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Eigenvalue Asymptotics for the Shrödinger Operator with Steplike Magnetic Field and Slowly Decreasing Electric Potential

By

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Abstract

In this paper we consider the two-dimensional Schrödinger operator of the form:

$$H_V = -\frac{\partial^2}{\partial x_1^2} + \left(\frac{1}{i}\frac{\partial}{\partial x_2} - b(x_1)\right)^2 + V(x_1, x_2),$$

where the magnetic field $B(x_1) = \mathrm{rot}(0, b(x_1))$ is monotone increasing and steplike, namely the limits $\lim_{x_1 \to \pm\infty} B(x_1) = B_\pm$ exist with $0 < B_- < B_+ < \infty$, and V is the slowly power-decaying electric potential. The spectrum $\sigma(H_0)$ of the unperturbed operator H_0 ($= H_V$ with $V = 0$) has the band structure and H_V has the discrete spectrum in the gaps of the essential spectrum $\sigma_{ess}(H_V) = \sigma(H_0)$. The aim of this paper is to study the asymptotic distribution of the eigenvalues near the edges of the spectral gaps. Using the min-max argument, we prove that the classical Weyl-type asymptotic formula is satisfied under suitable assumptions on B and V.

§1. Introduction

In this paper we investigate the asymptotic distribution of eigenvalues of the two-dimensional magnetic Schrödinger operator. We consider the operator acting in $L^2(\mathbf{R}^2)$ of the form:

$$H_V = -\frac{\partial^2}{\partial x_1^2} + \left(\frac{1}{i}\frac{\partial}{\partial x_2} - b(x_1)\right)^2 + V(x_1, x_2), \tag{1.1}$$

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where $(0, b(x_1))$ is the magnetic vector potential and V is the electric scalar potential decaying at infinity.

We consider the case where the magnetic field $B = B(t)$ depends only on one variable. The relation between the magnetic vector potential and the magnetic field is given by

$$b(x_1) = \int_0^{x_1} B(t)dt.$$

When the vector potential $(0, b(x_1))$ gives a constant magnetic field and the electric potential decays at infinity, the eigenvalue distribution around the essential spectrum (the Landau levels) has been investigated by several authors (see, [Rai1], [Rai2], [Tam]).

The purpose of the present paper is to obtain similar eigenvalue asymptotics in the spectral gaps of the operator (1.1) in the case of certain non-constant, non-vanishing, bounded magnetic field (see the condition (B.1) below) and slowly power-decaying electric scalar potential (see the condition (V) below). The main strategy is the min-max argument as in [Col], [Tam]. The case where V decays like $|x|^{-m}$ $(m \geq 1)$ at infinity will be discussed in a future work.

In what follows, we denote $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial x_2}$ by ∂_1, ∂_2, respectively, and we set $D_1 = -i\partial_1$, $D_2 = -i\partial_2$. Set $\langle z \rangle = (1 + |z|^2)^{1/2}$ for $z \in \mathbf{R}^n$. For each open subset Ω of $\mathbf{R}^2$, we denote by $C_0^\infty(\Omega)$ the space of smooth functions with compact support in Ω.

We introduce the conditions for the magnetic field B:

(B.1) B is a real-valued C^2-function on $\mathbf{R}$. Moreover, B is monotone increasing and there exist positive numbers $B_\pm > 0$ such that $B_- < B_+$ and $\lim_{x_1 \to \pm\infty} B(x_1) = B_\pm$, respectively.

(B.2)$_\pm$ In addition to (B.1), there exist positive constants M, M' and C such that $|B(x_1) - B_\pm| \leq C\langle x_1 \rangle^{-M}$ and $|\partial_1 B(x_1)| \leq C\langle x_1 \rangle^{-M'}$ hold as $x_1 \to \infty$.

The Schrödinger operator with magnetic field satisfying (B.1) is also known as *the Iwatsuka model* (see [Iwa]). The various spectral properties of the Iwatsuka model have been investigated under weaker assumptions on B by [Iwa], [M-P], [E-K]. In particular, under the condition (B.1), the unperturbed operator

$$H_0 = -\frac{\partial^2}{\partial x_1^2} + \left(\frac{1}{i}\frac{\partial}{\partial x_2} - b(x_1)\right)^2$$

is essentially self-adjoint on $C_0^\infty(\mathbf{R}^2)$ and the spectrum of H_0 is purely absolutely continuous and has a band structure, i.e.,

$$\sigma(H_0) = \bigcup_{n=1}^{\infty} [\Lambda_n^-, \Lambda_n^+] \tag{1.2}$$

holds for $\Lambda_n^\pm = (2n-1)B_\pm$ $(n = 1, 2, \ldots)$. Here $\sigma(\cdot)$ denotes the spectrum of an operator.

Next, we introduce conditions for the electric scalar potential V:

(V) V is a real-valued C^2-function on $\mathbf{R}^2$ and there exist positive numbers m, m' and C such that $0 < m < 1$, $2m < m'$,

$$|V(x)| \le C\langle x\rangle^{-m},$$
$$|\,\partial_1 V(x)| + |\,\partial_2 V(x)| \le C\langle x\rangle^{-m'}$$

for all $x \in \mathbf{R}^2$.

The condition (V) implies that V is a relatively compact perturbation with respect to H_0 and then H_V is essentially self-adjoint on $C_0^\infty(\mathbf{R}^2)$ ([A-H-S], [L-S]). Thus, one expects that H_V has discrete spectra, i.e., isolated eigenvalues of finite multiplicity, in the spectral gaps of H_0 and they may accumulate to the tips of the gap.

For $\mu > 0$ and $a_0 \in \mathbf{R}$, we introduce the volume functions

$$\nu_\pm(\mu; a_0) = \frac{1}{2\pi}\mathrm{vol}\{(x_1, x_2) \in \mathbf{R}^2 |\, \pm x_1 > a_0,\ \ \pm V(x_1, x_2) > \mu\}.$$

For a positive, decreasing function f, we introduce a condition by

(T) The homogeneity condition

$$\lim_{\varepsilon\downarrow 0} \limsup_{\mu\downarrow 0} \mu^{2/m}\left(f((1-\varepsilon)\mu) - f((1+\varepsilon)\mu)\right) = 0$$

holds and there exist positive numbers C, μ_0 such that

$$f(\mu) \ge C\mu^{-\frac{2}{m}}$$

holds for all $\mu \in (0, \mu_0)$.

For a self-adjoint operator A acting in a Hilbert space, set

$$N((a,b)|A) = \dim\left(\mathrm{Ran}\, E_A((a,b))\right),$$

where $E_A(I)$ denotes the spectral projection of A on an interval I.

In what follows, we shall devote ourselves to obtaining the eigenvalue distribution in a gap $(\Lambda_n^+, \Lambda_{n+1}^-)$ when the gap is not empty, i.e.,

$$\Lambda_n^+ < \Lambda_{n+1}^- \tag{1.3}$$

holds. For notational convenience, we put $\Lambda_0^+ = -\infty$.

The main result of this paper is the following theorem.

Theorem 1.1. *Let $n \geq 0$ be an integer which satisfies* (1.3). *Suppose that* $(\mathrm{B.2})_+$ *(resp.* $(\mathrm{B.2})_-$*) and* (V) *hold with*

$$M > m, \qquad M' > 3m. \tag{1.4}$$

Moreover, suppose that $\nu_+(\mu; a_0)$ (resp. $\nu_-(\mu; a_0)$) satisfies (T) *for some a_0. Then we have*

$$N((\Lambda_n^+ + \mu, M_n)|H_V) = B_+\nu_+(\mu; a_0)(1 + o(1)) \quad as \quad \mu \downarrow 0$$
$$(resp.\ N((M_n, \Lambda_{n+1}^- - \mu)|H_V) = B_-\nu_-(\mu; a_0)(1 + o(1)) \quad as \quad \mu \downarrow 0),$$

where we put $M_n = (\Lambda_n^+ + \Lambda_{n+1}^-)/2$ for each $n \geq 1$ and $M_0 = -\infty$.

We remark that, as $\mu \downarrow 0$, the leading term of asymptotics of $\nu_\pm(\mu; a_0)$ does not depend on the choice of a_0. In fact, if we replace a_0 by $a_1(\mu) = a_0 + o(\mu^{-\frac{1}{m}})$ as $\mu \downarrow 0$, then we have

$$|\nu_+(\mu; a_0) - \nu_+(\mu; a_1(\mu))| = o(\nu_+(\mu; a_0))$$

as $\mu \downarrow 0$, because of the conditions (V) and (T) for $\nu_+(\mu; a_0)$. A similar assertion holds for $\nu_-(\mu; a_0)$. Thus, in the sequel, we shall denote $\nu_\pm(\mu; a_0)$ by $\nu_\pm(\mu)$ for simplicity.

We shall give a proof only for the asymptotics for $N((\Lambda_n^+ + \mu, M_n)|H_V)$, since we can obtain the result for $N((M_n, \Lambda_{n+1}^- - \mu)|H_V)$ in a similar way.

The plan of the paper is as follows. In Section 2, we shall state some results preparatory to the succeeding sections. In Section 3, we shall introduce a tessellation of $\mathbf{R}^2$. In Section 4, we shall introduce several quadratic forms and reduce the problem to the ones on each regions of the tessellation and, using the min-max argument, we shall give an upper estimate for the number of eigenvalues. In Section 5, we shall give a lower estimate and complete the proof of the main theorem, accepting Proposition 2.1 in Section 2 below. In Section 6, we shall give a proof of Proposition 2.1.

§2. Preliminaries

In what follows, we use the symbol "C" to denote various positive constants in the place where any confusions do not occur. Its value may change from line to line. In the place where we emphasize the dependence on some variable, we denote C_ε, $C(\alpha), \dots$ and so on.

Let $q = q[u, v]$ be a closable, symmetric sesqui-linear form bounded from below with domain $D(q)$, and let H_q be the self-adjoint operator associated with the form closure $\bar{q}$ ([R-S1], Theorem VIII.15, [R-S2], Theorem X.23). For simplicity, we denote $N((-\infty, \mu)|H_q)$ by $N_q(\mu)$ and denote the quadratic form $q[u, u]$ by $q[u]$. In the sequel, we shall identify a sesqui-linear form with the associated quadratic form.

Lemma 2.1. *For $j = 1, 2$, let $(\mathcal{H}_j, q_j, D(q_j))$ be a triplet of a Hilbert space $\mathcal{H}_j$, a quadratic form q_j and its form domain $D(q_j)$, and let J an isometric operator from $D(q_1)$ to $D(q_2)$ with respect to the norms of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Suppose that there exist positive constants C_1 and C_2 such that*

$$q_1[u] \geq C_1\, q_2[Ju] - C_2\, \|u\|^2_{\mathcal{H}_1} \tag{2.1}$$

holds for all $u \in D(q_1)$. Then we have, for any $\mu \in \mathbf{R}$,

$$N_{q_1}(\mu) \leq N_{q_2}((\mu + C_2)/C_1). \tag{2.2}$$

Proof. This is an easy consequence of the variational principle (see, e.g., [R-S4], Theorem XIII.2). □

In what follows, we denote the operator domain by $\mathrm{Dom}(\cdot)$.

Lemma 2.2. *Let D_0 be a dense subspace in a separable Hilbert space. Assume that a linear operator A is closable on D_0. Define a quadratic form a on $D(a) = D_0$ by $a[u] = \|Au\|^2$. Then we have:*

(i) *The form closure is given by $\bar{a}[u] = \|\bar{A}u\|^2$ with form domain $D(\bar{a}) = \mathrm{Dom}(\bar{A})$ and the self-adjoint operator associated with $\bar{a}$ coincides with $(\bar{A})^*\bar{A}$. Here $\bar{A}$ is the operator closure of A on D_0.*

(ii) *Moreover, if A is essentially self-adjoint on D_0, then the identity*

$$N_a(\mu^2) = N((-\mu, \mu)|\bar{A})$$

holds for any $\mu > 0$.

Proof. Set $\tilde{a}[u] = \|\bar{A}u\|^2$ with domain $D(\tilde{a}) = \mathrm{Dom}(\bar{A})$. Then it is easy to see that the closedness of the operator $\bar{A}$ implies the closedness of the form $\tilde{a}$ and that the fact that D_0 is an operator core for $\bar{A}$ implies that $\tilde{a}$ coincides with $\bar{a}$. The rest of assertion (i) is obvious.

