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Polarized States on the Weyl Algebra

By

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Abstract

A state σ on the Weyl algebra $A(V, \Omega)$ over a real symplectic vector space (V, Ω) is *polarized* when the vectors v for which the absolute value of σ on the corresponding Weyl generator δ_v is unity constitute a maximal Ω-integral additive subgroup of V; such a state is necessarily pure, but has inseparable carrier space and is not regular. We determine fundamental properties of such states: in particular, we decide precisely when the GNS representations associated to a pair of polarized states are unitarily equivalent and decide precisely when a given symplectic automorphism is unitarily implemented in the GNS representation associated to a given polarized state.

§1. Introduction

In this paper, we complete and extend the study of certain states on the Weyl algebra that was initiated in [1]. To introduce these states, let (V, Ω) be a real symplectic vector space and $A(V, \Omega)$ its Weyl algebra, this being the twisted complex group algebra of (V, Ω) with basis $\{\delta_v : v \in V\}$ satisfying $\delta_x \delta_y = e^{i\pi\Omega(x,y)}\delta_{x+y}$ whenever $x, y \in V$. For any state σ on the Weyl algebra, $\Lambda_\sigma = \{v \in V : |\sigma(\delta_v)| = 1\}$ is an additive subgroup of V on which Ω takes integer values and the resulting map $\chi_\sigma : \Lambda_\sigma \longrightarrow \mathbf{T} : v \longmapsto \sigma(\delta_v)$ satisfies $\chi_\sigma(x)\chi_\sigma(y) = (-1)^{\Omega(x,y)}\chi_\sigma(x + y)$ whenever $x, y \in \Lambda_\sigma$. We call the state σ *polarized* when Λ_σ is maximal; two extreme cases are those in which Λ_σ is a Lagrangian (when σ is also called a plane wave) and those in which Λ_σ is a lattice (when σ is also called a Zak wave). At the outset, it should perhaps be remarked that while automatically pure, a polarized state σ on $A(V, \Omega)$ is necessarily ill-behaved in some respects: on the one hand, the state σ fails to be regular, in that the function $V \longrightarrow \mathbf{C} : v \longmapsto \sigma(\delta_v)$ fails to be continuous on

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finite-dimensional subspaces; on the other hand, the Hilbert space $\mathbf{H}_\sigma$ carrying the GNS representation π_σ associated to σ is always inseparable.

Our chief aim is to determine precise necessary and sufficient conditions on the polarized states σ_1 and σ_2 of $A(V,\Omega)$ in order that the associated GNS representations π_{σ_1} and π_{σ_2} be unitarily equivalent: we show that the precise conditions are that the intersection $\Lambda_{\sigma_1} \cap \Lambda_{\sigma_2}$ should have finite index in Λ_{σ_1} and/or Λ_{σ_2} and that there should exist $u \in V$ such that $\chi_2/\chi_1 = e^{2\pi i \Omega(u,\cdot)}$ on $\Lambda_{\sigma_1} \cap \Lambda_{\sigma_2}$. As a direct consequence, we find that the symplectic automorphism $g \in Sp(V,\Omega)$ is unitarily implemented in the GNS representation π_σ associated to the polarized state σ of $A(V,\Omega)$ exactly when the index $[\Lambda_\sigma : \Lambda_\sigma \cap g^{-1}\Lambda_\sigma]$ is finite and on $\Lambda_\sigma \cap g^{-1}\Lambda_\sigma$ the ratio $\chi_\sigma \circ g/\chi_\sigma$ equals $e^{2\pi i \Omega(u,\cdot)}$ for some $u \in V$. When (V,Ω) is finite-dimensional, matters simplify somewhat. Consider implementation, as being more cleanly stated: if Λ_σ is a lattice, then g is implemented in π_σ if and only if $[\Lambda_\sigma : \Lambda_\sigma \cap g^{-1}\Lambda_\sigma]$ is finite if and only if g has rational matrix relative to Λ_σ; if Λ_σ is a Lagrangian and χ_σ is continuous, then g is implemented in π_σ if and only if g preserves Λ_σ. As a particular feature of our approach, we realize the various representations on function spaces in such a way that the representations share precisely the same functional form, facilitating comparisons.

As noted above, the work in this paper stems from [1]: in that paper, the authors confine their attention to finite dimensions and their discussion of the equivalence problem for Zak waves treats only lattices that are parallel in having proportional bases. For a detailed account of the Weyl algebra and its structure we refer to [4] and [6]; for general background on C*-algebras, their states and their representations we recommend [2] and [5].

§2. Polarized States on the Weyl Algebra

Let (V,Ω) be a real symplectic vector space: thus, V is a real vector space upon which Ω is a nonsingular alternating bilinear form. Its Weyl algebra $A(V,\Omega)$ is the set comprising all finitely-supported functions $V \longrightarrow \mathbf{C}$ equipped with the pointwise linear structure, with the (twisted) convolution product given by

$$[\xi\eta](z) = \sum_{x+y=z} e^{i\pi\Omega(x,y)}\xi(x)\eta(y)$$

when $\xi, \eta \in A(V,\Omega)$ and $z \in V$, and with the involution given by

$$\zeta^*(z) = \overline{\zeta(-z)}$$

when $\zeta \in A(V,\Omega)$ and $z \in V$. If for $v \in V$ we define $\delta_v : V \longrightarrow \mathbf{C}$ by $\delta_v(z) = 1$ when $z = v$ and $\delta_v(z) = 0$ otherwise, then $\{\delta_v : v \in V\} \subset A(V,\Omega)$ is a basis of unitary elements satisfying

$$x, y \in V \Longrightarrow \delta_x \delta_y = e^{i\pi\Omega(x,y)}\delta_{x+y}.$$

The Weyl algebra $A(V,\Omega)$ admits a canonical pre-C*-norm $\|\cdot\|$: the norm of $\zeta \in A(V,\Omega)$ is well-defined to be the operator norm $\|\pi(\zeta)\|$ for any nonzero Hilbert space representation π of $A(V,\Omega)$ as an involutive algebra. The corresponding completion of $A(V,\Omega)$ is called the (minimal) C* Weyl algebra $A[V,\Omega]$.

Let $\sigma : A(V,\Omega) \longrightarrow \mathbf{C}$ be a state (a normalized positive linear functional) on the Weyl algebra. Let $L_\sigma = \{\zeta \in A(V,\Omega) : \sigma(\zeta^*\zeta) = 0\}$ be its left kernel and equip the quotient vector space $H_\sigma = A(V,\Omega)/L_\sigma$ with the inner product well-defined by the requirement that if $\xi, \eta \in A(V,\Omega)$ then $\langle \xi + L_\sigma | \eta + L_\sigma \rangle = \sigma(\xi^*\eta)$. The resulting inner product space carries a representation π_σ of $A(V,\Omega)$ as an involutive algebra, well-defined by the rule that if $a, \zeta \in A(V,\Omega)$ then $\pi_\sigma(a)(\zeta + L_\sigma) = a\zeta + L_\sigma$. These structures pass to the appropriate completions: thus, π_σ extends uniquely to a representation of the C*-algebra completion $A[V,\Omega]$ on the Hilbert space completion $\mathbf{H}_\sigma$. Of course, this is just the standard GNS representation associated to σ. As usual, the generating vector $s_0 = \delta_0 + L_\sigma$ plays a distinguished role.

