On a Certain Semiclassical Problem
on Wiener Spaces

By

Shigeki AIDA[*]

Abstract

We study asymptotic behavior of the spectrum of a Schrödinger type operator $L_V^\lambda = L - \lambda^2 V$ on the Wiener space as $\lambda \to \infty$. Here L denotes the Ornstein-Uhlenbeck operator and V is a nonnegative potential function which has finitely many zero points. For some classes of potential functions, we determine the divergence order of the lowest eigenvalue. Also tunneling effect is studied.

§1. Introduction

Let Δ be the Laplace operator on $\mathbb{R}^n$ and consider a Schrödinger operator $H_\lambda = \Delta - \lambda^2 V$, where V is a smooth nonnegative potential function. The study on the spectral set of $-H_\lambda$ when λ tends to infinity is called a semiclassical problem and many results were obtained ([18], [19], [21], [30], [31] and references therein).

We recall basic results in semiclassical problem on $\mathbb{R}^n$. Now we assume that V has finitely many zero points and the Hessian is nondegenerate there and $\liminf_{|x| \to \infty} V(x) > 0$. Then the divergence order of the low lying eigenvalues is the same as λ and the coefficients are determined by the harmonic oscillators which are obtained by approximating V by the quadratic functions near the zero points. Also the ground state (= positive normalized eigenfunction corresponding to the lowest eigenvalue) localizes near zero points as $\lambda \to \infty$. If the

[*]Department of Mathematical Science, Graduate School of Engineering Science, Osaka University, Toyonaka, 560-8531, Japan.
e-mail: `aida@sigmath.es.osaka-u.ac.jp`

ground state can localize in more than two wells in positive probability, then the difference of the lowest two eigenvalues is exponentially small like $e^{-c\lambda}$ under semiclassical limit and the coefficient c is determined by the Agmon distance between the two zero points.

On the other hand, second order differential operators in infinite dimensional spaces naturally arose in quantum field theory and the spectral properties were studied from such view points and basic properties of Schrödinger type operators in abstract Boson Fock spaces can be found in Simon and Hoegh-Krohn [33]. More generally, let $-L$ be a nonnegative self-adjoint operator which generates a diffusion semigroup on $L^2(X, \mu)$. Here (X, μ) denotes a probability space. Take a nonnegative integrable function V on X and consider a Schrödinger type operator $-L_V = -L + V$ which is an abstract version of a perturbed Hamiltonian in Boson Fock space. Let $E_0 = \inf\{\sigma(-L_V)\}$ and $E_1 = \inf\{\sigma(-L_V) \setminus \{E_0\}\}$. Actually it is not trivial when E_0 and E_1 are eigenvalues and $E_1 \neq E_0$. By the result in [14], if $-L_V$ generates a hyperbounded semigroup, then E_0 is an eigenvalue. In addition to the assumption, if $-L$ satisfies the weak spectral gap property (see [27], [4], [23], [13]), $E_1 - E_0 > 0$ holds which was proved recently by Gong, Röckner and Wu [12]. This is a generalization of a part of Theorem 4.5 in Simon and Hoegh-Krohn [33]. The results in [12] can be applied to Schrödinger type operator on loop spaces over compact Riemannian manifolds which were studied in [16], [3] and [11]. However, differently from finite dimensional cases, little is known about semiclassical analysis in infinite dimensional spaces. It might be a good time to study semiclassical analysis in infinite dimensional spaces taking the developments above into account.

In this paper, we will consider a Schrödinger type operator $-L_V^\lambda = -L + \lambda^2 V$ on an abstract Wiener space X and we will study the asymptotic behavior of the lowest eigenvalue and the gap of the lowest two eingenvalues as $\lambda \to \infty$. Here L denotes the Ornstein-Uhlenbeck operator. We just consider the case where X is a Hilbert space and V is a C^3 function on X in this paper. In Euclidean quantum field theory, the different scaled Hamiltonian $-L + \lambda V(\lambda^{-1/2}\phi)$ should be studied and Arai [8] studied the semiclassical limit of the partition function of the Hamiltonian. In the forthcoming paper [6], we will study the behavior of the lowest eigenvalue of a Schrödinger operator of this scaling. Also see [9], [20], [34] and [35] for the semiclassical problem in high dimensions. One of my motivation to study semiclassical analysis in infinite dimensional spaces is to study the semiclassical behavior of "Witten complex" on loop spaces. See [36], [7], [6].

The structure of this paper is as follows. First, we consider a Schrödinger operator with a quadratic potential function. Differently from finite dimensional cases, the asymptotic order of the lowest eigenvalue may be the same as λ^α, where α is any positive number in $[1, 2)$. Next we consider a general potential function V and we prove that the asymptotics of the lowest eigenvalue of L_V^λ is the same as that of the Schrödinger operator with the approximate quadratic potential. Finally, we will consider double well type potential function. In such a case, tunneling phenomena occur and we will give an upper bound estimate on the asymptotics of the gap of the two lowest eigenvalues by using the Agmon distance. At the moment, I do not have any results on the lower bound estimate. For that purpose, probably, we need good estimates on the semigroup $e^{tL_V^\lambda/\lambda}$ which is related with Schilder type large deviation and pointwise estimate on the ground state near the localized points under the limit $\lambda \to \infty$ as one can see in the proof in [30].

§2. Quadratic Potential

Let H be a real separable Hilbert space. Let T be a trace class strictly positive self-adjoint operator on H. Let X be the Hilbert space which is the completion of H with respect to the norm $\|h\|_X = \|\sqrt{T}h\|_H$. Then naturally $H \subset X$ and the embedding is a Hilbert-Schmidt operator. For simplicity, we may denote $(\ ,\ )$ instead of $(\ ,\ )_H$ by omitting the subscript H. By the definition, $\sqrt{T}$ can be extended to an isometry uniquely from X onto H. We denote the operator by the same notation. It is standard that there exists a unique Gaussian measure μ on X such that for any $h \in X^*$

$$\int_X \exp\left[\sqrt{-1}\,_{X^*}(h, \phi)_X\right] d\mu(\phi) = \exp\left(-\frac{1}{2}\|h\|_H^2\right).$$

Here we use the natural identification and the embedding $H \simeq H^* \supset X^*$. The triplet (X, H, μ) is called an abstract Wiener space. The space X is not necessary a Hilbert space in general. But in the calculus below, we use the Hilbert space structure. So we consider the Hilbert space case only in this paper. For $h \in H$, we denote by $(\phi, h)_H$ a Wiener integral. Note that the functional is defined just for almost all $\phi \in X$. We denote by $L(H, H)$ the set of bounded linear operators on H. Now we introduce a subset $L_T(H, H)$ of $L(H, H)$. We define $K \in L(H, H)$ belongs to $L_T(H, H)$ if and only if the following (1) and (2) hold.

(1) K is a strictly positive trace class operator on H.

(2) $\sqrt{K}$ can be extended to a bounded linear operator from X to H,

Again we denote the extension by $\sqrt{K}$. Of course, $T \in L_T(H, H)$. In this section, we denote the eigenvalues and normalized eigenvectors of K by $\{\xi_i\}_{i=1}^{\infty}$ and $\{e_i\}_{i=1}^{\infty}$ such that $Ke_i = \xi_i e_i$. Note that $e_i \in X^*$ for all i. This is obvious because $|(e_i, h)_H| \leq \xi_i^{-1/2} \|\sqrt{K}\|_{L(X,H)} \|h\|_X$ for all $h \in H$. Therefore (ϕ, e_i) is not a Wiener integral but a continuous linear functional on X. Let us consider a quadratic Wiener functional $V_{K,h}(\phi) = \|\sqrt{K}(\phi - h)\|_H^2 \ \ (\phi \in X)$. It holds that $DV_{K,h}(\phi) = 2(K\phi - Kh)$ and $|DV_{K,h}(\phi)|_H \leq 2\|\sqrt{K}\|_{L(H,H)} \cdot \sqrt{V_{K,h}(\phi)}$ for all ϕ. Here D denotes the H-derivative. Below we will use several notations in the Malliavin calculus. $D^{k,p}$ denotes the space of real valued functions on X whose H-derivatives up to k-times and themselves are in L^p. We refer the precise definition to [10]. Let L be the Ornstein-Uhlenbeck operator on $L^2(X, \mu)$. Let us consider a nonpositive symmetric operator $L_{K,h}^{\lambda} = L - \lambda^2 V_{K,h}$ on $\mathcal{F}C_b^{\infty}$. Here $\mathcal{F}C_b^{\infty}$ denotes the space of smooth cylindrical functions. Then by Theorem X.58 in [28], this operator is essentially self-adjoint. We use the same notation to denote the self-adjoint extension. We can see the spectral property of $L_{K,h}^{\lambda}$ in Theorem 2.2 below. Before doing so, we prepare the following.

Lemma 2.1. *For any $h \in H$, we have*

$$(2.1) \qquad 0 \leq \left(\left(\sqrt{I_H + 4K} - I_H \right) h, h \right) \leq 2 \min \left\{ (Kh, h), \left(\sqrt{K} h, h \right) \right\}.$$

In particular $\sqrt{I_H + 4\lambda^2 K} - I_H$ is a trace class operator for any λ.

Proof. Let $h_i = (h, e_i)_H$. Then for any $h \in H$, it holds

$$(2.2) \qquad \left(\left(\sqrt{I_H + 4K} - I_H \right) h, h \right) = \sum_{i=1}^{\infty} \left(\sqrt{1 + 4\xi_i} - 1 \right) h_i^2$$

$$= \sum_{i=1}^{\infty} \frac{4\xi_i h_i^2}{1 + \sqrt{1 + 4\xi_i}}$$

$$\leq \min \left\{ 2 \sum_{i=1}^{\infty} \xi_i h_i^2, 2 \sum_{i=1}^{\infty} \sqrt{\xi_i} h_i^2 \right\}.$$

$\square$

Theorem 2.2. (1) *The lowest eigenvalue of $-L_{K,h}^{\lambda}$ is given by*

$$(2.3) \qquad E_0(\lambda, K, h) = \sum_{i=1}^{\infty} \left\{ \frac{1}{2} \left(\sqrt{1 + 4\lambda^2 \xi_i} - 1 \right) + \frac{h_i^2 \xi_i \lambda^2}{1 + 4\lambda^2 \xi_i} \right\}.$$

(2) *The spectral set of $\sigma(-L_{K,h}^{\lambda})$ consists of eigenvalues*

(2.4)
$$\sigma_{pp}(-L_{K,h}^{\lambda}) = \left\{ E_0(\lambda, K, h) + \sum_{i=1}^{\infty} k_i \sqrt{1 + 4\lambda^2 \xi_i} \,\middle|\, k_i \in \mathbb{N} \cup \{0\}, \sum_{i=1}^{\infty} k_i < \infty \right\}$$

and the essential spectrum

(2.5)
$$\sigma_{ess}(-L_{K,h}^{\lambda}) = \left\{ x + n \,\middle|\, x \in \sigma_{pp}(-L_{K,h}^{\lambda}), n \in \mathbb{N} \right\}.$$

The set $\sigma_{pp}(-L_{K,h}^{\lambda})$ is given by counting multiplicities. Also the multiplicity of each eigenvalue is finite and the eigenfunctions constitute a complete orthonormal system.