The assertion (ii) follows since $N_a(\mu^2) = N((-\infty, \mu^2)|(\bar{A})^2) = N((-\mu, \mu)|\bar{A})$. □

In the following, we denote by $Q(z_0, R)$ the square in $\mathbf{R}^2$ centered at $z_0 \in \mathbf{R}^2$, of size $R \times R$, with sides parallel to the coordinate axes.

For $B_0 > 0$, we introduce the operators

$$\Pi_1(B_0) = D_1 - \frac{B_0}{2}x_2, \qquad \Pi_2(B_0) = D_2 + \frac{B_0}{2}x_1$$

and set

$$H(B_0) = \Pi_1(B_0)^2 + \Pi_2(B_0)^2. \tag{2.3}$$

For each $z_0 \in \mathbf{R}^2$, $R > 0$ and $\lambda \in \mathbf{R}$, we define the quadratic form $q = q(B_0, \lambda, Q(z_0, R))$ on $D(q) = C_0^\infty(Q(z_0, R))$ by

$$q[u] = \|(H(B_0) + \lambda)u\|^2_{L^2(Q(z_0,R))}.$$

It is easy to see that q is a closable form in $L^2(Q(z_0, R))$ by Lemma 2.2.

The next proposition shall play an important role in the following sections. (We shall give a proof in Section 6.)

Proposition 2.1. *Let $q = q(B_0, \lambda, Q(z_0, R))$ be the form as above. We have*:

(i) *For any $\mu > 0$,*

$$N_q(\mu^2) \le \frac{1}{2\pi} B_0 R^2 \sharp\{n \in \boldsymbol{N} \,|\, |(2n-1)B_0 + \lambda| < \mu\}$$

holds. Here $\sharp$ denotes the cardinal number of a set.

(ii) *There exist constants $C > 0$, $\delta_1 > 0$ such that, for any η and δ with $0 < \eta < R/2$, $0 < \delta < \delta_1$,*

$$N_q(\mu^2) \ge \frac{1}{2\pi} B_0 (R-\eta)^2 \sharp\{n \in \boldsymbol{N} \,|\, |(2n-1)B_0 + \lambda| < \mu_{\lambda\eta\delta}\}$$

holds for any $\mu > 0$, where $\mu_{\lambda\eta\delta} > 0$ is a number defined by the relation

$$\mu^2_{\lambda\eta\delta} = (1-\delta)\mu^2 - C\left(\delta\lambda^2 + \frac{1}{\eta^4\delta^3}\right).$$

Lemma 2.3. *Set $X_1 = D_1$ and $X_2 = D_2 - b(x_1)$. For $u \in C_0^\infty(\mathbf{R}^2)$, the estimate*

$$\sum_{j=1}^{2} \|X_j^2 u\|^2 \le C(\|H_0 u\|^2 + \|u\|^2)$$

holds for some constant $C > 0$.

Proof. Set $A = X_1 + iX_2$ and $A^* = X_1 - iX_2$. Then we have

$$\|X_j^2 u\|^2 \le C\|(A \pm A^*)^2 u\|^2 \le C \sum \|A^{\sharp_1} A^{\sharp_2} u\|^2, \tag{2.4}$$

where $A^{\sharp_j}$ denotes A or A^*.

By direct computations, we have, on $C_0^\infty(\mathbf{R}^2)$,

$$\begin{aligned} AA^* &= H_0 + B, \quad A^*A = H_0 - B, \\ [A, A^*] &= 2B, \quad [A, B] = -iB', \quad [A^*, B] = -iB'. \end{aligned}$$

Then it follows from the above relations that

$$\|AA^* u\|^2 \le C(\|H_0 u\|^2 + \|u\|^2), \tag{2.5}$$

$$\|A^* A u\|^2 \le C(\|H_0 u\|^2 + \|u\|^2), \tag{2.6}$$

and

$$\begin{aligned} \|A^2 u\|^2 &= ((A^*A)^2 u, u) - 2(BA^*Au, u) + 2i(B'Au, u) \\ &= ((H_0 - B)^2 u, u) - 2(B(H_0 - B)u, u) + 2i(Au, B'u) \\ &\le C(\|H_0 u\|^2 + \|u\|^2). \end{aligned} \tag{2.7}$$

We can estimate $\|(A^*)^2 u\|^2$ in the same way. Hence the statement follows from (2.4)–(2.7). □

§3. Partition of $\mathbf{R}^2$

In order to employ the mini-max arguments, we introduce a tessellation and an associated open covering of $\mathbf{R}^2$.

For small $\delta_0 > 0$, set

$$\sigma_1 = 1/m - 2\delta_0, \quad \sigma_2 = 1/m + \delta_0, \quad \sigma_3 = 1 + \delta_0^2, \tag{3.1}$$

where m is the constant as in (V). Then we observe that, for sufficiently small $\delta_0 > 0$, the relation

$$1 < \sigma_3 < \sigma_1 < 1/m < \sigma_2 \tag{3.2}$$

holds. Also, we take and fix δ_0 sufficiently small depending only on m, m', M, M' in the way which will be specified later (see, (4.12), (4.33), (4.35)).

Set

$$\begin{aligned}\Omega_0 &= \{(x_1,x_2)\in\mathbf{R}^2 | |x_1|<\mu^{-\sigma_1}, |x_2|<\mu^{-\sigma_2}\},\\ \Omega_{1\pm} &= \{(x_1,x_2)\in\mathbf{R}^2 | \pm x_1>\mu^{-\sigma_1}, |x_2|<\mu^{-\sigma_2}\},\\ \Omega_{2\pm} &= \{(x_1,x_2)\in\mathbf{R}^2 | \pm x_2>\mu^{-\sigma_2}\}\end{aligned}$$

and

$$\begin{aligned}\widetilde{\Omega}_0 &= \{(x_1,x_2)\in\mathbf{R}^2 | |x_1|<\mu^{-\sigma_1}+\mu^{-\sigma_3}, |x_2|<\mu^{-\sigma_2}+\mu^{-\sigma_3}\},\\ \widetilde{\Omega}_{1\pm} &= \{(x_1,x_2)\in\mathbf{R}^2 | \pm x_1>\mu^{-\sigma_1}-\mu^{-\sigma_3}, |x_2|<\mu^{-\sigma_2}+\mu^{-\sigma_3}\},\\ \widetilde{\Omega}_{2\pm} &= \{(x_1,x_2)\in\mathbf{R}^2 | \pm x_2>\mu^{-\sigma_2}-\mu^{-\sigma_3}\}.\end{aligned}$$

If we set $K=\{0,1+,1-,2+,2-\}$, then $\{\widetilde{\Omega}_k\}_{k\in K}$ forms an open covering of $\mathbf{R}^2$. Let $\{\varphi_k\}_{k\in K}$ be a set of functions such that $\operatorname{supp}\varphi_k\subset\widetilde{\Omega}_k$ and $0\le\varphi_k\le 1$ hold for each k, $\sum_{k\in K}\varphi_k^2=1$, and for each $k\in K$ and for each multi-index α,

$$\|\partial^\alpha\varphi_k\|_\infty \le C_\alpha\mu^{\sigma_3|\alpha|}, \tag{3.3}$$

holds, where the constant $C_\alpha>0$ is independent of μ, k and $\|\cdot\|_\infty$ denotes the supremum norm on $\mathbf{R}^2$.

§4. Upper bound for the quantity $N_h(d_n^2)$

§4.1. Quadratic forms

Let Λ_n^+, M_n be as in Section 1. For each $\mu>0$, set

$$a_n(\mu)=\frac{(\Lambda_n^++2\mu)+M_n}{2},\qquad d_n=(M_n-\Lambda_n^+)/2. \tag{4.1}$$

and define the form h on $D(h)=C_0^\infty(\mathbf{R}^2)$ by

$$h[u]=\|(H_V-a_n(\mu))u\|^2. \tag{4.2}$$

Lemma 4.1. *For any $\mu>0$ sufficiently small, we have*

$$|N((\Lambda_n^++\mu,M_n)|H_V)-N_h(d_n^2)|\le C_n,$$

where C_n is independent of μ.

Proof. We first note that

$$a_n(\mu) - d_n = \Lambda_n^+ + \mu, \qquad a_n(\mu) + d_n = M_n + \mu. \tag{4.3}$$

Applying Lemma 2.2 (ii) with $A = H_V - a_n(\mu)$, $D_0 = C_0^\infty(\mathbf{R}^2)$, we have

$$\begin{aligned} N_h(d_n^2) &= N((-d_n, d_n)|H_V - a_n(\mu)) \\ &= N((\Lambda_n^+ + \mu, M_n + \mu)|H_V) \\ &= N((\Lambda_n^+ + \mu, M_n)|H_V) + N([M_n, M_n + \mu)|H_V), \end{aligned}$$

where we used (4.3) in the last equality. Then the assertion follows from the fact that the eigenvalues of H_V do not accumulate near M_n. □

Lemma 4.2. *Let $\varepsilon_0 > 0$ be sufficiently small. Then, there exists a constant $C > 0$, independent of ε_0, such that the estimate*

$$h[u] \geq \sum_{k \in K} (1 - C\varepsilon_0)\|(H_V - a_n(\mu))\varphi_k u\|^2 - C\left(\varepsilon_0 + \mu^{4\sigma_3}/\varepsilon_0^3\right)\|u\|^2$$

holds for all $\mu > 0$ small enough and for any $u \in C_0^\infty(\mathbf{R}^2)$. Here $\{\varphi_k\}_{k \in K}$ is as in the previous section.

Proof. In the following, we shall frequently use the elementary inequality

$$\|a + b\|^2 \geq (1 - \varepsilon_0)\|a\|^2 - C\varepsilon_0^{-1}\|b\|^2, \tag{4.4}$$

where the constant $C > 0$ is independent of a, b and ε_0.

For $u \in C_0^\infty(\mathbf{R}^2)$ and for small $\varepsilon_0 > 0$, we have

$$\begin{aligned} h[u] &= \|(H_V - a_n(\mu))u\|^2 \\ &= \sum_{k \in K} \|\varphi_k(H_V - a_n(\mu))u\|^2 \\ &= \sum_{k \in K} \|(H_V - a_n(\mu))\varphi_k u + [\varphi_k, H_V]u\|^2 \\ &\geq \sum_{k \in K} \left((1 - \varepsilon_0)\|(H_V - a_n(\mu))\varphi_k u\|^2 - \frac{C}{\varepsilon_0}\|[\varphi_k, H_V]u\|^2\right). \end{aligned} \tag{4.5}$$

Let X_j $(j = 1, 2)$ be as in Lemma 2.3. Using the relation $H_V = X_1^2 + X_2^2 + V$, we have

$$\begin{aligned} \sum_{k \in K} \|[\varphi_k, H_V]u\|^2 &= \sum_{k \in K} \|\sum_{j=1}^{2}(2i(\partial_j\varphi_k)(X_j u) + (\partial_j^2\varphi_k)u)\|^2 \\ &\leq C \sum_{k \in K}\sum_{j=1}^{2}\left(\|(\partial_j\varphi_k)X_j u\|^2 + \|(\partial_j^2\varphi_k)u\|^2\right). \end{aligned} \tag{4.6}$$

Since it follows that

$$\begin{aligned}
&\sum_{k\in K} \|(\partial_j\varphi_k)X_j u\|^2 \\
&\quad = \sum_{k\in K}((\partial_j\varphi_k)^2 X_j^2 u, u) + \sum_{k\in K}([X_j, (\partial_j\varphi_k)^2]X_j u, u) \\
&\quad = \sum_{k\in K}(X_j^2 u, (\partial_j\varphi_k)^2 u) - 2i\sum_{k\in K}((\partial_j\varphi_k)X_j u, (\partial_j^2\varphi_k)u) \\
&\quad = \left(X_j^2 u, \sum_{k\in K}(\partial_j\varphi_k)^2 u\right) - 2i\sum_{k\in K}((\partial_j\varphi_k)X_j u, (\partial_j^2\varphi_k)u) \\
&\quad \le \varepsilon_0^2\|X_j^2 u\|^2 + \frac{C}{\varepsilon_0^2}\|\sum_{k\in K}(\partial_j\varphi_k)^2 u\|^2 + \varepsilon'\sum_{k\in K}\|(\partial_j\varphi_k)X_j u\|^2 \\
&\qquad + C_{\varepsilon'}\sum_{k\in K}\|(\partial_j^2\varphi_k)u\|^2,
\end{aligned}$$

holds for any small $\varepsilon' > 0$, we have

$$\begin{aligned}
(4.7)\quad &\sum_{k\in K}\sum_{j=1}^{2}\|(\partial_j\varphi_k)X_j u\|^2 \\
&\qquad \le C\sum_{j=1}^{2}\left(\varepsilon_0^2\|X_j^2 u\|^2 + \frac{1}{\varepsilon_0^2}\|\sum_{k\in K}(\partial_j\varphi_k)^2 u\|^2 + \sum_{k\in K}\|(\partial_j^2\varphi_k)u\|^2\right).
\end{aligned}$$