Our interest centres on such states σ as are *polarized* in a sense that we proceed to elaborate. Let $v \in V$: if $\sigma(\delta_v)$ lies in the unit circle $\mathbf{T}$ then a routine calculation places $\delta_v - \sigma(\delta_v)\mathbf{1}$ in L_σ so that $\pi_\sigma(\delta_v)$ has s_0 as eigenvector with $\sigma(\delta_v)$ as eigenvalue; conversely, if $\pi_\sigma(\delta_v)$ has s_0 as eigenvector then $\sigma(\delta_v) \in \mathbf{T}$. Define

$$\Lambda_\sigma = \{v \in V : \sigma(\delta_v) \in \mathbf{T}\}$$

and

$$\chi_\sigma : \Lambda_\sigma \longrightarrow \mathbf{T} : v \longmapsto \sigma(\delta_v).$$

Theorem 1. *Λ_σ is an additive subgroup of V on which Ω takes integer values; if $x, y \in \Lambda_\sigma$ then*

$$\chi_\sigma(x)\chi_\sigma(y) = (-1)^{\Omega(x,y)}\chi_\sigma(x+y).$$

Proof. From $\pi_\sigma(\delta_x)s_0 = \chi_\sigma(x)s_0$ and $\pi_\sigma(\delta_y)s_0 = \chi_\sigma(y)s_0$ it follows that

$$\pi_\sigma(\delta_{x+y})s_0 = \pi_\sigma(e^{-i\pi\Omega(x,y)}\delta_x\delta_y)s_0$$

$$
\begin{aligned}
&= e^{-i\pi\Omega(x,y)}\pi_\sigma(\delta_x)\pi_\sigma(\delta_y)s_0 \\
&= e^{-i\pi\Omega(x,y)}\chi_\sigma(x)\chi_\sigma(y)s_0
\end{aligned}
$$

whence $x + y \in \Lambda_\sigma$ and

$$\chi_\sigma(x + y) = e^{-i\pi\Omega(x,y)}\chi_\sigma(x)\chi_\sigma(y)$$

which forces $\Omega(x, y)$ to be integral since Ω is alternating. □

We say that the state σ of $A(V, \Omega)$ is *polarized* if and only if the Ω-integral subgroup Λ_σ is maximal: that is, if and only if

$$\{v \in V : \Omega(v, \Lambda_\sigma) \subset \mathbf{Z}\} = \Lambda_\sigma.$$

At one extreme, Λ_σ may be a Lagrangian (real polarization): a maximal subspace of V on which Ω vanishes identically; at the other extreme, Λ_σ may be a lattice.

To aid the investigation of these states, we associate to each $\Lambda \subset V$ the subgroup $\Lambda' \subset V$ defined by

$$\Lambda' = \{v \in V : \Omega(v, \Lambda) \subset \mathbf{Z}\}.$$

A routine argument establishes that if $A(\Lambda)$ denotes the subalgebra of $A(V, \Omega)$ generated by $\{\delta_v : v \in \Lambda\}$ and if $B' \subset A(V, \Omega)$ denotes the commutant of $B \subset A(V, \Omega)$ then

$$A(\Lambda)' = A(\Lambda').$$

In terms of this notation, we claim that if σ is any state on the Weyl agebra $A(V, \Omega)$ then

$$v \in V - {\Lambda_\sigma}' \Longrightarrow \sigma(\delta_v) = 0.$$

Indeed, let $w \in \Lambda :$ as $\delta_w - \chi_\sigma(w)\mathbf{1} \in L_\sigma$ so

$$
\begin{aligned}
\sigma(\delta_v) &= \sigma(\overline{\chi_\sigma(w)}\delta_v\chi_\sigma(w)) \\
&= \sigma(\delta_{-w}\delta_v\delta_w) \\
&= e^{2\pi i\Omega(v,w)}\sigma(\delta_v)
\end{aligned}
$$

whence arranging $\Omega(v, w) \notin \mathbf{Z}$ reveals that $\sigma(\delta_v) = 0$ as claimed. In particular, a polarized state σ is uniquely determined by the pair $(\Lambda_\sigma, \chi_\sigma)$.

Theorem 2. *Each polarized state σ on the Weyl algebra $A(V,\Omega)$ is pure.*

Proof. Assume that $\sigma = (1-t)\sigma_0 + t\sigma_1$ for $0 < t < 1$ and for states σ_0 and σ_1. If $v \in \Lambda_\sigma$ then

$$(1-t)\sigma_0(\delta_v) + t\sigma_1(\delta_v) = \sigma(\delta_v) \in \mathbf{T}$$

whence $\sigma_0(\delta_v) = \sigma_1(\delta_v) = \sigma(\delta_v)$ because points of $\mathbf{T}$ are extreme in the unit disc. The claim established prior to the theorem now shows that the equality of σ_0 and σ_1 with σ extends from the maximal abelian subalgebra $A(\Lambda)$ to the whole of $A(V,\Omega)$. $\square$

It is perhaps worth pointing out that the process of assigning to a polarized state σ the pair $(\Lambda_\sigma, \chi_\sigma)$ is fully reversible: indeed, if $\Lambda = \Lambda'$ and if $\chi : \Lambda \longrightarrow \mathbf{T}$ satisfies $\chi(x)\chi(y) = (-1)^{\Omega(x,y)}\chi(x+y)$ whenever $x, y \in \Lambda$ then χ determines a (multiplicative, hence) pure state on the maximal abelian subalgebra $A(\Lambda)$ which extends to a (unique, hence) pure state σ on $A(V,\Omega)$ itself.

Until further notice, let us fix a polarized state σ on $A(V,\Omega)$. It proves convenient to identify H_σ with a certain space of complex functions on V. To be specific, we shall say that $s : V \longrightarrow \mathbf{C}$ is σ-quasiperiodic (or just quasiperiodic) precisely when it satisfies the following condition: that if $z \in V$ and $\lambda \in \Lambda$ then

$$s(z-\lambda) = \chi_\sigma(\lambda)e^{i\pi\Omega(\lambda,z)}s(z).$$

For such a function, the points at which it is nonzero constitute a Λ_σ-invariant set; we may regard the corresponding subset of V/Λ_σ as its support. We shall write F_σ for the space comprising all finitely-supported σ-quasiperiodic functions $V \longrightarrow \mathbf{C}$.

Theorem 3. *A canonical isomorphism from H_σ to F_σ is induced by mapping $\zeta \in A(V,\Omega)$ to the function*

$$s : V \longrightarrow \mathbf{C} : z \longmapsto \sigma(\delta_{-z}\zeta) = \langle \delta_z + L_\sigma | \zeta + L_\sigma \rangle.$$

Proof. The assignment $\zeta \longmapsto s$ is plainly linear with L_σ as its kernel. If $\lambda \in \Lambda$ then $\delta_\lambda - \chi_\sigma(\lambda)\mathbf{1} \in L_\sigma$ whence if also $z \in V$ then

$$\begin{aligned} s(z-\lambda) &= \sigma(\delta_{\lambda-z}\zeta) \\ &= \sigma(e^{-i\pi\Omega(\lambda,-z)}\delta_\lambda\delta_{-z}\zeta) \\ &= e^{i\pi\Omega(\lambda,z)}\sigma(\chi_\sigma(\lambda)\delta_{-z}\zeta) \\ &= \chi_\sigma(\lambda)e^{i\pi\Omega(\lambda,z)}s(z) \end{aligned}$$

so that s is quasiperiodic. Moreover, when $\zeta = \sum_{j=1}^{n} c_j \delta_{v_j}$ a direct calculation shows that if $z \in V$ then

$$s(z) = \sum_{j=1}^{n} c_j e^{i\pi\Omega(v_j,z)} \sigma(\delta_{v_j - z})$$

thus $s(z) = 0$ unless $z \in \bigcup_{j=1}^{n} (v_j + \Lambda_\sigma)$ and so supp $s \subset V/\Lambda_\sigma$ is finite. In the opposite direction, if $s : V \longrightarrow \mathbf{C}$ is quasiperiodic and its support consists of the distinct points $v_1 + \Lambda_\sigma, \ldots, v_n + \Lambda_\sigma$ then it may be verified that it arises from $\zeta = \sum_{j=1}^{n} s(v_j)\delta_{v_j}$. □

Naturally, we use this isomorphism to transport the canonical inner product from H_σ to F_σ. We remark that the GNS representation π_σ of $A(V,\Omega)$ on H_σ assumes the following form on F_σ: if $s \in F_\sigma$ and $v, z \in V$ then

$$[\pi_\sigma(\delta_v)s](z) = e^{i\pi\Omega(v,z)} s(z - v).$$

Certain special elements of F_σ play a distinguished role. For $v \in V$ we shall denote by s_v the function in F_σ corresponding to the vector $\delta_v + L_\sigma$ in H_σ : thus, if also $z \in V$ then

$$s_v(z) = \sigma(\delta_{-z}\delta_v) = \sigma(e^{i\pi\Omega(-z,v)}\delta_{v-z}) = e^{i\pi\Omega(v,z)}\sigma(\delta_{v-z})$$

so that $s_v(z) = e^{i\pi\Omega(v,z)}\chi_\sigma(v - z)$ when $z \in v + \Lambda_\sigma$ and $s_v(z) = 0$ otherwise.

