N)$. Fortunately, if N is a simply connected, nilpotent Lie group, then $\mathrm{sr}(C^*(N)) = \dim_{\mathbb{C}} \hat{N}_1$ ([19]). Furthermore, this formula is extended to the connected case ([20]). On the other hand, it is obtained that

Proposition 3.6. *Let $\mathfrak{A}$ be a C^*-algebra of continuous fields over a locally compact T_2-space X with fibers $\mathbb{K}(H_x)$ on separable Hilbert spaces H_x, $x \in X$. Then $\mathfrak{A}$ has continuous trace, and it is stable, i.e. $\mathfrak{A} \cong \mathfrak{A} \otimes \mathbb{K}$.*

Remark. By local triviality of continuous fields [4, Theorem 10.8.8], $\mathfrak{A}$ in the statement is assumed to be an inductive limit of $C_0(X_k) \otimes \mathbb{K}$ with $\{X_k\}_{k=1}^{\infty}$ open subspaces of X. This implies that $\mathfrak{A}$ satisfies Fell's condition ([4, Definition 10.5.7]). If necessary, by using Hjelmborg and Rørdam's result [8, Corollary 4.1], the latter claim is obtained.

Combining [16, Theorem 3] with the above proposition and (F1), it follows that

Proposition 3.7. *If N is a connected nilpotent Lie group, then*

$$\mathrm{csr}(C^*(N)) \leq 2 \vee \mathrm{csr}(C_0(\hat{N}_1)) = [(\dim \hat{N}_1 + 1)/2] + 1.$$

Proof. If N is simply connected, we use the structure of $C^*(N)$ in [19], Proposition 3.6 and (F1). Also, the inequality in the statement is valid in the connected case because if N is connected, then $C^*(N)$ is regarded as a quotient of $C^*(\tilde{N})$ of the universal covering group $\tilde{N}$ of N, so that the structure of $C^*(N)$ is inherited from that of $C^*(\tilde{N})$. □

Applying the above estimates and (F1-F3) to Theorem 3.1, it is obtained that

Corollary 3.8. *In Theorem 3.1, if N is nilpotent, then*

$$\begin{cases} \mathrm{sr}(C^*(G)) = \dim_{\mathbb{C}} \hat{G}_1 & \text{if } \dim \hat{G}_1 \text{ is even,} \\ \dim_{\mathbb{C}} \hat{G}_1 \leq \mathrm{sr}(C^*(G)) \leq \dim_{\mathbb{C}} \hat{G}_1 + 1 & \text{if } \dim \hat{G}_1 \text{ is odd, and} \end{cases}$$
$$\mathrm{csr}(C^*(G)) \leq 2 \vee \mathrm{csr}(C_0(\hat{G}_1)) = [(\dim \hat{G}_1 + 1)/2] + 1.$$

Proof. In Theorem 3.1, notice that $C_0(\mathbb{C}^{n_0}) \otimes C^*(N) \cong C^*(\mathbb{C}^{n_0} \times N)$, and $\mathbb{C}^{n_0} \times N$ is a connected, nilpotent Lie group. By Theorem 3.1, $\hat{G}_1$ is homeomorphic to the product space $\mathbb{C}^{n_0} \times \hat{N}_1$. Thus it follows from the rank estimates given above that

$$\begin{cases} \mathrm{sr}(C_0(\mathbb{C}^{n_0}) \otimes C^*(N)) = \dim_{\mathbb{C}} (\mathbb{C}^{n_0} \times \hat{N}_1) = \dim_{\mathbb{C}} \hat{G}_1, \\ \mathrm{csr}(C_0(\mathbb{C}^{n_0}) \otimes C^*(N)) \leq 2 \vee \mathrm{csr}(C_0(\mathbb{C}^{n_0} \times \hat{N}_1)) = [(\dim \hat{G}_1 + 1)/2] + 1. \end{cases}$$

□

On the other hand, it is obtained that

Corollary 3.9. *In Theorem* 3.1*, if N is a Lie semi-direct product $\mathbb{R}^m \rtimes \mathbb{R}$, then the same conclusion as Corollary* 3.8 *is obtained.*

Proof. If $N = \mathbb{R}^m \rtimes \mathbb{R}$, the rank estimates of Corollary 3.8 hold for $C^*(N)$ ([18]). □

Remark. As an example, let M_5 the Mautner group defined by the Lie semi-direct product $\mathbb{C}^2 \rtimes_\gamma \mathbb{R}$ with $\gamma_t(z_1, z_2) = (e^{it}z_1, e^{i\theta t}z_2)$ for $t \in \mathbb{R}$, $z_1, z_2 \in \mathbb{C}$, and an irrational number θ. Then $M_5/[M_5, M_5] \cong \mathbb{R}$. Define G by the Lie semi-direct product of $\mathbb{C}$ by M_5. If the action of M_5 on $\mathbb{C}$ is nontrivial, then $\mathrm{sr}(C^*(G)) = 2$ and $\mathrm{csr}(C^*(G)) = 2$ (cf. [18]). If the action is trivial, these stable ranks of $C^*(G)$ are 2 or 3.

