

Lie Subalgebras of Differential Operators on the Super Circle

By

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Abstract

We classify anti-involutions of Lie superalgebra $\widehat{\mathcal{SD}}$ preserving the principal gradation, where $\widehat{\mathcal{SD}}$ is the central extension of the Lie superalgebra of differential operators on the super circle $S^{1|1}$. We clarify the relations between the corresponding subalgebras fixed by these anti-involutions and subalgebras of $\widehat{gl}_{\infty|\infty}$ of types OSP and P . We obtain a criterion for an irreducible highest weight module over these subalgebras to be quasifinite and construct free field realizations of a distinguished class of these modules. We further establish dualities between them and certain finite-dimensional classical Lie groups on Fock spaces.

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§1. Introduction

Extended symmetries have played an important role in conformal field theories. This leads one to study \mathcal{W} algebras, which are higher-spin extensions of the Virasoro algebra, and their superanalogues, if one takes supersymmetry into account, cf. [BS], [FeF] and the vast literature therein. A fundamental example of \mathcal{W} algebras, now known as the $\mathcal{W}_{1+\infty}$ algebra, appears as the limit, in an appropriate sense, of the \mathcal{W}_N algebras, as N goes to ∞ [PRS1]. It has been realized [PRS2] that the $\mathcal{W}_{1+\infty}$ algebra can also be interpreted as the central extension of the Lie algebra of differential operators on the circle [KP]. Such an interpretation potentially links $\mathcal{W}_{1+\infty}$ to geometry in a direct way, which has yet to be developed. On the other hand, the super $\mathcal{W}_{1+\infty}$ algebra, which is a central extension of the Lie algebra \mathcal{SD} of differential operators on the super circle $S^{1|1}$ and which will be denoted by $\widehat{\mathcal{SD}}$ throughout this paper, is also promising to have connection with mirror geometry, as it contains the $N = 2$ superconformal algebra as a subalgebra.

The difficulty for the development of a reasonable representation theory for such Lie (super)algebras lies in the fact that the graded subspaces are infinite-dimensional in spite of the existence of a natural principle gradation and thus of a triangular decomposition. The Cartan subalgebras of such Lie algebras are also infinite-dimensional. A sensible physical theory however often requires finite-dimensionality of the graded subspaces in a representation (which will be referred to as the *quasifiniteness* condition). In [KR1], Kac and Radul developed a powerful machinery and initiated a systematic study of

quasifinite representations of $\mathcal{W}_{1+\infty}$. Subsequently, there have been further development and extension, cf. [AFMO1], [FKRW], [AFMO2], [AFMO3], [KR2], [W2], [KWY] and the references therein. In particular, the Kac-Radul method was further extended to study the representation theory of the Lie superalgebra $\widehat{\mathcal{SD}}$ in [AFMO1].

As explored in [KR1], the study of representations of $\mathcal{W}_{1+\infty}$ is closely related to that of the well-known Lie algebra $\widehat{\mathfrak{gl}}_\infty$. It is well known that $\widehat{\mathfrak{gl}}_\infty$ admits natural subalgebras of types B, C, D . In the paper [KWY], the authors identified certain distinguished subalgebras of $\mathcal{W}_{1+\infty}$, which correspond to these classical subalgebras of $\widehat{\mathfrak{gl}}_\infty$, by first classifying all anti-involutions of $\mathcal{W}_{1+\infty}$ preserving the principal gradation. Quasifinite representations and free field realizations of these subalgebras were studied systematically, and dualities between these subalgebras and certain finite-dimensional Lie groups were obtained by applying dualities for classical Lie subalgebras of $\widehat{\mathfrak{gl}}_\infty$ in [W1]. We remark that the free field realizations of $\mathcal{W}_{1+\infty}$ and dualities between $\mathcal{W}_{1+\infty}$ and finite dimensional general linear Lie groups were earlier established in [FKRW] and [KR2].

The goal of the present paper is to generalize the results of [KWY] to the $\widehat{\mathcal{SD}}$ setting. Note that $\widehat{\mathcal{SD}}$ affords a canonical principal $\frac{1}{2}\mathbb{Z}$ -gradation. More explicitly, we first classify all anti-involutions of the \mathcal{SD} preserving the principal gradation. It turns out the theory is much richer in the super case and we find five families of such anti-involutions. This is done in Section 3. The anti-involutions within each family is related to each other by a spectral flow, so there are essentially five distinct anti-involutions of \mathcal{SD} . In Section 4 we give explicit descriptions of the central extensions of these five subalgebras fixed by these anti-involutions, denoted by ${}^0\widehat{\mathcal{SD}}$ and ${}^{\pm\pm}\widehat{\mathcal{SD}}$.