(3) *The ground state of $-L_{K,h}^{\lambda}$ is given by*

(2.6)
$$\Omega_{K,h}(\lambda, \phi)$$
$$= \det\left(I_H + 4\lambda^2 K\right)^{1/8} \exp\left[-\frac{1}{4}\left(\left(\sqrt{I_H + 4\lambda^2 K} - I_H\right)(\phi - h), (\phi - h)\right)\right]$$
$$\times \exp\left[-\frac{1}{2}\left(\phi - h, \left\{\left(I_H + 4\lambda^2 K\right)^{-1/2} - I_H\right\} h\right)\right.$$
$$\left. + \frac{1}{4}\left(\left\{I_H - (I_H + 4\lambda^2 K)^{-3/2}\right\} h, h\right)\right].$$

Proof. Let $h_i = (h, e_i)$. Define $-L_{K,h,n}^{\lambda} = -\sum_{i=1}^{n} \bar{L}_i$, where

$$-\bar{L}_i = -\frac{d^2}{dx_i^2} + x_i \frac{d}{dx_i} + \lambda^2 \xi_i (x_i - h_i)^2.$$

$\bar{L}_i$ is a 1-dimensional version of $-L_{K,h}^{\lambda}$. Namely it is a self-adjoint operator on $L^2(\mathbb{R}, \mu_i)$, where $d\mu_i = \frac{e^{-\frac{x_i^2}{2}}}{\sqrt{2\pi}} dx_i$. Let us consider the spectral set of $\bar{L}_i$. By using the unitary transformation $U_i : L^2(\mathbb{R}, d\mu_i) \to L^2(\mathbb{R}, dx_i)$ such that $(U_i f)(x_i) = \rho_i(x_i) f(x_i)$, we have $U_i \bar{L}_i U_i^{-1} = H_i$ where $\rho_i(x) = \frac{e^{-\frac{x_i^2}{4}}}{(2\pi)^{1/4}}$ and $-H_i = -\frac{d^2}{dx_i^2} + \frac{x_i^2}{4} - \frac{1}{2} + \lambda^2 \xi_i (x_i - h_i)^2$. Let S_h be the unitary transformation on $L^2(\mathbb{R}, dx)$ such that $S_h f(x) = f(x + h)$. Then for $\tilde{h}_i = \frac{4\lambda^2 \xi_i h_i}{1 + 4\lambda^2 \xi_i}$

(2.7)
$$S_{\tilde{h}_i} H_i S_{-\tilde{h}_i} = -\frac{d^2}{dx_i^2} + \frac{1}{4}(1 + 4\lambda^2 \xi_i) x_i^2 - \frac{1}{2} + \frac{h_i^2 \lambda^2 \xi_i}{1 + 4\lambda^2 \xi_i}.$$

Note that $H_\eta = -\frac{d^2}{dx^2} + \frac{1}{4}\eta^2 x^2$ has the pure point spectrum with multiplicity 1 and it is given by

$$(2.8) \qquad \left\{ |\eta| \left(\frac{1}{2} + k \right) \;\middle|\; k \in \mathbb{N} \cup \{0\} \right\}.$$

Normalized eigenfunction with eigenvalue $|\eta| \left(\frac{1}{2} + k \right)$ is

$$(2.9) \qquad e_{k,\eta}(x) = H_k(\sqrt{\eta}x) \frac{\exp\left(-\frac{|\eta|}{4}x^2 \right)}{(2\pi|\eta|^{-1})^{1/4}},$$

where $H_k(x) = \frac{(-1)^k}{\sqrt{k!}} e^{x^2/2} \frac{d^k}{dx^k} \left(e^{-x^2/2} \right)$ is the Hermite polynomial of degree k. Thus $-L^\lambda_{K,h,n}$ has the eigenvalues

$$(2.10) \qquad E_{k_1,\dots,k_n} = \sum_{i=1}^{n} \left\{ \sqrt{1 + 4\lambda^2\xi_i} \left(\frac{1}{2} + k_i \right) - \frac{1}{2} + \frac{h_i^2 \xi_i \lambda^2}{1 + 4\lambda^2\xi_i} \right\}$$

and the corresponding eigenfunction is

$$(2.11) \qquad \mathbf{e}_{k_1,\dots,k_n}(x_1,\dots,x_n) = \prod_{i=1}^{n} \rho_i(x_i)^{-1} e_{k_i, \sqrt{1+4\lambda^2\xi_i}}(x_i - \tilde{h}_i).$$

For $\phi \in X$,

$$(2.12) \qquad \mathbf{e}_{k_1,\dots,k_n,\mathbf{0}}(\phi) = \lim_{m \to \infty} \mathbf{e}_{k_1,\dots,k_n,\mathbf{0}_m} \left((\phi, e_1), \dots, (\phi, e_{m+n}) \right),$$

where $\mathbf{0}_m$ is the m-dimensional zero vector. This limit exists for almost all ϕ and explicitly we have

$$(2.13) \qquad \mathbf{e}_{k_1,\dots,k_n,\mathbf{0}}(\phi) = \Omega_{K,h}(\lambda, \phi) \prod_{i=1}^{n} H_{k_i} \left((1 + 4\lambda^2\xi_i)^{1/4} \left((\phi, e_i) - \tilde{h}_i \right) \right).$$

It is easy to check that $\mathbf{e}_{k_1,\dots,k_n,\mathbf{0}}$ is an eigenfunction of $L^\lambda_{K,h}$ with the eigenvalue $E_{k_1,\dots,k_n,\mathbf{0}}$, where

$$E_{k_1,\dots,k_n,\mathbf{0}} = \sum_{i=1}^{\infty} \left\{ \sqrt{1 + 4\lambda^2\xi_i} \left(\frac{1}{2} + k_i \right) - \frac{1}{2} + \frac{h_i^2 \xi_i \lambda^2}{1 + 4\lambda^2\xi_i} \right\}$$

and $k_j = 0$ for $j \geq n+1$. In order to complete the proof of (2.3) and (2.4), it is sufficient to show that the set of eigenfunctions $\mathbf{e}_{k_1,\dots,k_n,\mathbf{0}}$ $(k_i \in \{0\} \cup \mathbb{N}, n \in \mathbb{N})$ is a complete orthonormal system of $L^2(X, \mu)$. To this end, let

$$(2.14) \qquad U_h f(\phi) = \Omega_{K,h}(\lambda, \phi) \cdot f \left((I_H + 4\lambda^2 K)^{1/4} (\phi - \tilde{h}) \right),$$

where $\tilde{h} = 4\lambda^2 K(I_H + 4\lambda^2 K)^{-1}h$. Let

$$(2.15) \qquad u(\phi) = \left\{(I_H + 4\lambda^2 K)^{1/4} - I_H\right\}\phi - (I_H + 4\lambda^2 K)^{1/4}\tilde{h}.$$

Then $u(\cdot)$ is an H-valued function and it holds that

$$(2.16) \qquad \Omega_{K,h}(\lambda,\phi)^2 = \det{}_2(I_H + Du(\phi))\exp\left(-D^*u(\phi) - \frac{1}{2}\|u(\phi)\|_H^2\right),$$

where $\det_2$ stands for the Carleman-Fredholm determinant. Consequently U_h is an unitary transformation on $L^2(X,\mu)$ by Theorem 6.4 in [25]. Since the set

$$\left\{\prod_{i=1}^n H_{k_i}((\phi,e_i)) \ \Big|\ k_i \in \mathbb{N}\cup\{0\}, n \in \mathbb{N}\right\}$$

constitutes an complete orthonormal system in $L^2(X,\mu)$, these prove (2.3), (2.4) and (2.6). Now we prove the multiplicity of the eigenvalue is finite and (2.5). To this end, it suffices to prove the following Lemma 2.3. $\qquad\square$

Lemma 2.3. *Let $\{a_i\}_{i=1}^\infty$ be a positive sequence such that $a_i > 1$ and $\lim_{i\to\infty} a_i = 1$. Let S be the set of all real numbers which are written as $\sum_{i=1}^\infty k_i a_i$ where $k_i \in \mathbb{N}\cup\{0\}$. For $s \in S$, we denote by $m(s)$ the total number of such representations of s. Then the following hold.*

(1) *For all s, $m(s) < \infty$.*

(2) *The set of accumulation points of S is $\{x + n \mid x \in S, n \in \mathbb{N}\}$.*

Proof. (1) Assume $m(s) = \infty$. Then there exists at least one natural number n and there exist countable family of finite sequence such that

- $p_1^i < \cdots < p_n^i$, $p_j^i \in \mathbb{N}$, $\{q_1^i,\ldots,q_n^i\} \subset \mathbb{N}\cup\{0\}$ and $s = \sum_{j=1}^n q_j^i a_{p_j^i}$ for all $i \in \mathbb{N}$.

- $(p_1^i,\ldots,p_n^i,q_1^i,\ldots,q_n^i) \neq (p_1^j,\ldots,p_n^j,q_1^j,\ldots,q_n^j)$ if $i \neq j$.

Since $a_i > 1$ for all i, there exists l $(1 \le l \le n)$, such that $\lim_{i\to\infty} p_j^i = \infty$ for $l \le j \le n$ and $\limsup_{i\to\infty} p_j^i < \infty$ for $1 \le j \le l-1$. Then there exists M such that $p_j^i \le M$ and $q_t^i \le M$ for all $1 \le j \le l-1$ and $t,i \in \mathbb{N}$. Therefore by choosing subsequence of i, we may assume $p_j^i = p_j$, $q_t^i = q_t$ for all $1 \le j \le l-1, 1 \le t \le n$. This implies for all i

$$s = \sum_{1\le j\le l-1} q_j a_{p_j} + \sum_{l\le j\le n} q_j a_{p_j^i} = \sum_{1\le j\le l-1} q_j a_{p_j} + \sum_{l\le j\le n} q_j.$$

But this is a contradiction since $a_{p_j^i} > 1$.