The complex multi-dimensional case.

Next suppose that $G = \mathbb{C}^s \rtimes_\alpha (\mathbb{R}^{n-m} \times \mathbb{T}^m)$ with α a complex multi-dimensional, multi-diagonal action on a direct sum $\mathbb{C}^s = \oplus_{i=1}^n \mathbb{C}^{s_i}$, that is,

$$\alpha_t = (\oplus_{i=1}^{n-m} \alpha_i(t_i)) \oplus (\oplus_{j=n-m+1}^{n} \alpha_j(e^{it_j})) = \begin{pmatrix} \alpha_1(t_1) & & 0 \\ & \ddots & \\ 0 & & \alpha_n(e^{it_n}) \end{pmatrix} \in GL_s(\mathbb{C})$$

with $t = ((t_i)_{i=1}^{n-m}, (e^{it_j})_{j=n-m+1}^n) \in \mathbb{R}^{n-m} \times \mathbb{T}^m$, where α_i $(1 \le i \le n-m)$ and α_i $(n-m+1 \le i \le n)$ are Lie actions of $\mathbb{R}$, $\mathbb{T}$ on $\mathbb{C}^{s_i}$ respectively. Then G is isomorphic to the direct product $(\Pi_{i=1}^{n-m}(\mathbb{C}^{s_i} \rtimes_{\alpha_i} \mathbb{R})) \times (\Pi_{j=n-m+1}^n (\mathbb{C}^{s_j} \rtimes_{\alpha_j} \mathbb{T}))$. Then

$$C^*(G) \cong (\otimes_{i=1}^{n-m} C^*(\mathbb{C}^{s_i} \rtimes_{\alpha_i} \mathbb{R})) \otimes (\otimes_{j=n-m+1}^n C^*(\mathbb{C}^{s_j} \rtimes_{\alpha_j} \mathbb{T})).$$

By [18], the structure of $C^*(\mathbb{C}^{s_i} \rtimes_{\alpha_i} \mathbb{R})$ is obtained from extensions by $\{\mathfrak{K}_{i,j}/\mathfrak{K}_{i,j-1}\}_{j=1}^{K_i}$ isomorphic to $C_0(\mathbb{R}^{2u_i+1})$ for $j = K_i$, and

$$C_0(\mathbb{R}^{2u_{i_j}+v_{i_j}} \times \mathbb{T}^{w_{i_j}}) \otimes \mathbb{K}, \quad \text{or } C_0(\mathbb{R}^{2u_{i_j}+v_{i_j}}) \otimes \mathbb{K} \otimes \mathfrak{A}_{\Theta_{i_j}}$$

for $1 \le j \le K_i - 1$ with $u_i, u_{i_j}, v_{i_j}, w_{i_j} \ge 0$.

Thus we now consider the structure of $C^*(\mathbb{C}^{s_j} \rtimes_{\alpha_j} \mathbb{T})$. Then it is obtained that

Proposition 3.10. *Let $G = \mathbb{C}^n \rtimes_\alpha \mathbb{T}$. Then there exists a finite composition series $\{\mathfrak{I}_j\}_{j=1}^{n-n_0+1}$ of $C^*(G)$ with $\mathfrak{I}_{n-n_0+1-k}/\mathfrak{I}_{n-n_0-k}$ isomorphic to $C_0(\mathbb{C}^{n_0} \times \mathbb{Z})$ for $k = n - n_0 + 1$, and*

$$\oplus_{1 \le i_1 < \cdots < i_k \le n-n_0} (C_0(\mathbb{C}^{n_0} \times \mathbb{R}_+^k \times \mathbb{T}^{k-1}) \otimes \mathbb{K}) \quad \text{for } 1 \le k \le n - n_0,$$

where $\mathbb{C}^{n_0}$ is the fixed point subspace under the action of $\mathbb{T}$.

