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A New Technique in Nonlinear Singularity Analysis

By

Pilar R. Gordoa*, Nalini Joshi** and Andrew Pickering***

Abstract

The test for singularity structure proposed by Ablowitz, Ramani and Segur (1980) has been widely used to obtain necessary conditions for non-linear ordinary differential equations (ODEs) to have the Painlevé property. In this paper, we provide an example of an ODE for which this standard test breaks down in a new and surprising way. We show how to overcome the problems presented by this equation by developing a new technique based on regular singular points of bilinear equations.

§1. Introduction

The connection between completely integrable partial differential equations (PDEs), i.e., those solvable through inverse scattering or spectral theory, and ordinary differential equations (ODEs) having the Painlevé property was first noted by Ablowitz and Segur [1].

Definition 1.1. Consider an ordinary differential equation of the form

$$y^{(n)} = \mathcal{F}\big(y^{(n-1)}, \dots, y', y, x\big), \qquad y = y(x), \quad x \in \mathbb{C}, \tag{1}$$

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*Area de Fisica Teórica, Facultad de Ciencias, Universidad de Salamanca, 37008 Salamanca, Spain.
e-mail: prg@sonia.usal.es
**School of Mathematics and Statistics, University of Sydney, NSW2006 Sydney, Australia.
e-mail: Nalini.Joshi@maths.usyd.edu.au
***Departamento de Matemáticas, Universidad de Salamanca, Plaza de la Merced 1, 37008 Salamanca, Spain.
e-mail: andrew@gugu.usal.es

where $\mathcal{F}$ is rational in the first n arguments and analytic in x, except for a set $\mathcal{S}$ of isolated points $x_i \in \mathbb{C}$, $i = 1, \ldots, k$, for integer k.

(1) If $y(x)$ solves Equation (1) and is a function of n arbitrary parameters independent of each other, of any parameters in Equation (1), and of x, then it is a *general solution.*

(2) If any of the points $x_1, \ldots, x_k$ is a singularity of a solution $y(x)$, it is called a *fixed singularity* of $y(x)$.

(3) If $x_0 \notin \mathcal{S}$ is a singularity of a solution $y(x)$, it is called a *movable singularity* of $y(x)$.

(4) If the general solution of Equation (1) is single-valued except in a neighbourhood of a fixed singularity, then the equation is said to have the *Painlevé property.*

(5) If all movable singularities of any of its solutions are poles, then Equation (1) is said to be of *Painlevé-type.*

Later, together with Ramani, they formulated the famous Ablowitz-Ramani-Segur (ARS) conjecture [2], [3].

Conjecture 1.2. *Every ODE obtained as a similarity reduction of a completely integrable PDE is of Painlevé-type, perhaps after a transformation of variables.*

Along with the conjecture, ARS also proposed a test for complete integrability, known as the ARS Painlevé test: find all ODE reductions of a given PDE and determine whether or not they are of Painlevé-type. Painlevé classified ODEs having his eponymous property [4], [5]. As is well known, the work of Painlevé's school led to the discovery of six new transcendental functions, known today as the Painlevé transcendents [4], [5], [6] (see also [7]).

ARS reignited modern interest in the work of classical authors, including Kowalevski, on the question of how to test an ODE for the Painlevé property. Kowalevski had, in her investigations into the motion of a rigid body rotating about a fixed point [8], [9], provided one answer to this question. Whilst other approaches were developed by Painlevé [4] and Bureau [10], it was the technique developed by Kowalevski that ARS fomulated into an algorithm, now called the ARS algorithm.

The algorithm is based on expanding a solution formally in the neighbourhood of a movable singularity x_0:

$$y(x) = \sum_{j=0}^{\infty} a_j (x - x_0)^{j+\alpha}, \tag{2}$$

where $\alpha \in \mathbb{C}$ is determined by the ODE. One necessary condition for the Painlevé property is that, in any formal solution (2) which represents the general solution, α be integer. Moreover, this should be the case for any general solution $y(x)$. To ensure that (2) represents the general solution, it is sufficient that it contains n arbitrary parameters as coefficients. In practice, the latter is checked by searching for *resonances*, i.e., indices j where the corresponding coefficients a_j are arbitrary parameters [3]. Resonances are usually revealed by a study of perturbations of such series. This algorithm remains the most commonly used approach to testing for the Painlevé property and forms the focus of this paper.