Theorem 4. *If $v \in V$ then the vector*

$$s_v = \pi_\sigma(\delta_v)s_0$$

in F_σ has the reproducing property

$$t \in F_\sigma \Longrightarrow t(v) = \langle s_v | t \rangle.$$

Proof. The identity $\pi_\sigma(\delta_v)s_0 = s_v$ in F_σ follows at once from the identity $\pi_\sigma(\delta_v)(\delta_0 + L_\sigma) = \delta_v + L_\sigma$ in H_σ. If $t \in F_\sigma$ arises from $\zeta \in A(V,\Omega)$ in the standard way then

$$t(v) = \sigma(\delta_{-v}\zeta) = \sigma(\delta_v^*\zeta) = \langle s_v | t \rangle.$$

□

These two properties of $\{s_v : v \in V\}$ yield separate proofs of a third: that if $\lambda \in \Lambda$ then

$$s_{v+\lambda} = \chi_\sigma(\lambda)e^{i\pi\Omega(\lambda,v)}s_v.$$

On the one hand,

$$\begin{aligned} s_{v+\lambda} &= \pi_\sigma(\delta_{v+\lambda})s_0 \\ &= \pi_\sigma(e^{-i\pi\Omega(v,\lambda)}\delta_v\delta_\lambda)s_0 \\ &= e^{i\pi\Omega(\lambda,v)}\pi_\sigma(\delta_v)\pi_\sigma(\delta_\lambda)s_0 \\ &= e^{i\pi\Omega(\lambda,v)}\pi_\sigma(\delta_v)\chi_\sigma(\lambda)s_0. \end{aligned}$$

On the other hand, if $z \in V$ then

$$s_{v+\lambda}(z) = \langle s_z | s_{v+\lambda}\rangle = \overline{\langle s_{v+\lambda} | s_z\rangle} = \overline{s_z(v+\lambda)}$$

and quasiperiodicity of s_z may be invoked to reach the same conclusion.

Of course, as $\{\delta_v : v \in V\}$ is a basis for $A(V,\Omega)$ so $\{s_v : v \in V\}$ spans F_σ. The foregoing property shows that the latter vectors are linearly dependent; in order to extract from them a basis, it is necessary to pick out one representative for each Λ-coset in V. With this in mind, by a transversal of $\Lambda \subset V$ we mean a full set Δ of distinct representatives for the Λ-cosets in V. In special cases, we may find it convenient to assume more of Δ: if Λ is a Lagrangian then we may require Δ to be a subspace; if Λ is a lattice then we may require Δ to be a domain.

Theorem 5. *If Δ is a transversal for $\Lambda \subset V$ then $\{s_v : v \in \Delta\}$ is a unitary basis for F_σ.*

Proof. As was just noted, the set $\{s_v : v \in \Delta\}$ spans F_σ because each vector in V is equivalent to a unique vector in the transversal Δ. Moreover, if $x, y \in \Delta$ are distinct then $y - x \notin \Lambda$ so that $\langle s_x | s_y\rangle = s_y(x) = 0$. $\square$

As a consequence, the Hilbert space completion $\mathbf{F}_\sigma$ of F_σ is isometrically isomorphic to $\ell^2(\Delta)$ via the map that associates to the series $\sum_{v\in\Delta} c_v s_v$ the sequence $(c_v : v \in \Delta)$. In particular, the Hilbert space $\mathbf{F}_\sigma$ is inseparable; of course, the same applies to the canonically isomorphic Hilbert space $\mathbf{H}_\sigma$.

Recall that on F_σ the representation π_σ assumes the following form: that if $s \in F_\sigma$ and $v, z \in V$ then

$$[\pi(\delta_v)s](z) = e^{i\pi\Omega(v,z)}s(z-v).$$

Of course, this formula makes sense whenever s is any complex function on V. Accordingly, this rule defines a representation π of the algebra $A(V,\Omega)$ on the space of all functions $V \longrightarrow \mathbf{C}$. Routine calculation from the definitions establishes that in this general context, the function $s : V \longrightarrow \mathbf{C}$ is σ-quasiperiodic if and only if it satisfies

$$\lambda \in \Lambda_\sigma \Longrightarrow \pi(\delta_\lambda)s = \chi_\sigma(\lambda)e^{2\pi i\Omega(\lambda,\cdot)}s.$$

For this reason among others, it is convenient to associate to each $v \in V$ the linear operator ε_v having effect on $s : V \longrightarrow \mathbf{C}$ given by

$$z \in V \Longrightarrow (\varepsilon_v s)(z) = e^{2\pi i\Omega(v,z)}s(z).$$

Theorem 6. *If $\lambda \in \Lambda_\sigma$ then ε_λ leaves F_σ invariant: it satisfies*

$$v \in V \Longrightarrow \varepsilon_\lambda(s_v) = e^{2\pi i\Omega(\lambda,v)}s_v.$$

Proof. This may of course be verified straight from the definitions. Alternatively, if $v \in V$ then from $\pi_\sigma(\delta_\lambda)(\delta_v + L_\sigma) = \delta_\lambda\delta_v + L_\sigma$ and $s_{v+\lambda} = \chi_\sigma(\lambda)e^{i\pi\Omega(\lambda,v)}s_v$ it follows that

$$\varepsilon_\lambda(s_v) = \overline{\chi_\sigma(\lambda)}\pi_\sigma(\delta_\lambda)s_v = \overline{\chi_\sigma(\lambda)}e^{\pi i\Omega(\lambda,v)}s_{v+\lambda} = e^{2\pi i\Omega(\lambda,v)}s_v.$$

□

We remark that by direct calculation, if $x, y \in V$ then

$$\pi(\delta_x)\varepsilon_y = e^{2\pi i\Omega(x,y)}\varepsilon_y\pi(\delta_x).$$

From this, it follows that if $u \in V$ then ε_u maps F_σ to $F_{\sigma'}$ where $\Lambda_{\sigma'} = \Lambda_\sigma$ and where $\chi_{\sigma'} = \chi_\sigma e^{2\pi i\Omega(\cdot,v)}$: indeed, if $s \in F_\sigma$ and $\lambda \in \Lambda_\sigma$ then

$$\begin{aligned}\pi_\sigma(\delta_\lambda)(\varepsilon_v s) &= e^{2\pi i\Omega(\lambda,v)}\varepsilon_v\pi_\sigma(\delta_\lambda)s\\ &= e^{2\pi i\Omega(\lambda,v)}\varepsilon_v\chi_\sigma(\lambda)\varepsilon_\lambda s\\ &= e^{2\pi i\Omega(\lambda,v)}\chi_\sigma(\lambda)\varepsilon_\lambda\varepsilon_v s\end{aligned}$$

A minor modification produces an intertwining operator.

Theorem 7. *If $u \in V$ then*

$$U_u = \pi(\delta_u)\varepsilon_{-u} = \varepsilon_{-u}\pi(\delta_u)$$

defines a unitary operator from F_σ to F_{σ^u} intertwining π_σ with π_{σ^u} where $\Lambda_{\sigma^u} = \Lambda_\sigma$ and where $\chi_{\sigma^u} = \chi_\sigma e^{2\pi i\Omega(u,\cdot)}$.