Proof. The argument before [18, Proposition 3.1] implicitly shows that α may be diagonal by taking a suitable base of $\mathbb{C}^n$. Otherwise, we have the contradiction against compactness of orbits under α. Then we may have the diagonal sum $\alpha(e^{it}) = \oplus_{k=1}^n e^{i\theta_k t}$ with $\theta_k = 0$ for $1 \le k \le n_0$ with some $0 \le n_0 \le n$ and $\theta_k \in \mathbb{R} \setminus \{0\}$ for $n_0 + 1 \le k \le n$, where θ_k $(n_0 + 1 \le k \le n)$ are linearly dependent over $\mathbb{Q}$. Then $C^*(\mathbb{C}^n \rtimes_\alpha \mathbb{T})$ is isomorphic to $C_0(\mathbb{C}^{n_0}) \otimes (C_0(\mathbb{C}^{n-n_0}) \rtimes \mathbb{T})$. Moreover, by the same way as Theorem 3.1, the tensor product on the right side has a finite composition series $\{\mathfrak{I}_k\}_{k=1}^{n-n_0+1}$ such that

$$\mathfrak{I}_{n-n_0+1-k}/\mathfrak{I}_{n-n_0-k} \cong \oplus_{1\le i_1<\cdots<i_k\le n-n_0}(C_0((\mathbb{C} \setminus \{0\})^k) \rtimes \mathbb{T}).$$

Each direct factor $C_0((\mathbb{C}\setminus\{0\})^k) \rtimes \mathbb{T}$ splits into $C_0(\mathbb{R}_+^k) \otimes (C(\mathbb{T}^k) \rtimes \mathbb{T})$. Since $\mathbb{T}^k$ is homeomorphic to $\mathbb{T}^{k-1} \times \mathbb{T}$ and an orbit of $\mathbb{T}$ is compatible with the action of $\mathbb{T}$, it follows that $C(\mathbb{T}^k) \rtimes \mathbb{T} \cong C(\mathbb{T}^{k-1}) \otimes (C(\mathbb{T}) \rtimes \mathbb{T}) \cong C(\mathbb{T}^{k-1}) \otimes \mathbb{K}$. □

Remark. The structure of group C^*-algebras of Lie semi-direct products $\mathbb{R}^n \rtimes_\alpha \mathbb{T}$ is obtained similarly by taking quotients of group C^*-algebras of $\mathbb{C}^n \rtimes_\beta \mathbb{T}$ with $\beta = \alpha + i\alpha$.

The following theorem is obtained from the above argument:

Theorem 3.11. *Let $G = \mathbb{C}^s \rtimes_\alpha (\mathbb{R}^{n-m} \times \mathbb{T}^m)$ with α a complex multi-dimensional, multi-diagonal action. Then there exists a finite composition series $\{\mathfrak{I}_j\}_{j=1}^K$ of $C^*(G)$ with $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ isomorphic to $C_0(\mathbb{R}^{2u+n-m} \times \mathbb{Z}^m) = C_0(\hat{G}_1)$ for $j = K$, and*

$$\begin{cases} C_0(\mathbb{R}^{2u_j+v_j} \times \mathbb{T}^{w_j}) \otimes \mathbb{K} & \text{or} \\ C_0(\mathbb{R}^{2u_j+v_j} \times \mathbb{T}^{w_j}) \otimes \mathbb{K} \otimes (\otimes_{l=1}^{k_j} \mathfrak{A}_{\Theta_l}) & \text{or} \\ C_0(\mathbb{R}^{2u_j+v_j}) \otimes \mathbb{K} \otimes (\otimes_{l=1}^{n} \mathfrak{A}_{\Theta_l}) & \text{for } 1 \le j \le K-1 \end{cases}$$

with $u, u_j, v_j, w_j \ge 0$, $1 \le k_j \le n-1$, and $\mathfrak{A}_{\Theta_l} \cong C(\mathbb{T}^{t_l}) \rtimes \mathbb{Z}$ a noncommutative torus.