It follows from Lemma 2.3 that

$$\begin{aligned}
(4.8)\quad &\sum_{j=1}^{2}\|X_j^2 u\|^2 \\
&\quad \le C(\|H_0 u\|^2 + \|u\|^2) \\
&\quad \le C(\|(H_V - a_n(\mu))u\|^2 + (a_n(\mu)^2 + \|V\|_\infty + 1)\|u\|^2) \\
&\quad \le C(\|(H_V - a_n(\mu))u\|^2 + \|u\|^2)
\end{aligned}$$

holds for some constant $C > 0$, independent of $\mu > 0$. Then, from (4.6)–(4.8), we have

$$\begin{aligned}
(4.9)\quad &\sum_{k\in K}\|[\varphi_k, H_V]u\|^2 \\
&\quad \le C\left(\varepsilon_0^2\|(H_V - a_n(\mu))u\|^2 + \varepsilon_0^2\|u\|^2\right) \\
&\qquad + \varepsilon_0^{-2}C\sum_{j=1}^{2}\left(\|\sum_{k\in K}(\partial_j\varphi_k)^2 u\|^2 + \sum_{k\in K}\|(\partial_j^2\varphi_k)u\|^2\right).
\end{aligned}$$

Since it follows from (4.5) and (4.9) that

$$h[u] \geq (1-\varepsilon_0)\sum_{k\in K}\|(H_V - a_n(\mu))\varphi_k u\|^2 \\ - C\left(\varepsilon_0 h[u] + \varepsilon_0\|u\|^2 + \varepsilon_0^{-3}\left(\|\sum_{k\in K}(\partial_j\varphi_k)^2 u\|^2 + \sum_{k\in K}\|(\partial_j^2\varphi_k)u\|^2\right)\right),$$

we have by (3.3)

$$(1+C\varepsilon_0)h[u] \\ \geq (1-\varepsilon_0)\sum_{k\in K}\|(H_V - a_n(\mu))\varphi_k u\|^2 - C\left(\varepsilon_0 + \mu^{4\sigma_3}/\varepsilon_0^3\right)\|u\|^2,$$

from which the lemma immediately follows. □

For each $k \in K$, define the form h_k on $D(h_k) = C_0^\infty(\widetilde{\Omega}_k)$ by

$$h_k[v] = \|(H_V - a_n(\mu))v\|^2 \qquad (v \in D(h_k)).$$

Lemma 4.3. *There exists a constant $C > 0$ such that*

$$N_h(d_n^2) \leq \sum_{k\in K} N_{h_k}((1+C\mu^{\sigma_3})d_n^2)$$

holds for any small $\mu > 0$.

Proof. If we set $\varepsilon_0 = \mu^{\sigma_3}$, then it follows from Lemma 4.2 that, for any $u \in C_0^\infty(\mathbf{R}^2)$,

$$\begin{aligned} h[u] &\geq \sum_{k\in K}\left((1-C\mu^{\sigma_3})\|(H_V - a_n(\mu))\varphi_k u\|^2 - C\mu^{\sigma_3}\|\varphi_k u\|^2\right) \\ &= \sum_{k\in K}((1-C\mu^{\sigma_3})h_k[\varphi_k u] - C\mu^{\sigma_3}\|\varphi_k u\|^2). \end{aligned} \tag{4.10}$$

Define the form q_1 on $D(q_1) = \oplus_{k\in K} C_0^\infty(\widetilde{\Omega}_k)$ by

$$q_1[\oplus_{k\in K} v_k] = \sum_{k\in K} h_k[v_k],$$

and the isometry J_1 from $C_0^\infty(\mathbf{R}^2)$ to $\oplus_{k\in K} C_0^\infty(\widetilde{\Omega}_k)$ by $J_1 u = \oplus_{k\in K}(\varphi_k u)$. Applying Lemma 2.1 to the pair of triplets $(L^2(\mathbf{R}^2),\, h,\, D(h))$ and $(\oplus_{k\in K} L^2(\widetilde{\Omega}_k),\, q_1,\, D(q_1))$, we deduce from (4.10) that $N_h(d_n^2) \leq N_{q_1}((1+C\mu^{\sigma_3})d_n^2)$. □

§4.2. Estimate of the form $h_{2,\pm}$

Lemma 4.4. *For any $\mu > 0$ small enough, we have*

$$N_{h_{2\pm}}((1 + C\mu^{\sigma_3})d_n^2) = 0.$$

Proof. Let $\mu > 0$ be sufficiently small and let $u \in D(h_{2\pm})$. Since $|x_2| > \mu^{-\sigma_2} - \mu^{-\sigma_3} \geq C\mu^{-\sigma_2}$ on $\widetilde{\Omega}_{2\pm}$ by (3.2), we see that

$$|V(x_1, x_2)| \leq C\langle x_2\rangle^{-m} \leq C\mu^{m\sigma_2}. \tag{4.11}$$

Then, by (4.4), we have, for $\varepsilon = \mu^{\sigma_3}$,

$$\begin{aligned} h_{2\pm}[u] &= \|(H_V - a_n(\mu))u\|^2 \\ &\geq (1-\varepsilon)\|(H_0 - a_n(\mu))u\|^2 - C\varepsilon^{-1}\|Vu\|^2 \\ &\geq (1-\varepsilon)\|(H_0 - a_n(\mu))u\|^2 - C\varepsilon^{-1}\sup_{\widetilde{\Omega}_{2\pm}}|V(x_1,x_2)|^2\|u\|^2. \end{aligned}$$

From (3.2) and (4.11), it follows that $1 - \varepsilon = 1 - o(\mu)$ and

$$\varepsilon^{-1}\sup_{\widetilde{\Omega}_{2\pm}}|V(x_1,x_2)|^2 \leq C\mu^{2m\sigma_2 - \sigma_3} = O(\mu^{1+(2m-\delta_0)\delta_0}) = o(\mu)$$

hold as $\mu \downarrow 0$ if the condition

$$\delta_0 < 2m \tag{4.12}$$

is satisfied.

If we define the form q_2 on $D(q_2) = C_0^\infty(\mathbf{R}^2)$ by

$$q_2[u] = \|(H_0 - a_n(\mu))u\|^2,$$

and denote by J_2 is the natural injection from $D(h_{2\pm})$ to $D(q_2)$, then we have

$$h_{2\pm}[u] \geq (1 - o(\mu))q_2[J_2u] - o(\mu)\|u\|^2 \qquad \text{as} \quad \mu \downarrow 0. \tag{4.13}$$

Hence it follows from Lemma 2.1 that

$$\begin{aligned} N_{h_{2\pm}}((1 + C\mu^{\sigma_3})d_n^2) &\leq N_{q_2}(\tilde{d_n}(\mu)^2) \\ &= N((a_n(\mu) - \tilde{d_n}(\mu),\ a_n(\mu) + \tilde{d_n}(\mu))|H_0), \end{aligned} \tag{4.14}$$

where $\tilde{d_n}(\mu)$ is given by the relation of the form $\tilde{d_n}(\mu)^2 = (d_n^2 + o(\mu))/(1 - o(\mu))$ as $\mu \downarrow 0$.

On the other hand, since $a_n(\mu)+\tilde{d}_n(\mu) = M_n+\mu+o(\mu)$ and $a_n(\mu)-\tilde{d}_n(\mu) = \Lambda_n^+ + \mu + o(\mu)$ hold as $\mu \downarrow 0$, and $(\Lambda_n^+, \Lambda_{n+1}^-) \cap \sigma(H_0) = \emptyset$, we have

$$(a_n(\mu) - \tilde{d}_n(\mu)\ ,\ a_n(\mu) + \tilde{d}_n(\mu)) \ \cap\ \sigma(H_0) \ =\ \emptyset. \tag{4.15}$$

Then the result follows from (4.14) and (4.15). □

§4.3. Estimate of the form h_0

For $i = (i_1, i_2) \in \boldsymbol{Z}^2$, set

$$Q(i) = \{(x_1, x_2) \in \mathbf{R}^2 | i_j < x_j < i_j + 1 \text{ for } j = 1, 2\}$$

and

$$I = \{i \in \boldsymbol{Z}^2 |\ |i_j| \le [[\mu^{-\sigma_j} + \mu^{-\sigma_3}]] + 2 \text{ for } j = 1, 2\},$$

where $[[N]]$ expresses the largest integer less than or equal to N. Then it is easy to see that

$$\widetilde{\Omega}_0 \subset \bigcup_{i \in I} \overline{Q(i)}$$

Here, $\overline{\cdot}$ denotes the closure with respect to the usual topology of $\mathbf{R}^2$.

We denote the center of $Q(i)$ by z_i and denote $Q(z_i, 1+\eta)$ by $Q_\eta(i)$ for $0 < \eta < 1$. Define the form q_3 on $D(q_3) = \bigoplus_{i \in I} C_0^\infty(Q_\eta(i))$ by

$$q_3[\oplus u_i] = \sum_{i \in I} \|(H_0 - a_n(\mu))u_i\|^2 \qquad (u_i \in C_0^\infty(Q_\eta(i))).$$

Lemma 4.5. *Let q_3 be as above and let d_n be as in* (4.1). *Then, for any $C_0 > 0$, we have $N_{q_3}(C_0 d_n^2) = o(\mu^{-2/m})$ holds as $\mu \ \downarrow\ 0$.*

Proof. For each $i \in I$, we denote the center of $Q(i)$ by $z_i = (c_i, d_i)$ and set

$$b_i(x_1) = \int_0^{c_i} B(t)dt + B(c_i)(x_1 - c_i),$$

and introduce the operator K_i acting in $L^2(\mathbf{R}^2)$ by

$$K_i = D_1^2 + (D_2 - b_i)^2.$$

Then, for a fixed $\varepsilon_1 > 0$ sufficiently small,

$$\|(H_0 - a_n(\mu))u\|^2 \ge (1 - \varepsilon_1)\|(K_i - a_n(\mu))u\|^2 - \frac{C}{\varepsilon_1}\|(H_0 - K_i)u\|^2 \tag{4.16}$$

holds for any $u \in C_0^\infty(Q_\eta(i))$.

We set

$$\begin{aligned} r_i(x_1) &= b(x_1) - b_i(x_1) \\ &= \int_{c_i}^{x_1} (B(t) - B(c_i))dt. \end{aligned}$$

Then it follows from the relation

$$H_0 - K_i = -2r_i(D_2 - b_i) + r_i^2$$

that, for any small $\varepsilon_2 > 0$,

$$\begin{aligned} (4.17) \quad & \|(H_0 - K_i)u\|^2 \\ & \le C\{\sup_{Q_\eta(i)} |r_i|^2 \|(D_2 - b_i)u\|^2 + \sup_{Q_\eta(i)} |r_i|^4 \|u\|^2\} \\ & \le C\varepsilon_2 \sup_{Q_\eta(i)} |r_i|^2 \|K_i u\|^2 + C\left(\varepsilon_2^{-1} \sup_{Q_\eta(i)} |r_i|^2 + \sup_{Q_\eta(i)} |r_i|^4\right) \|u\|^2 \\ & \le C\varepsilon_2 \sup_{Q_\eta(i)} |r_i|^2 \|(K_i - a_n(\mu))u\|^2 \\ & \quad + C'\left(\varepsilon_2^{-1} \sup_{Q_\eta(i)} |r_i|^2 + \sup_{Q_\eta(i)} |r_i|^4\right) \|u\|^2, \end{aligned}$$

where we used the fact that

$$\|(D_2 - b_i)u\|^2 = ((D_2 - b_i)^2 u, u) \le (K_i u, u)$$

in the second inequality and the fact that $a_n(\mu)$ is uniformly bounded in small $\mu > 0$ in the third inequality.