Proof. The operator $\pi(\delta_u)$ leaves each F-space invariant, while ε_{-u} maps F_σ to F_{σ^u} by the preceding discussion. The intertwining nature of U_u is readily apparent: if $v \in V$ then $\pi(\delta_u)\pi(\delta_v) = e^{2\pi i\Omega(u,v)}\pi(\delta_v)\pi(\delta_u)$ and $\varepsilon_{-u}\pi(\delta_v) = e^{-2\pi i\Omega(u,v)}\pi(\delta_v)\varepsilon_{-u}$. To see that U_u is unitary, temporarily write t_w for the function in F_{σ^u} corresponding to the vector $\delta_w + L_{\sigma^u}$ in H_{σ^u} when $w \in V$. In particular, $t_0 = \varepsilon_{-u}s_0$ so that if $v \in V$ then

$$\begin{aligned} U_u\pi(\delta_v)s_0 &= \pi(\delta_v)U_u s_0 \\ &= \pi(\delta_v)\pi(\delta_u)t_0 \\ &= e^{i\pi\Omega(v,u)}\pi(\delta_{v+u})t_0 \end{aligned}$$

or

$$U_u s_v = e^{i\pi\Omega(v,u)}t_{v+u}.$$

All that remains is to observe that just as $\{s_v : v \in \Delta\}$ is a unitary basis for F_σ so $\{t_w : w \in u + \Delta\}$ is a unitary basis for F_{σ^u}. $\square$

Of course, if $u \in V$ then U_u extends to a unitary intertwining operator $\mathbf{F}_\sigma \longrightarrow \mathbf{F}_{\sigma^\mathbf{u}}$ corresponding to a unitary intertwining operator $\mathbf{H}_\sigma \longrightarrow \mathbf{H}_{\sigma^\mathbf{u}}$.

Recall that a state σ of the Weyl algebra $A(V,\Omega)$ is said to be regular if and only if $\sigma(\delta_{tv})$ depends continuously on $t \in \mathbf{R}$ whenever $v \in V$. We close this section by pointing out that polarized states lack this property. In fact, rather more is true.

Theorem 8. *Let σ be a state of the Weyl algebra $A(V,\Omega)$. If σ is regular then Λ_σ is zero.*

Proof. Suppose $u \in \Lambda_\sigma$ to be nonzero. For $v \in V$ let

$$f : \mathbf{R} \longrightarrow \mathbf{C} : t \longmapsto \sigma(\delta_{tv})$$

and note that (as established just prior to Theorem 2)

$$(e^{2\pi i\Omega(u,v)t} - 1)f(t) = 0$$

so that

$$\Omega(u,v)t \notin \mathbf{Z} \Longrightarrow f(t) = 0$$

while of course $f(0) = 1$. Now, if $\Omega(u,v) = 1$ then the restriction $f|(-1,1)$ vanishes away from 0 at which point it is therefore discontinuous. $\square$

§3. Equivalence and Implementation

Our primary aim is to relate the GNS representations associated to two polarized states of the Weyl algebra $A(V,\Omega)$. For convenience, let us agree to refer to a subgroup $\Lambda \subset V$ such that $\Lambda = \Lambda'$ as a *polarization* and to a map $\chi : \Lambda \longrightarrow \mathbf{T}$ such that $\chi(x)\chi(y) = (-1)^{\Omega(x,y)}\chi(x+y)$ whenever $x, y \in \Lambda$ as a *quasicharacter.*

Let $\Lambda_1 \subset V$ and $\Lambda_2 \subset V$ be polarizations and consider the restriction of

$$\omega : V \times V \longrightarrow \mathbf{T} : (x,y) \longmapsto e^{2\pi i\Omega(x,y)}$$

to $\Lambda_1 \times \Lambda_2$. This descends to a map

$$\underline{\omega} : \underline{\Lambda_1} \times \underline{\Lambda_2} \longrightarrow \mathbf{T}$$

where $\underline{\Lambda_1} = \Lambda_1/(\Lambda_1 \cap \Lambda_2)$ and $\underline{\Lambda_2} = \Lambda_2/(\Lambda_1 \cap \Lambda_2)$ since the values of Ω on Λ_1 and Λ_2 separately are integral. Moreover, if $\lambda_1 \in \Lambda_1 - (\Lambda_1 \cap \Lambda_2)$ then $\lambda_1 \notin \Lambda_2 = \Lambda_2{}'$ so there exists $\lambda_2 \in \Lambda_2$ such that $\Omega(\lambda_1, \lambda_2) \notin \mathbf{Z}$ and therefore $\omega(\lambda_1, \lambda_2) \neq 1$. This shows that the map

$$\underline{\Lambda_1} \longrightarrow \underline{\Lambda_2}^* : \underline{\lambda_1} \longmapsto \underline{\omega}(\underline{\lambda_1}, \cdot)$$

embeds $\underline{\Lambda_1}$ in the dual $\underline{\Lambda_2}^* = \hom(\underline{\Lambda_2}, \mathbf{T})$; similarly, $\underline{\omega}$ induces an embedding of $\underline{\Lambda_2}$ in the dual $\underline{\Lambda_1}^*$. In this connexion, recall that a finite abelian group and its dual have the same cardinality.

Now, we declare the polarizations (meaning, maximal Ω-integral subgroups) Λ_1 and Λ_2 to be equivalent (equivalently, *commensurable*) and write $\Lambda_1 \equiv \Lambda_2$ if and only if either (hence each) of the following equivalent conditions is satisfied:

1. $\Lambda_1 \cap \Lambda_2$ has finite index in Λ_1;
2. $\Lambda_1 \cap \Lambda_2$ has finite index in Λ_2.

To see the equivalence, assume (1): from $|\underline{\Lambda_1}| < \infty$ and $|\underline{\Lambda_2}| \leq |\underline{\Lambda_1}^*| = |\underline{\Lambda_1}|$ it follows that $|\underline{\Lambda_2}| < \infty$; hence (2). Note that the argument by symmetry reveals a little more: namely, that if $\Lambda_1 \equiv \Lambda_2$ then in fact

$$|\Lambda_1/(\Lambda_1 \cap \Lambda_2)| = |\Lambda_2/(\Lambda_1 \cap \Lambda_2)|.$$

It is perhaps worth establishing that commensurability is indeed an equivalence relation. Thus, let $\Lambda_1 \equiv \Lambda_2$ and $\Lambda_2 \equiv \Lambda_3$. The natural composite

$$\Lambda_1 \cap \Lambda_2 \longrightarrow \Lambda_2 \longrightarrow \Lambda_2/(\Lambda_2 \cap \Lambda_3)$$

has kernel $\Lambda_1 \cap \Lambda_2 \cap \Lambda_3$ whence (by $\Lambda_3 \equiv \Lambda_2$) it follows that $[\Lambda_1 \cap \Lambda_2 : \Lambda_1 \cap \Lambda_2 \cap \Lambda_3] < \infty$ and hence (by $\Lambda_2 \equiv \Lambda_1$) that $[\Lambda_1 : \Lambda_1 \cap \Lambda_2 \cap \Lambda_3] < \infty$. Thus $[\Lambda_1 : \Lambda_1 \cap \Lambda_3] < \infty$ and so $\Lambda_1 \equiv \Lambda_3$.

Let $\Lambda \subset V$ be an arbitrary (not necessarily maximal) subgroup on which Ω takes integer values and denote by Ξ_Λ the set comprising all functions $\chi : \Lambda \longrightarrow \mathbf{T}$ such that $\chi(x)\chi(y) = (-1)^{\Omega(x,y)}\chi(x+y)$ whenever $x, y \in \Lambda$. Note that Ξ_Λ is plainly a principal homogeneous space for the dual $\Lambda^* = \hom(\Lambda, \mathbf{T})$ acting by pointwise multiplication. Note also that associating to $u \in V$ and $\chi \in \Xi_\Lambda$ the quasicharacter

$$\chi^u : \Lambda \longrightarrow \mathbf{T} : \lambda \longmapsto e^{2\pi i \Omega(u,\lambda)}\chi(\lambda)$$

descends to an action of the quotient V/Λ' on Ξ_Λ. These actions are compatible: indeed, they are related by the embedding $V/\Lambda' \longrightarrow \Lambda^* : u + \Lambda' \longmapsto e^{2\pi i \Omega(u,\cdot)}$ that is induced by the pairing $\omega = e^{2\pi i \Omega}$. Let us declare the quasicharacters χ_1 and χ_2 in Ξ_Λ to be equivalent and write $\chi_1 = \chi_2$ if and only if they lie in the same V/Λ'-orbit.