By further developing the machinery in [KR1], we analyze the structure of parabolic subalgebras of ${}^0\widehat{\mathcal{SD}}$ and ${}^{\pm\pm}\widehat{\mathcal{SD}}$, and obtain criterion in terms of certain differential equations for an irreducible highest weight module of ${}^0\widehat{\mathcal{SD}}$ and ${}^{\pm\pm}\widehat{\mathcal{SD}}$ to be quasifinite. This is the content of Section 5. Our analysis has a somewhat different flavor from the one in [KR1].

The quasifinite representations of ${}^0\widehat{\mathcal{SD}}$ and ${}^{\pm\pm}\widehat{\mathcal{SD}}$ turn out to be intimately related to that of the Lie superalgebra $\widehat{\mathfrak{gl}}_{\infty|\infty}$ and its subalgebras, $\widehat{\mathcal{B}}$ of type OSP and $\mathcal{P}_{\pm\pm}$ of type P . It is interesting to note the strange Lie superalgebra of type P (cf. e.g. [K]) makes a natural appearance here. We develop such a link by exhibiting certain Lie superalgebra homomorphisms among them in Section 6 (compare [KR1], [AFMO1], [KWY]). Via these homomorphisms, the pullback of irreducible quasifinite modules of $\widehat{\mathfrak{gl}}_{\infty|\infty}, \widehat{\mathcal{B}}$

and $\mathcal{P}_{\pm\pm}$ remain irreducible and quasifinite as modules over ${}^0\widehat{\mathcal{SD}}$ and ${}^{\pm\pm}\widehat{\mathcal{SD}}$, respectively.

In Section 8 we study free field realization of the Lie superalgebras $\widehat{\mathcal{SD}}$ and ${}^i\widehat{\mathcal{SD}}$ in the Fock space $\mathfrak{F}^{\otimes l}$ of l pairs of free fermionic fields (i.e. bc fields) and l pairs of bosonic ghosts (i.e. $\beta\gamma$ fields), also cf. [AFMO1]. We further obtain a duality in the sense of Howe [H1], [H2] between $\widehat{\mathcal{SD}}$ and the general linear group $GL(l)$. The multiplicity-free decomposition of $\mathfrak{F}^{\otimes l}$ under the joint action of $\widehat{\mathcal{SD}}$ and $GL(l)$ is explicitly presented. In particular we obtain explicit formulas for the joint highest weight vectors in each isotypic subspace under such a joint action. To achieve this, we use the technique developed in [CW1] to study in Section 7 a finite-dimensional duality between a general linear Lie superalgebra and $GL(l)$ with respect to certain non-standard Borel subalgebras, which is then adapted to our infinite-dimensional setting. A similar duality between ${}^0\widehat{\mathcal{SD}}$ and the Lie group $\text{Pin}(2l)$ is also obtained. We remark that the vacuum modules of $\widehat{\mathcal{SD}}$ and ${}^{\pm\pm}\widehat{\mathcal{SD}}$ carry a natural vertex superalgebra structure. Finally we conclude this paper with discussion of some open problems.

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Notation. \mathbb{N} stands for the set of positive intergers, \mathbb{Z} for the set of integers, \mathbb{Z}_+ for the set of non-negative integers, \mathbb{Z}_2 for the two-element group $\mathbb{Z}/2\mathbb{Z}$, and \mathbb{C} the field of complex numbers.

§2. Lie Superalgebra \mathcal{SD} of Differential Operators on $S^{1|1}$

In this section we set up notation and review the definition of the Lie superalgebras \mathcal{SD} and $\widehat{\mathcal{SD}}$, cf. [KR1], [AFMO1].