Since the side length of $Q_\eta(i)$ is less than 2 units, we have

$$\sup_{Q_\eta(i)} |r_i| \le C\|B'\|_\infty \tag{4.18}$$

for some constant $C > 0$. Then it follows from (4.16)–(4.18) that

$$\begin{aligned} & \|(H_0 - a_n(\mu))u\|^2 \\ & \quad \ge (1 - \varepsilon_1)\|(K_i - a_n(\mu))u\|^2 - C\varepsilon_2\varepsilon_1^{-1}\|B'\|_\infty^2 \|(K_i - a_n(\mu))u\|^2 \\ & \qquad - C\varepsilon_1^{-1}(\varepsilon_2^{-1}\|B'\|_\infty^2 + \|B'\|_\infty^4)\|u\|^2 \\ & \quad \ge (1 - 2\varepsilon_1)\|(K_i - a_n(\mu))u\|^2 - C(\varepsilon_1^{-3}\|B'\|_\infty^2 + \varepsilon_1^{-1}\|B'\|_\infty^4)\|u\|^2, \end{aligned}$$

where we set $\varepsilon_2 = \varepsilon_1^2/C$ in the second inequality. Then it is easy to see that

$$q_3(\oplus_i u_i) \ge \sum_{i \in I} \{C\|(K_i - a_n(\mu))u_i\|^2 - C'\|u_i\|^2\} \tag{4.19}$$

for some positive constants C, $C' > 0$, independent of μ and i.

Finally it follows from (4.19), Lemma 2.1 and Proposition 2.1 that

$$\begin{aligned}
&N_{q_3}(C_0 d_n^2) \\
&\quad \le \sum_{i\in I} \frac{1}{2\pi} B(c_i)(1+\eta)^2 \sharp\{n \in \boldsymbol{N} |\ |(2n-1)B(c_i) - a_n(\mu)|^2 \le C d_n^2 + C\} \\
&\quad \le \sum_{i\in I} B(c_i) \frac{C}{B(c_i)} \\
&\quad \le C\mu^{-\sigma_1-\sigma_2} = o(\mu^{-\frac{2}{m}})
\end{aligned}$$

as $\mu \downarrow 0$, where we used (3.2) in the last equality. □

Lemma 4.6. *Let d_n be as in* (4.1). *For any $C > 0$, we have*

$$N_{h_0}((1 + C\mu^{\sigma_3})d_n^2) = o(\mu^{-2/m})$$

as $\mu \downarrow 0$.

Proof. Let $\{\psi_i\}_{i\in Z^2}$ be a partition of unity of $\mathbf{R}^2$ such that $\sum_{i\in Z^2} \psi_i^2 = 1$, and, for each $i \in I$, $\psi_i \in C_0^\infty(Q_\eta(i))$, $0 \le \psi_i \le 1$, $\psi_i(z) = \psi_0(z - i)$ hold and $|\partial^\alpha \psi_i| \le C_\alpha \eta^{-|\alpha|}$ holds for each $\alpha \ge 0$.

We define the isometry J_3 from $D(h_0) = C_0^\infty(\widetilde{\Omega_0})$ to $D(q_3) = \oplus_{i\in I} C_0^\infty(Q_\eta(i))$ by setting $J_3 u = \oplus_{i\in I}(\psi_i u)$ for $u \in D(h_0)$. Since as in the proof of Lemma 4.2, we can show that, for a small $\varepsilon > 0$,

$$h_0[u] \ge (1-\varepsilon) q_3[\oplus_{i\in I}(\psi_i u)] - C\|u\|^2$$

for some constant C which depends only on ε, η, V and B, applying Lemma 2.1 to q_3, h_0 and J_3, we have

$$N_{h_0}((1 + C\mu^{\sigma_3})d_n^2) \le N_{q_3}\left(\frac{1+C}{1-\varepsilon} d_n^2\right).$$

Then the lemma obeys from Lemma 4.5. □

§4.4. Estimate of the form h_{1-}

Lemma 4.7. *Let d_n be as in* (4.1). *Then, for any $C > 0$, we have $N_{h_{1-}}((1 + C\mu^{\sigma_3})d_n^2) = 0$ for any $\mu > 0$ sufficiently small.*

Proof. Since $x_1 \le -\mu^{\sigma_1} + \mu^{-\sigma_3} \le -C\mu^{-\sigma_1}$ holds on $\widetilde{\Omega}_{1-}$, it follows from (B.1) and (V) that, as $\mu \downarrow 0$,

$$|V(x_1, x_2)| \le C\langle x_1\rangle^{-m} \le C\mu^{m\sigma_1},$$
$$B_- \le B(x_1) \le B(-\mu^{\sigma_1} + \mu^{-\sigma_3}) = B_- + o(1).$$

By (4.4), we see that, as $\mu \downarrow 0$,

$$\begin{aligned} h_{1-}[u] &= \|(H_V - a_n(\mu))u\|^2 \\ &\ge (1-\mu^{m\sigma_1})\|(H_0 - a_n(\mu))u\|^2 - C\mu^{-m\sigma_1} \sup_{\widetilde{\Omega}_{1-}} |V|^2 \|u\|^2 \\ &\ge (1-\mu^{m\sigma_1})\|(H_0 - a_n(\mu))u\|^2 - O(\mu^{m\sigma_1})\|u\|^2. \end{aligned} \tag{4.20}$$

Let β be a number which satisfies

$$B_- < \beta < B_+ \tag{4.21}$$

and let $\tilde{B}$ be a real-valued, monotonically increasing, smooth function on $\mathbf{R}$ which satisfies

$$\tilde{B}(x_1) = \begin{cases} B(x_1), & \text{if } x_1 \le -\mu^{-\sigma_1} + \mu^{-\sigma_3}, \\ \beta, & \text{if } x_1 \text{ is sufficiently large,} \end{cases}$$

and the modified operator $\tilde{H}_0$ acting in $L^2(\mathbf{R}^2)$ by

$$\tilde{H}_0 = D_1^2 + (D_2 - \tilde{b})^2,$$

where we set

$$\tilde{b}(x_1) = \int_0^{x_1} \tilde{B}(t)dt + \int_{-\mu^{-\sigma_1}+\mu^{-\sigma_3}}^{0} (\tilde{B}(t) - B(t))dt.$$

Then we can find that

$$H_0 u = \tilde{H}_0 u \qquad \text{for} \quad u \in C_0^\infty(\widetilde{\Omega}_{1-}) \tag{4.22}$$

and

$$\sigma(\tilde{H}_0) = \bigcup_{n=1}^{\infty} [(2k-1)B_-,\ (2k-1)\beta]. \tag{4.23}$$

We denote by q_4 (resp. by q_5) the form $\|(\tilde{H}_0 - a_n(\mu))u\|^2$ on $C_0^\infty(\widetilde{\Omega}_{1-})$ (resp. on $C_0^\infty(\mathbf{R}^2)$). Then we have, from (4.20) and (4.22),

$$\begin{aligned} &N_{h_{1-}}((1+C\mu^{\sigma_3})d_n^2) \\ &\le N_{q_4}((1+C\mu^{m\sigma_1})(1+C\mu^{\sigma_3})d_n^2)) \\ &\le N_{q_5}((1+C\mu^{m\sigma_1})(1+C\mu^{\sigma_3})d_n^2)) \\ &\le N((a_n(\mu) - (1+C\mu^{m\sigma_1})d_n, a_n(\mu) + (1+C\mu^{m\sigma_1})d_n)|\ \tilde{H}_0), \end{aligned} \tag{4.24}$$

where we used Lemma 2.1 for the pair h_{1-} and q_4 in the first inequality, and for the pair q_4 and q_5 in the second inequality.

On the other hand, since

$$a_n(\mu) - (1 + C\mu^{m\sigma_1})d_n = \Lambda_n^+ + O(\mu^{m\sigma_1}),$$
$$a_n(\mu) + (1 + C\mu^{m\sigma_1})d_n = M_n + O(\mu^{m\sigma_1})$$

hold as $\mu \downarrow 0$, we deduce from (4.21), (4.23) that

$$(a_n(\mu) - (1 + C\mu^{m\sigma_1})d_n, a_n(\mu) + (1 + C\mu^{m\sigma_1})d_n) \ \cap \ \sigma(\tilde{H}_0) = \emptyset \tag{4.25}$$

holds for any small $\mu > 0$. The lemma follows from (4.24) and (4.25). □

§4.5. Estimate of the form h_{1+}

For $i = (i_1, i_2) \in \boldsymbol{Z}^2$, set

$$Q_i = \{(x_1, x_2) \in \mathbf{R}^2 |\ i_j < \mu^{\sigma_3} x_j < i_j + 1 \text{ for } j = 1, 2\}.$$

For η with $0 < \eta < 1$, we denote the center of Q_i by $z_i = (c_i, d_i)$ and we denote $Q_i(z_i, \mu^{-\sigma_3}(1 + \eta))$ by $Q_{i\eta}$.

Let $\{\psi_i\}_{i \in Z^2}$ be a partition of unity associated with the covering $\{Q_{i\eta}\}_{i \in Z^2}$ of $\mathbf{R}^2$ such that $\sum_{i \in Z^2} \psi_i^2 = 1$, and, for each $i \in \boldsymbol{Z}^2$, $\psi_i \in C_0^\infty(Q_{i\eta})$, $0 \le \psi_i \le 1$, $\psi_i(x_1, x_2) = \psi_0(x_1 - i_1\mu^{-\sigma_3}, x_2 - i_2\mu^{-\sigma_3})$ and $|\partial^\alpha \psi_i| \le C(\mu^{\sigma_3}/\eta)^{|\alpha|}$ holds for each $\alpha \ge 0$. If we set

$$I' = \{i \in \boldsymbol{Z}^2 | Q_{i\eta} \cap \widetilde{\Omega}_{1+} \ne \emptyset\},$$

then $\sum_{i \in I'} \psi_i^2 = 1$ holds on $\widetilde{\Omega}_{1+}$.

For each $i \in I'$, we set

$$\begin{aligned} B_i &= B(c_i), \\ b_i(x_1) &= \int_0^{c_i} B(t)dt + B(c_i)(x_1 - c_i), \\ V_i(x_1, x_2) &= V(z_i) \quad (= V(c_i, d_i)) \end{aligned} \tag{4.26}$$

and introduce the operator $H_{V,i}$ acting in $L^2(\mathbf{R}^2)$ by

$$H_{V,i} = D_1^2 + (D_2 - b_i)^2 + V_i. \tag{4.27}$$

Lemma 4.8. *For any $u \in D(h_{1+})$, the estimate*

$$h_{1+}[u] \ge \sum_{i \in I'} (1 - C\mu^{\sigma_3}) \|(H_{V,i} - a_n(\mu))\psi_i u\|^2 - \eta^{-4} o(\mu) \|u\|^2$$

holds as $\mu \downarrow 0$ for some $C > 0$. Here the term $o(\mu)$ is uniformly bounded with respect to small $\eta > 0$.

Proof. As in the proof of Lemma 4.2, we have, for $u \in C_0^\infty(\widetilde{\Omega}_{1+})$ and for small $\varepsilon_1 = \mu^{\sigma_3} > 0$,

$$
\begin{aligned}
(4.28)\qquad h_{1+}[u] &= \|(H_V - a_n(\mu))u\|^2 \\
&\geq (1-\varepsilon_1)\sum_{i\in I'} \|(H_V - a_n(\mu))\psi_i u\|^2 - C\varepsilon_1\|u\|^2 \\
&\quad - C\frac{1}{\varepsilon_1^3}\sum_{j=1}^{2}\left(\|\sum_{k\in I'}(\partial_j\psi_k)^2 u\|^2 + \sum_{k\in I'}\|(\partial_j^2\psi_k)u\|^2\right) \\
&\geq (1-\varepsilon_1)\sum_{i\in I'} \|(H_V - a_n(\mu))\psi_i u\|^2 - C\varepsilon_1\|u\|^2 \\
&\quad - C\mu^{4\sigma_3}/(\eta^4\varepsilon_1^3)\|u\|^2 \\
&\geq (1-\mu^{\sigma_3})\sum_{i\in I'} \|(H_V - a_n(\mu))\psi_i u\|^2 - C\eta^{-4}\mu^{\sigma_3}\|u\|^2.
\end{aligned}
$$

For each $i \in I'$, set

$$
r_i = b(x_1) - b_i(x_1) = \int_{c_i}^{x_1} (B(t) - B(c_i))dt.
$$

Then, since the condition $(x_1, x_2) \in Q_{i\eta}$ implies that $x_1 \geq C\mu^{-\sigma_1}$, using the mean value theorem, we have

$$
(4.29)\qquad \sup_{Q_{i\eta}} |r_i| \leq \sup_{Q_{i\eta}} \sup_{t\geq C\mu^{-\sigma_1}} |B'(t)|\,|x_1 - c_i|^2 \leq C\mu^{\sigma_1 M' - 2\sigma_3}
$$

and

$$
\begin{aligned}
(4.30)\qquad \sup_{Q_{i\eta}} |V(x_1,x_2) - V_i| &\leq C \sup_{x_1\geq C\mu^{-\sigma_1}} |\nabla V(x_1,x_2)|\mu^{-\sigma_3} \\
&\leq C\mu^{\sigma_1 m' - \sigma_3},
\end{aligned}
$$

where we used the conditions $(\mathrm{B}.2)_+$ and (V).