Having made these preparations, we now address the unitary equivalence of the GNS representations associated to the polarized states σ_1 and σ_2 of $A(V, \Omega)$. Explicitly, we wish to determine precise necessary and sufficient conditions in order that the unitary representations $\pi_1 = \pi_{\sigma_1}$ on $\mathbf{H}_1 = \mathbf{H}_{\sigma_1}$ and $\pi_2 = \pi_{\sigma_2}$ on $\mathbf{H}_2 = \mathbf{H}_{\sigma_2}$ be unitarily equivalent in the sense that there exists a unitary isomorphism $U : \mathbf{H}_1 \longrightarrow \mathbf{H}_2$ that intertwines them:

$$a \in A(V, \Omega) \Longrightarrow U\pi_1(a) = \pi_2(a)U$$

or simply

$$v \in V \Longrightarrow U\pi_1(\delta_v) = \pi_2(\delta_v)U.$$

For convenience, we shall work in the equivalent $\mathbf{F}$-space picture, where U is a unitary intertwiner from $\mathbf{F}_1 = \mathbf{F}_{\sigma_1}$ to $\mathbf{F}_2 = \mathbf{F}_{\sigma_2}$. Among other things, this has the virtue that π_1 and π_2 have exactly the same functional form.

Theorem 9. *Let σ_1 and σ_2 be polarized states of $A(V, \Omega)$. If π_{σ_1} and π_{σ_2} are unitarily equivalent then $\Lambda_{\sigma_1} \equiv \Lambda_{\sigma_2}$ and $\chi_{\sigma_1}|(\Lambda_{\sigma_1} \cap \Lambda_{\sigma_2}) \equiv \chi_{\sigma_2}|(\Lambda_{\sigma_1} \cap \Lambda_{\sigma_2})$.*

Proof. To avoid an overabundance of labels, let us agree to write $\{s_v : v \in V\}$ for the standard vectors in $\mathbf{F}_1$ and $\{t_v : v \in V\}$ for their counterparts in $\mathbf{F}_2$. Further, let Δ_2 be a transversal for Λ_2 in V.

Suppose $U : \mathbf{F}_1 \longrightarrow \mathbf{F}_2$ to be a unitary isomorphism intertwining π_1 with π_2. Write $f = Us_0$ and choose $u \in V$ so that $f(u) = \langle t_u|f\rangle \neq 0$. If $\lambda_1 \in \Lambda_1$ then

$$\pi(\delta_{\lambda_1})f = \pi(\delta_{\lambda_1})Us_0 = U\pi(\delta_{\lambda_1})s_0 = U\chi_1(\lambda_1)s_0 = \chi_1(\lambda_1)f$$

whence

$$e^{i\pi\Omega(\lambda_1,u)}f(u-\lambda_1) = \chi_1(\lambda_1)f(u)$$

and therefore $|f(u-\lambda_1)| = |f(u)|$. From this it follows that $|f|$ takes the same value $|f(u)| > 0$ at each point of the subset of Δ_2 comprising all points that are Λ_2-equivalent to points in $\{u-\lambda_1 : \lambda_1 \in \Lambda_1\}$; this subset of Δ_2 is parametrized by $\Lambda_1/(\Lambda_1 \cap \Lambda_2)$. In view of the fact that

$$\sum_{v\in\Delta_2} |f(v)|^2 = \sum_{v\in\Delta_2} |\langle t_v|f\rangle|^2 = \|f\|^2 < \infty$$

it follows that $\Lambda_1/(\Lambda_1 \cap \Lambda_2)$ is finite. Finally, let $\lambda \in \Lambda_1 \cap \Lambda_2$: evaluate

$$\chi_1(\lambda)f = \pi(\delta_\lambda)f = \chi_2(\lambda)\varepsilon_\lambda f$$

at u and cancel $f(u) \neq 0$ to obtain

$$\chi_1(\lambda) = \chi_2(\lambda)e^{2\pi i\Omega(\lambda,u)}.$$

$\square$

Upon reflection, this proof actually gives a little more: it identifies the vectors $u \in V$ for which $\chi_2 = {\chi_1}^u$ on $\Lambda_1 \cap \Lambda_2$ as being those points in V at which Us_0 is nonzero.

Theorem 10. *Let σ_1 and σ_2 be polarized states of $A(V,\Omega)$. If $\Lambda_{\sigma_1} \equiv \Lambda_{\sigma_2}$ and $\chi_{\sigma_1}|(\Lambda_{\sigma_1}\cap\Lambda_{\sigma_2}) \equiv \chi_{\sigma_2}|(\Lambda_{\sigma_1}\cap\Lambda_{\sigma_2})$ then π_{σ_1} and π_{σ_2} are unitarily equivalent.*

Proof. The vector $u \in V$ provides us with a unitary isomorphism $U_u = \pi(\delta_u)\varepsilon_{-u}$ from $\mathbf{F}_{\sigma_1}$ to $\mathbf{F}_{\sigma_1^u}$ that intertwines π_{σ_1} with $\pi_{\sigma_1^u}$. This being so, we may assume that $\chi_1 = \chi_2$ on $\Lambda_1 \cap \Lambda_2$ and exhibit a unitary intertwiner under this assumption. In fact, we claim that an intertwining operator $U = U_{21} : \mathbf{F}_1 \longrightarrow \mathbf{F}_2$ is given by the formula

$$U_{21} = \sum_{\lambda\in\underline{\Lambda_2}} \overline{\chi_2(\lambda)}\pi(\delta_\lambda)\varepsilon_{-\lambda}$$

where summation takes place over a full set of distinct coset representatives for Λ_2 over $\Lambda_1 \cap \Lambda_2$. First, U_{21} is well-defined: if $\lambda \in \Lambda_2$ and $\mu \in \Lambda_1 \cap \Lambda_2$ then for $f \in \mathbf{F}_1$

$$\begin{aligned}\overline{\chi_2(\lambda+\mu)}\pi(\delta_{\lambda+\mu})\varepsilon_{-(\lambda+\mu)}f &= \overline{e^{-i\pi\Omega(\lambda,\mu)}\chi_2(\lambda)\chi_2(\mu)} \\ &\quad \times e^{-i\pi\Omega(\lambda,\mu)}\pi(\delta_\lambda)\pi(\delta_\mu)\varepsilon_{-\lambda}\varepsilon_{-\mu}f \\ &= \overline{\chi_2(\lambda)\chi_2(\mu)}\pi(\delta_\lambda)\pi(\delta_\mu)\varepsilon_{-\lambda}\varepsilon_{-\mu}f \\ &= \overline{\chi_2(\lambda)}\pi(\delta_\lambda)\varepsilon_{-\lambda}f\end{aligned}$$

since $\chi_2(\mu) = \chi_1(\mu)$ and $\pi(\delta_\mu)f = \chi_1(\mu)\varepsilon_\mu f$. Next, U maps $\mathbf{F}_1$ to $\mathbf{F}_2$: if $\lambda_0 \in \Lambda_2$ is fixed and if $\lambda \in \Lambda_2$ then

$$\begin{aligned}\overline{\chi_2(\lambda_0)}\pi(\delta_{\lambda_0})\varepsilon_{-\lambda_0}\overline{\chi_2(\lambda)}\pi(\delta_\lambda)\varepsilon_{-\lambda} &= \overline{\chi_2(\lambda_0)\chi_2(\lambda)}\pi(\delta_{\lambda_0})\pi(\delta_\lambda)\varepsilon_{-\lambda_0}\varepsilon_{-\lambda} \\ &= \overline{e^{i\pi\Omega(\lambda_0,\lambda)}\chi_2(\lambda_0+\lambda)}e^{i\pi\Omega(\lambda_0,\lambda)}\pi(\delta_{\lambda_0+\lambda}) \\ &\quad \times \varepsilon_{-(\lambda_0+\lambda)} \\ &= \overline{\chi_2(\lambda_0+\lambda)}\pi(\delta_{\lambda_0+\lambda})\varepsilon_{-(\lambda_0+\lambda)};\end{aligned}$$

since $\lambda_0 + \lambda$ runs over a full set of distinct coset representatives as λ does, it follows by summation that if $f \in \mathbf{F}_1$ then

$$\overline{\chi_2(\lambda_0)}\pi(\delta_{\lambda_0})\varepsilon_{-\lambda_0}(Uf) = Uf.$$