(2) Clearly $x + n$ where $x \in S, n \in \mathbb{N}$ is an accumulation point. Let c be an accumulation point of S. Then $c = \lim_{i \to \infty} \sum_{j=1}^n q_j^i a_{p_j^i}$, where q_j^i, p_j^i are numbers satisfying the same property as in the proof of (1). Then by the same argument as in (1), we have $c = \sum_{1 \leq j \leq l-1} q_j a_{p_j} + \sum_{l \leq j \leq n} q_j$ which completes the proof. $\square$

Remark 2.4.   A similar fact of the essential spectrum of the Dirichlet operator for a weighted Wiener measure was obtained by Hino [22].

Now let us consider the divergence order of the lowest eigenvalue. For $L_{K,h,n}^\lambda$, by (2.10)

$$(2.17) \qquad \lim_{\lambda \to \infty} \frac{\inf \sigma(-L_{K,h,n}^\lambda)}{\lambda} = \sum_{i=1}^n \sqrt{\xi_i}.$$

This kind of asymptotics is well-known in semiclassical analysis in finite dimensional spaces. Also it is known that the growth order of the lowest eigenvalue of the Schrödinger operator $-\Delta + \lambda^2 |x|^\alpha$ is not linear if $\alpha \neq 2$. In infinite dimensional cases, the effect of the dimension appears. That is, differently from finite dimensional cases, $E_0(\lambda, K, h)$ is an infinite sum and the divergence order changes according to the decreasing speed of $\{\xi_i\}$. We will prove it in the lemma below.

Lemma 2.5.   (1) *Let $1 < \alpha \leq 2$. If $\sum_{i=1}^\infty \xi_i^{\alpha/2} < \infty$, then*

$$(2.18) \qquad \lim_{\lambda \to \infty} \frac{E_0(\lambda, K, h)}{\lambda^\alpha} = 0.$$

(2) *Assume that $\{\xi_n\}$ is a strictly decreasing sequence and $\limsup_{\lambda \to \infty} \lambda^{-\alpha} E_0(\lambda, K, h) < \infty$. Here $1 \leq \alpha \leq 2$. Then it holds that $\sup_n n^{2/\alpha} \xi_n < \infty$.*
(3) *We have*

$$(2.19) \qquad \lim_{\lambda \to \infty} \frac{E_0(\lambda, K, h)}{\lambda} = \sum_{i=1}^\infty \sqrt{\xi_i}.$$

In particular, if $\sum_{i=1}^\infty \sqrt{\xi_i} = \infty$, then the left-hand side diverges.
(4) *Let $1 < \alpha < 2$. Set $\xi_n = \frac{1}{n^\beta}$, where $\beta = 2/\alpha$. Then*

$$(2.20) \qquad \lim_{\lambda \to \infty} \frac{E_0(\lambda, K, h)}{\lambda^\alpha} = \int_0^\infty \frac{2}{t^\beta + \sqrt{t^{2\beta} + 4t^\beta}} dt < \infty.$$

(5) *Let $\xi_n = 1/n^2$. Then*

$$(2.21) \qquad \lim_{\lambda \to \infty} \frac{E_0(\lambda, K, h)}{2\lambda \log \lambda} = 1.$$

Proof. First note that

$$(2.22) \qquad \sum_{i=1}^{\infty} \frac{h_i^2 \xi_i \lambda^2}{1 + 4\lambda^2 \xi_i} \le \frac{1}{4} \|h\|_H^2.$$

So we may omit this term in the calculation below.

(1) Let

$$(2.23) \qquad g_i(\lambda) = \frac{1}{2}\left(\sqrt{1 + 4\lambda^2 \xi_i} - 1\right) = \frac{2\xi_i \lambda^2}{1 + \sqrt{1 + 4\lambda^2 \xi_i}}$$

and set $h_i(\lambda) = \frac{g_i(\lambda)}{\lambda^\alpha}$. Then by elementary calculations, we see that $h_i(\lambda)$ has the maximum value at $\lambda = \frac{\sqrt{\alpha(2-\alpha)\xi_i^{-1}}}{2(\alpha-1)}$. So for any $\lambda \ge 0$,

$$(2.24) \qquad h_i(\lambda) \le \frac{2(\alpha-1)}{\alpha}\left(\frac{\sqrt{\alpha(2-\alpha)}}{2(\alpha-1)}\right)^{2-\alpha} \xi_i^{\alpha/2}.$$

Therefore if $\sum_{i=1}^{\infty} \xi_i^{\alpha/2} < \infty$, then by the Lebesgue dominated convergence theorem, we are done.

(2) Let $f(x)$ be a smooth strictly decreasing function on $[0, \infty)$ such that $f(i) = \xi_{i+1}$ for $i \in \mathbb{N} \cup \{0\}$. For $0 < t \le \xi_1$, set $\varphi(t) = f^{-1}(t)$. For $R > 0$, let

$$I_R(\lambda) = \int_0^R \frac{2\lambda^2 f(x)}{1 + \sqrt{1 + 4\lambda^2 f(x)}} dx.$$

Then for any R,

$$(2.25) \qquad I_R(\lambda) \le \sum_{i=1}^{\infty} g_i(\lambda).$$

Let ε be a continuous point of φ'. Then

$$\begin{aligned} I_{\varphi(\varepsilon)}(\lambda) &= -\int_\varepsilon^{\xi_1} \frac{2t\lambda^2}{1 + \sqrt{1 + 4\lambda^2 t}} \varphi'(t) dt \\ &\ge \frac{2\varepsilon\lambda^2 \varphi(\varepsilon)}{1 + \sqrt{1 + 4\lambda^2 \varepsilon}} \end{aligned}$$

Then by the assumption, we have

$$\limsup_{\lambda\to\infty} \lambda^{-\alpha} I_{\varphi(\lambda^{-2})}(\lambda) < \infty.$$

This implies that there exists $C > 0$ such that $\varphi(t) \le C \cdot t^{-\alpha/2}$ for sufficiently small t. By the definition of $\varphi(t)$, for sufficiently large x, $f(x) \le \left(\frac{C}{x}\right)^{2/\alpha}$. This completes the proof.

(3) We have

$$(2.26) \qquad \frac{g_i(\lambda)}{\lambda} = \frac{2\xi_i}{\sqrt{\lambda^{-2} + 4\xi_i} + \lambda^{-1}} \le \sqrt{\xi_i}.$$

Hence by the Lebesgue dominated convergence theorem, we have

$$\lim_{\lambda\to\infty} \frac{E_0(\lambda, K, h)}{\lambda} = \sum_{i=1}^{\infty} \sqrt{\xi_i}.$$

(4) By the monotone decreasing property of $f(x) = \frac{4x^{-\beta}\lambda^{2-\alpha}}{1+\sqrt{1+4\lambda^2 x^{-\beta}}}$ $(x > 0)$, we have

$$(2.27) \int_1^\infty \frac{4x^{-\beta}\lambda^{2-\alpha}}{1+\sqrt{1+4\lambda^2 x^{-\beta}}}dx \le \sum_{k=1}^{\infty} \frac{4k^{-\beta}\lambda^{2-\alpha}}{1+\sqrt{1+4\lambda^2 k^{-\beta}}}$$

$$\le \frac{4\lambda^{2-\alpha}}{1+\sqrt{1+4\lambda^2}} + \int_1^\infty \frac{4x^{-\beta}\lambda^{2-\alpha}}{1+\sqrt{1+4\lambda^2 x^{-\beta}}}dx.$$

and

$$I(\lambda) = \int_1^\infty \frac{4x^{-\beta}\lambda^{2-\alpha}}{1+\sqrt{1+4\lambda^2 x^{-\beta}}}dx = \int_{\lambda^{-2/\beta}}^\infty \frac{4t^{-\beta}}{1+\sqrt{1+4t^{-\beta}}}dt.$$

Since the improper integral $\lim_{\lambda\to\infty} I(\lambda)$ converges and $2 - \alpha < 1$, the proof is completed.

(5) In this case, $I'(\lambda) = \dfrac{4}{1+\sqrt{1+4\lambda^2}}$. So by L'hospital's theorem, $\lim_{\lambda\to\infty} \frac{I(\lambda)}{2\log\lambda} = 1$. $\qquad\qquad\square$

§3. Estimates on Eigenfunctions

In this section, we consider a potential function V which satisfies the following assumptions **(A1)**–**(A4)**.
(A1) V is a nonnegative C^3 function on X in the sense of Fréchet and it holds that for some $p, L > 0$,

$$(3.1) \qquad |V(\phi)| + \sum_{i=1}^{3} |D^i V(\phi)|_{\otimes^i X} \leq L \cdot (1 + \|\phi\|_X)^p \qquad \text{for all} \quad \phi \in X.$$

Note that the notation D above denotes the usual Fréchet derivative. We will use the positive number p and L for an estimate on an Agmon distance which we will introduce later.

(A2) $\{\phi \in X \mid V(\phi) = 0\} = \{h_1, \dots, h_n\}$, where $h_j \in H$. We denote $N := \{h_1, \dots, h_n\}$.

The second derivative $(D^2 V)(h_j)$ defines a continuous symmetric form on X. We assume the following.

(A3) There exists $\varepsilon_j > 0$ such that for all $\phi \in X$,

$$(3.2) \qquad\qquad (D^2 V)(h_j)(\phi, \phi) \geq \varepsilon_j \|\phi\|_X^2.$$

The bounded symmetric operator on H corresponding to the continuous symmetric form $\frac{1}{2}(D^2 V)(h_j)|_{H \times H}$ is a trace class operator. We denote it by K_j. Note that $K_j \in L_T(H, H)$ and $\|\sqrt{K_j}\phi\|_H^2 = \frac{1}{2}(D^2 V)(h_j)(\phi, \phi)$ for all $\phi \in X$. $L_T(H, H)$ was introduced in Section 2.

(A4) There exists a positive constant $C(V) > 0$ such that

$$(3.3) \qquad \inf \{V(\phi) \mid \|\phi - h_j\|_X > \delta \ (1 \leq j \leq n)\} \geq C(V) \min(\delta^2, 1).$$

All assumptions above hold for

$$(3.4) \qquad\qquad V(\phi) = \prod_{i=1}^{n} \|\phi - h_i\|_X^2$$

where $h_i \neq h_j$ if $i \neq j$. When $n = 1$, it gives a quadratic Wiener functional studied in Section 2. When $n = 2$, the corresponding potential function is an infinite dimensional version of so called double well potential. Let us consider $-L_V^\lambda = -L + \lambda^2 V$ on $\mathcal{F}C_b^\infty$. Then again by Theorem X.58 in [28], this is essentially self-adjoint. Approximate quadratic potential function V_{K_j, h_j} also satisfies all assumptions above. In the next section, we will use the Schrödinger operator $-L + \lambda^2 V_{K_j, h_j}$ and the lowest eigenvalue $E_0(\lambda, K_j, h_j)$ to describe the asymptotic property of $-L_V^\lambda$.

Now we introduce the Agmon distance which is related with the Schrödinger operator $-L_V^\lambda$.