From the relation

$$
H_V - H_{V,i} = -2r_i(D_2 - b_i) + r_i^2 + (V - V_i),
$$

we have, for any $v \in C_0^\infty(Q_{i\eta})$ and for any small $\varepsilon_2 = \mu^{2\sigma_3} > 0$,

$$
\begin{aligned}
(4.31)\quad &\|(H_V - H_{V,i})v\|^2 \\
&\leq C\left(\sup_{Q_{i\eta}} |r_i|^2\|(D_2 - b_i)v\|^2 + \sup_{Q_{i\eta}} |r_i|^4\|v\|^2 + \sup_{Q_{i\eta}} |V - V_i|^2\|v\|^2\right) \\
&\leq C\left(\varepsilon_2\|(H_{V,i} - a_n(\mu))v\|^2 + \varepsilon_2^{-1}\sup_{Q_{i\eta}} |r_i|^4\|v\|^2 + \sup_{Q_{i\eta}} |r_i|^4\|v\|^2\right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \sup_{Q_{i\eta}} |V - V_i|^2 \|v\|^2 \Bigg) \\
&\le C\mu^{2\sigma_3} \|(H_{V,i} - a_n(\mu))v\|^2 \\
&\quad + C(\mu^{4\sigma_1 M' - 10\sigma_3} + \mu^{4\sigma_1 M' - 8\sigma_3} + \mu^{2\sigma_1 m' - 2\sigma_3})\|v\|^2,
\end{aligned}
$$

where we used (4.29) and (4.30) in the third inequality. Then we have, for any $v \in C_0^\infty(Q_{i\eta})$ and for small $\varepsilon_3 = \mu^{\sigma_3} > 0$,

$$
\begin{aligned}
&\|(H_V - a_n(\mu))v\|^2 \\
&\quad \ge (1 - \varepsilon_3)\|(H_{V,i} - a_n(\mu))v\|^2 - C\varepsilon_3^{-1}\|(H_V - H_{V,i})v\|^2 \\
&\quad \ge (1 - \varepsilon_3 - C\varepsilon_3^{-1}\mu^{2\sigma_3})\|(H_{V,i} - a_n(\mu))v\|^2 \\
&\qquad - C\varepsilon_3^{-1}(\mu^{4\sigma_1 M' - 10\sigma_3} + \mu^{2\sigma_1 m' - 2\sigma_3})\|v\|^2 \\
&\quad \ge (1 - C\mu^{\sigma_3})\|(H_{V,i} - a_n(\mu))v\|^2 - C\mu^{\sigma_4}\|v\|^2,
\end{aligned}
$$

where we used (4.31) in the second inequality, and we set

$$
\sigma_4 = \min\{4\sigma_1 M' - 11\sigma_3,\ 2\sigma_1 m' - 3\sigma_3\} \tag{4.32}
$$

in the third inequality. Then since it follows from (1.4), (3.1) that

$$
\begin{aligned}
4\sigma_1 M' - 11\sigma_3 &= 4\frac{M'}{m} - 11 - \delta_0(8M' + 11\delta_0) > 1, \\
2\sigma_1 m' - 3\sigma_3 &= 2\frac{m'}{m} - 3 - \delta_0(4m' + 3\delta_0) > 1
\end{aligned} \tag{4.33}
$$

for sufficiently small $\delta_0 > 0$, the lemma follows from (4.28), (4.32). □

Lemma 4.9. *Let d_n be as in (4.1). Then, for any $C > 0$, we have*

$$
\begin{aligned}
&N_{h_{1+}}((1 + C\mu^{\sigma_3})d_n^2) \\
&\quad \le \sum_{i \in I'} \frac{B_i}{2\pi}\mu^{-2\sigma_3}(1 + \eta)^2 \cdot \\
&\qquad \sharp\{k \in \boldsymbol{N} \,|\, |(2k - 1)B_i + V_i - a_n(\mu)| \le (1 + \eta^{-4}o(\mu))d_n\}
\end{aligned}
$$

as $\mu \downarrow 0$, where $B_i = B(c_i)$, $V_i = V(z_i)$. Here the term $o(\mu)$ is uniformly bounded with respect to small $\eta > 0$ and $i \in I'$.

Proof. For each $i \in I'$, define the form $q_{6,i}$ on $D(q_{6,i}) = C_0^\infty(Q_{i\eta})$ by

$$
q_{6,i}[u] = \|(H_{V,i} - a_n(\mu))v_i\|^2 \qquad (u \in D(q_{6,i}))
$$

and define the form q_6 on $D(q_6) = \oplus_{i\in I'} C_0^\infty(Q_{i\eta})$ by

$$q_6[\oplus_{i\in I'}(v_i)] = \sum_{i\in I'} q_{6,i}[v_i] \qquad (v_i \in C_0^\infty(Q_{i\eta})) \tag{4.34}$$

and define the isometry J_6 from $D(h_{1+}) = C_0^\infty(\widetilde{\Omega}_{1+})$ to $D(q_6) = \oplus_{i\in I'} C_0^\infty(Q_{i\eta})$ by $J_6 u = \oplus_{i\in I'}(\psi_i u)$ $(u \in D(h_{1+}))$. Then applying Lemma 2.1 to h_{1+}, q_6 and J_6, we have, by Lemma 4.8,

$$\begin{aligned}
&N_{h_{1+}}((1+C\mu^{\sigma_3})d_n^2) \\
&\quad \le N_{q_6}\left(\frac{1+\eta^{-4}o(\mu)}{1-C\mu^{\sigma_3}}(1+C\mu^{\sigma_3})d_n^2\right) \\
&\quad = \sum_{i\in I'} N_{q_{6,i}}\left(\frac{1+\eta^{-4}o(\mu)}{1-C\mu^{\sigma_3}}(1+C\mu^{\sigma_3})d_n^2\right) \\
&\quad \le \sum_{i\in I'} \frac{B_i}{2\pi}\mu^{-2\sigma_3}(1+\eta)^2 \cdot \\
&\qquad \sharp\{k \in \boldsymbol{N} \,|\, |(2k-1)B_i + V_i - a_n(\mu)| < (1+\eta^{-4}o(\mu))d_n\},
\end{aligned}$$

where we used (4.34) and Proposition 2.1 in the third inequality. □

To the end of this subsection, the quantities of the form $(1+\eta^{-4}o(\mu))d_n$ are often abbreviated simply as $\tilde{d}(\mu;\eta)$.

Lemma 4.10. *Let $\tilde{d}(\mu;\eta)$ be as above. Then*

$$\begin{aligned}
&\sum_{i\in I'} \frac{B_i}{2\pi}\,\sharp\{k \in \boldsymbol{N} \,|\, |(2k-1)B_i + V_i - a_n(\mu)| < \tilde{d}(\mu;\eta)\} \\
&\qquad \le \sum_{i\in I'} \frac{B_i}{2\pi}\, F((2n-1)B_i + V_i > a_n(\mu) - \tilde{d}(\mu;\eta))
\end{aligned}$$

holds as $\mu \downarrow 0$, where $F(P) = 1$ if a statement P is true, $F(P) = 0$ if P is false.

Proof. We first note that the condition $|(2k-1)B_i + V_i - a_n(\mu)| < \tilde{d}(\mu;\eta)$ is equivalent to

$$\begin{aligned}
(2k-1)B_i + V_i &\in (a_n(\mu) - \tilde{d}(\mu;\eta),\ a_n(\mu) + \tilde{d}(\mu;\eta)) \\
&= (\Lambda_n^+ + \mu - \eta^{-4}o(\mu),\ M_n + \mu + \eta^{-4}o(\mu)).
\end{aligned}$$

Since the conditions (B.1) and (V) implies the inequalities

$$\begin{aligned}
|V_i| &\le C\mu^{m\sigma_1}, \\
B_+ - C\mu^{M\sigma_1} \le B(\mu^{-\sigma_1} - \mu^{-\sigma_3}) &\le B_i \le B_+,
\end{aligned}$$

we can deduce that, for small $\mu \downarrow 0$,

$$\begin{aligned}
&(2k-1)B_i + V_i \le (2n-3)B_i + V_i < \Lambda_n^+ \qquad \text{if } k \le n-1,\\
&(2k-1)B_i + V_i \ge (2n+1)B_i + V_i \ge \Lambda_{n+1}^+ - o(1) \qquad \text{if } k \ge n+1.
\end{aligned}$$

Then it follows that $|(2k-1)B_i + V_i - a_n(\mu)| < \tilde{d}(\mu;\eta)$ does not hold for any $k \ge n+1$, and $F((2k-1)B_i + V_i > a_n(\mu) - \tilde{d}(\mu;\eta)) = 0$ holds for all $k \le n-1$. Thus it is enough to consider only the case of $k = n$. It is easy to see that, for small $\mu > 0$,

$$\begin{aligned}
&\sharp\{k \in \boldsymbol{N} |\ |(2k-1)B_i + V_i - a_n(\mu)| < \tilde{d}(\mu;\eta)\}\\
&\quad = \begin{cases} 1, & \text{if} \quad (2n-1)B_i + V_i \in (a_n(\mu) - \tilde{d}(\mu;\eta),\ a_n(\mu) + \tilde{d}(\mu;\eta)),\\ 0, & \text{if} \quad (2n-1)B_i + V_i \notin (a_n(\mu) - \tilde{d}(\mu;\eta),\ a_n(\mu) + \tilde{d}(\mu;\eta)), \end{cases}\\
&\quad \le F((2n-1)B_i + V_i > a_n(\mu) - \tilde{d}(\mu;\eta))
\end{aligned}$$

This completes the proof. □

For sufficiently small $\delta_1 > 0$, set

$$F_{\delta_1}(\mu) = \{(x_1, x_2) \in \widetilde{\Omega}_{1+} |\ V(x_1, x_2) > \mu(1-\delta_1)\}$$

and set

$$\begin{aligned}
I_1 &= \{i \in I' |\ Q_i \subset F_{\delta_1}(\mu)\}\ ,\\
I_2 &= I' \setminus I_1.
\end{aligned}$$

Lemma 4.11. *For small $\mu > 0$,*

$$F((2n-1)B_i + V_i > a_n(\mu) - \tilde{d}(\mu;\eta)) = 0$$

holds uniformly in $i \in I_2$.