An active current area of research concerns the search for new higher transcendental functions that arise as solutions of higher order ODEs with the Painlevé property. One approach has been to consider reductions of well known completely integrable hierarchies of PDEs. A more general approach, based on the connection noted in [12] between non-isospectral scattering problems for PDEs and monodromy problems for ODEs, has been developed in [13]–[16]. In this approach we obtain hierarchies of ODEs that are more general than those that would be obtained by straightforward similarity reduction, together with their underlying linear problems; examples include new hierarchies based on the second and fourth Painlevé equations.

It turns out that one of the ODEs obtained in [16] presents a surprising and previously unseen difficulty for the ARS algorithm. To our knowledge, it provides the first example for which resonances fail to be defined. In particular, as we noted in [16], this means that any standard searches for integrable equations, based on the ARS algorithm, would fail to identify this equation and possibly many other classes of ODEs for which a similar failure occurs. To find higher order ODEs with the Painlevé property, it is of vital importance that techniques are available with which to overcome this failure.

The layout of the paper is as follows. In Section 2 we present our example and explain the difficulty that it poses for the ARS algorithm. In Section 3 we explain how this difficulty can be overcome, and how a set of resonances can in fact be determined for this equation. This is done using a new approach based on a study of the regular singular points of bilinear equations, and linearizations thereof. We obtain the new and at first surprising result that, for some leading order behaviours, it is possible to have more than one corresponding set of resonances. Section 4 is devoted to a summary and conclusions.

§2. An Important and Surprising Example

In a recent paper [16] we derived, using the approach developed in [13], [14], [15], the sequence of coupled ODEs

$$\mathcal{R}^n \mathbf{u}_x + \sum_{i=0}^{n-2} c_i \mathcal{R}^i \mathbf{u}_x + g_{n-1}\mathcal{R}^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + g_n \mathcal{R} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + g_{n+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3}$$

where $\mathbf{u} = (u, v)$, $\mathcal{R}$ is the recursion operator of the dispersive water wave (DWW) hierarchy [17]–[18] ($\partial_x = \partial/\partial x = d/dx$ in our ODE case (3)),

$$\mathcal{R} = \frac{1}{2}\begin{pmatrix} \partial_x u \partial_x^{-1} - \partial_x & 2 \\ 2v + v_x \partial_x^{-1} & u + \partial_x \end{pmatrix}, \tag{4}$$

and where (using a shift on u) we have taken $c_{n-1} = 0$. This hierarchy represents a generalized $P_{IV} - P_{II}$ hierarchy; special cases yield both a hierarchy based on the fourth Painlevé equation (P_{IV}), and a hierarchy based on the second Painlevé equation (P_{II}). In [16] we also presented, following the approach in [12], a corresponding hierarchy of underlying linear problems for our ODE hierarchy.

Here we consider the special case $g_{n-1} = 0$ and $g_n \neq 0$, which yields our P_{IV} hierarchy. A shift on x allows us to set $c_0 = 0$ and, in the case $n = 2$, we obtain the second member of our P_{IV} hierarchy (here $' = d/dx$),

$$u'' = 3uu' - u^3 - 6uv - 2g_2xu - 4g_3x + 4\alpha_2 - 8(g_3/g_2)^3, \tag{5}$$

$$\begin{aligned} v'' = {} & 2\left(\frac{\left[uv + \frac{1}{2}v' - (g_3/g_2)v - \alpha_2 + \frac{1}{2}g_2\right]^2 - \frac{1}{4}\beta_2^2}{v + \frac{1}{2}u^2 - \frac{1}{2}u' + g_2x - (g_3/g_2)u + 2(g_3/g_2)^2}\right) \\ & - 2v\left(v + \frac{1}{2}u^2 - \frac{1}{2}u' + g_2x - (g_3/g_2)u + 2(g_3/g_2)^2\right) \\ & - 2(uv)' + 2(g_3/g_2)v', \end{aligned} \tag{6}$$

where α_2 and β_2 are two independent constants of integration. It is our system (5), (6) that poses problems for the ARS algorithm, as we will now describe.

Equation (5) can be solved for v; substitution in (6) then yields a fourth order scalar ODE for u. This ODE admits the leading order behaviour $u \sim$

$u_0(x-x_0)^{-1}$ in the neighbourhood of a movable singular point x_0, with dominant terms

$$(7)\quad K[u] \equiv 6u^2(u''-2u^3)u'''' - 3u^2(u''')^2 - 6uu'(u''-8u^3)u''' - 8u(u'')^3 + 9((u')^2-u^4)(u'')^2 - 12u^3(6(u')^2-5u^4)u'' - 20u^{10}.$$

It is in fact the equation $K[u]=0$ that we will be dealing with in this paper, since the problem of defining resonances is associated with the dominant terms. In particular, we will explore the leading order behaviour $u \sim u_0(x-x_0)^{-1}$ of $K[u]=0$.