Lastly, U intertwines: if $v \in V$ and $\lambda \in \Lambda_2$ then passing $\pi(\delta_v)$ to the left of $\varepsilon_{-\lambda}$ picks up a factor $e^{-2\pi i\Omega(\lambda,v)}$ which is dropped on passing $\pi(\delta_v)$ to the left of $\pi(\delta_\lambda)$. Denoting by N the common index $[\Lambda_1 : \Lambda_1 \cap \Lambda_2] = [\Lambda_2 : \Lambda_1 \cap \Lambda_2]$ we claim that

$$(1/\sqrt{N})U_{21} : \mathbf{F}_1 \longrightarrow \mathbf{F}_2$$

is a unitary isomorphism. To justify this claim, we enlist the intertwiner $U_{12} : \mathbf{F}_2 \longrightarrow \mathbf{F}_1$ given analogously by

$$U_{12} = \sum_{\mu\in\underline{\Lambda_1}} \overline{\chi_1(\mu)}\pi(\delta_\mu)\varepsilon_{-\mu}.$$

Note that

$$(U_{21}s_0)(0) = \sum_{\lambda\in\underline{\Lambda_2}} \overline{\chi_2(\lambda)}s_0(-\lambda)$$

in which sum only the term with $\lambda \in \Lambda_1 \cap \Lambda_2$ will contribute; taking this λ to be 0 without loss, it follows that $(U_{21}s_0)(0) = 1$. In exactly the same way, $(U_{12}t_0)(0) = 1$ so that

$$(U_{12}^* s_0)(0) = \langle t_0|U_{12}^* s_0\rangle = \overline{\langle s_0|U_{12}t_0\rangle} = 1.$$

As the irreducibility of π forces U_{12}^* and U_{21} to be proportional, it now follows that they are equal: $U_{12}^* = U_{21}$. In each term of the sum

$$U_{12}U_{21} = \sum_{\mu \in \underline{\Lambda_1}} \sum_{\lambda \in \underline{\Lambda_2}} \overline{\chi_1(\mu)}\pi(\delta_\mu)\varepsilon_{-\mu}\overline{\chi_2(\lambda)}\pi(\delta_\lambda)\varepsilon_{-\lambda}$$

pass $\overline{\chi_1(\mu)}\pi(\delta_\mu)\varepsilon_{-\mu}$ to the right of $\overline{\chi_2(\lambda)}\pi(\delta_\lambda)\varepsilon_{-\lambda}$: passing $\varepsilon_{-\mu}$ beyond $\pi(\delta_\lambda)$ picks up a factor $e^{-2\pi i\Omega(\mu,\lambda)}$ while $\pi(\delta_\mu)$ commutes with $\pi(\delta_\lambda)\varepsilon_{-\lambda}$; moreover, $\overline{\chi_1(\mu)}\pi(\delta_\mu)\varepsilon_{-\mu}$ acts on $\mathbf{F}_1$ as the identity. Thus

$$U_{12}U_{21} = \sum_{\mu \in \underline{\Lambda_1}} \sum_{\lambda \in \underline{\Lambda_2}} e^{-2\pi i\Omega(\mu,\lambda)}\overline{\chi_2(\lambda)}\pi(\delta_\lambda)\varepsilon_{-\lambda}$$

and so

$$\begin{aligned}(U_{12}U_{21}s_0)(0) &= \sum_{\mu \in \underline{\Lambda_1}} \sum_{\lambda \in \underline{\Lambda_2}} e^{-2\pi i\Omega(\mu,\lambda)}\overline{\chi_2(\lambda)}s_0(-\lambda) \\ &= \sum_{\mu \in \underline{\Lambda_1}} e^{-2\pi i\Omega(\mu,0)}\overline{\chi_2(0)}s_0(0) \\ &= \sum_{\mu \in \underline{\Lambda_1}} 1 \\ &= N\end{aligned}$$

since again only the terms with $\lambda \in \Lambda_1 \cap \Lambda_2$ contribute. As the irreducibility of π forces $U_{12}U_{21}$ to be a scalar operator, it now follows that $U_{12}U_{21} = NI$; of course, $U_{21}U_{12} = NI$ likewise. Finally, the mutually adjoint operators $(1/\sqrt{N})U_{21}$ and $(1/\sqrt{N})U_{12}$ are unitary. $\square$

Once again, the proof yields more: an explicit unitary intertwining operator from F_1 (or its completion $\mathbf{F}_1$) to F_2 (or its completion $\mathbf{F}_2$) is given by

$$(1/\sqrt{|\underline{\Lambda_2}|}) \sum_{\lambda \in \underline{\Lambda_2}} \overline{\chi_2(\lambda)}\pi(\delta_\lambda)\varepsilon_{-\lambda}\pi(\delta_u)\varepsilon_{-u}.$$

There is an alternative, instructive and illuminating way to view intertwining operators between the GNS representations associated to polarized states. To begin, let σ be a polarized state of $A(V,\Omega)$. Let $\Lambda \subset V$ be an Ω-integral lattice contained in Λ_σ with $[\Lambda_\sigma : \Lambda]$ finite and let $\chi = \chi_\sigma|\Lambda$. Denote by τ the state on $A(V,\Omega)$ defined by prescribing that $\tau(\delta_v) = \chi(v)$ when $v \in \Lambda$ and $\tau(\delta_v) = 0$ otherwise; of course, $\Lambda_\tau = \Lambda$ and $\chi_\tau = \chi$. Of course, F_σ is contained in F_τ but the inclusion $Q_{\tau\sigma} : F_\sigma \longrightarrow F_\tau$ is not isometric: indeed, it may be verified that if $s \in F_\sigma$ then $\|Q_{\tau\sigma}s\| = \sqrt{[\Lambda_\sigma : \Lambda]}\|s\|$. It may also be verified that the formula

$$P_{\sigma\tau} = (1/[\Lambda_\sigma : \Lambda]) \sum_{\lambda \in \Lambda_\sigma/\Lambda} \overline{\chi_\sigma(\lambda)}\pi(\delta_\lambda)\varepsilon_{-\lambda}$$

well-defines the selfadjoint projection operator from F_τ to F_σ; of course, this extends to the selfadjoint projection operator $P_{\sigma\tau} : \mathbf{F}_\tau \longrightarrow \mathbf{F}_\sigma$. In fact, a routine calculation shows that $Q_{\tau\sigma} = \sqrt{[\Lambda_\sigma : \Lambda]}P^*_{\sigma\tau}$. With these preparations, we may now offer the following reformulation.

Theorem 11. *Let σ_1 and σ_2 be polarized states of $A(V,\Omega)$. Assume that $\Lambda = \Lambda_{\sigma_1} \cap \Lambda_{\sigma_2}$ has finite index in Λ_{σ_1} and/or Λ_{σ_2} and that $\chi = \chi_{\sigma_1}|\Lambda = \chi_{\sigma_2}|\Lambda$. If τ denotes the state on $A(V,\Omega)$ given by $\tau(\delta_v) = \chi(v)$ when $v \in \Lambda$ and $\tau(\delta_v) = 0$ otherwise, then a unitary operator $U_{21} : F_{\sigma_1} \longrightarrow F_{\sigma_2}$ intertwining π_{σ_1} with π_{σ_2} is defined by the formula*

$$U_{21} = \sqrt{[\Lambda_\sigma : \Lambda]}P_{\sigma_2\tau} \circ Q_{\tau\sigma_1}.$$

Proof. It only remains to note that the indicated formula actually reproduces the operator appearing in the proof of the previous theorem. ☐

Thus, we have a completely satisfactory solution to the equivalence problem; we now turn to a consideration of the implementation problem.