Proof. We choose $\delta_0 > 0$ sufficiently small so that

$$\sigma_1 m' - \sigma_3 > 1, \qquad \sigma_1 M > 1. \tag{4.35}$$

Then, since $\sup_{Q_{i\eta}} |V - V_i| \le C\mu^{\sigma_1 m' - \sigma_3} = o(\mu)$ holds as $\mu \downarrow 0$, we have

$$\begin{aligned}
\sup_{i \in I_2} ((2n-1)B_i + V_i) &\le \Lambda_n^+ + C\mu^{M\sigma_1} + \mu(1-\delta_1) + o(\mu)\\
&\le \Lambda_n^+ + \mu(1-\delta_1) + o(\mu)\\
&< a_n(\mu) - \tilde{d}(\mu;\eta)
\end{aligned}$$

as $\mu \downarrow 0$, where we used $(\mathrm{B}.2)_+$ in the first inequality and used (1.4) in the second inequality. This proves the lemma. □

Lemma 4.12. *Let $C > 0$. Then we have*

$$\limsup_{\mu \downarrow 0} N_{h_{1+}}((1 + C\mu^{\sigma_3})d_n^2)/B_+\nu_+(\mu) \leq 1.$$

Proof. It follows from Lemmas 4.9–4.11 that

$$\begin{aligned} N_{h_{1+}}((1 + C\mu^{\sigma_3})d_n^2) &\leq \sum_{i \in I_1} \frac{B_i}{2\pi}\mu^{-2\sigma_3}(1+\eta)^2 \\ &\leq \frac{B_+}{2\pi}\mu^{-2\sigma_3}(1+\eta)^2(\sharp I_1) \\ &\leq \frac{B_+}{2\pi}\mathrm{vol}\ F_{\delta_1}(\mu)(1+\eta)^2, \end{aligned}$$

where we used the fact that $\mu^{-2\sigma_3}(\sharp I_1) = \mathrm{vol}\ (\bigcup_{i \in I_1} Q_i) \leq \mathrm{vol}\ F_{\delta_1}(\mu)$. Hence, taking Remark 1.1 into account, we have

$$\begin{aligned} \limsup_{\mu \downarrow 0} N_{h_{1+}}((1 + C\mu^{\sigma_3})d_n^2)/B_+\nu_+(\mu) &\leq \limsup_{\mu \downarrow 0} \frac{1}{2\pi}\frac{\mathrm{vol}\ F_{\delta_1}(\mu)}{\nu_+(\mu)}(1+\eta)^2 \\ &\leq \limsup_{\mu \downarrow 0} \frac{\nu_+(\mu(1-\delta_1))}{\nu_+(\mu)}(1+\eta)^2. \end{aligned}$$

Therefore, using the condition (T) for ν_+ and the arbitrariness of the choice of η, δ_1, we complete the proof. □

Combining Lemmas 4.3, 4.4, 4.6, 4.7 and 4.12, we have:

Proposition 4.1.

$$\limsup_{\mu \downarrow 0} N_h(d_n^2)/B_+\nu_+(\mu) \leq 1.$$

§5. Lower Bound for the Quantity $N_h(d_n^2)$

For each $i = (i_1, i_2) \in \boldsymbol{Z}^2$, set

$$Q_i = \{(x_1, x_2) \in \mathbf{R}^2 |\ i_j < x_j\mu^{\sigma_3} < i_j + 1 \text{ for } j = 1, 2\}$$

and set, for a fixed small number $\delta_2 > 0$,

$$\begin{aligned} F'_{\delta_2}(\mu) &= \{(x_1, x_2) \in \mathbf{R}^2 |\ x_1 > \mu^{-\sigma_1}\ , V(x_1, x_2) > \mu(1+\delta_2)\}, \\ I_0 &= \{i \in \boldsymbol{Z}^2 | Q_i \cap F'_{\delta_2}(\mu) \neq \emptyset\}. \end{aligned}$$

Let the operator $H_{V,i}$ be as in (4.27). For each $i \in I_0$, we introduce the form $q_{7,i}$ on $D(q_{7,i}) = C_0^\infty(Q_i)$ by

$$q_{7,i}[u] = \|(H_{V,i} - a_n(\mu))u\|^2 \qquad (u \in D(q_{7,i}))$$

and define the form q_7 on $D(q_7) = \oplus_{i\in I_0} C_0^\infty(Q_i)$ by

$$q_7(\oplus_i u_i) = \sum_{i\in I_0} q_{7,i}[u_i] \qquad (\oplus_{i\in I_0}(u_i) \in D(q_7)).$$

Lemma 5.1. *Let h be the form as in* (4.2) *and let d_n be the number as in* (4.1)*. Let B_i, V_i and b_i be as in* (4.26)*. For any small $\eta > 0$, we have*

$$\begin{aligned} &N_h(d_n^2) \\ &\quad \geq \sum_{i\in I_0} \frac{B_i}{2\pi}(1-\eta)^2\mu^{-2\sigma_3}\sharp\{k \in \boldsymbol{N} \mid |(2k-1)B_i + V_i - a_n(\mu)| < (1+o(\mu))d_n\} \end{aligned}$$

as $\mu \downarrow 0$, where the term $o(\mu)$ is uniformly bounded with respect to $i \in I_0$.

Proof. Let J_7 be the natural isometry from $\oplus_{i\in I_0} C_0^\infty(Q_i)$ to $C_0^\infty(\mathbf{R}^2)$ defined by

$$J_7(\oplus_i u_i) = \sum_{i\in I_0} u_i \qquad (u_i \in C_0^\infty(Q_i)).$$

Then we have, as in the proof of Lemma 4.2,

$$\begin{aligned} (5.1)\quad &h[J_7(\oplus u_i)] \\ &= \sum_{i\in I_0} \|(H_V - a_n(\mu))u_i\|^2 \\ &= \sum_{i\in I_0} \|(H_{V,i} - a_n(\mu))u_i + (H_V - H_{V,i})u_i\|^2 \\ &\leq \sum_{i\in I_0} \left((1+\mu^{\sigma_3})\|(H_{V,i} - a_n(\mu))u_i\|^2 + C/\mu^{\sigma_3}\|(H_V - H_{V,i})u_i\|^2\right) \end{aligned}$$

and, as in the proof of Lemma 4.8, we obtain

$$\begin{aligned} (5.2)\qquad &\|(H_V - H_{V,i})u_i\|^2 \\ &\leq \mu^{2\sigma_3}\|(H_{V,i} - a_n(\mu))u_i\|^2 \\ &\quad + C\left(\mu^{-2\sigma_3}\sup_{Q_i}|r_i|^4 + \sup_{Q_i}|r_i|^4 + \sup_{Q_i}|V - V_i|^2\right)\|u_i\|^2 \\ &\leq C\mu^{2\sigma_3}\|(H_{V,i} - a_n(\mu))u_i\|^2 + C\mu^{\sigma_4+\sigma_3}\|u_i\|^2, \end{aligned}$$

where $\sigma_4 > 1$ is the constant defined by (4.32).

Hence it follows from (5.1), (5.2) that

$$\begin{aligned} h[J_7(\oplus u_i)] &\le \sum_{i\in I_0} \left((1+C\mu^{\sigma_3})\|(H_{V,i}-a_n(\mu))u_i\|^2 + O(\mu^{\sigma_4})\|u_i\|^2\right) \\ &= (1+C\mu^{\sigma_3})q_7[\oplus u_i] + O(\mu^{\sigma_4})\sum_{i\in I_0}\|u_i\|^2. \end{aligned} \tag{5.3}$$

By using Lemma 2.1 and Proposition 2.1 (ii), we deduce from (5.3) that, as $\mu \downarrow 0$,

$$\begin{aligned} N_h(d_n^2) &\ge N_{q_7}((1+o(\mu))d_n^2) \\ &= \sum_{i\in I_0} N_{q_{7,i}}((1+o(\mu))d_n^2) \\ &\ge \sum_{i\in I_0} \frac{B_i}{2\pi}\mu^{-2\sigma_3}(1-\eta)^2\,\sharp\{k\in \boldsymbol{N}|\ |(2k-1)B_i+V_i-a_n(\mu)|<d_\eta(\mu)\}, \end{aligned}$$

where $\eta > 0$ is a sufficiently small number and $d_\eta(\mu)$ is given by the relation

$$\begin{aligned} d_\eta(\mu)^2 &= (1-\mu^{\sigma_3})(1-o(\mu))d_n^2 - C\left(\mu^{\sigma_3} + (\eta\mu^{-\sigma_3})^{-4}\mu^{-3\sigma_3}\right) \\ &= (1-o(\mu))d_n^2 - C\mu^{\sigma_3}(1+\eta^{-4}) \\ &= (1+o(\mu))d_n^2 + o(\mu) = (1+o(\mu))d_n^2. \end{aligned}$$

as $\mu \downarrow 0$. The lemma follows from this. □

Proposition 5.1. *Let h be the form as in* (4.2) *and let d_n be the number as in* (4.1). *Then we have*

$$\liminf_{\mu\downarrow 0} N_h(d_n^2)/B_+\nu_+(\mu) \ge 1.$$

Proof. We see that, as $\mu \downarrow 0$,

$$\begin{aligned} B_+ \ge B_i > B(\mu^{-\sigma_1}) = B_+ - O(\mu^{\sigma_1 M}) = B_+ - o(\mu), \\ o(1) > V_i > \mu(1+\delta_2) - o(\mu) \end{aligned}$$

uniformly in $i \in I_0$, where we used the condition $(\mathrm{B.2})_+$ in the first expression, and used (V) and the mean value theorem in the second one. Then we have, as $\mu \downarrow 0$,

$$\begin{aligned} \Lambda_n^+ + o(1) &> (2n-1)B_i + V_i \\ &> (2n-1)(B_+ - o(\mu)) + \mu(1+\delta_2) - o(\mu) \\ &= \Lambda_n^+ + \mu(1+\delta_2) + o(\mu) \end{aligned} \tag{5.4}$$

and

$$a_n(\mu) + d_\eta(\mu) = M_n + \mu + o(\mu), \quad a_n(\mu) - d_\eta(\mu) = \Lambda_n^+ + \mu + o(\mu). \tag{5.5}$$

Then we have from (5.4), (5.5) that, for any $\mu > 0$ sufficiently small,

$$a_n(\mu) - d_\eta(\mu) < (2n-1)B_i + V_i < a_n(\mu) + d_\eta(\mu)$$

holds as $\mu \downarrow 0$. From this, as in the proof of Lemma 4.10, we can deduce that

$$\sharp\{k \in \boldsymbol{N} \mid |(2k-1)B_i + V_i - a_n(\mu)| < d_\eta(\mu)\} \geq 1. \tag{5.6}$$

Then it follows from Lemma 5.1 that

$$\begin{aligned} N_h(d_n^2) &\geq \sum_{i \in I_0} \frac{B_i}{2\pi}(1-\eta)^2 \mu^{-2\sigma_3} \\ &\geq \frac{1}{2\pi}(B_+ - C\mu^{\sigma_1 M})(1-\eta)^2 \mu^{-2\sigma_3}(\sharp I_0) \\ &\geq \frac{1}{2\pi}(B_+ - C\mu^{\sigma_1 M})(1-\eta)^2 \operatorname{vol} F'_{\delta_2}(\mu), \end{aligned}$$

where we used (5.6) in the first inequality and the fact that

$$\mu^{-2\sigma_3}(\sharp I_0) = \operatorname{vol}\left(\bigcup_{i \in I_0} Q_i\right) \geq \operatorname{vol} F'_{\delta_2}(\mu)$$

in the third inequality.

On the other hand, since

$$\begin{aligned} F'_{\delta_2}(\mu) = &\{(x_1, x_2) \in \mathbf{R}^2 | x_1 > a_0, V(x_1, x_2) > \mu(1+\delta_2)\} \\ &\setminus \{(x_1, x_2) \in \mathbf{R}^2 | a_0 < x_1 < \mu^{-\sigma_1}, V(x_1, x_2) > \mu(1+\delta_2)\}, \end{aligned}$$

we obtain, as $\mu \downarrow 0$,

$$\begin{aligned} &\frac{1}{2\pi}\operatorname{vol} F'_{\delta_2}(\mu) \\ &\quad = \nu_+(\mu(1+\delta_2)) \\ &\qquad - \frac{1}{2\pi}\operatorname{vol}\{(x_1, x_2) \in \mathbf{R}^2 | a_0 < x_1 < \mu^{-\sigma_1}, V(x_1, x_2) > \mu(1+\delta_2)\} \\ &\quad \geq \nu_+(\mu(1+\delta_2)) - C\mu^{-\sigma_1 - 1/m} \\ &\quad = \nu_+(\mu(1+\delta_2)) + o(\mu^{-2/m}). \end{aligned} \tag{5.7}$$

Thus we have

$$\begin{aligned} \liminf_{\mu \downarrow 0} N_h(d_n^2)/B_+\nu_+ &\geq (1-\eta) \liminf_{\mu \downarrow 0} \frac{B_+ - C\mu^{\sigma_1 M}}{B_+} \frac{1}{2\pi} \frac{\operatorname{vol} F'_{\delta_2}(\mu)}{\nu_+(\mu)} \\ &\geq (1-\eta) \liminf_{\mu \downarrow 0} \nu_+(\mu(1+\delta_2))/\nu_+(\mu). \end{aligned}$$

Therefore the lemma follows from the condition (T) for ν_+ and the arbitrariness of the choice of $\eta > 0$, $\delta_2 > 0$. □

Accepting Proposition 2.1, we now obtain Theorem 1.1 because of Lemma 4.1, Propositions 4.1 and 5.1.