The leading order (and so nonzero) coefficient u_0 is obtained from

$$(8)\qquad K[u_0\xi^{-1}] = -20\xi^{-10}u_0^4(u_0^2-4)(u_0^2-1)^2 = 0,$$

where $\xi = x-x_0$, which then yields $u_0 = \pm 2$ and $u_0 = \pm 1$ (each of these last being a double root). We then have to determine the resonances corresponding to each of the leading order behaviours $u \sim \xi^{-1}$ and $u \sim 2\xi^{-1}$ (the invariance of the dominant terms (7) under $u \to -u$ means that we need only consider $u_0 > 0$).

According to the ARS algorithm, the resonances are determined by substituting $u = u_0\xi^{-1} + u_r\xi^{r-1}$ into the dominant terms, and isolating the coefficient of the term linear in u_r. The zeroes of the resulting polynomial in r are the resonances; these give the locations in the corresponding series solution at which we expect arbitrary coefficients to enter. Equivalently, we can determine the resonance polynomial as

$$(9)\qquad P(r;u_0) = \xi^{10-r}K'[u_0\xi^{-1}]\xi^{r-1}$$

where $K'[u]$ is the Fréchet derivative of $K[u]$; in our case

$$(10)\quad K'[u] = 6u^2[u''-2u^3]\frac{d^4}{dx^4} + 6u[8u^3u' - u'u'' - uu''']\frac{d^3}{dx^3} + 6[10u^7 - 12u^3(u')^2 - 3u^4u'' + 3(u')^2u'' - 4u(u'')^2 - uu'u''' + u^2u'''']\frac{d^2}{dx^2} - 6[24u^3u'u'' - 3u'(u'')^2 - 8u^4u''' + uu''u''']\frac{d}{dx} - 2[100u^9 - 210u^6u'' + 108u^2(u')^2u'' + 18u^3(u'')^2 + 4(u'')^3 - 96u^3u'u''' + 3u'u''u''' + 3u(u''')^2 + 30u^4u'''' - 6uu''u''''].$$

In this way, for example, we obtain

$$(11)\qquad P(r;2) = \xi^{10-r}K'[2\xi^{-1}]\xi^{r-1} = -288(r+2)(r+1)(r-4)(r-5),$$

which tells us that for the leading order behaviour $u \sim 2\xi^{-1}$ we have resonances $r = -2, -1, 4, 5$. Thus this leading order behaviour has negative resonances (other than $r = -1$ once), and can be dealt with using the perturbative Painlevé test [19].

However, for the leading order behaviour $u \sim \xi^{-1}$, we find that all coefficients of the operator (10) vanish, i.e.

$$K'[\xi^{-1}] \equiv 0, \tag{12}$$

and so we are unable to determine the resonances corresponding to this leading order behaviour. This represents a fundamental problem for the ARS algorithm. It also represents a fundamental problem for Painlevé classification based on a requirement that all resonances be integer; such ODEs will have to be included, but are apparently untestable. Our ODE is the first example that has been found to exhibit this kind of behaviour, although clearly it will not be the only such example.

§3. A New Technique in Singularity Analysis

We now show how "resonances" corresponding to the leading order behaviour $u \sim \xi^{-1}$ of (7) can be determined. We find in fact that there are two sets of resonances corresponding to this leading order behaviour.

Our first comment however is with regard to the occurrence of $u_0 = 1$ (and similarly $u_0 = -1$) as a double root of the equation that determines the leading order coefficient. Normally this would imply that $r = 0$ is a resonance and would automatically lead to branching [3], [20]. However for our example (7) the demonstration that $r = 0$ must be a resonance breaks down precisely because $K'[\xi^{-1}] \equiv 0$.

Clearly the barrier to being able to determine resonances is our reliance on using the linearization of the dominant terms (this reliance is implicit in the ARS algorithm, and is made explicit by equation (9)). This linearization corresponds of course to a perturbation about the exact solution $u = \xi^{-1}$ of $K[u] = 0$,

$$K[\xi^{-1} + \varepsilon U] = K[\xi^{-1}] + \varepsilon K'[\xi^{-1}]U + \cdots, \tag{13}$$

and so in this case where $K'[\xi^{-1}] \equiv 0$ it seems natural to ask what information is given at higher orders of perturbation. That is, we make the perturbation

$$u = \xi^{-1} + \varepsilon U + \varepsilon^2 V + \varepsilon^3 W + \varepsilon^4 X + \cdots, \tag{14}$$

which yields a sequence of equations (with no contributions at orders ε^0 or ε):

(15) at order ε^2 : $F[\xi, U] = 0,$

(16) at order ε^3 : $G[\xi, U, V] \equiv F'[\xi, U]V + \tilde{G}[\xi, U] = 0,$

(17) at order ε^4 : $H[\xi, U, V, W] \equiv F'[\xi, U]W + \tilde{H}[\xi, U, V] = 0,$

and similarly at higher orders of ε, and where we use $F'[\xi, U]$ to denote the Fréchet derivative of $F[\xi, U]$ with respect to U (the linearization of $F[\xi, U]$ about U).