Definition 3.1.   (1) Let $\phi_1, \phi_2 \in X$. Define

(3.5)
$$
\rho(\phi_1, \phi_2) = \inf \left\{ \int_0^1 \sqrt{V(\phi_1 + h(t))} \|\dot{h}(t)\|_H dt \,\Big|\, h \in \bar{H} \text{ and } \phi_1 + h(1) = \phi_2 \right\},
$$

where

(3.6) $\bar{H} = \left\{ h \;:\; [0,1] \to H \,\Big|\, h(0) = 0 \text{ and } h(\cdot) \text{ is an absolutely} \right.$

continuous path on H and satisfies that $\left. \int_0^1 \|\dot{h}(t)\|_H^2 dt < \infty \right\}.$

In (3.5), if the set on the right-hand side is empty set, we define $\rho(\phi_1, \phi_2) = \infty$.
(2) For an open subset G of X, we define

(3.7) $\rho(x, G) = \inf \left\{ \rho(x, y) \,\Big|\, y \in G \right\}.$

We can prove the measurability of $\rho(\cdot, G)$. To this end, we introduce some notations. Let S be a countable dense subset of H and for $v \in S$, let C_v be a countable dense subset of $H^1([0,1] \to H \mid c(0) = 0, \; c(1) = v)$ consisting of smooth curves. Also let $C_S = \cup_{v \in S} C_v = \{c_n(\cdot) \mid n = 1, 2, \dots\}$.

Lemma 3.2.   *For any non empty open set G, $\rho(\cdot, G)$ is a measurable function on X and for any $\phi \in G$, $\rho(\phi, G) < \infty$.*

Proof.   Let $\psi_G(\phi)$ be a measurable function on X such that $\psi_G(\phi) = 0$ for $\phi \in G$ and $\psi_G(\phi) = \infty$ for $\phi \notin G$. For each $c_n \in C_S$, define

(3.8) $\rho_n(\phi, G) = \int_0^1 \sqrt{V(\phi + c_n(t))} \|\dot{c}_n(t)\|_H dt + \psi_G(\phi + c_n(1)).$

By the continuity of V and the open property of G, it holds that for each $\phi \in X$,
$$
\lim_{n \to \infty} \min_{1 \le k \le n} \rho_k(\phi, G) = \rho(\phi, G).
$$

Noting $\rho_k(\phi, G)$ is a measurable function, we see $\rho(\phi, G)$ is a measurable function. The latter part is obvious since for any $\phi \in X$, there exists $h \in H$ such that $\phi + h \in G$. □

Let us denote
$$
U_\delta(N) = \cup_{j=1}^n U_\delta(h_j),
$$
$$
U_\delta(h_j) = \left\{ \phi \in X \mid \|\phi - h_j\|_X < \delta \right\}.
$$

Since $U_\delta(N)$ is an open set, $\rho\left(\cdot, U_\delta(N)\right)$ is a measurable function.

We prove some elementary properties of the Agmon distance below. Recall that a measurable function F on X is called an H-continuous function if $F(\phi + \cdot) : H \to \mathbb{R}$ is continuous for all $\phi \in X$.

Lemma 3.3. *For each $\delta > 0$, the following hold.*

(1) $\rho\left(\cdot, U_\delta(N)\right)$ *is an H-continuous function and $\rho\left(\cdot, U_\delta(N)\right) \in D^{1,2}$ and satisfies that*

$$(3.9) \qquad |D\rho(\phi, U_\delta(N))| \leq \sqrt{V(\phi)}, \qquad \mu\text{-a.e. } \phi.$$

(2) *There exists a polynomial $P(x, y, z)$ whose degree is at most $2p$ such that for all $\phi \in X$,*

$$(3.10) \qquad |V(\phi)| \leq P\left(\delta, \delta^{-1}, \rho\left(\phi, U_\delta(N)\right)\right).$$

The coefficients of P are positive number and determined by K, h_j, L, $C(V)$ and ε_i.

(3) *Let us consider the case where $V(\phi) = V_{K,h}(\phi)$. Then, for all $\phi \in X$, we have*

$$(3.11) \qquad \left(V_{K,h}(\phi) - \delta^2\right)^+ \leq 2\|\sqrt{K}\|_{L(H,H)}\rho(\phi, U_\delta(h)).$$

where $a^+ := \max(a, 0)$.

Proof. (1) First, we prove the H-continuity of $\rho\left(\cdot, U_\delta(N)\right)$. Let $h \in H$. Then by the polynomial growth condition on V,

$$(3.12) \quad \rho(\phi + h, U_\delta(N)) \leq \rho\left(\phi, U_\delta(N)\right) + \int_0^1 \sqrt{V\left(\phi + h - th\right)}\|h\|_H dt$$
$$\leq \rho\left(\phi, U_\delta(N)\right) + L^{1/2}(1 + \|\phi\|_X + \|h\|_X)^{p/2}\|h\|_H.$$

This proves the H-continuity of $\rho\left(\cdot, U_\delta(N)\right)$. If V is a bounded function, then by the same argument above, we see that $\rho\left(\cdot, U_\delta(N)\right)$ is an H-Lipschitz continuous function. That is, there exists $C > 0$ such that for all $\phi \in X$ and $h \in H$, $|\rho(\phi + h, U_\delta(N)) - \rho\left(\phi, U_\delta(N)\right)| \leq C\|h\|_H$. By 5.4.10. Example in [10], we obtain $\rho\left(\cdot, U_\delta(N)\right) \in D^{1,2}$. Also by the same calculation as in (3.12), we see that

$$(3.13) \qquad \limsup_{\varepsilon \to 0} \left|\frac{\rho(\phi + \varepsilon h, U_\delta(N)) - \rho(\phi, U_\delta(N))}{\varepsilon}\right| \leq \sqrt{V(\phi)}\|h\|_H.$$

By 5.7.2. Theorem in [10], we obtain (3.9).

Now let us consider the unbounded case. Let $V_m(\phi) = \min(V(\phi), m)$ and denote the Agmon distance by $\rho_m(\phi, U_\delta(N))$ which is defined by the potential function $V_m(\phi)$. Then we have $|D\rho_m(\phi, U_\delta(N))| \leq \sqrt{V_m(\phi)}$ a.e. ϕ and for any $\phi \in X$,

$$(3.14) \qquad \lim_{m \to \infty} \rho_m(\phi, U_\delta(N)) = \rho(\phi, U_\delta(N)).$$

These imply (3.9).

(2) For simplicity, we write $r = \rho(\phi, U_\delta(N))$. Note that

$$(3.15) \qquad \rho(\phi, U_\delta(N)) = \min\left\{ \rho(\phi, U_\delta(h_j)) \mid 1 \leq j \leq n \right\}.$$

First we consider the case where $\phi \notin U_\delta(N)$. Then there exists $h \in \bar{H}$, (by reparametrization if necessary) $\phi + h(t) \in U_{\delta/2}(N)^c$ for all $0 \leq t \leq 1$, $\phi + h(1) \in U_\delta(h_j)$ for some $1 \leq j \leq n$ and

$$(3.16) \qquad \int_0^1 \sqrt{V(\phi + h(t))} \|\dot{h}(t)\|_H dt \leq r + 1.$$

Since $\sqrt{V(\phi + h(t))} \geq \sqrt{C(V)} \min(\delta/2, 1)$ for $0 \leq t \leq 1$, this proves

$$\|h(1)\|_H \leq \left(\frac{2}{\delta} + 1\right) \frac{r+1}{\sqrt{C(V)}}.$$

$\phi + h(1) \in U_\delta(h_j)$ implies $\|\phi + h(1) - h_j\|_X < \delta$. Hence we obtain

$$
\begin{aligned}
(3.17) \quad |V(\phi)| &\leq L \left(1 + \|\phi\|_X\right)^p \\
&\leq L \left(1 + \|\phi + h(1) - h_j\|_X + \|h(1)\|_X + \|h_j\|_X\right)^p \\
&\leq L \left(1 + \delta + \frac{\|\sqrt{K}\|_{L(H,H)}}{\sqrt{C(V)}} (1 + r)\left(\frac{2}{\delta} + 1\right) + \|h_j\|_X\right)^p.
\end{aligned}
$$

Now we consider the case $\phi \in U_\delta(N)$. Then there exists h_j such that $\|\phi - h_j\|_X \leq \delta$. Therefore

$$|V(\phi)| \leq L \left(1 + \|\phi\|_X\right)^p \leq L \left(1 + \delta + \|h_j\|_X\right)^p.$$

(3) First let V be a general potential function satisfying **(A1)**–**(A4)**. Take u such that for any $\phi \in X$, $u(\phi + \cdot) : H \to \mathbb{R}$ is C^1 function and $\|Du(\phi)\|_H \leq \sqrt{V(\phi)}$ and $u(\phi) = 0$ for all $\phi \in U_\delta(h)$. Then we prove that

$$(3.18) \qquad u(\phi) \leq \rho(\phi, U_\delta(h)) \qquad \text{for all } \phi.$$

For arbitrary $\varepsilon > 0$, there exists $\phi + h(\cdot)$ with $h(\cdot) \in \bar{H}$ and $\phi + h(1) \in U_\delta(h)$ such that

$$(3.19) \qquad \int_0^1 \sqrt{V(\phi + h(t))} \|\dot{h}(t)\|_H dt \leq \rho(\phi, U_\delta(h)) + \varepsilon.$$

Then

$$u(\phi) = u(\phi + h(1)) - \int_0^1 \left(Du(\phi + h(t)), \dot{h}(t) \right) dt$$

$$\leq \int_0^1 \sqrt{V(\phi + h(t))} \|\dot{h}(t)\|_H dt$$

$$\leq \rho(\phi, U_\delta(h)) + \varepsilon$$

which is a desired result. Now we prove (3.11). Note that $(V_{K,h}(\phi) - \delta^2)^+ = 0$ on $U_\delta(h)$ and

$$|DV_{K,h}(\phi)| \leq 2\|\sqrt{K}\|_{L(H,H)} \sqrt{V_{K,h}(\phi)}.$$

So by approximating $(x - \delta^2)^+$ by a smooth function and applying (3.18), we are done. $\qquad \square$

We prove the decay estimate on eigenfunctions of L_V^λ in the rest of this section. This kind of estimate is common in semiclassical analysis and originally is due to Agmon [1], [2]. See also Theorem 3.1.1 in [19], p. 105–107 in [31].