Let t be an even indeterminate and let $\partial_t = \frac{d}{dt}$. We denote by \mathcal{D}_{as} the associative algebra of regular differential operators on the circle S^1 . It has a linear basis given by

$$J_k^l = -t^{l+k}(\partial_t)^l, \quad l \in \mathbb{Z}_+, k \in \mathbb{Z}.$$

A different choice of basis of \mathcal{D}_{as} is given by $t^k D^l, l \in \mathbb{Z}_+, k \in \mathbb{Z}$, where $D = t\partial_t$. Note that $f(D)t = tf(D+1)$ for $f(w) \in \mathbb{C}[w]$ and hence $J_k^l = -t^k[D]_l$, where here and further we use the notation

$$(2.1) \quad [x]_l = x(x-1)\dots(x-l+1).$$

Let \mathcal{D} denote the Lie algebra obtained from \mathcal{D}_{as} by taking the usual bracket of operators $[a, b] = ab - ba$.

Denote by \mathcal{SD}_{as} the associative superalgebra of regular differential operators on the supercircle $S^{1|1}$. Then the following elements

$$t^{k+l}(\partial_t)^l \theta \partial_\theta, t^{k+l}(\partial_t)^l \partial_\theta \theta, t^{k+l}(\partial_t)^l \theta, t^{k+l}(\partial_t)^l \partial_\theta, \quad l \in \mathbb{Z}_+, k \in \mathbb{Z}$$

form a linear basis of \mathcal{SD}_{as} , where θ is an odd indeterminate. Here of course θ and ∂_θ commute with t and ∂_t . Note that the odd elements θ and ∂_θ generate the four-dimensional Clifford superalgebra with relation $\theta \partial_\theta + \partial_\theta \theta = 1$. Clearly \mathcal{SD}_{as} is isomorphic to the tensor algebra of \mathcal{D}_{as} and this Clifford superalgebra.

Denote by $M(1, 1)$ the set of all 2×2 matrices of the form

$$(2.2) \quad \begin{bmatrix} \alpha^0 & \alpha^+ \\ \alpha^- & \alpha^1 \end{bmatrix},$$

where $\alpha^a \in \mathbb{C}, a = 0, 1, \pm$, viewed as the associative superalgebra of linear transformations on the complex $(1|1)$ -dimensional superspace $\mathbb{C}^{1|1}$. Namely, letting M_a ($a = 0, 1, \pm$) be the matrix of the form (2.2) with $\alpha^a = 1$ and 0 elsewhere, we declare M_0, M_1 to be even and M_+, M_- to be odd elements. This equips $M(1, 1)$ with a \mathbb{Z}_2 -gradation. The supertrace Str of the matrix (2.2) is then defined to be $\alpha^0 - \alpha^1$.

Note that the four-dimensional Clifford superalgebra generated by θ and ∂_θ can be identified with $M(1, 1)$ as associative superalgebras by making the following identification:

$$M_0 = \partial_\theta \theta, \quad M_1 = \theta \partial_\theta, \quad M_+ = \partial_\theta, \quad M_- = \theta.$$

It follows therefore that one can canonically identify the associative superalgebra \mathcal{SD}_{as} with the associative superalgebra of 2×2 (super)matrices with entries in \mathcal{D}_{as} . By taking the usual super-bracket $[\ , \]$ we make \mathcal{SD}_{as} into a Lie superalgebra, which we denote by \mathcal{SD} . We will adopt the convention that the capital letter $F(D)$ denotes the matrix

$$(2.3) \quad \begin{bmatrix} f_0(D) & f_+(D) \\ f_-(D) & f_1(D) \end{bmatrix},$$

where $f_a(w) \in \mathbb{C}[w], a = 0, 1, \pm$, so that we may regard $F(D)$ as an element of \mathcal{SD} .

Denote by $\widehat{\mathcal{SD}}$ the central extension of \mathcal{SD} by a one-dimensional vector space with a specified generator C . The commutation relation for homogeneous

$F(D)$ and $G(D)$ in $\widehat{\mathcal{SD}}$ is given by

$$(2.4) \quad [t^r F(D), t^s G(D)] = t^{r+s}(F(D+s)G(D) - (-1)^{|F||G|}F(D)G(D+r)) + \Psi(t^r F(D), t^s G(D))C,$$

where the two-cocycle Ψ is given by:

$$(2.5) \quad \Psi(t^r F(D), t^s G(D)) = \begin{cases} \sum_{-r \leq j \leq -1} \text{Str}(F(j)G(j+r)), & r = -s \geq 0 \\ 0, & r + s \neq 0. \end{cases}$$

$$= \sum_{-r \leq j \leq -1} (f_0(j)g_0(j+r) + f_+(j)g_-(j+r) - f_-(j)g_+(j+r) - f_1(j)g_1(j+r)).$$

Here $|F|$ and $|G|$ stand for the degree of $F(D)$ and $G(D)$, respectively.