Here equation (15),

$$\begin{aligned}(18)\quad F[\xi, U] = {} & 6\frac{U''U''''}{\xi^2} - 36\frac{UU''''}{\xi^4} - 3\frac{(U''')^2}{\xi^2} + 6\frac{U''U'''}{\xi^3} + 36\frac{U'U'''}{\xi^4} \\ & - 108\frac{UU'''}{\xi^5} - 48\frac{(U'')^2}{\xi^4} + 108\frac{U'U''}{\xi^5} + 216\frac{UU''}{\xi^6} - 108\frac{(U')^2}{\xi^6} \\ & - 216\frac{UU'}{\xi^7} = 0,\end{aligned}$$

is (necessarily) a bilinear equation in U, and can be obtained as

(19)

$$F[\xi, U] = \frac{1}{2}\left(\frac{d^2}{d\varepsilon^2}K[\xi^{-1} + \varepsilon U]\right)\Bigg|_{\varepsilon=0} = \frac{1}{2}\sum_{i=0}^{4}\sum_{j=0}^{4}\frac{\partial^2 K}{\partial u^{(i)}\partial u^{(j)}}[\xi^{-1}]U^{(i)}U^{(j)} = 0.$$

Furthermore, we can recognise, by analogy with linear equations, that $\xi = 0$ can be defined as a *regular singular point* of this bilinear equation; each coefficient of $U^{(i)}U^{(j)}$ in (18) is such that it is analytic at $\xi = 0$ when multiplied by ξ^{8-i-j}. Thus the substitution $U = \xi^\alpha$ yields an equation for α (the "indicial equation"), which is

$$(20)\qquad (\alpha + 2)^2\alpha(\alpha - 3)^2(\alpha - 4) = 0.$$

Seeking a series solution[1] starting at the lowest root of the indicial equation then leads us to the general solution of (18)

$$(21)\qquad U = U_{-1}\xi^{-2} + U_1 + U_3\xi^2 + U_4\xi^3 + U_5\xi^4,$$

where U_1, U_3 and U_5 are subject to the constraint

$$(22)\qquad U_3^2 + 9U_1U_5 = 0.$$

[1]In fact the construction of such a series solution does not proceed entirely in the way that one would expect; this seems to be due to the nature of the particular example we are dealing with.

Some remarks are in order. First, the indicial equation has six roots; this is because (18) is bilinear and includes terms having a total of six derivatives. Repeated roots do not imply logarithmic terms, in contrast to the case for linear equations. Second, the roots of the indicial equation give us the powers of ξ which are solutions of the equation; however, and again unlike the usual linear analysis, these do not correspond to the (leading) powers of ξ in the series solution whose coefficients are left arbitrary. There is in fact an ambiguity in the solution (21) as to which of the five coefficients are to be considered arbitrary. As we shall see, the answer to this question lies in the solutions of the equations appearing at higher orders of ε. Third, we have labelled the coefficients of the expansion for U relative to the dominant behaviour of u, i.e. $u = \xi^{-1} + \varepsilon\xi^{-1}[U_{-1}\xi^{-1} + U_1\xi + U_3\xi^3 + U_4\xi^4 + U_5\xi^5] + O(\varepsilon^2)$.

It is equation (18) that forms the basis of our approach to understanding the singularity structure of $K[u] = 0$ (where $K[u]$ is given by (7)). This approach, whereby we study the regular singular points of a bilinear equation in order to determine resonances, rather than those of a linear equation (see (9)), is new. However, as indicated above, in order to understand the information obtained from the ODE (18) — and in particular in order to understand which of the coefficients in (21) correspond to resonances — we need to go to higher orders of perturbation.