To state the results, we introduce some functions. For $R > 0$, let $\Theta_R(t)$ be the piecewise linear function such that $\Theta_R(t) = t$ for $t \leq R$ and $\Theta_R(t) = R$ for $t > R$. Let us define for $0 < q < 1$,

$$(3.20) \qquad \varphi_{q,R,\delta}(\phi) = q \cdot \Theta_R \left(\rho \left(\phi, U_\delta(N) \right) \right).$$

Lemma 3.4. *Take $\eta \in D^{1,2} \cap L^\infty(X, \mu)$ such that $D\eta \in L^{\infty-}(X, \mu)$. Let $\Psi(\lambda, \phi)$ be an eigenfunction such that*

$$(3.21) \qquad -L_V^\lambda \Psi(\lambda, \phi) = E(\lambda) \Psi(\lambda, \phi).$$

Then

$$(3.22) \qquad \int_X \left\{ \lambda^2 (1 - q^2) V(\phi) - E(\lambda) \right\} e^{2\lambda \varphi_{q,R,\delta}(\phi)} \eta(\phi)^2 \Psi(\lambda, \phi)^2 d\mu(\phi)$$

$$\leq \int_X e^{2\lambda \varphi_{q,R,\delta}(\phi)} \Psi(\lambda, \phi)^2 |D\eta|^2 d\mu(\phi)$$

$$+ 2\lambda \int_X e^{2\lambda \varphi_{q,R,\delta}} \eta(D\eta, D\varphi_{q,R,\delta}) \Psi(\lambda, \phi)^2 d\mu(\phi).$$

Proof.   The proof is similar to the argument in p. 106 and 107 in [31]. For the completeness of the paper, we include it. Let $F, G \in D^\infty \cap L^\infty(X, \mu)$. By using the integration by parts, the nonnegativity of $-L$ and (3.9), we have

$$
(3.23) \qquad \left(e^{\lambda F}G, (-L_V^\lambda - E(\lambda))\left(e^{-\lambda F}G\right)\right)
$$
$$
= (G, -LG) + (G, 2\lambda(DF, DG)) + \left((\lambda^2 V - E(\lambda))G, G\right)
$$
$$
+ (G, \lambda LF \cdot G) - (G, \lambda^2 |DF|^2 G)
$$
$$
\geq \left((\lambda^2(V - |DF|^2) - E(\lambda))G, G\right).
$$

Let $\eta \in D^\infty \cap L^\infty(X, \mu)$. Since $\Psi(\lambda, \phi) \in L^{\infty-}(X, \mu)$ (this follows from the hyperbounded property of $e^{tL_V^\lambda}$), (3.23) holds for $G = \Psi(\lambda, \phi) \cdot e^{\lambda F}\eta$. Therefore we have

$$
(3.24) \qquad \int_X \left\{\lambda^2(V(\phi) - |DF(\phi)|^2 - E(\lambda)\right\} e^{2\lambda F(\phi)}\eta(\phi)^2 \Psi(\lambda, \phi)^2 d\mu(\phi)
$$
$$
\leq \int_X e^{2\lambda F(\phi)}\Psi(\lambda, \phi)^2 |D\eta(\phi)|^2 d\mu(\phi)
$$
$$
+ 2\lambda \int_X e^{2\lambda F(\phi)}\eta(\phi)(D\eta(\phi), DF(\phi))\Psi(\lambda, \phi)^2 d\mu(\phi).
$$

Let $F = \varphi_{q,R,\delta}$ and by approximating η, the proof is completed. $\qquad \square$

Theorem 3.5.   *Assume that $\|\Psi(\lambda, \cdot)\|_{L^2(X, \mu)} = 1$. Let $g(\lambda)$ be a positive number such that $g(\lambda) > 1$ and $\lambda^{-2}E(\lambda)g(\lambda) < C(V)$.*

(1) Let $0 < q < 1$ be a real number such that $\lambda^{-2}E(\lambda)g(\lambda) < C(V)(1-q^2)$. Then it holds that

$$
(3.25) \qquad \int_{\rho(\phi, U_\delta(N)) \geq d(\lambda)} \Psi(\lambda, \phi)^2 d\mu(\phi)
$$
$$
\leq q^2 P\left(\delta, \delta^{-1}, 3q^{-1}g(\lambda)^{-1}\right)(g(\lambda) + 2\lambda)
$$
$$
\times \frac{g(\lambda)^2}{\lambda^2}\left\{C(V)(1-q^2) - \frac{E(\lambda)g(\lambda)}{\lambda^2}\right\}^{-1} e^{6\lambda/g(\lambda) - 2q\lambda d(\lambda)},
$$

where $d(\lambda) \geq 3q^{-1}g(\lambda)^{-1}$ and δ is a positive number such that $\delta \geq g(\lambda)^{-1/2}$. Also P is the polynomial in (3.10).

(2) Let us consider the case where $V = V_{K,h}$ and $\Psi(\lambda, \phi) = \Omega_{K,h}(\lambda, \phi)$. In this case, $C(V) = 1$ holds. Also for $\lambda > 0$ and $0 < q < 1$ satisfying

$\lambda^{-2}E_0(\lambda, K, h)g(\lambda) < 1 - q^2$ *and* $g(\lambda) > 1$, *it holds that*

$$(3.26) \qquad \int_{\left\|\sqrt{K}(\phi-h)\right\|_H \geq d(\lambda)^{1/2}} \Omega_{K,h}(\lambda, \phi)^2 d\mu(\phi)$$

$$\leq \frac{q^2 g(\lambda)}{\lambda} \left(\frac{g(\lambda)}{\lambda} + 2 \right) \left\{ 1 - q^2 - \frac{E_0(\lambda, K, h)g(\lambda)}{\lambda^2} \right\}^{-1}$$

$$\times (6q^{-1}\|\sqrt{K}\|_{L(H,H)} + 1)e^{6q\lambda/g(\lambda) - 2q\lambda d(\lambda)},$$

where $d(\lambda) \geq (6q^{-1}\|\sqrt{K}\|_{L(H,H)} + 1)g(\lambda)^{-1}$.

Proof. (1) Let $\chi_\lambda(t)$ be the function on $\mathbb{R}$ such that $\chi_\lambda(t) = 0$ for $t \leq 2g(\lambda)^{-1}$, $\chi_\lambda(t) = \left(t - 2g(\lambda)^{-1}\right)g(\lambda)$ for $2g(\lambda)^{-1} \leq t \leq 3g(\lambda)^{-1}$ and $\chi_\lambda(t) = 1$ for $t \geq 3g(\lambda)^{-1}$. Define $\eta(\phi) = \chi_\lambda\left(\varphi_{q,R,\delta}(\phi)\right)$, where we assume $R > 3q^{-1}$. Noting

$$\inf \{V(\phi) \mid \eta(\phi) > 0\} \geq C(V)g(\lambda)^{-1},$$

we have

$$(3.27) \quad \lambda^2(1 - q^2)V(\phi) - E(\lambda)$$
$$\geq C(V)(1 - q^2)\lambda^2 g(\lambda)^{-1} - E(\lambda) \qquad \text{for } \phi \text{ on } \{\eta(\phi) \neq 0\}.$$

Thus by (3.22), we botain

$$(3.28)$$
$$\int_{\varphi_{q,R,\delta} \geq 3g(\lambda)^{-1}} e^{2\lambda\varphi_{q,R,\delta}(\phi)} \Psi(\lambda, \phi)^2 d\mu(\phi) \leq h(\lambda) \times \text{right-hand side of (3.22)},$$

where

$$(3.29) \qquad h(\lambda) = \frac{g(\lambda)}{\lambda^2} \left\{ C(V)(1 - q^2) - \frac{E(\lambda)g(\lambda)}{\lambda^2} \right\}^{-1}.$$

Now let us consider the right-hand side of (3.22). By (3.9), we have for almost all $\phi \in X$,

$$|D\varphi_{q,R,\delta}(\phi)| \leq q\sqrt{V(\phi)},$$
$$|D\eta(\phi)| \leq qg(\lambda)\sqrt{V(\phi)}.$$

Therefore

(3.30)
$$\int_{\varphi_{q,R,\delta} \geq 3g(\lambda)^{-1}} e^{2\lambda\varphi_{q,R,\delta}} \Psi(\lambda,\phi)^2 d\mu$$
$$\leq h(\lambda) \int_{2g(\lambda)^{-1} \leq \varphi_{q,R,\delta} \leq 3g(\lambda)^{-1}} \left\{ (g(\lambda) + 2\lambda)\, g(\lambda) q^2 V(\phi) \right\} \Psi(\lambda,\phi)^2 e^{2\lambda\varphi_{q,R,\delta}} d\mu$$

(3.31)
$$\leq q^2 P\left(\delta, \delta^{-1}, 3q^{-1}g(\lambda)^{-1}\right) (g(\lambda) + 2\lambda)g(\lambda)h(\lambda)e^{6q\lambda/g(\lambda)}.$$

In (3.31), we used that $qR > 3 \geq 3g(\lambda)^{-1}$ and (3.10). In (3.31), taking the limit $R \to \infty$,

(3.32)
$$\int_{q\rho(\phi,U_\delta(N)) \geq 3g(\lambda)^{-1}} e^{2\lambda q\rho(\phi,U_\delta(N))} \Psi(\lambda,\phi)^2 d\mu$$
$$\leq q^2 P\left(\delta, \delta^{-1}, 3q^{-1}g(\lambda)^{-1}\right) (g(\lambda) + 2\lambda)g(\lambda)h(\lambda)e^{6q\lambda/g(\lambda)}.$$

This implies (3.25).

(2) $C(V) = 1$ is obvious by the definition. Let $\delta = g(\lambda)^{-1/2}$. In (3.30), by using the estimate $V_{K,h}(\phi) \leq 2\|\sqrt{K}\|_{L(H,H)}\rho(\phi,U_\delta(h)) + g(\lambda)^{-1}$ which follows from (3.11), we have for λ satisfying $\frac{E_0(\lambda,K,h)g(\lambda)}{\lambda^2} < 1 - q^2$,

(3.33)
$$\int_{\varphi_{q,R,\delta} \geq 3g(\lambda)^{-1}} e^{2\lambda\varphi_{q,R,\delta}} \Omega_{K,h}(\lambda,\phi)^2 d\mu$$
$$\leq h(\lambda) \int_{2g(\lambda)^{-1} \leq \varphi_{q,R,\delta} \leq 3g(\lambda)^{-1}}$$
$$\times \left\{ q^2 \left(g(\lambda) + 2\lambda\right) g(\lambda) \left(6q^{-1}\|\sqrt{K}\|_{L(H,H)}g(\lambda)^{-1} + g(\lambda)^{-1}\right) \right\}$$
$$\times \Omega_{K,h}(\lambda,\phi)^2 e^{2\lambda\varphi_{q,R,\delta}} d\mu$$
$$\leq q^2 \left(g(\lambda) + 2\lambda\right) \left(6q^{-1}\|\sqrt{K}\|_{L(H,H)} + 1\right)h(\lambda)e^{6q\lambda/g(\lambda)}.$$

Here take the limit $R \to \infty$. Next noting

(3.34)
$$\{V_{K,h}(\phi) \geq d(\lambda)\} \subset \left\{q\rho(\phi,U_\delta(h)) \geq 3g(\lambda)^{-1}\right\},$$

we complete the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

The following corollary immediately follows from Theorem 3.5 (2).