Proof. Note that $C^*(G)$ splits into the tensor product of $C_0(\mathbb{C}^{s_i}) \rtimes \mathbb{R}$ $(1 \le i \le n-m)$ and $C_0(\mathbb{C}^{s_i}) \rtimes \mathbb{T}$ $(n-m+1 \le i \le n)$. Each tensor factor is built up by a finite number of extensions by subquotients $\{\mathfrak{K}_{i,j}/\mathfrak{K}_{i,j-1}\}_{j=1}^{K_i}$ given above. Then $C^*(G)$ is built up by a finite number of extensions by subquotients $\otimes_{i=1}^n (\mathfrak{K}_{i,j_i}/\mathfrak{K}_{i,j_i-1})$. □

Remark. This theorem is a generalization for the case $n = 1$, $m = 0$ obtained in [18]. If $\mathbb{C}^s$ is replaced by $\mathbb{R}^s$, the structure of $C^*(G)$ of $G =$

$\mathbb{R}^s \rtimes_\alpha (\mathbb{R}^{n-m} \times \mathbb{T}^m)$ is obtained similarly by taking a quotient of $C^*(\mathbb{C}^s \rtimes_\beta (\mathbb{R}^{n-m} \times \mathbb{T}^m))$ with $\beta = \alpha + i\alpha$.

Theorem 3.12. *Let G be a Lie semi-direct product of $\mathbb{C}^s$ by a connected, nilpotent Lie group N or a Lie semi-direct product $N = \mathbb{R}^m \rtimes \mathbb{R}$ with a complex multi-dimensional, multi-diagonal action induced from that of $N/[N,N]$. Then we obtain the same rank estimates as in Corollary* 3.8.

Proof. By assumption, we have the decomposition $\mathbb{C}^s = \oplus_{i=1}^n \mathbb{C}^{s_i}$ with $n \geq 1$ such that $N/[N,N] \cong \mathbb{R}^{n-m} \times \mathbb{T}^m$ with $m \geq 0$. Then the restriction of the action of N to $\mathbb{C}^{s_i}$ gives the action of $\mathbb{R}$ or $\mathbb{T}$ on $\mathbb{C}^{s_i}$. Taking invariant subspaces Ω_{k_i} of $\mathbb{C}^{s_i}$ under the action of $\mathbb{R}$ or $\mathbb{T}$ corresponding to subquotients given above ([18]), we can construct a finite composition series of $C^*(G)$ such that each subquotient is isomorphic to $C_0(\Pi_{i=1}^n \Omega_{k_i}) \rtimes N$. Moreover, notice that dimension of stabilizer of any point of $\Pi_{i=1}^n \Omega_{k_i}$ under the action of N is fixed. Hence, each subquotient is assumed to be a foliation C^*-algebra $C_r^*((\Pi_{i=1}^n \Omega_{k_i}) \times N)$ of the groupoid $(\Pi_{i=1}^n \Omega_{k_i}) \times N$ by orbits of N [9, p. 39 and Proposition 6.5]. Furthermore, it follows from [9, Theorem 6.14] that each subquotient is stable if the action of N is nontrivial. Thus by the same argument as in the proof of Corollary 3.8, the proof is complete. □

Remark. The same result as above can be deduced in the case of Lie semi-direct products of $\mathbb{R}^s$ by connected nilpotent Lie groups or $\mathbb{R}^m \rtimes \mathbb{R}$ (cf. Remark of Theorem 3.11).

§4. The Lie Semi-direct Products of $\mathbb{R}^u \times \mathbb{C}^v$ by Connected Lie Groups with Diagonal Actions

Let $G = (\mathbb{R}^u \times \mathbb{C}^v) \rtimes_\alpha (\mathbb{R}^{n-m} \times \mathbb{T}^m)$ with a diagonal action α. We may assume that α_g for $g = ((g_i)_{i=1}^{n-m}, (e^{ig_j})_{j=n-m+1}^n) \in \mathbb{R}^{n-m} \times \mathbb{T}^m$ is defined by the diagonal sum:

$$\begin{pmatrix} e^{(\sum_{j=1}^{p_1} g_{i_{1j}})} & & 0 \\ & \ddots & \\ 0 & & e^{(\sum_{j=1}^{p_u} g_{i_{uj}})} \end{pmatrix} \oplus \begin{pmatrix} e^{(\sum_{j=1}^{q_1} w_{i_{1j}} g_{i_{1j}})} & & 0 \\ & \ddots & \\ 0 & & e^{(\sum_{j=1}^{q_v} w_{i_{vj}} g_{i_{vj}})} \end{pmatrix}$$

with $g_{i_{kj}} \in \{g_i\}_{i=1}^{n-m}$ for $0 \leq j \leq p_k \leq n-m$ $(1 \leq k \leq u)$, and $w_{i_{kj}} \in \mathbb{C}$, $g_{i_{kj}} \in \{g_i\}_{i=1}^n$ for $0 \leq j \leq q_k \leq n$ $(1 \leq k \leq v)$. If $g_{i_{kj}} \in \{g_i\}_{i=n-m+1}^n$, then $w_{i_{kj}} = i1$. Thus, we may assume that the action of $\mathbb{R}^{n-m} \times \mathbb{T}^m$ on each direct factor is nontrivial. Then