At order ε^3 we obtain a (nonhomogeneous) linear equation for V,

$$
\begin{aligned}
(23)\qquad & 12\left(-\frac{3}{\xi^4}U_1 - \frac{2}{\xi^2}U_3 + 3U_5\right)V'''' - 12\left(\frac{9}{\xi^5}U_1 + \frac{2}{\xi^3}U_3 + \frac{3}{\xi}U_5\right)V''' \\
& + 24\left(\frac{9}{\xi^6}U_1 + \frac{10}{\xi^4}U_3 - \frac{9}{\xi^2}U_5\right)V'' - 216\left(\frac{1}{\xi^7}U_1 + \frac{2}{\xi^5}U_3 - \frac{5}{\xi^3}U_5\right)V' \\
& - \left(\frac{1728}{\xi^4}U_5\right)V + \frac{3240}{\xi^{11}}U_{-1}^2U_1 + \frac{3024}{\xi^9}U_{-1}^2U_3 + \frac{72}{\xi^4}(15U_1^2 \\
& - 16U_{-1}U_3)U_4 \\
& - \frac{216}{\xi^7}(U_1^3 + 2U_{-1}U_1U_3 + 35U_{-1}^2U_5) - \frac{48}{\xi^5}(9U_1^2U_3 + 34U_{-1}U_3^2 \\
& + 144U_{-1}U_1U_5) \\
& - \frac{24}{\xi^3}(17U_1U_3^2 + 45U_1^2U_5 + 54U_{-1}U_3U_5) + \frac{6048}{\xi^2}U_1U_3U_4 \\
& - \frac{8}{\xi}(86U_3^3 - 945U_1U_4^2 - 1476U_1U_3U_5) + 48(26U_3^2 + 531U_1U_5)U_4 \\
& + 24\xi(126U_3U_4^2 + 253U_3^2U_5 + 873U_1U_5^2) + 12384\xi^2U_3U_4U_5 \\
& - 3240\xi^3(U_4^2 - 2U_3U_5)U_5 - 16200\xi^4U_4U_5^2 - 16200\xi^5U_5^3 = 0,
\end{aligned}
$$

this being (16) with U replaced by (21) (and so with U_1, U_3 and U_5 subject to (22)).

The behaviour of the solution of (23) depends on the coefficients of the terms linear in V, and so on U_1, U_3 and U_5. We consider first the case $U_1 \neq 0$, for which choice the dominant linear part of (23) is

$$-\frac{36}{\xi^4}U_1\left(V'''' + \frac{3}{\xi}V''' - \frac{6}{\xi^2}V'' + \frac{6}{\xi^3}V'\right). \tag{24}$$

For this dominant linear part the corresponding indicial equation, labelling once again relative to the leading order behaviour of u, is determined by

$$\xi^{5-r}\left(\frac{d^4}{dx^4} + \frac{3}{\xi}\frac{d^3}{dx^3} - \frac{6}{\xi^2}\frac{d^2}{dx^2} + \frac{6}{\xi^3}\frac{d}{dx}\right)\xi^{r-1} = (r+1)(r-1)(r-3)(r-4) = 0, \tag{25}$$

and thus in the series solution of (23) the coefficients V_{-1}, V_1, V_3 and V_4 (of ξ^{-2}, ξ^0, ξ^2 and ξ^3 respectively), will be arbitrary. The leading order behaviour of V is however determined by the term in ξ^{-11} in (23), which tells us that $V \sim U_{-1}^2\xi^{-3}$.

We are thus, in this case $U_1 \neq 0$, able to determine the general solution of (23),

$$\begin{aligned} V = \xi^{-1}\Big[&U_{-1}^2\xi^{-2} + V_{-1}\xi^{-1} + V_1\xi - (U_1^2 + 2U_{-1}U_3)\xi^2 + V_3\xi^3 + V_4\xi^4 \\ &+ \left(\frac{5}{8}U_1U_4 - \frac{2}{3}\frac{U_{-1}U_3U_4}{U_1} - \frac{U_5}{U_1}V_1 - \frac{2}{9}\frac{U_3}{U_1}V_3\right)\xi^5 + \frac{3}{7}U_1U_5\xi^6 \\ &+ \frac{7}{24}U_3U_4\xi^7 + \left(\frac{1}{6}U_4^2 + \frac{8}{21}U_3U_5\right)\xi^8 + \frac{3}{8}U_4U_5\xi^9 + \frac{15}{77}U_5^2\xi^{10}\Big], \end{aligned} \tag{26}$$

where once again U_1, U_3 and U_5 are subject to (22), and where V_{-1}, V_1, V_3 and V_4 are left arbitrary with satisfied compatibility conditions.