Recall that the symplectic group $Sp(V,\Omega)$ comprises all linear automorphisms g of V such that $\Omega(gx,gy) = \Omega(x,y)$ whenever $x,y \in V$. When $g \in Sp(V,\Omega)$ the formula

$$\theta_g : A(V,\Omega) \longrightarrow A(V,\Omega) : a \longmapsto a \circ g^{-1}$$

defines a (so-called Bogoliubov) automorphism of $A(V,\Omega)$ which extends continuously to an automorphism of $A[V,\Omega]$. Recall also that $g \in Sp(V,\Omega)$ is said to be unitarily implemented in the GNS representation π_σ associated to the state σ of $A(V,\Omega)$ in case there exists a unitary operator U on $\mathbf{H}_\sigma$ such that

$$a \in A(V,\Omega) \Longrightarrow U\pi_\sigma(a) = \pi_\sigma(\theta_g a)U$$

or simply

$$v \in V \Longrightarrow U\pi_\sigma(\delta_v) = \pi_\sigma(\delta_{gv})U.$$

Theorem 12. *Let σ be a polarized state of $A(V,\Omega)$. The symplectic automorphism $g \in Sp(V,\Omega)$ is unitarily implemented in the GNS representation π_σ associated to σ precisely when $\Lambda_\sigma \equiv g^{-1}\Lambda_\sigma$ and $\chi_\sigma \circ g|(\Lambda_\sigma \cap g^{-1}\Lambda_\sigma) \equiv \chi_\sigma|(\Lambda_\sigma \cap g^{-1}\Lambda_\sigma)$.*

Proof. This may be either established directly or deduced from our solution to the equivalence problem as follows. It is readily verified that the rule

$$U_g : \mathbf{F}_{\sigma\circ\theta_g} \longrightarrow \mathbf{F}_\sigma : f \longmapsto f \circ g^{-1}$$

determines a unitary operator intertwining $\pi_{\sigma\circ\theta_g}$ with $\pi_\sigma \circ \theta_g$. Thus, g is unitarily implemented in π_σ precisely when π_σ and $\pi_{\sigma\circ\theta_g}$ are unitarily equivalent, which is the case precisely when $\Lambda_\sigma \equiv g^{-1}\Lambda_\sigma$ and $\chi_\sigma \circ g \equiv \chi_\sigma$ on $\Lambda_\sigma \cap g^{-1}\Lambda_\sigma$. □

As shown by a closer examination, if g is unitarily implemented in π_σ then a specific implementer U may be fashioned for it as follows: choose $u \in V$ such that $\chi_\sigma \circ g = {\chi_\sigma}^u$ on $\Lambda_\sigma \cap g^{-1}\Lambda_\sigma$, write $\sigma_1 = \sigma^u$ and write $\sigma_2 = \sigma \circ \theta_g$ so that $\chi_1 = \chi_2$ on $\Lambda_1 \cap \Lambda_2$; then let

$$U = U_g \circ U_{21} \circ U_u$$

where the isomorphisms

$$\begin{aligned} U_u &: \mathbf{F}_\sigma \longrightarrow \mathbf{F}_{\sigma_1} \\ U_{21} &: \mathbf{F}_{\sigma_1} \longrightarrow \mathbf{F}_{\sigma_2} \\ U_g &: \mathbf{F}_{\sigma_2} \longrightarrow \mathbf{F}_\sigma \end{aligned}$$

have the meanings assigned to them previously. Of course, a unitary intertwiner is obtained upon dividing U by $\sqrt{[\Lambda_\sigma : \Lambda_\sigma \cap g^{-1}\Lambda_\sigma]}$.

Here is perhaps a convenient place at which to discuss the unitary implementation of more general automorphisms of the Weyl algebra. The automorphisms of $A(V, \Omega)$ as an involutive algebra were determined precisely in [6]: to each $\Theta \in AutA(V, \Omega)$ there correspond an additive automorphism G of V preserving Ω and a character $\Phi \in \hom(V, \mathbf{T})$ such that

$$v \in V \Longrightarrow \Theta(\delta_v) = \Phi(v)\delta_{Gv}$$

and each such pair (G, Φ) gives rise to an automorphism Θ of $A(V, \Omega)$ in this way. Instrumental in this determination was the observation that the nonzero scalar multiples of the standard basis vectors $\{\delta_v : v \in V\}$ are precisely the units of $A(V, \Omega)$ and each automorphism of $A(V, \Omega)$ leaves these units collectively invariant.

Theorem 13. *Let σ be a polarized state on $A(V, \Omega)$ and let Φ be a character on V. The automorphism Θ of $A(V, \Omega)$ defined by*

$$v \in V \Longrightarrow \Theta(\delta_v) = \Phi(v)\delta_v$$

is unitarily implemented in the GNS representation π_σ *if and only if there exists* $u \in V$ *such that* $\Phi|\Lambda_\sigma = e^{2\pi i \Omega(u,\cdot)}$.

Proof. Let $U : \mathbf{F}_\sigma \longrightarrow \mathbf{F}_\sigma$ implement Θ; write $f = Us_0$ and choose $u \in V$ at which f is nonzero. If $\lambda \in \Lambda_\sigma$ then

$$\chi_\sigma(\lambda) f = U\pi_\sigma(\delta_\lambda)s_0 = \Phi(\lambda)\pi_\sigma(\delta_\lambda)Us_0 = \Phi(\lambda)\chi_\sigma(\lambda)\varepsilon_\lambda f$$

whence evaluation at u and cancellation of $f(u)$ yield

$$\Phi(\lambda) = e^{2\pi i \Omega(u,\lambda)}.$$

In the opposite direction, let $\Phi|\Lambda_\sigma = e^{2\pi i \Omega(u,\cdot)}$ for some $u \in V$. Let $U : F_\sigma \longrightarrow F_\sigma$ be defined by linear extension of the rule

$$v \in V \Longrightarrow Us_v = \Phi(v)e^{i\pi\Omega(v,u)}s_{v+u}.$$

This rule implies (and is, up to a uniform scalar multiple, forced by) the property that U intertwine π_σ and $\pi_\sigma \circ \theta_g$. Direct calculation reveals that if $x, y \in V$ then

$$\langle Us_x|Us_y\rangle = \overline{\Phi(x)e^{-2\pi i\Omega(u,x)}}\Phi(y)e^{-2\pi i \Omega(u,y)}\langle s_x|s_y\rangle$$

whence it follows that U is unitary in light of the assumption $\Phi|\Lambda_\sigma = e^{2\pi i \Omega(u,\cdot)}$. $\square$

Upon reconsideration, our solution to the unitary implementation problem for symplectic automorphisms of (V, Ω) does not require them to be linear. Thus, the additive symplectic automorphism G of (V, Ω) is unitarily implemented in the GNS representation π_σ associated to the polarized state σ of $A(V, \Omega)$ exactly when $[\Lambda_\sigma : \Lambda_\sigma \cap G^{-1}\Lambda_\sigma] < \infty$ and $(\chi_\sigma \circ G/\chi_\sigma)|(\Lambda_\sigma \cap G^{-1}\Lambda_\sigma) = e^{2\pi i\Omega(u,\cdot)}$ for some $u \in V$.

§4. Closing Remarks

For the purpose of discussing illustrative special cases, let us begin by supposing that the underlying symplectic vector space (V, Ω) is finite-dimensional.