Theorem 4.1. *Let G be a Lie semi-direct product $(\mathbb{R}^u \times \mathbb{C}^v) \rtimes_\alpha (\mathbb{R}^{n-m} \times \mathbb{T}^m)$ with a diagonal action α. Then $C^*(G)$ has a finite composition series $\{\mathfrak{I}_j\}_{j=1}^K$ such that*

$$\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong \begin{cases} C_0(\mathbb{R}^{u_0+n-m} \times \mathbb{C}^{v_0} \times \mathbb{Z}^m) & \text{for } j = K, \\ C_0(\mathbb{R}^{p_j} \times \mathbb{T}^{q_j} \times \mathbb{Z}^{r_j} \times \Omega_j) \otimes \mathbb{K}, & \text{or} \\ C_0(\mathbb{R}^{p_j} \times \mathbb{T}^{q_j} \times \mathbb{Z}^{r_j}) \otimes \mathfrak{A}_{\Theta_j} \otimes \mathbb{K} & \text{for } 1 \le j \le K-1 \end{cases}$$

with $p_j, q_j, r_j \ge 0$, where the fixed point subspace under $\hat{\alpha}$ is homeomorphic to $\mathbb{R}^{u_0} \times \mathbb{C}^{v_0}$, each Ω_j is an orbit subspace on whose preimage $\hat{\alpha}$ is wandering, and $\mathfrak{A}_{\Theta_j}$ is a higher dimensional noncommutative torus.

Proof. By the similar argument as before Theorem 2.1, we obtain a finite composition series $\{\mathfrak{I}_j\}_{j=1}^{u+v+uv+1}$ of $C^*(G)$ with subquotients $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ isomorphic to

$$\oplus_{1\le k_1<\cdots<k_{u_j}\le u} \oplus_{1\le l_1<\cdots<l_{v_j}\le v} (C_0((\mathbb{R}\setminus\{0\})^{u_j} \times (\mathbb{C}\setminus\{0\})^{v_j}) \rtimes (\mathbb{R}^{n-m} \times \mathbb{T}^m))$$

with $u_j, v_j \ge 0$. From the analysis of actions of $\mathbb{R}^{n-m} \times \mathbb{T}^m$ on $\mathbb{C}\setminus\{0\}$ in the previous section, each direct factor is isomorphic to the direct sum of tensor products

$$\oplus^{2^{u_j}} (C_0(\mathbb{T}^{v_{j1}} \times \mathbb{R}_+^{v_{j2}}) \otimes (C_0(\mathbb{R}_+^{u_j+v_{j1}} \times \mathbb{T}^{v_{j2}} \times (\mathbb{C}\setminus\{0\})^{v_j-v_{j1}-v_{j2}}) \rtimes (\mathbb{R}^{n-m} \times \mathbb{T}^m))$$

with $0 \le v_{j1} + v_{j2} \le v_j$, where $\mathbb{R}^{n-m} \times \mathbb{T}^m$ acts on direct factors of $\mathbb{R}_+^{u_j+v_{j1}}$ by translation, on those of $\mathbb{T}^{v_{j2}}$ by rotation and on those of $(\mathbb{C}\setminus\{0\})^{v_j-v_{j1}-v_{j2}}$ transitively. Put $X_j = \mathbb{R}_+^{u_j+v_{j1}} \times \mathbb{T}^{v_{j2}} \times (\mathbb{C}\setminus\{0\})^{v_j-v_{j1}-v_{j2}}$. Note that if a direct factor of $\mathbb{R}^{n-m} \times \mathbb{T}^m$ acts on X_j trivially, $C_0(X_j) \rtimes (\mathbb{R}^{n-m} \times \mathbb{T}^m)$ has the tensor factor $C_0(\mathbb{R})$ or $C_0(\mathbb{Z})$. Thus we assume that each direct factor of $\mathbb{R}^{n-m} \times \mathbb{T}^m$ acts on X_j nontrivially.