We introduce the *principal gradation* of \mathcal{SD} and $\widehat{\mathcal{SD}}$ by letting the weight of C be 0, the weights of $t^n f(D)\partial_\theta$ and $t^n f(D)\theta\partial_\theta$ be n , and the weights of $t^n f(D)\theta$ and $t^{n+1}f(D)\partial_\theta$ be $n + 1/2$, that is, the grading is uniquely determined by assigning $wt(C) = 0, wt(D) = 0, wt(t) = 1, wt(\theta) = 1/2$, and $wt(\partial_\theta) = -1/2$. This equips \mathcal{SD} and $\widehat{\mathcal{SD}}$ with $\frac{1}{2}\mathbb{Z}$ -gradations compatible with their \mathbb{Z}_2 -gradations. Namely, if the \mathbb{Z}_2 -graded decompositions of $\mathcal{SD}, \widehat{\mathcal{SD}}$ are given by

$$\mathcal{SD} = \mathcal{SD}_0 \oplus \mathcal{SD}_1, \quad \widehat{\mathcal{SD}} = \widehat{\mathcal{SD}}_0 \oplus \widehat{\mathcal{SD}}_1,$$

then

$$\mathcal{SD}_0 = \bigoplus_{n \in \mathbb{Z}} \mathcal{SD}_n, \quad \mathcal{SD}_1 = \bigoplus_{n \in \mathbb{Z} + 1/2} \mathcal{SD}_n,$$

$$\widehat{\mathcal{SD}}_0 = \bigoplus_{n \in \mathbb{Z}} \widehat{\mathcal{SD}}_n, \quad \widehat{\mathcal{SD}}_1 = \bigoplus_{n \in \mathbb{Z} + 1/2} \widehat{\mathcal{SD}}_n.$$

Note that each graded subspace $\widehat{\mathcal{SD}}_n$ is still infinite-dimensional.

We also have the following triangular decompositions of \mathcal{SD} and $\widehat{\mathcal{SD}}$:

$$\mathcal{SD} = \mathcal{SD}_+ \oplus \mathcal{SD}_0 \oplus \mathcal{SD}_-, \quad \widehat{\mathcal{SD}} = \widehat{\mathcal{SD}}_+ \oplus \widehat{\mathcal{SD}}_0 \oplus \widehat{\mathcal{SD}}_-,$$

where

$$\mathcal{SD}_\pm = \bigoplus_{j \in \pm\mathbb{N}/2} \mathcal{SD}_j, \quad \widehat{\mathcal{SD}}_\pm = \bigoplus_{j \in \pm\mathbb{N}/2} \widehat{\mathcal{SD}}_j.$$

§3. Anti-involutions of \mathcal{SD} Preserving the Principal Gradation

In this section we classify anti-involutions of \mathcal{SD}_{as} which preserve the principal gradation. An *anti-involution* σ of the superalgebra \mathcal{SD}_{as} is an involutive anti-automorphism of \mathcal{SD}_{as} , i.e. $\sigma^2 = I, \sigma(aX + bY) = a\sigma(X) + b\sigma(Y)$ and $\sigma(XY) = (-1)^{|X||Y|}\sigma(Y)\sigma(X)$, where $a, b \in \mathbb{C}, X, Y \in \mathcal{SD}_{as}$. We have the following description of anti-involutions of \mathcal{SD}_{as} . The classification here is much more involved than in the case of $W_{1+\infty}$ as done in [KWY].

Theorem 3.1. *Any anti-involution σ of \mathcal{SD}_{as} which preserves the principal gradation is one of the following ($a \in \mathbb{C}, b \in \mathbb{C}^\times$):*