For our first special case, let $\Lambda \subset V$ be a lattice (a discrete cocompact subgroup) on which Ω takes integer values; for the present, do not assume Λ to be maximal. In this case, we claim that the canonical embedding $V/\Lambda' \longrightarrow \Lambda^*$ is an isomorphism. To justify this claim, given $\phi \in \Lambda^*$ we must produce a vector $u \in V$ such that $\phi = e^{2\pi i \Omega(u,\cdot)}$. For this, let $(\lambda_1, \dots, \lambda_{2m})$ be a basis for Λ and therefore for V. When $1 \leq k \leq 2m$ choose $r_k \in \mathbf{R}$ so that $\phi(\lambda_k) = e^{2\pi i r_k}$ and then the linear functional $V \longrightarrow \mathbf{R} : c_1\lambda_1 + \cdots + c_{2m}\lambda_{2m} \longmapsto c_1 r_1 + \cdots + c_{2m} r_{2m}$ is $\Omega(u, \cdot)$ for a unique $u \in V$. Thus, if $n_1, \ldots, n_{2m} \in \mathbf{Z}$ then

$$\begin{aligned}\phi\left(\sum_{k=1}^{2m} n_k \lambda_k\right) &= \prod_{k=1}^{2m} \phi(\lambda_k)^{n_k} \\ &= \prod_{k=1}^{2m} e^{2\pi i n_k r_k} \\ &= e^{2\pi i \sum_{k=1}^{2m} n_k r_k} \\ &= e^{2\pi i \Omega\left(u, \sum_{k=1}^{2m} n_k \lambda_k\right)}\end{aligned}$$

and so $\phi = e^{2\pi i \Omega(u, \cdot)}$ as required. As a consequence, if χ_1 and χ_2 are quasicharacters in Ξ_Λ then $\chi_2 = {\chi_1}^u$ for a suitable $u \in V$ that is unique modulo Λ'.

Now, by a lattice polarized state of $A(V, \Omega)$ we shall mean a polarized state σ such that Λ_σ is an Ω-integral lattice as above; in [1] these are referred to as Zak waves.

Theorem 14. *Let (V, Ω) be finite-dimensional; let σ_1 and σ_2 be lattice polarized states of $A(V, \Omega)$. The GNS representations π_{σ_1} and π_{σ_2} are unitarily equivalent if and only if Λ_{σ_1} and Λ_{σ_2} are commensurable.*

Proof. We know already that π_{σ_1} and π_{σ_2} are unitarily equivalent precisely when $\Lambda_{\sigma_1} \equiv \Lambda_{\sigma_2}$ and $\chi_{\sigma_1}|(\Lambda_{\sigma_1} \cap \Lambda_{\sigma_2}) \equiv \chi_{\sigma_2}|(\Lambda_{\sigma_1} \cap \Lambda_{\sigma_2})$: see Theorems 9 and 10. All we need add is that in the present finite-dimensional context, all quasicharacters on the intersection lattice $\Lambda = \Lambda_{\sigma_1} \cap \Lambda_{\sigma_2}$ are automatically equivalent, as noted prior to the theorem. $\square$

Thus, the solution to the unitary equivalence problem simplifies for lattice polarized states in finite dimensions. Of course, the same is true of the unitary implementation problem: explicitly, let (V, Ω) be finite-dimensional and let σ be a lattice polarized state of $A(V, \Omega)$; the symplectic automorphism $g \in Sp(V, \Omega)$ is unitarily implemented in the GNS representation π_σ if and only if the lattices $g^{-1}\Lambda_\sigma$ and Λ_σ are commensurable. We remark that this condition may be reformulated: when (V, Ω) is finite-dimensional, $g^{-1}\Lambda_\sigma$ is commensurable with Λ_σ if and only if g has rational matrix relative to a symplectic basis in Λ_σ.

For our second special case, let Λ be a Lagrangian (maximal isotropic subspace) in (V, Ω). In this case, a quasicharacter $\chi : \Lambda \longrightarrow \mathbf{T}$ is exactly a homomorphism: $\Xi_\Lambda = \Lambda^* = \hom(\Lambda, \mathbf{T})$. Notice that if χ is continuous and $f : \Lambda \longrightarrow \mathbf{R}$ is a continuous lift (so that $\chi = e^{2\pi i f}$) then stipulating that $f(0) = 0$ forces f to be linear; moreover, $f = \Omega(u, \cdot)$ for some $u \in V$. To summarize, if χ is continuous then $\chi = e^{2\pi i \Omega(u,\cdot)}$ for some $u \in V$ that is unique modulo Λ.

Now, by a Lagrangian polarized state of $A(V, \Omega)$ we shall mean a polarized state σ such that Λ_σ is a Lagrangian and χ_σ is continuous; in [1] these are referred to as plane waves.

Theorem 15. *Let (V, Ω) be finite-dimensional; let σ_1 and σ_2 be Lagrangian polarized states of $A(V, \Omega)$. The GNS representations π_{σ_1} and π_{σ_2} are unitarily equivalent if and only if Λ_{σ_1} and Λ_{σ_2} are equal.*

Proof. Again, we need only add a little to Theorems 9 and 10: namely, that the subspaces Λ_{σ_1} and Λ_{σ_2} are commensurable precisely when they are equal and that the condition $\chi_{\sigma_1} \equiv \chi_{\sigma_2}$ is automatic in light of the observation preceding the theorem. $\square$

Regarding unitary implementation, this has the following consequence: let (V, Ω) be finite-dimensional and let σ be a Lagrangian polarized state of $A(V, \Omega)$; the symplectic automorphism $g \in Sp(V, \Omega)$ is unitarily implemented in the GNS representation π_σ if and only if g preserves the Lagrangian Λ_σ.

Returning now to the general case in which (V, Ω) has arbitrary dimension, there are a number of directions in which our work can be extended; we plan to pursue some of these in future publications. For example, let σ be a polarized state of $A(V, \Omega)$. Certain natural questions arise regarding the group $Sp(V, \Omega)_\sigma$ comprising all symplectic automorphisms of (V, Ω) that are unitarily implemented in π_σ. Thus, the group of all unitary operators on $\mathbf{H}_\sigma$ implementing elements of $Sp(V, \Omega)_\sigma$ is a central extension of $Sp(V, \Omega)_\sigma$ by the unit circle $\mathbf{T}$ on account of irreducibility. We may demand an explicit rendering of the cocycle for this extension of $Sp(V, \Omega)_\sigma$; in particular, we may ask for special subgroups of $Sp(V, \Omega)_\sigma$ over which the extension splits or is a product. Of course, if $g \in Sp(V, \Omega)$ is such that the Bogoliubov automorphism θ_g preserves σ then g is unitarily implemented in π_σ by a canonical unitary operator (fixing s_0) as the GNS construction guarantees: explicitly, θ_g on $A(V, \Omega)$ descends to a unitary operator on H_σ whose extension U_g to $\mathbf{H}_\sigma$ implements g. To take another example, we may consider rather more generally the structure of suitable classes of states σ on $A(V, \Omega)$ for which Λ_σ is not maximal.

In fact, such states σ for which Λ_σ is a (necessarily isotropic) subspace and $\chi_\sigma \equiv 1$ have already been studied in [3] under the name of Dirac states: each Dirac state σ corresponds to a state $\check{\sigma}$ on the Weyl algebra $A({\Lambda_\sigma}^\perp/\Lambda_\sigma)$ where $^\perp$ signifies symplectic polarity so that ${\Lambda_\sigma}^\perp/\Lambda_\sigma$ is the symplectic quotient (or normal); explicitly, the assumption $\chi_\sigma \equiv 1$ guarantees that we may well-define $\check{\sigma}(\delta_v + \Lambda_\sigma) = \sigma(\delta_v)$ for each $v \in {\Lambda_\sigma}^\perp$. A similar study should be made in general: an arbitrary state σ extends from its restriction to $A(\Lambda_\sigma)$ by data on the quotient $\Lambda'_\sigma/\Lambda_\sigma$; the precise relationships merit further scrutiny.

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