§6. Proof of Proposition 2.3

In order to complete the proof of Theorem 1.1, we give a proof of Proposition 2.1 in this section.

For $B_0 > 0$, let $H(B_0)$ be the Schrödinger operator with the constant magnetic field B_0 defined by (2.3). For $R > 0$ fixed, we introduce the function space $\mathcal{E}(B_0, R)$ by

$$\mathcal{E}(B_0, R) = \{u \in C^\infty(\mathbf{R}^2) |\ S_j u = u \quad (j = 1, 2)\}, \tag{6.1}$$

where we set

$$S_1 u(x_1, x_2) = u(x_1 + R, x_2) e^{i\frac{B_0}{2} R x_2}, \quad S_2 u(x_1, x_2) = u(x_1, x_2 + R) e^{-i\frac{B_0}{2} R x_1}. \tag{6.2}$$

It is easy to see that $B_0 R^2 \in 2\pi \boldsymbol{Z}$ if and only if $S_1 S_2 = S_2 S_1$, and that $H(B_0)$, $\Pi_j(B_0)$ $(j = 1, 2)$ leave $\mathcal{E}$ invariant.

In what follows we fix $z_0 \in \mathbf{R}^2$ and denote $Q(z_0, R')$ simply by $Q_{R'}$ for $R' > 0$. We denote by $H(B_0, R)$ the operator closure of $H(B_0)|_{\mathcal{E}_{Q_R}}$, where we set

$$\mathcal{E}_{Q_R} = \{\text{the restriction of } f \text{ on } Q_R |\ f \in \mathcal{E}(B_0, R)\} \qquad (\subset L^2(Q_R)).$$

Proposition 6.1. *Assume $B_0 R^2 = 2\pi N_0$ holds for some integer N_0. Then $H(B_0, R)$ is a self-adjoint operator. Moreover $H(B_0, R)$ has eigenvalues $\{(2k-1)B_0\}_{k \in N}$ and each of them has multiplicity N_0.*

Proof. For the proof, see, e.g., [Col], Section 2. □

For each $\lambda \in \mathbf{R}$, we define the quadratic form $q = q(B_0, \lambda, Q_R)$ on $D(q) = C_0^\infty(Q_R)$ by

$$q[u] = \|(H(B_0) + \lambda)u\|^2_{L^2(Q_R)} \qquad (u \in D(q)), \tag{6.3}$$

We prove Proposition 2.3 (i). Let $R > 0$ and take $R_1 > 0$ large enough with $B_0 R_1^2 \in 2\pi \boldsymbol{Z}$, and let $l \geq 0$ be the largest integer which satisfies $lR \leq R_1$.

We divide a square $Q' = Q(z_0, lR)$ to mutually disjoint l^2 squares of size $R \times R$, and denote them by $\{Q'_j\}_{j=1}^{l^2}$.

Define the form q_8 on $D(q_8) = \bigoplus_{j=1}^{l^2} C_0^\infty(Q'_j)$ by

$$q_8[\oplus_{j=1}^{l^2} f_j] = \sum_{j=1}^{l^2} q[f_j] \qquad (f_j \in C_0^\infty(Q'_j)),$$

where the form q is as in (6.3), and define the form q_9 on $D(q_9) = \mathcal{E}(B_0, R_1)|_{Q_{R_1}}$ by

$$q_9[f] = \|(H(B_0) + \lambda)f\|^2_{L^2(Q_{R_1})} \qquad (f \in D(q_9)).$$

Then it follows from Lemma 2.1 and the definition of Q'_j's that

$$N_{q_8}(\mu^2) = l^2 N_q(\mu^2). \tag{6.4}$$

If we define the isometry J_8 by

$$J_8 : D(q_8) \ni \oplus_{j=1}^{l^2} f_j \mapsto \sum_{j=1}^{l^2} f_j \in \mathcal{E}(B_0, R_1)|_{Q_{R_1}},$$

then we have $q_8[f] = q_9[J_6 f]$ for all $f \in D(q_8)$. Then, applying Lemma 2.1, we deduce from (6.4) that

$$l^2 N_q(\mu^2) \le N_{q_9}(\mu^2). \tag{6.5}$$

Lemma 2.2 implies that the self-adjoint operator associated with q_9 is $(H(B_0, R_1) + \lambda)^2$, so it follows from Proposition 6.1 that

$$N_{q_9}(\mu^2) = \frac{B_0 R_1^2}{2\pi} \sharp\{k \in \boldsymbol{N} |\ |(2k-1)B_0 + \lambda| < \mu\}. \tag{6.6}$$

Then, from (6.4)–(6.6), we have

$$N_q(\mu^2) \le \frac{B_0 R_1^2}{2\pi l^2} \sharp\{k \in \boldsymbol{N} |\ |(2k-1)B_0 + \lambda| < \mu\}. \tag{6.7}$$

Since, by the definition of l, we have

$$\frac{B_0 R_1^2}{2\pi l^2} \longrightarrow \frac{R^2 B_0}{2\pi} \qquad \text{as} \quad R_1 \to \infty,$$

the lemma follows from (6.7). Then we complete the proof of Proposition 2.1 (i).

In the rest of this section, we devote ourselves to proving Proposition 2.1 (ii). For any $\eta > 0$ fixed, choose $R > 0$ sufficiently large so that $0 < \eta < R/2$. For a large integer $l > 0$, choose $R_2 > 0$ as the largest number such that $B_0 R_2^2 \in 2\pi \boldsymbol{Z}$ and $R_2 \le l(R-\eta)$ hold. By the definition of R_2, we observe that

$$\frac{B_0}{2\pi} l^2 (R-\eta)^2 - 1 \le \frac{B_0}{2\pi} R_2^2,$$

then, for any given $\varepsilon > 0$ and for sufficiently large l,

$$\begin{aligned} R_2 &> \{l^2(R-\eta)^2 - 2\pi/B_0\}^{1/2} \\ &\ge l(R-\eta)\left(1 - \frac{2\pi C}{B_0 l^2 (R-\eta)^2}\right) \\ &\ge (l-\varepsilon)(R-\eta). \end{aligned}$$

Thus we have, for any given $\varepsilon > 0$ and for sufficiently large l,

$$l(R-\eta) \ge R_2 > (l-\varepsilon)(R-\eta). \tag{6.8}$$

Let $z_0 \in \mathbf{R}^2$ fixed and we consider a partition of a square $Q(z_0, l(R-\eta))$ into mutually disjoint l^2 squares $Q'_j = Q(z_j, R-\eta)$ $(j = 1, 2, \dots, l^2)$, where z_j is the center of Q'_j. Note that the condition (6.8) implies that $\overline{Q(z_0, R_2)} \subset \cup_{j=1}^{l^2} \overline{Q'_j}$.

Let Ω be a subset of $\mathbf{R}^2$. For each $n = (n_1, n_2) \in \boldsymbol{Z}^2$ and for any $R > 0$, we denote by $\Omega + nR$ the set $\{(z_1 + n_1 R,\ z_2 + n_2 R) \in \mathbf{R}^2 | z = (z_1, z_2) \in \Omega\}$.

Lemma 6.1. *Let η, R, R_2 and z_j be as above and set $Q_j = Q(z_j, R)$ $(j = 1, \dots, l^2)$. Then there exists a partition of unity of $\mathbf{R}^2$, $\{\psi_j^\sharp\}_{j=1}^{l^2}$, such that $0 \le \psi_j^\sharp(z) \le 1$,*

$$\sum_{j=1}^{l^2} \psi_j^\sharp(z)^2 = 1 \quad \text{on} \quad \mathbf{R}^2, \qquad \operatorname{supp} \psi_j^\sharp \subset \bigcup_{n \in Z^2} (Q_j + nR_2) \tag{6.9}$$

for $j = 1, \dots, l^2$ and, for each multi-index α,

$$|\partial_z^\alpha \psi_j^\sharp(z)| \le C_\alpha \eta^{-|\alpha|} \tag{6.10}$$

hold, where the constant C_α is independent of z, η. Moreover, $\psi_j^\sharp(x_1, x_2)$ is periodic with periods $(R_2, 0)$, $(0, R_2)$.

Proof. Let $\theta(z)$ be a smooth function on $Q(0, R)$ with compact support such that $0 \le \theta(z) \le 1$, $\theta(z) = 1$ on $Q(0, R-\eta)$, and $|\partial^\alpha \theta(z)| \le C_\alpha \eta^{-|\alpha|}$ holds

for each multi-index $\alpha = (\alpha_1, \alpha_2)$. If we set $\theta_j(z) = \theta(z - z_j)$ for $z \in \mathbf{R}^2$, then $\theta_j \in C_0^\infty(Q_j)$ and $\theta_j = 1$ holds on $\overline{Q'_j}$.

Define

$$\theta_j^\sharp(z) = \sum_{n \in Z^2} \theta_j(z - nR_2),$$

then we have

$$\theta_j^\sharp = 1 \quad \text{on} \quad \bigcup_{n \in Z^2} (\overline{Q'_j} + nR_2), \qquad \operatorname{supp} \theta_j^\sharp \subset \bigcup_{n \in Z^2} (Q_j + nR_2). \tag{6.11}$$

It follows from (6.8) for sufficiently large l that $(Q_j + nR_2) \cap (Q_j + n'R_2) = \emptyset$ if $n \neq n'$, and, for each $k = 1, \ldots, l^2$,

$$\sharp\{j \mid \bigcup_{n \in Z^2} (Q_j + nR_2) \cap \bigcup_{n' \in Z^2} (Q_k + n'R_2) \neq \emptyset\} \leq 9. \tag{6.12}$$

If we set $\Theta = \sum_{j=1}^{l^2} (\theta_j^\sharp)^2$, then we can deduce from (6.11) and (6.12) that $1 \leq \Theta \leq 9$ holds.

Finally, we obtain the desired functions $\psi_j^\sharp = \theta_j^\sharp / \sqrt{\Theta}$ for $j = 1, \ldots, l^2$. $\square$

We introduce the forms

$$\begin{aligned} q_{10}[u] &= \|(H(B_0) + \lambda)u\|^2_{L^2(Q_{R_2})}, \\ D(q_{10}) &= \mathcal{E}(B_0, R_2)|_{Q_{R_2}} \qquad (\subset L^2(Q_{R_2})) \end{aligned}$$

and

$$\begin{aligned} q_{11}[\oplus_{k=1}^{l^2} f_k] &= \sum_{k=1}^{l^2} \|(H(B_0) + \lambda) f_k\|^2_{L^2(Q_k)}\ , \\ D(q_{11}) &= \bigoplus_{k=1}^{l^2} C_0^\infty(Q_k) \quad \left(\subset \bigoplus_{k=1}^{l^2} L^2(Q_k) \right), \end{aligned}$$

and define a map J_{10} by

$$J_{10} : D(q_{10}) \ni u \mapsto \oplus_{k=1}^{l^2} (\psi_k^\sharp \cdot u^\sharp)|_{Q_k} \in D(q_{11}),$$

where $u^\sharp$ is the element of $\mathcal{E}(B_0, R_2)$ such that $u^\sharp|_{Q_{R_2}} = u$.