We are now in a position to interpret our results in this case $U_1 \neq 0$. First of all we note that in this case the natural interpretation of (21) is that the coefficients U_{-1} U_1, U_3 and U_4 are arbitrary, and that U_5 is determined as $U_5 = -U_3^2/(9U_1)$. Fixing now on the arbitrary coefficients in our perturbation expansion (14), we see that at each order of perturbation ≥ 1 we have that for $r = -1, 1, 3, 4$ the coefficients of ξ^{r-1} are arbitrary (recall that the linear parts of the equations obtained at perturbation orders ≥ 3 are all the same). These arbitrary coefficients, labelled relative to the leading order behaviour of u, $u \sim \xi^{-1}$, are precisely the sought-after resonances.

That is, we have obtained that corresponding to the leading order behaviour $u \sim \xi^{-1}$, we have resonances $r = -1, 1, 3, 4$. Thus we see that it is possible to determine resonances for leading order behaviours such as the one being studied here, in spite of the fact that an equation corresponding to (12) holds. This then means that we are able to impose the condition that any resonances determined in this way, where these resonances give us information about the general solution of the equation, must be integer if the equation is to have the Painlevé property. That is, we are able to extend the usual integer resonance condition [19] to systems of the type under discussion. (The same condition can be imposed on the roots of the indicial equation of the corresponding bilinear equation, again where we know that these roots give us information about the general solution of the original equation.)

We can, without loss of generality of the solution, set all but one of the arbitrary coefficients which enter at each resonant power of ξ to be zero. One choice, when as here a full complement of arbitrary coefficients enters at order ε, would be to set all arbitrary coefficients at perturbation order ≥ 2 to be zero. However this is not necessarily the best choice. For example, for the above case $U_1 \neq 0$ we could take $U_3 = U_4 = 0$; U_5 is then determined as $U_5 = 0$. For this choice, the essential structure of equation (23) is preserved. We obtain

$$-\frac{36}{\xi^4}U_1\left(V'''' + \frac{3}{\xi}V''' - \frac{6}{\xi^2}V'' + \frac{6}{\xi^3}V'\right) + \frac{3240}{\xi^{11}}U_{-1}^2U_1 - \frac{216}{\xi^7}U_1^3 = 0, \tag{27}$$

which has general solution

$$V = \xi^{-1}\left[U_{-1}^2\xi^{-2} + V_{-1}\xi^{-1} + V_1\xi - U_1^2\xi^2 + V_3\xi^3 + V_4\xi^4\right]. \tag{28}$$

Thus we see that in any case we introduce arbitrary data corresponding to all resonances at order ε^2 in (14); we may now set $V_{-1} = V_1 = 0$, in which case we have at orders ε and ε^2, when taken together, arbitrary coefficients corresponding to all resonances $r = -1, 1, 3, 4$. If we wish, all other arbitrary coefficients at higher orders of perturbation in (14) can then be set to zero. (We note in passing that the choice $U_3 = U_4 = 0$ greatly simplifies our analysis; for this choice, the equation for W at order ε^4 also becomes very simple to solve).

We now consider the case $U_1 = 0$; the condition (22) then gives $U_3 = 0$, and equation (23) becomes

$$\begin{aligned}36U_5\left(V'''' - \frac{1}{\xi}V''' - \frac{6}{\xi^2}V'' + \frac{30}{\xi^3}V' - \frac{48}{\xi^4}V\right) - \frac{7560}{\xi^7}U_{-1}^2U_5 \\ - 3240\xi^3U_4^2U_5 - 16200\xi^4U_4U_5^2 - 16200\xi^5U_5^3 = 0.\end{aligned} \tag{29}$$

We assume $U_5 \neq 0$; the corresponding indicial equation of (29), labelling once again relative to the leading order behaviour of u, is determined by

$$\xi^{5-r}\left(\frac{d^4}{dx^4} - \frac{1}{\xi}\frac{d^3}{dx^3} - \frac{6}{\xi^2}\frac{d^2}{dx^2} + \frac{30}{\xi^3}\frac{d}{dx} - \frac{48}{\xi^4}\right)\xi^{r-1} = (r+1)(r-3)(r-4)(r-5) = 0, \tag{30}$$

and thus in the series solution of (29) the coefficients V_{-1}, V_3, V_4 and V_5 (of ξ^{-2}, ξ^2, ξ^3, and ξ^4 respectively), will be arbitrary. The leading order behaviour of V is determined by the term in ξ^{-7} as $V \sim U_{-1}^2\xi^{-3}$. The general solution of (29) is

$$V = \xi^{-1}\left[\frac{U_{-1}^2}{\xi^2} + \frac{V_{-1}}{\xi} + V_3\xi^3 + V_4\xi^4 + V_5\xi^5 + \frac{U_4^2}{6}\xi^8 + \frac{3}{8}U_4U_5\xi^9 + \frac{15}{77}U_5^2\xi^{10}\right], \tag{31}$$

where V_{-1}, V_3, V_4 and V_5 are left arbitrary with satisfied compatibility conditions. We note in passing that the equation for W at order ε^4 is also simple to solve.