Suppose that the action of $\mathbb{R}^{n-m} \times \mathbb{T}^m$ on X_j is wandering. We can analyze the orbit space $\Omega_j = X_j/(\mathbb{R}^{n-m} \times \mathbb{T}^m)$ under the action of $\mathbb{R}^{n-m} \times \mathbb{T}^m$, and every orbit in this subspace has the same type. Thus X_j is homeomorphic to the product space of Ω_j and an orbit. Thus, Green's result [6] implies that $C_0(X_j) \rtimes (\mathbb{R}^{n-m} \times \mathbb{T}^m)$ is isomorphic to

$$\begin{aligned} C_0(\Omega_j) \otimes C^*((\mathbb{R}^{n-m} \times \mathbb{T}^m)/(\mathbb{R}^{n-m} \times \mathbb{T}^m)_{1_{u_j+v_j}}) \rtimes (\mathbb{R}^{n-m} \times \mathbb{T}^m) \\ \cong C_0(\Omega_j) \otimes C^*((\mathbb{R}^{n-m} \times \mathbb{T}^m)_{1_{u_j+v_j}}) \otimes \mathbb{K} \end{aligned}$$

where $(\mathbb{R}^{n-m} \times \mathbb{T}^m)_{1_{u_j+v_j}}$ is the stabilizer of $1_{u_j+v_j} \in (\mathbb{R}\setminus\{0\})^{u_j} \times (\mathbb{C}\setminus\{0\})^{v_j}$, and it is isomorphic to a product group of either $\mathbb{R}$, $\mathbb{T}$ or $\mathbb{Z}$.

Next suppose that the action of $\mathbb{R}^{n-m} \times \mathbb{T}^m$ on X_j is not wandering. Then $X_j = \mathbb{T}^{v_{j2}}$. If the action is 1-dimensionally multi-diagonal, then

$$\begin{aligned} C(X_j) \rtimes (\mathbb{R}^{n-m} \times \mathbb{T}^m) &= C(\mathbb{T}^n) \rtimes (\mathbb{R}^{n-m} \times \mathbb{T}^m) \\ &\cong (\otimes^{n-m}(C(\mathbb{T}) \rtimes \mathbb{R})) \otimes (\otimes^m (C(\mathbb{T}) \rtimes \mathbb{T})) \cong C(\mathbb{T}^{n-m}) \otimes \mathbb{K}. \end{aligned}$$

If the action is multi-dimensionally multi-diagonal, then

$$\begin{aligned} C(X_j) \rtimes (\mathbb{R}^{n-m} \times \mathbb{T}^m) &= C(\Pi_{k=1}^n \mathbb{T}^{l_k}) \rtimes (\mathbb{R}^{n-m} \times \mathbb{T}^m) \\ &\cong (\otimes_{k=1}^{n-m}(C(\mathbb{T}^{l_k}) \rtimes \mathbb{R})) \otimes (\otimes_{k=n-m+1}^{n}(C(\mathbb{T}^{l_k}) \rtimes \mathbb{T})) \end{aligned}$$

with $\sum_{k=1}^n l_k = v_{j2}$. Moreover, each direct factor $C(\mathbb{T}^{l_k}) \rtimes \mathbb{R}$ is assumed to be a foliation C^*-algebra. Thus $C(\mathbb{T}^{l_k}) \rtimes \mathbb{R} \cong (C(\mathbb{T}^{l_k - 1}) \rtimes \mathbb{Z}) \otimes \mathbb{K}$, where $C(\mathbb{T}^{l_k-1}) \rtimes \mathbb{Z}$ is a special case of higher dimensional noncommutative tori, say $\mathfrak{A}_\Theta$ (cf. [2], [18]). For other direct factors, it is obtained that $C(\mathbb{T}^{l_k}) \rtimes \mathbb{T} \cong C(\mathbb{T}^{l_k-1}) \otimes (C(\mathbb{T}) \rtimes \mathbb{T}) \cong C(\mathbb{T}^{l_k-1}) \otimes \mathbb{K}$ since the action of $\mathbb{T}$ on $\mathbb{T}^{l_k}$ is periodic. More generally, since dimension of stabilizers of points of X_j is fixed, $C(X_j) \rtimes (\mathbb{R}^{n-m} \times \mathbb{T}^m)$ is also assumed to be a foliation C^*-algebra. If the action of $\mathbb{R}^{n-m} \times \mathbb{T}^m$ on X_j is transitive, we obtain the same conclusion as the case of wandering actions. The other cases can be treated the similar way as the case of multi-dimensionally multi-diagonal actions. In fact, since the action on each direct factor of X_j is explicitly given, we can find an invariant torus $\mathbb{T}^{w_j}$ transversal to every orbits under $\mathbb{R}^{n-m} \times \mathbb{T}^m$ such that $C(X_j) \rtimes (\mathbb{R}^{n-m} \times \mathbb{T}^m) = C(\mathbb{T}^{v_{j2}}) \rtimes (\mathbb{R}^{n-m} \times \mathbb{T}^m)$ is isomorphic to $C(\mathbb{T}^{n_{j1}} \times \mathbb{R}^{n_{j2}}) \otimes (C(\mathbb{T}^{w_j}) \rtimes \mathbb{Z}^{n_{j3}}) \otimes \mathbb{K}$ for some $n_{j1}, n_{j2}, n_{j3} \geq 0$, where $C(\mathbb{T}^{w_j}) \rtimes \mathbb{Z}^{n_{j3}}$ is a special case of $\mathfrak{A}_\Theta$. $\square$