Lemma 6.2. *The map J_{10} is an isometry from $D(q_{10})$ to $D(q_{11})$ with respect to norms of $L^2(Q_{R_2})$ and $\bigoplus_{k=1}^{l^2} L^2(Q_k)$, respectively.*

Proof. By the definitions, we have

$$\|J_{10}u\|^2_{\oplus L^2(Q_k)} = \sum_{k=1}^{l^2} \|\psi_k^\sharp u^\sharp\|^2_{L^2(Q_k)} = \sum_{k=1}^{l^2} \int_{Q_k} \psi_k^\sharp(z)^2 |u^\sharp(z)|^2 dz, \tag{6.13}$$

and

$$\begin{aligned} &\int_{Q_k} \psi_k^\sharp(z)^2 |u^\sharp(z)|^2 dz \\ &= \sum_{n\in Z^2} \int_{Q_k\cap(Q_{R_2}+nR_2)} \psi_k^\sharp(z)^2 |u^\sharp(z)|^2 dz \\ &= \sum_{n\in Z^2} \int_{(Q_k-nR_2)\cap Q_{R_2}} \psi_k^\sharp(z'+nR_2)^2 |u^\sharp(z'+nR_2)|^2 dz' \\ &= \int_{Q_{R_2}} \psi_k^\sharp(z)^2 |u^\sharp(z)|^2 dz, \end{aligned} \tag{6.14}$$

since it follows from Lemma 6.1 that $\psi_k^\sharp(z)^2|u^\sharp(z)|^2$ is a function with periods $(R_2, 0)$, $(0, R_2)$ and $\operatorname{supp}\psi_k^\sharp \subset \bigcup_{n\in Z^2}(Q_k - nR_2)$. Hence it follows from (6.13) and (6.14) that

$$\begin{aligned} \|J_{10}u\|^2_{\oplus L^2(Q_k)} &= \sum_{k=1}^{l^2} \int_{Q_{R_2}} \psi_k^\sharp(z)^2 |u^\sharp(z)|^2 dz \\ &= \int_{Q_{R_2}} |u^\sharp(z)|^2 dz \\ &= \|u\|^2_{L^2(Q_{R_2})}, \end{aligned}$$

where we used (6.9) in the second inequality. This shows that J_{10} is an isometry. □

Lemma 6.3. *There exist $C > 0$ and $\delta_1 > 0$ such that*

$$q_{10}[u] \geq (1-2\delta)\ q_{11}[J_{10}u] - C\left(\delta\lambda^2 + \frac{1}{\delta^3\eta^4} + \frac{1}{\delta\eta^4}\right)\|u\|^2_{L^2(Q_{R_2})}$$

holds for any positive numbers $\eta < R/2$ and $\delta < \delta_1$ and for all $u \in D(q_1)$.

Proof. By the same method as in the preceding sections, we have, for small $\delta > 0$,

$$
\begin{aligned}
q_{10}[u] &= \|(H(B_0)+\lambda)u\|^2_{L^2(Q_{R_2})} \\
&= \|J_{10}(H(B_0)+\lambda)u)\|^2_{\oplus L^2(Q_k)} \\
&= \sum_{k=1}^{l^2} \|\psi_k^\sharp (H(B_0)+\lambda)u^\sharp\|^2_{L^2(Q_k)} \\
&\geq (1-\delta)\sum_{k=1}^{l^2} \|(H(B_0)+\lambda)\psi_k^\sharp u^\sharp\|^2_{L^2(Q_k)} \\
&\quad - \frac{C}{\delta}\sum_{k=1}^{l^2} \|[\psi_k^\sharp, H(B_0)]u^\sharp\|^2_{L^2(Q_k)} \\
&= (1-\delta)q_{11}[J_{10}u] - \frac{C}{\delta}\sum_{k=1}^{l^2} \|[\psi_k^\sharp, H(B_0)]u^\sharp\|^2_{L^2(Q_k)}.
\end{aligned} \tag{6.15}
$$

For the sum in the R.H.S. of (6.15), we have

$$
[\psi_k^\sharp, H(B_0)] = \sum_{j=1}^{2}\{2i(\partial_j\psi_k^\sharp)\Pi_j(B_0) + (\partial_j^2\psi_k^\sharp)\}.
$$

Then it follows from (6.10) that

$$
\begin{aligned}
&\sum_{k=1}^{l^2} \|[\psi_k^\sharp, H(B_0)]u^\sharp\|^2_{L^2(Q_k)} \\
&\quad \leq \frac{C}{\eta^2}\sum_{k=1}^{l^2}\sum_{j=1}^{2} \|\Pi_j(B_0)u^\sharp\|^2_{L^2(Q_k)} + \frac{C}{\eta^4}\sum_k \|\chi_{Q_k}u^\sharp\|^2_{L^2(Q_k)}.
\end{aligned} \tag{6.16}
$$

For the second sum in the R.H.S. of (6.16), we have

$$
\begin{aligned}
\sum_{k=1}^{l^2} \|\chi_{Q_k}u^\sharp\|^2_{L^2(Q_k)} &= \sum_{k=1}^{l^2} \int_{Q_k} |u^\sharp(z)|^2 dz \\
&= \sum_{k=1}^{l^2}\sum_{n\in Z^2} \int_{Q_k\cap(Q_{R_2}+nR_2)} |u^\sharp(z)|^2 dz \\
&= \sum_{k=1}^{l^2}\sum_{n\in Z^2} \int_{(Q_k-nR_2)\cap Q_{R_2}} |u^\sharp(z)|^2 dz
\end{aligned} \tag{6.17}
$$

$$= \sum_{k=1}^{l^2} \int_{\cup_n (Q_k + nR_2) \cap Q_{R_2}} |u^\sharp(z)|^2 dz$$
$$\leq 9 \int_{Q_{R_2}} |u(z)|^2 dz,$$

where we used the periodicity of $|u^\sharp(z)|$ in the third equality and used (6.12) in the last inequality.

For the first term in the R.H.S. of (6.16), we have, for small $\delta' > 0$,

$$\begin{aligned}
(6.18)\quad & \sum_{j=1}^{2} \sum_{k=1}^{l^2} \|\Pi_j(B_0) u^\sharp\|^2_{L^2(Q_k)} \\
& \leq \sum_{j=1}^{2} 9 \int_{Q_{R_2}} |\Pi_j(B_0) u(z)|^2 dz \quad \text{(by (6.12))} \\
& = 9 \sum_{j=1}^{2} (\Pi_j(B_0)^2 u, u)_{L^2(Q_{R_2})} \quad \text{(by (6.1)–(6.2))} \\
& \leq \delta' \|H(B_0) u\|^2_{L^2(Q_{R_2})} + \frac{C}{\delta'} \|u\|^2_{L^2(Q_{R_2})} \\
& \leq 2\delta' \left(\|(H(B_0) + \lambda) u\|^2_{L^2(Q_{R_2})} + \lambda^2 \|u\|^2_{L^2(Q_{R_2})} \right) + \frac{C}{\delta'} \|u\|^2_{L^2(Q_{R_2})}.
\end{aligned}$$

From (6.16)–(6.18), we have

$$\begin{aligned}
(6.19)\quad & \sum_k \|[\psi_k, H(B_0)] u^\sharp\|^2 \\
& \leq C \left(\frac{1}{\eta^2} \left(\delta' \|(H(B_0) + \lambda) u\|^2_{L^2(Q_{R_2})} + \left(\frac{1}{\delta'} + \delta' \lambda^2 \right) \|u\|^2_{L^2(Q_{R_2})} \right) \right. \\
& \quad \left. + \frac{1}{\eta^4} \|u\|^2_{L^2(Q_{R_2})} \right).
\end{aligned}$$

Then it follows from (6.15) and (6.19) that

$$q_{10}[u] \geq (1 - \delta)\ q_{11}[J_{10} u] - \frac{C\delta'}{\delta \eta^2}\ q_{10}[u] - C \left(\frac{\delta' \lambda^2}{\delta \eta^2} + \frac{1}{\delta \eta^2 \delta'} + \frac{1}{\delta \eta^4} \right) \|u\|^2_{L^2(Q_{R_2})}.$$

Finally, if we set $\delta' = \delta^2 \eta^2 / C$, we have

$$\begin{aligned}
q_{10}[u] & \geq \frac{1 - \delta}{1 + \delta}\ q_{11}[J_{10} u] - \frac{C}{1 + \delta} \left(\delta \lambda^2 + \frac{1}{\delta^3 \eta^4} + \frac{1}{\delta \eta^4} \right) \|u\|^2_{L^2(Q_{R_2})} \\
& \geq (1 - 2\delta)\ q_{11}[J_{10} u] - C \left(\delta \lambda^2 + \frac{1}{\delta^3 \eta^4} + \frac{1}{\delta \eta^4} \right) \|u\|^2_{L^2(Q_{R_2})}
\end{aligned}$$

for sufficiently small $\delta > 0$. This completes the proof. $\square$

Applying Lemma 2.1 to the following pair of triplets:

$$\left(L^2(Q_{R_2}),\ q_{10},\ \mathcal{E}(B_0, R_2)|_{Q_{R_2}}\right), \qquad (\bigoplus_{k=1}^{l^2} L^2(Q_k),\ q_{11},\ \bigoplus_{k=1}^{l^2} C_0^\infty(Q_k)),$$

we have from Lemmas 6.2 and 6.3, for small $\mu > 0$,

$$N_{q_{10}}(\tilde{\mu}^2_{\lambda\eta\delta}) \le N_{q_{11}}(\mu^2), \tag{6.20}$$

where $\tilde{\mu}_{\lambda\eta\delta}$ is defined by the relation

$$\tilde{\mu}^2_{\lambda\eta\delta} = (1-2\delta)\ \mu^2 - C\left(\delta\lambda^2 + \frac{1}{\delta^3\eta^4} + \frac{1}{\delta\eta^4}\right).$$

Then Proposition 6.1 implies that

$$N_{q_{10}}(\tilde{\mu}^2_{\lambda\eta\delta}) = \frac{B_0 R_2^2}{2\pi}\sharp\{k \in \boldsymbol{N} |\ |(2k-1)B_0 + \lambda| < \tilde{\mu}_{\lambda\eta\delta}\} \tag{6.21}$$

and

$$N_{q_{11}}(\mu^2) = l^2 N_q(\mu^2), \tag{6.22}$$

where the form q is as in (6.3). From (6.20)–(6.22), it follows that, as $l \to \infty$,

$$\begin{aligned} N_q(\mu^2) &\ge \frac{B_0 R_2^2}{2\pi l^2}\sharp\{k \in \boldsymbol{N} |\ |(2k-1)B_0 + \lambda| < \tilde{\mu}_{\lambda\eta\delta}\} \\ &\ge (l-\varepsilon)^2 \frac{B_0(R-\eta)^2}{2\pi l^2}\sharp\{k \in \boldsymbol{N} |\ |(2k-1)B_0 + \lambda| < \tilde{\mu}_{\lambda\eta\delta}\}, \end{aligned} \tag{6.23}$$

where we used the relation (6.8) in the last inequality.

Now we complete the proof of Proposition 2.1 (ii), since

$$\delta\lambda^2 + \frac{1}{\delta^3\eta^4} + \frac{1}{\delta\eta^4} \le C\left(\delta\lambda^2 + \frac{1}{\delta^3\eta^4}\right) \tag{6.24}$$

holds for $\delta > 0$ sufficiently small and hence $\mu_{\lambda\eta\delta} \le \tilde{\mu}_{\lambda\eta\delta}$.

References

[A-H-S] Avron, J., Herbst, I. and Simon, B., Schrödinger operators with magnetic fields I. General interactions, *Duke Math. J.*, **45**(4) (1978), 847-883.

[Col] Colin de Verdiere, Yves., L'asymptotique de Weyl pour les bouteilles magnétiques, *Comm. Math. Phys.*, **105** (1986), 327-335.

[E-K] Exner, P. and Kovarik, H., Magnetic strip waveguides, *J. Phys. A.*, **33** (16) (2000), 3297-3311.

[Iwa] Iwatsuka, A., Examples of absolutely continuous Schrödinger operators in magnetic fields, *Publ. RIMS, Kyoto Univ.*, **21** (1985), 385-401.

[Kat] Kato, T., *Perturbation Theory for Linear Operator*, Springer, Berlin-Heidelberg-New York, 1976.

[L-S] Leinfelder, H. and Simader, C. G., Schrödinger operators with singular magnetic potentials, *Math. Z.*, **176** (1981), 1-19.

[M-P] Mântoiu, M. and Purice, R., Some propagation properties of the Iwatsuka model, *Comm. Math. Phys.*, **188** (1997), 691-708.

[Rai1] Raikov, G. D., Eigenvalue asymptotics for the Schrödinger operator with homogeneous magnetic potential and decreasing electric potential I. Behaviour near the essential spectrum tips, *Comm. in P.D.E.*, **15**(3) (1990), 407-434.

[Rai2] ———, Border-line eigenvalue asymptotics for the Schrödinger operator with electromagnetic potential, *Integral Eq. and Operator Theory*, **14** (1991), 875-888.

[R-S1] Reed, M. and Simon, B., *Methods of Modern Mathematical Physics*, Vol. I, Academic Press, New York, 1972.

[R-S2] ———, *Methods of Modern Mathematical Physics*, Vol. II, Academic Press, New York, 1975.

[R-S4] ———, *Methods of Modern Mathematical Physics*, Vol. IV, Academic Press, New York, 1978.

[Tam] Tamura, H., Asymptotic distribution of eigenvalues for Schrödinger operators with homogeneous magnetic fields, *Osaka J. Math.*, **25** (1988), 633-647.