We now turn to the interpretation of our results, in this case $U_1 = U_3 = 0$ and $U_5 \neq 0$. We see that at order ε we have three arbitrary coefficients, U_{-1}, U_4 and U_5; at all subsequent orders of ε we have four arbitrary coefficients, namely the coefficients of ξ^{r-1}, $r = -1, 3, 4, 5$ (thus for example V_{-1}, V_3, V_4 and V_5 at order ε^2). These arbitrary coefficients, labelled relative to the leading order behaviour of u, form a distinct set of resonances corresponding to the leading order behaviour $u \sim \xi^{-1}$. We note that this is the first example that has been given where a particular leading order behaviour has been found to have two distinct sets of resonances.

Again we note that we can, without losing the generality of the solution, set to zero all but one of the arbitrary coefficients that enter at each resonant power of ξ.

We now turn to the case $U_1 = U_3 = U_5 = 0$. In this case the equation at order ε^3 vanishes ($G[\xi, U_{-1}\xi^{-2} + U_4\xi^3, V] \equiv 0$) and V is instead determined by the equation at order ε^4, which now no longer depends on W, and has a structure very similar to the structure of (18); it has the same bilinear terms, but also terms linear in V, and nonhomogeneous terms. The general solution of this equation for V is

$$V = \xi^{-1}\left[\frac{U_{-1}^2}{\xi^2} + \frac{V_{-1}}{\xi} + V_1\xi + V_3\xi^3 + V_4\xi^4 + V_5\xi^5 + \frac{U_4^2}{6}\xi^8\right], \tag{32}$$

where V_1, V_3 and V_5 are subject to $(V_3 + 3U_{-1}U_4)^2 + 9V_1V_5 = 0$.

The equation which arises at ε^5 is now a nonhomogeneous linear equation for W, rather than for X, and has the same linear part as (23) but with U_1, U_3 and U_5 replaced by V_1, $V_3 + 3U_{-1}U_4$ and V_5, respectively (and V by W). Thus we see that setting $U_1 = U_3 = U_5 = 0$ simply means that arbitrary data corresponding to our resonances is introduced at higher orders of perturbation. Since it is (resonances in) representations of the general solution that we are interested in here, and not a particular solution of the form (14) having arbitrary coefficients only at powers ξ^{-2} and ξ^3, we see that we do not need to consider further the case $U_1 = U_3 = U_5 = 0$.

In the above determination of the resonances corresponding to the leading order behaviour $u \sim \xi^{-1}$, we note that, at least in the second case ($U_1 = U_3 = 0$, $U_5 \neq 0$), the linearization of our bilinear equation played a decisive role. In this case we have three arbitrary coefficients in our solution for U, (U_{-1}, U_4 and U_5), and it is only when we consider our linear equation for V that we can be certain where the fourth resonance is. As we now see, this is important when considering series solutions.

Let us consider the construction of series solutions for the equation $K[u] = 0$, where $K[u]$ is given by (7), for the leading order behaviour $u \sim u_0\xi^{-1}$. That is, we consider the construction of series solutions (Painlevé expansions) of the form

$$u = \xi^{-1} \sum_{j=0}^{\infty} u_j \xi^j, \tag{33}$$

where all u_j are constant and $\xi = x - x_0$. We note that a necessary condition for the original system (5), (6) to have the Painlevé property is that $K[u] = 0$ have it.

First of all we remark that the case $u_0 = 2$ presents no problems; the Painlevé expansion can be constructed as usual and we find that in keeping with the resonances $r = -2, -1, 4, 5$, the coefficients u_4 and u_5 are left arbitrary with satisfied compatibility conditions. As noted earlier, a more complete treatment of this leading order behaviour can be given using the perturbative Painlevé test [19].

We now consider the case $u_0 = 1$. This choice leads to very unusual behaviour. Since $K'[\xi^{-1}] \equiv 0$, the coefficients u_j are no longer determined by a series of linear algebraic equations. Instead, they are determined by a series of nonlinear algebraic equations, with each u_j being determined by an equation at a level higher than that which would be expected (this might make one think of treating such equations as failed compatibility conditions, and of introducing logarithmic terms into (33), but all of our attempts to introduce such logarithmic terms have proved unsuccessful).