Remark. The proof of this theorem suggests that each Ω_j is also homeomorphic to a product space $\mathbb{T}^{k_j} \times \mathbb{R}^{s_j} \times \mathbb{Z}^{t_j}$ for some $k_j, s_j, t_j \geq 0$.

Similarly, it is obtained that

Theorem 4.2. *Let G be a Lie semi-direct product of $\mathbb{R}^u \times \mathbb{C}^v$ by a connected Lie group N with a diagonal action. Then there exists a finite composition series $\{\mathfrak{I}_j\}_{j=1}^K$ of $C^*(G)$ such that*

$$\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong \begin{cases} C_0(\mathbb{R}^{u_0} \times \mathbb{C}^{v_0}) \otimes C^*(N) & \text{for } j = K, \\ C_0(\mathbb{R}^{p_j} \times \mathbb{T}^{q_j} \times \mathbb{Z}^{r_j} \times \Omega_j) \otimes C^*(N_{z_j}) \otimes \mathbb{K} & \text{or} \\ C_0(\mathbb{R}^{p_j} \times \mathbb{T}^{q_j} \times \mathbb{Z}^{r_j}) \otimes C_r^*(W_j) \otimes \mathbb{K} & \text{for } 1 \leq j \leq K-1 \end{cases}$$

with $p_j, q_j, r_j \geq 0$, where $\mathbb{R}^{u_0} \times \mathbb{C}^{v_0}$ is the fixed point subspace under the dual action of N, each Ω_j is an orbit subspace on whose preimage the dual action of

N is wandering, and N_{z_j} means the stabilizer of a point z_j of an N-invariant subspace of $\mathbb{R}^u \times \mathbb{C}^v$, and $C_r^*(W_j)$ means the reduced C^*-algebra of a reduced groupoid W_j associated with orbits on an N-invariant torus.

Proof. Note that a diagonal action of N is reduced to that of $N/[N,N]$. Thus we can use the setting of Theorem 4.1. It suffices to consider the crossed product $C_0(X_j) \rtimes N$. If X_j is the fixed point subspace under the action of N, it is homeomorphic to $\mathbb{R}^{u_0} \times \mathbb{C}^{v_0}$. Then $C_0(X_j) \rtimes N \cong C_0(\mathbb{R}^{u_0} \times \mathbb{C}^{v_0}) \otimes C^*(N)$. If the action of N on X_j is wandering,

$$C_0(X_j) \rtimes N \cong C_0(X_j/N) \otimes (C^*(N/N_{z_j}) \rtimes N) \cong C_0(\Omega_j) \otimes C^*(N_{z_j}) \otimes \mathbb{K}$$

where N_{z_j} is the stabilizer of $z_j \in X_j$ and $\Omega_j = X_j/N$. If the action of N on X_j is not wandering, it follows from some techniques of foliation C^*-algebras that

$$C_0(X_j) \rtimes N \cong C_r^*(X_j \times N) \cong C_r^*(W_j) \otimes \mathbb{K}$$

where $C_r^*(X_j \times N)$ means the reduced (foliation) C^*-algebra of the groupoid $X_j \times N$ arising from the action of N of X_j, and $C_r^*(W_j)$ means the reduced C^*-algebra of the reduced groupoid W_j of $X_j \times N$ (cf. [9]). □

Remark. If $N = [N,N]$ in the above setting, then $C^*(G) \cong C_0(\mathbb{R}^u \times \mathbb{C}^v) \otimes C^*(N)$.