Corollary 3.6.   *Let* $V = V_{K,h}$, $\Psi(\lambda,\phi) = \Omega_{K,h}(\lambda,\phi)$.

(1) *Assume that* $\limsup_{\lambda \to \infty} \frac{E_0(\lambda, K, h)}{\lambda^\alpha} < \infty$, *where* $1 \le \alpha \le 2$. *Then there exist positive constants* C_1, C_2 *such that for sufficiently large* R *and* λ, *it holds that*

$$(3.35) \qquad \int_{\|\sqrt{K}(\phi - h)\|_H \ge R\lambda^{(\alpha/2)-1}} \Omega_{K,h}(\lambda, \phi)^2 d\mu(\phi) \le C_1 e^{-C_2 R^2 \lambda^{\alpha-1}}.$$

(2) *Let* $0 < p \le 1$ *and assume that*

$$\lim_{\lambda \to \infty} \lambda^{1-p} \left(\frac{E_0(\lambda, K, h)}{\lambda} \right)^p = \infty.$$

Let $0 < C_1 < 3/4$ *and set* $R > C_1^{-1}(3\|\sqrt{K}\|_{L(H,H)} + 1)^{1/p}$. *Then there exists* $0 < C_2 < 1$ *such that for sufficiently large* λ,

$$(3.36) \qquad \int_{\|\sqrt{K}(\phi - h)\|_H \ge \left(\frac{E_0(\lambda, K, h)R}{\lambda^2} \right)^{p/2}} \Omega_{K,h}(\lambda, \phi)^2 d\mu(\phi)$$

$$\le \exp \left[-C_2 \left(R^p - C_1^{-p} \right) \lambda^{1-p} \left(\frac{E_0(\lambda, K, h)}{\lambda} \right)^p \right].$$

Proof. (1) For $\delta, C > 0$, set $g(\lambda) = \delta\lambda^{2-\alpha}$ and $d(\lambda) = \lambda^{(\alpha/2)-1}\delta^{-1}C$. Then for sufficiently small δ, q and large C, we have

$$(3.37) \qquad \int_{\|\sqrt{K}(\phi - h)\|_H \ge (C/\delta)^{1/2}\lambda^{(\alpha/2)-1}} \Omega_{K,h}(\lambda, \phi)^2 d\mu(\phi)$$

$$\le C' \exp \left[-2q\lambda^{\alpha-1} \frac{C}{\delta} \left((6q^{-1}\|\sqrt{K}\|_{L(H,H)} + 1) - \frac{3}{C} \right) \right].$$

By setting $R = C/\delta$, we complete the proof.

(2) Let $g(\lambda) = \left(\frac{C_1 \lambda^2}{E_0(\lambda, K, h)} \right)^p$, $d(\lambda) = \left(\frac{R E_0(\lambda, K, h)}{\lambda^2} \right)^p$ and $q = 1/2$. Then

$$6q \frac{\lambda}{g(\lambda)} - 2q\lambda d(\lambda) = \lambda^{1-p} \left(\frac{E_0(\lambda, K, h)}{\lambda} \right)^p (3R^p - C_1^p).$$

So (3.36) holds. $\qquad\qquad\square$

§4. Asymptotics of Lowest Eigenvalue of $-L_V^\lambda$

By [33] and [14], L_V^λ has the lowest eigenvalue and the ground state. Let us denote by $E_0(\lambda)$ and $\Omega(\lambda, \phi)$ the lowest eigenvalue and the positive ground state of $-L_V^\lambda$. In this section, we determine the divergence order of $E_0(\lambda)$ by using $E_0(\lambda, K_j, h_j)$. Let us denote

(4.1) $$E_0(\lambda, N) = \min \left\{ E_0(\lambda, K_j, h_j) \mid 1 \leq j \leq n \right\}.$$

Main theorem in this section is the following.

Theorem 4.1. *It holds that*

(4.2) $$\lim_{\lambda \to \infty} \frac{E_0(\lambda)}{E_0(\lambda, N)} = 1.$$

Consequently we have

(4.3) $$\lim_{\lambda \to \infty} \frac{E_0(\lambda)}{\lambda^2} = 0.$$

The proof is divided into two parts. The following proof is a modification of Simon [30].

Lemma 4.2 (Upper bound). *It holds that*

(4.4) $$\limsup_{\lambda \to \infty} \frac{E_0(\lambda)}{E_0(\lambda, N)} \leq 1.$$

Proof of Lemma 4.2. We construct an approximate ground state of L_V^λ near h_j by using the ground state $\Omega_{K_j, h_j}(\lambda, \phi)$ of L_{K_j, h_j}^λ. Let $\chi(t)$ be a smooth nonnegative function such that $\chi(t) = 1$ for $|t| \leq 2$, $\chi(t) = 1 - \exp\left(-\frac{1}{t^2 - 4}\right)$ for $2 \leq t \leq 3$ and $\chi(t) = 0$ for $|t| \geq 4$. Also we assume $\chi'(t) \leq 0$ for $t \geq 0$. Set

(4.5) $$\Phi_j(\lambda, \phi) = C_\lambda^{-1} \Omega_{K_j, h_j}(\lambda, \phi) \chi \left(\|\phi - h_j\|_X^2 l(\lambda) \right),$$

where C_λ is the normalized constant and

(4.6) $$l(\lambda) = \left(\frac{\lambda^2}{E_0(\lambda, K_j, h_j)} \right)^{5/6}.$$

By (3.36), $\lim_{\lambda \to \infty} C_\lambda = 1$. Let us calculate the energy of $\Phi_j(\lambda, \phi)$.

(4.7) $$\begin{aligned}
&\left(-L_V^\lambda \Phi_j(\lambda, \cdot), \Phi_j(\lambda, \cdot) \right) \\
&= \left(-L_{K_j, h_j}^\lambda \Phi_j(\lambda, \cdot), \Phi_j(\lambda, \cdot) \right) + \lambda^2 \left((V - V_{K_j, h_j}) \Phi_j(\lambda, \cdot), \Phi_j(\lambda, \cdot) \right) \\
&= E_0(\lambda, K_j, h_j) + 4l(\lambda)^2 C_\lambda^{-2} \\
&\quad \times \left(\|T(\phi - h_j)\|_H^2 \Omega_{K_j, h_j}(\lambda, \phi) \chi' \left(\|\phi - h_j\|_X^2 l(\lambda) \right)^2, \Omega_{K_j, h_j}(\lambda, \phi) \right) \\
&\quad + \lambda^2 \left((V - V_{K_j, h_j}) \Phi_j(\lambda, \phi), \Phi_j(\lambda, \phi) \right) \\
&= E_0(\lambda, K_j, h_j) + I_1(\lambda) + I_2(\lambda).
\end{aligned}$$

First let us consider I_1. We have

$$(4.8) \qquad |I_1(\lambda)| \leq C \cdot l(\lambda) \int_{\|\phi - h_j\|_X \geq l(\lambda)^{-1/2}} \Omega_{K_j, h_j}(\lambda, \phi)^2 d\mu.$$

Here we have used that $\|T(\phi - h_j)\|_H^2 \leq \|\sqrt{T}\|_{L(H,H)}^2 \cdot \|\phi - h_j\|_X^2$. By (3.36) and (3.2), $\lim_{\lambda \to \infty} I_1(\lambda) = 0$. Let us consider $I_2(\lambda)$. By applying Taylor's theorem for V around h_j, we get

$$(4.9) \qquad V(\phi) = V_{K_j, h_j}(\phi) + \frac{1}{3!} (D^3 V) (h_j + \theta(\phi - h_j)) \left(\otimes^3 (\phi - h_j) \right).$$

Thus on $\{\Phi_j(\lambda, \phi) \neq 0\}$, it holds that

$$(4.10) \qquad \lambda^2 \left| V(\phi) - V_{K_j, h_j}(\phi) \right| \leq C \cdot \lambda^2 l(\lambda)^{-3/2}$$
$$= C \cdot \lambda^2 \left(\frac{E_0(\lambda, K_j, h_j)}{\lambda^2} \right)^{5/4}$$
$$= C \cdot E_0(\lambda, K_j, h_j) \left(\frac{E_0(\lambda, K_j, h_j)}{\lambda^2} \right)^{1/4}.$$

Consequently, we have

$$(4.11) \qquad E_0(\lambda) \leq E_0(\lambda, K_j, h_j)(1 + o(1))$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Lemma 4.3 (Lower bound).

$$(4.12) \qquad \liminf_{\lambda \to \infty} \frac{E_0(\lambda)}{E_0(\lambda, N)} \geq 1.$$

To prove the lower bound estimate, we need IMS localization formula. This can be proved in the same way as in [30].

Lemma 4.4. *Let $J_1, \ldots, J_n \in D^\infty$ and assume that*

$$(4.13) \qquad \sum_{k=1}^{n} J_k^2 = 1.$$

Then

$$(4.14) \qquad -L_V^\lambda = -\sum_{k=1}^{n} J_k L_V^\lambda J_k - \sum_{k=1}^{n} |D J_k|^2.$$

Proof of Lemma 4.3.   Let

$$(4.15) \qquad J_k^\lambda(\phi) = \chi\left(\|\phi - h_k\|_X^2 \frac{\varepsilon\lambda^2}{E_0(\lambda, N)}\right) \qquad (1 \le k \le n)$$

$$(4.16) \qquad J_0^\lambda(\phi) = \left\{1 - \sum_{k=1}^n J_k^\lambda(\phi)^2\right\}^{1/2},$$

where ε is a small positive number and χ is the function in the proof of Lemma 4.2. Note that since $h_1, \ldots, h_n$ are distinct points in H, for sufficiently large λ,

$$(4.17) \qquad J_0^\lambda(\phi) = 0 \qquad \text{for } \phi \in U_{\left(\frac{E_0(\lambda,N)}{\varepsilon\lambda^2}\right)^{1/2}}(N).$$

By IMS localization formula,

$$(4.18) \qquad -L_V^\lambda = -J_0^\lambda L_V^\lambda J_0^\lambda - \sum_{k=1}^n J_k^\lambda L_{K_k,h_k}^\lambda J_k^\lambda$$

$$+ \lambda^2 \sum_{k=1}^n J_k^\lambda(V - V_{K_k,h_k})J_k^\lambda - \sum_{k=0}^n |DJ_k^\lambda|^2.$$