Thus the determination of the coefficients u_j of (33) splits into a series of subcases. If we assume $u_1 \neq 0$, we obtain a solution (33) having u_1, u_3 and u_4 arbitrary, and all other u_j apparently determined in terms of these three. This clearly corresponds to our first case above, where we obtained resonances $r = -1, 1, 3, 4$ (and with x_0 being the arbitrary constant corresponding to the resonance $r = -1$).

When $u_1 = 0$ we obtain that $u_2 = u_3 = 0$, u_4 is left arbitrary, and that the determination of subsequent coefficients u_j splits according as to whether $u_5 \neq 0$ or $u_5 = 0$. In the first of these cases we obtain a solution having u_4 and u_5 arbitrary, and with all other coefficients u_j seemingly determined in terms of these two. In the second case we obtain a solution seemingly having only u_4 arbitrary, and with all other coefficients apparently determined in terms of this one. Our interpretation of these solutions is that the first ($u_5 \neq 0$) corresponds to our second case discussed above, and that the second ($u_5 = 0$) corresponds to a particular solution.

We cannot here rule out a further splitting into subsubcases; the determination of the coefficients of (33) is a highly nonlinear process, and it is difficult to know what happens at higher powers of ξ. This is made much more difficult by the fact that coefficients are not determined at their usual level in Painlevé analysis, but instead (and especially for $u_1 = 0$) at levels significantly shifted from these.

However, we see here that our equation has given us one last surprise. Given the behaviour of our equation, we might have thought that one solution to the problem of determining resonances was to simply seek a solution as a Painlevé expansion (33), with the idea that we would at least be able to find any positive integer resonances, since we would at least obtain arbitrary coefficients corresponding to these. However, as we see here, this cannot be done; for the leading order behaviour $u \sim \xi^{-1}$ we know that there are two distinct sets of resonances. The first of these ($r = -1, 1, 3, 4$) we can find by seeking a solution as a Painlevé expansion, but the second ($r = -1, 3, 4, 5$) we cannot, because u_3 is determined as $u_3 = 0$ instead of being left arbitrary. This means that $r = 3$ behaves like a negative resonance; arbitrary data can only be assigned to it through a perturbation expansion.

Thus, if we had not determined our second set of resonances as $r = -1, 3, 4, 5$, we would not know from our Painlevé expansion (33) what the fourth resonance was. This would mean that we would be unable to rule out branching due to a rational resonance; the expansion (33) is of no use in detecting such a resonance. That is, without performing our resonance calculation, we would not know if the expansion (33), in the case $u_1 = 0$, $u_5 \neq 0$, needs to be modified so as to include rational branching. It is only by knowing the full set of resonances that we are able to rule out such branching. In short, any attempt to explore the leading order behaviour $u \sim \xi^{-1}$ of the equation $K[u] = 0$ which did not somehow determine its corresponding sets of resonances would be both incomplete and deeply flawed.

§4. Conclusions

In this paper we have presented an equation for which the ARS algorithm breaks down in a previously unseen way: it leaves the resonances undetermined. The equation presenting this difficulty is one which would arise in the application of the ARS Painlevé test; it should also arise in any Painlevé classification of rational equations. Although the problem presented by this equation has not been seen before, it is clear that this is not because the equation is unique, but rather because it is the first such equation encountered. That is, it is important to understand how to define the resonances for the equation studied in this paper.

This we have done using a new technique based on a study of the regular singular points of bilinear equations, and linearizations thereof. Once we have defined the resonances in this way, corresponding arbitrary coefficients are then encountered both amongst the arbitrary coefficients of the corresponding solution of the bilinear equation, and also (for all resonances) amongst the arbitrary coefficients of the corresponding solutions of the linear equations which occur at higher perturbation orders. It is possible to have more than one set of resonances corresponding to a particular leading order behaviour. It is also possible that, corresponding to some resonances, even though positive, no arbitrary coefficient is introduced in the Painlevé expansion when we come to consider its construction.

Once we know how to determine resonances, the usual integer resonance condition can be imposed, where the resonances give information about the general solution. It is also worth noting that, in order to test $K[u] = 0$ for logarithmic branching, instead of attempting to construct Painlevé expansions directly, a whole hierarchy of necessary conditions for the absence of logarithmic branching is provided by the sequence of equations obtained from our perturbation expansion. Finally, we remark that, just as here we have considered bilinear equations, so might it be useful in future to consider regular singular points of N-linear equations, $N \geq 3$.

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