Corollary 4.3. *In Theorem* 4.2*, if N is a connected, nilpotent Lie group, or a Lie semi-direct product $\mathbb{R}^n \rtimes \mathbb{R}$, then the same rank estimates as in Corollary* 3.8 *hold.*

Example 4.4. Let G be a Lie semi-direct product of $\mathbb{R}^u \times \mathbb{C}^v$ by the Mautner group M_5 with a diagonal action. Note that $M_5/[M_5, M_5] \cong \mathbb{R}$. Then $C^*(G)$ has a finite composition series $\{\mathfrak{I}_j\}_{j=1}^K$ with each subquotient $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ isomorphic to

$$\begin{cases} C_0(\mathbb{R}^{u_0} \times \mathbb{C}^{v_0}) \otimes C^*(M_5) & \text{for } j = K, \\ C_0(\mathbb{R}^{p_j+2u_j} \times \mathbb{T}^{q_j}) \otimes C^*(\mathbb{C}^2) \otimes \mathbb{K} & \text{or} \\ C_0(\mathbb{R}^{p_j+2u_j}) \otimes (C(\mathbb{T}^{q_j}) \rtimes M_5) & \text{for } 1 \le j \le K-1 \end{cases}$$

where the second, third cases respectively correspond to that the action of $M_5/[M_5, M_5]$ is free, the multi-rotation on an invariant subspace of $\mathbb{R}^u \times \mathbb{C}^v$. Moreover,

$$C(\mathbb{T}^{q_j}) \rtimes M_5 \cong \begin{cases} C(\mathbb{T}^{q_j}) \rtimes (\mathbb{C}^2 \rtimes \mathbb{R}) \cong C_0(\mathbb{T}^{q_j} \times \mathbb{C}^2) \rtimes \mathbb{R}, \\ C_r^*(\mathbb{T}^{q_j} \times M_5) \cong C_r^*(\mathbb{T}^{q_j} \times \mathbb{C}^2 \times \mathbb{Z}) \otimes \mathbb{K} \end{cases}$$

with $C_r^*(\mathbb{T}^{q_j} \times \mathbb{C}^2 \times \mathbb{Z}) \cong C_0(\mathbb{T}^{q_j} \times \mathbb{C}^2) \rtimes \mathbb{Z}$, where the lower isomorphism is obtained by some techniques of foliation C^*-algebras (cf. [9]). In the upper case, since $\mathbb{T}^{q_j} \times \{0_2\}$ is invariant under the action of $\mathbb{R}$, it is obtained that

$$0 \to C_0(\mathbb{T}^{q_j} \times (\mathbb{C}^2 \setminus \{0_2\})) \rtimes \mathbb{R} \to C_0(\mathbb{T}^{q_j} \times \mathbb{C}^2) \rtimes \mathbb{R} \to C(\mathbb{T}^{q_j}) \rtimes \mathbb{R} \to 0$$

with the quotient isomorphic to $(C(\mathbb{T}^{q_j-1}) \rtimes \mathbb{Z}) \otimes \mathbb{K}$, and

$$\begin{aligned} 0 \to C_0(\mathbb{T}^{q_j} \times (\mathbb{C} \setminus \{0\})^2) \rtimes \mathbb{R} \to C_0(\mathbb{T}^{q_j} \times (\mathbb{C}^2 \setminus \{0_2\})) \rtimes \mathbb{R} \\ \to \oplus^2 C_0(\mathbb{T}^{q_j} \times (\mathbb{C} \setminus \{0\})) \rtimes \mathbb{R} \to 0 \end{aligned}$$

where two direct factors of the quotient, and the ideal are respectively isomorphic to

$$\begin{cases} C_0(\mathbb{R}_+) \otimes (C(\mathbb{T}^{q_j+1}) \rtimes \mathbb{R}) \cong C_0(\mathbb{R}_+) \otimes (C(\mathbb{T}^{q_j}) \rtimes \mathbb{Z}) \otimes \mathbb{K} \\ C_0(\mathbb{R}_+^2) \otimes (C(\mathbb{T}^{q_j+2}) \rtimes \mathbb{R}) \cong C_0(\mathbb{R}_+^2) \otimes (C(\mathbb{T}^{q_j+1}) \rtimes \mathbb{Z}) \otimes \mathbb{K} \end{cases}$$

(cf. [18]). On the other hand, the structure of $C^*(M_5)$ is given by [18]. Moreover, $C^*(G)$ has no nontrivial projections.

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