First we consider $|DJ_k^\lambda|^2$ $(k \ne 0)$. We have

$$(4.19) \quad |DJ_k^\lambda(\phi)|^2 = 4\chi'\left(\|\phi - h_k\|_X^2 \frac{\varepsilon\lambda^2}{E_0(\lambda, N)}\right)^2 \|T(\phi - h_k)\|_H^2 \left(\frac{\varepsilon\lambda^2}{E_0(\lambda, N)}\right)^2$$

$$(4.20) \qquad \le 4C\chi'\left(\|\phi - h_k\|_X^2 \frac{\varepsilon\lambda^2}{E_0(\lambda, N)}\right)^2 \frac{\varepsilon\lambda^2}{E_0(\lambda, N)}.$$

Here noting that $\limsup_{\lambda\to\infty} \frac{\lambda^2}{E_0(\lambda,N)^2} < \infty$, this term is negligible if ε is small. Noting $\chi'(t)^2 \le C_R(1 - \chi(t))$ for $|t| \le R$, by the calculation similar to the above, we see that $|DJ_0^\lambda(\phi)|^2$ is also negligible. Let us consider the third term. By Taylor's expansion in (4.9),

$$(4.21) \qquad |\lambda^2 \sum_{i=1}^n J_k^\lambda(V - V_{K_k,h_K})J_k^\lambda| \le C\lambda^2 \left(\frac{E_0(\lambda, N)^{1/2}}{\varepsilon^{1/2}\lambda}\right)^3$$

$$\le CE_0(\lambda, N)\varepsilon^{-3/2}\frac{E_0(\lambda, N)^{1/2}}{\lambda}.$$

Since $\lim_{\lambda\to\infty} \frac{E_0(\lambda,N)}{\lambda^2} = 0$, this term is negligible. Let us consider the first term in (4.18). By the nonnegativity of $-L$ and **(A4)**, in operator sense, it holds that

$$(4.22) \qquad -J_0^\lambda L_V^\lambda J_0^\lambda \ge \lambda^2 J_0^\lambda V J_0^\lambda \ge C(V)E_0(\lambda, N)\varepsilon^{-1}(J_0^\lambda)^2.$$

Consequently, combining the estimates above, the following estimate holds as operators.

$$
-L_V^\lambda \geq C(V) E_0(\lambda, N) \varepsilon^{-1} (J_0^\lambda)^2 + \sum_{k=1}^n E_0(\lambda, K_k, h_k)(J_k^\lambda)^2 - C\varepsilon E_0(\lambda, N).
\tag{4.23}
$$

Since $\sum_{k=0}^n (J_k^\lambda)^2 = 1$, this shows the lower bound estimate. $\square$

§5. Upper Bound on the Gap of Spectrum

Let

$$
E_1(\lambda) = \inf \left\{ \sigma \left(-L_V^\lambda \right) \setminus \{E_0(\lambda)\} \right\}.
\tag{5.1}
$$

In this section, we consider the situation where the gap of spectrum $E_1(\lambda) - E_0(\lambda)$ is exponentially small and we give an upper bound by using the Agmon distance. In this case, for sufficiently large λ, $E_1(\lambda) - E_0(\lambda) < 1$ and so by the result in [33], $E_1(\lambda)$ is an eigenvalue. This result may be useful to get lower bound estimate on $E_1(\lambda) - E_0(\lambda)$.

In this section, we need the following additional assumptions.

(A5) $N = \{h_1, h_2\}$. Moreover, for any $\delta > 0$ and $\varepsilon > 0$,

$$
\liminf_{\lambda \to \infty} \min \left\{ \int_{\rho(\phi, U_\delta(h_1)) < \varepsilon} \Omega(\lambda, \phi)^2 d\mu, \int_{\rho(\phi, U_\delta(h_2)) < \varepsilon} \Omega(\lambda, \phi)^2 d\mu \right\} > 0.
\tag{5.2}
$$

(A6)

$$
\lim_{\delta \to 0} \rho \left(U_\delta(h_1), U_\delta(h_2) \right) = \rho(h_1, h_2),
\tag{5.3}
$$

where

$$
\rho(U_\delta(h_1), U_\delta(h_2)) = \inf \left\{ \rho(x, y) \mid x \in U_\delta(h_1), y \in U_\delta(h_2) \right\}.
\tag{5.4}
$$

We prove that the assumptions above hold for double well potentials.

Lemma 5.1. (1) *Assume that* $V(\phi) = V(-\phi)$ *for any* $\phi \in X$ *and* $\{V = 0\} = \{h_0, -h_0\}$, *where* $h_0(\neq 0) \in H$. *Then* $N = \{h_0, -h_0\}$ *and the assumption* **(A5)** *holds.*
(2) *For* $h_1, h_2 \in H$, *let* $V(\phi) = \|\phi - h_1\|_X^2 \|\phi - h_2\|_X^2$. *Assume* $h_1 \neq h_2$. *Then* (5.3) *holds.*
(3) *Let* $h_1, h_2 \in H$ *and assume* $h_1 \neq h_2$. *For* $V(\phi) = \|\phi - h_1\|_X^2 \|\phi - h_2\|_X^2$, $\rho(h_1, h_2) > 0$ *holds.*

Proof. (1) By the invariance property of the Wiener measure under the transformation $\phi \to -\phi$, (5.2) holds. Now we prove (2). Let $\{e_i\}_{i=1}^{\infty}$ and $\{\xi_i\}_{i=1}^{\infty}$ be eigenvalues and eigenvectors of T such that $Te_i = \xi_i e_i$. Let $h_\alpha^i = (h_\alpha, e_i)$, where $\alpha = 1, 2$. Let us define a map F from X to H by $F(\phi) = \sum_{i=1}^{\infty} F_i\left((\phi, e_i)\right) e_i$. F_i is a real valued function on $\mathbb{R}$ as follows. Suppose $h_\alpha^i \le h_\beta^i$ $(\alpha, \beta = 1, 2)$. Then $F_i(u) = h_\beta^i$ for $u \ge h_\beta^i$, $F_i(u) = u$ for $h_\alpha^i \le u \le h_\beta^i$ and $F_i(u) = h_\alpha^i$ for $u \le h_\alpha^i$. Noting $e_i \in X^*$, it is easy to check that F is a continuous map on X. Also note that if $\phi - \psi \in H$, $\|F(\phi) - F(\psi)\|_H \le \|\phi - \psi\|_H$ and $\|F(\phi)\|_H^2 \le \|h_1\|_H^2 + \|h_2\|_H^2$ for all $\phi \in X$. By the definition of F_i, it holds that

$$
\begin{aligned}
(5.5) \qquad \|F(\phi) - h_\alpha\|_X^2 &= \sum_{i=1}^{\infty} \xi_i (F_i\left((\phi, e_i)\right) - h_\alpha^i)^2 \\
&\le \sum_{i=1}^{\infty} \xi_i \left((\phi, e_i) - h_\alpha^i\right)^2 \\
&= \|\phi - h_\alpha\|_X^2 \qquad (\alpha = 1, 2).
\end{aligned}
$$

Therefore $V(F(\phi)) \le V(\phi)$ for any $\phi \in X$. Let us take a smooth path $c(t)$ such that $c(0) \in U_\delta(h_1)$, $c(1) \in U_\delta(h_2)$, $c(\cdot) - c(0) \in \bar{H}$ and satisfying that

$$
\int_0^1 \sqrt{V(c(t))} \|\dot{c}(t)\|_H dt \le \rho(U_\delta(h_1), U_\delta(h_2)) + \varepsilon.
$$

Let $p(t) = F(c(t))$. Then for any t, $V(p(t)) \le V(c(t))$. Noting $\|p(t) - p(s)\|_H \le \|c(t) - c(s)\|_H$, we have for almost all t,

$$
\|\dot{p}(t)\|_H \le \|\dot{c}(t)\|_H.
$$

Therefore

$$
(5.6) \qquad \int_0^1 \sqrt{V(p(t))} \|\dot{p}(t)\|_H dt \le \rho(U_\delta(h_1), U_\delta(h_2)) + \varepsilon.
$$

On the other hand, by (5.5)

$$
(5.7) \qquad \max\left\{\|p(0) - h_1\|_X, \|p(1) - h_2\|_X\right\} \le \delta.
$$

Consider two segments $q(t) = h_1 + t(p(0) - h_1) \in H$ and $r(t) = p(1) + t(h_2 - p(1))$. Then it holds that

$$
(5.8) \qquad \int_0^1 \sqrt{V(q(t))} \|\dot{q}(t)\|_H dt + \int_0^1 \sqrt{V(r(t))} \|\dot{r}(t)\|_H dt \le C \cdot \delta,
$$

where C depends only on $\|h_1\|_H$ and $\|h_2\|_H$. Consequently we have

$$\rho(h_1, h_2) \leq \rho(U_\delta(h_1), U_\delta(h_2)) + \varepsilon + C \cdot \delta$$

which implies $\rho(h_1, h_2) \leq \liminf_{\delta \to 0} \rho(U_\delta(h_1), U_\delta(h_2))$. Now we prove the converse direction. For any ε, there exists a smooth curve $h(\cdot) \in \bar{H}$ such that $h_1 + h(1) = h_2$ and

$$\int_0^1 \sqrt{V(h_1 + h(t))} \|\dot{h}(t)\|_H dt \leq \rho(h_1, h_2) + \varepsilon.$$

Take a point $\phi \in U_\delta(h_1)$. Then

(5.9) $$\|\phi + h(1) - h_2\|_X = \|\phi - h_1\|_X < \delta$$

This shows $\phi + h(1) \in U_\delta(h_2)$. Thus

$$\rho(U_\delta(h_1), U_\delta(h_2)) \leq \rho(h_1, h_2) + \varepsilon.$$

This completes the proof.

(3) Without loss of generality, we may assume that $h_1^1 < h_2^1$. Then $V(\phi) \geq V_1(h^1)$, where $h^1 = (h, h_1)_H$ and $V_1(h^1) = \xi_1^2 (h^1 - h_1^1)^2 (h^1 - h_2^1)^2$. For $h(\cdot) \in \bar{H}$ satisfying $h(1) + h_1 = h_2$, set $h^1(t) = (h(t), e_1)_H$. Then

(5.10) $$\int_0^1 \sqrt{V(h_1 + h(t))} \|\dot{h}(t)\|_H dt \geq \int_0^1 \sqrt{V_1(h_1^1 + h^1(t))} |\dot{h}^1(t)| dt$$

$$\geq -\int_{h_1^1}^{h_2^1} \xi_1 (u - h_1^1)(u - h_2^1) du$$

$$= \frac{\xi_1}{6} (h_2^1 - h_1^1)^3 > 0.$$

$\square$

Remark 5.2. By the calculation similar to the above, we can prove that **(A6)** holds for the potential function defined by

(5.11) $$V(\phi) = F_n (\|\phi - h_1\|_X, \dots, \|\phi - h_n\|_X).$$

where F_n is nonnegative C^∞ increasing function in the sense that for any $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ such that $x_i \geq y_i$ $(1 \leq i \leq n)$, $F_n(x) \geq F_n(y)$ holds.

We need the following elementary lemma.

Lemma 5.3.   *Let $(X, \mathfrak{B}, m)$ be a probability space and $\varphi \in L^2(X, m)$. Then for any $\delta > 0$, it holds that*

$$(5.12) \qquad \int_X \left(\varphi - \int_X \varphi dm \right)^2 dm \geq 2\delta^2 m\left(\varphi \geq \delta\right) \cdot m\left(\varphi \leq -\delta\right).$$

Proof.   For φ, we have

$$\int_X \varphi^2 dm - \left(\int_X \varphi dm \right)^2 = \frac{1}{2} \iint_{X \times X} \left(\varphi(x) - \varphi(y) \right)^2 dm(x) dm(y)$$
$$\geq 2\delta^2 m\left(\varphi \geq \delta\right) \cdot m\left(\varphi \leq -\delta\right).$$

$\square$

Theorem 5.4.   *In addition to* **(A1)–(A4)**, *we assume that* (5.2) *and* (5.3) *hold. Then it holds that*

$$(5.13) \qquad \limsup_{\lambda \to \infty} \frac{\log\left(E_1(\lambda) - E_0(\lambda)\right)}{\lambda} \leq -\rho(h_1, h_2).$$

Proof.   Let $\psi_\varepsilon(t)$ be the piecewise linear function on $\mathbb{R}$ such that $\psi_\varepsilon(t) = 1$ for $t \leq (\rho(h_1, h_2) - 2\varepsilon)/2$, $\psi_\varepsilon(t) = 0$ for $t \geq (\rho(h_1, h_2) - \varepsilon)/2$. Let us define a trial function by

$$(5.14) \qquad \varphi_\varepsilon(\phi) = \psi_\varepsilon\left(\rho(\phi, U_\delta(h_1))\right) - \psi_\varepsilon\left(\rho(\phi, U_\delta(h_2))\right).$$

By (5.3), for sufficiently small $\delta > 0$, it holds that for any $\phi \in X$

$$(5.15) \qquad \rho(\phi, U_\delta(h_1)) + \rho(\phi, U_\delta(h_2)) \geq \rho(h_1, h_2) - \frac{\varepsilon}{2}.$$

Therefore $\psi_\varepsilon(\rho(\phi, U_\delta(h_1))) \cdot \psi_\varepsilon(\rho(\phi, U_\delta(h_2))) = 0$ for any ϕ. By the assumption (5.2) and Lemma 5.3, we see

$$(5.16) \qquad \liminf_{\lambda \to \infty} \int_X \left(\varphi_\varepsilon(\phi) - \int_X \varphi_\varepsilon(\phi) \Omega(\lambda, \phi)^2 d\mu \right)^2 \Omega(\lambda, \phi)^2 d\mu(\phi) > 0.$$

Also by (3.9), (3.10) and (3.25), there exists a certain polynomial $P(\lambda, \delta, \delta^{-1}, \varepsilon^{-1})$ such that

$$(5.17) \qquad \int_X |D\varphi_\varepsilon(\phi)|^2 \Omega(\lambda, \phi)^2 d\mu(\phi)$$
$$\leq \int_X \frac{2}{\varepsilon^2} |V(\phi)| \Big(\mathbf{1}_{\frac{\rho(h_1, h_2)}{2} - \varepsilon \leq \rho(\phi, U_\delta(h_1)) \leq \frac{\rho(h_1, h_2) - \varepsilon}{2}}$$
$$+ \mathbf{1}_{\frac{\rho(h_1, h_2)}{2} - \varepsilon \leq \rho(\phi, U_\delta(h_2)) \leq \frac{\rho(h_1, h_2) - \varepsilon}{2}} \Big) \Omega(\lambda, \phi)^2 d\mu(\phi)$$
$$\leq P(\lambda, \delta, \delta^{-1}, \varepsilon^{-1}) \exp\left(-q\lambda(\rho(h_1, h_2) - 3\varepsilon)\right),$$

where we applied (3.25) in the case where $d(\lambda) = \frac{\rho(h_1,h_2)}{2} - \varepsilon$ and $g(\lambda) = C \cdot \sqrt{\frac{\lambda^2}{E_0(\lambda,N)}}$. Since $\lim_{\lambda\to\infty} \frac{E_0(\lambda)}{\lambda^2} = 0$, we can take q to be arbitrarily close to 1 in (3.25). This completes the proof. $\square$

Acknowledgement

References

[1] Agmon, S., Spectral properties of Schrödinger operators and scattering theory, *Ann. Scuola Norm Sup. Pisa*, **2** (1975), 152-218.

[2] ———, Lectures on Exponential decay of solutions of second order elliptic equations. *Bounds on eigenfunctions of N-body Schrödinger operators*, Mathematical Notes, Princeton Univ. Press, Princeton, N. J., 1982.

[3] Aida, S., Logarithmic Sobolev Inequalities on Loop Spaces over compact Riemannian Manifolds, I. M. Davies, A. Truman and K. D. Elworthy, (eds.), *Proceedings of the Fifth Gregynog Symposium, Stochastic Analysis and Applications*, World Scientific, pp. 1-19, 1996.

[4] ———, Uniform Positivity Improving Property, Sobolev Inequality and Spectral Gaps, *J. Funct. Anal.*, **158** (1998), 152-185.

[5] ———, An estimate of the gap of spectrum of Schrödinger operators which generate hyperbounded semigroups, *J. Funct. Anal.*, **185** (2001), 474-526.

[6] ———, Semiclassical limit of the lowest eigenvalue of a Schrödinger operator on a Wiener space, *to appear in J. Funct. Anal.*

[7] ———, Witten Laplacian on pinned path group and its expected semiclassical behavior, *to appear in the special issue "Probability and Geometry" of IDAQP*.

[8] Arai, A., Trace formula, a Golden-Thompson inequality and classical limit in Boson Fock space, *J. Funct. Anal.*, **136** (1996), 510-547.

[9] Barbulyak, V. S. and Kondrat'ev, Yu. G., The quasiclassical limit for the Schrödinger operator and phase transition in quantum statistical physics, *Funct. Anal. Appl.*, **26** (1992), 61-64.

[10] Bogachev, V. I., *Gaussian measures, Mathematical Surveys and Monographs*, **62**, American Mathematical Society, Providence, RI, 1998.

[11] Gong, F.-Z. and Ma, Z.-M., The log-Sobolev Inequality on Loop Space Over a Compact Riemannian Manifold, *J. Funct. Anal.*, **157** (1998), 599-623.

[12] Gong, F.-Z., Röckner, M. and Wu, L. M., Poincaré inequality for weighted first order Sobolev spaces on loop spaces, *J. Funct. Anal.*, **185** (2001), 527-563.

[13] Gong, F.-Z. and Wu, L. M., Spectral gap of positive operators and applications, *C. R. Acad. Sci. Paris Sr. I Math.*, **331** (2000), 983-988.

[14] Gross, L., Existence and uniqueness of physical ground states, *J. Funct. Anal.*, **10** (1972), 52-109.

[15] ———, Logarithmic Sobolev inequalities, *Amer. J. Math.*, **97** (1975), 1061-1083.

[16] ———, Logarithmic Sobolev Inequalities on Loop Groups, *J. Funct. Anal.*, **102** (1991), 268-313.

[17] ———, Uniqueness of ground states for Schrödinger operators over loop groups, *J. Funct. Anal.*, **112** (1993), 373-441.

[18] Harrell, E. M., Double Wells, *Comm. Math. Phys.*, **75** (1980), 239-261.

[19] Helffer, B., Semi-classical analysis for the Schrödinger operator and applications, *Lecture Notes in Math.*, **1336**, Springer-Verlag, Berlin, 1988.

[20] ———, Splitting in large dimension and infrared estimates. II. Moment inequalities, *J. Math. Phys.*, **39** (1998), 760-776.

[21] Helffer, B. and Sjöstrand, J., Multiple wells in the semiclassical limit. I, *Comm. Partial Diffrential Equation*, **9** (1984), 337-408.

[22] Hino, M., Spectral properties of Laplacians on an abstract Wiener space with a weighted Wiener measure, *J. Funct. Anal.*, **147** (1997), 485-520.

[23] ———, Exponential decay of positivity preserving semigroups on L^p, *Osaka J. Math.*, **37** (2000), 603-624.

[24] Kuo, H.-H., Gaussian measures in Banach spaces, *Lecture Notes in Math.*, **43**, Springer-Verlag, 1975.

[25] Kusuoka, S., The nonlinear transformation of Gaussian measure on Banach space and its absolute continuity, *J. Fac. Sci., Univ. Tokyo*, **29** (1982), 567-598.

[26] ———, On the foundations of Wiener-Riemannian manifolds, Stochastic analysis, path integration and dynamics, *Pitman Res. Notes in Math. Series*, **200** (1989), pp. 130-164, K. D. Elworthy and J.-C. Zambrini, (eds.), Longman Scientific & Technical.

[27] ———, Analysis on Wiener Spaces, II, Differential Forms, *J. Funct. Anal.*, **103** (1992), 229-427.

[28] Reed, M. and Simon, B., *Methods of Modern Mathematical Physics*, II *Fourier Analysis*, Self-Adjointness, Academic Press, 1975.

[29] Simon, B., *The $P(\phi)_2$ Euclidean (quantum) field theory*, Princeton University Press, New Jersey, 1974.

[30] ———, Semiclassical Analysis of Low Lying Eigenvalues I. Nondegenerate Minima: Asymptotic Expansions, *Ann. Inst. Henri Poincaré*, **XXXVIII**, (1983), 309-347.

[31] ———, Semiclassical analysis of low lying eigenvalues, II. Tunneling, *Ann. of Math.*, **120** (1984), 89-118.

[32] ———, Semiclassical analysis of low lying eigenvalues, IV. The flea on the elephant, *J. Funct. Anal.*, **63** (1985), 123-136.

[33] Simon, B. and Hoegh-Krohn, R., Hypercontractive semigroups and two dimensional self-coupled Bose fields, *J. Funct. Anal.*, **9** (1972), 121-180.

[34] Sjöstrand, J., Potential wells in high dimensions I, *Ann. Inst. H. Poincaré, Phys. Théor.*, **58** (1993), 1-41.

[35] ———, Potential wells in high dimensions II, more about the one well case, *Ann. Inst. H. Poincaré, Phys. Théor.*, **58** (1993), 43-53.

[36] Witten, E., Supersymmetry and Morse theory, *J. Diff. Geom.*, **17** (1982), 661-692.