- 1) $\sigma_{a,b}(t) = -t, \sigma_{a,b}(D) = -D + a + \partial_\theta \theta, \sigma_{a,b}(\theta) = bt\partial_\theta, \sigma_{a,b}(\partial_\theta) = -(bt)^{-1}\theta;$
- 2) $\sigma_{++,a}(t) = t, \sigma_{++,a}(D) = -D + a, \sigma_{++,a}(\theta) = \theta, \sigma_{++,a}(\partial_\theta) = -\partial_\theta;$
- 3) $\sigma_{+-,a}(t) = t, \sigma_{+-,a}(D) = -D + a, \sigma_{+-,a}(\theta) = -\theta, \sigma_{+-,a}(\partial_\theta) = \partial_\theta;$
- 4) $\sigma_{-+,a}(t) = -t, \sigma_{-+,a}(D) = -D + a, \sigma_{-+,a}(\theta) = \theta, \sigma_{-+,a}(\partial_\theta) = -\partial_\theta;$
- 5) $\sigma_{--,a}(t) = -t, \sigma_{--,a}(D) = -D + a, \sigma_{--,a}(\theta) = -\theta, \sigma_{--,a}(\partial_\theta) = \partial_\theta.$

The rest of this section is devoted to the proof of Theorem 3.1. The following simple lemma is often used in computations.

Lemma 3.1. *Given commuting variables x, y and a polynomial f of one variable, the following identities hold.*

$$\begin{aligned} f(x + \theta\partial_\theta y) &= f(x) + \theta\partial_\theta(f(x + y) - f(x)), \\ f(\partial_\theta\theta x + \theta\partial_\theta y) &= \partial_\theta\theta f(x) + \theta\partial_\theta f(y), \\ f(x + \theta\partial_\theta y)\partial_\theta\theta &= f(x)\partial_\theta\theta, \\ f(x + \partial_\theta\theta y)\theta\partial_\theta &= f(x)\theta\partial_\theta. \end{aligned}$$

Proof. We will prove the first identity. Other identities are obtained similarly.

It suffices to check for a monomial $f(z) = z^n$. But in this case one has

$$\begin{aligned} (x + \theta\partial_\theta y)^n &= \sum_{k=0}^n \binom{n}{k} x^k (\theta\partial_\theta y)^{n-k} = x^n + \sum_{k=1}^n \binom{n}{k} \theta\partial_\theta x^k y^{n-k} \\ &= x^n + \theta\partial_\theta ((x + y)^n - x^n), \end{aligned}$$

which is the first identity. □

Lemma 3.2. *Let σ be an anti-involution of \mathcal{SD}_{as} preserving the principal gradation. We have the following two possibilities:*

- 1) $\sigma(\theta\partial_\theta) = \theta\partial_\theta,$
- 2) $\sigma(\theta\partial_\theta) = \partial_\theta\theta.$

Proof. As σ preserves the principal gradation, we may assume

$$(3.1) \quad \sigma(\theta\partial_\theta) = \theta\partial_\theta k_0(D) + \partial_\theta\theta k_1(D)$$

for some $k_0(w), k_1(w) \in \mathbb{C}[w]$. Since $(\theta\partial_\theta)^2 = \theta\partial_\theta$, we have

$$(3.2) \quad \sigma(\theta\partial_\theta) = \sigma((\theta\partial_\theta)^2) = (\sigma(\theta\partial_\theta))^2 = k_0(D)^2\theta\partial_\theta + \partial_\theta\theta k_1(D)^2$$

by using Lemma 3.1 and (3.1). It follows by comparing (3.1) with (3.2) that

$$(3.3) \quad k_0(w) = k_0(w)^2,$$

$$(3.4) \quad k_1(w) = k_1(w)^2.$$

The solutions of the equations (3.3) and (3.4) are $(k_0, k_1) = (0, 1), (1, 0), (0, 0)$ or $(1, 1)$. The first two solutions give rise to what we have listed in 1) and 2). The last two solutions are easily ruled out since an involution can not send $\theta\partial_\theta$ to 0 or 1. □

Note that the case 1) in Lemma 3.2 is equivalent to $\sigma(\partial_\theta\theta) = \partial_\theta\theta$ while the case 2) in Lemma 3.2 is equivalent to $\sigma(\partial_\theta\theta) = \theta\partial_\theta$.

In the remainder of this section, we will continue to assume that σ is an anti-involution of \mathcal{SD}_{as} preserving the principal gradation. We will analyze the two cases that occurred in Lemma 3.2 one by one.

§3.1. The case when $\sigma(\theta\partial_\theta) = \theta\partial_\theta$

This subsection is devoted to the proof of the following.

Proposition 3.1. *Assume that an anti-involution σ of \mathcal{SD} preserving the principal gradation satisfies $\sigma(\theta\partial_\theta) = \theta\partial_\theta$. Then σ is one of the $\sigma_{a,b}$ in part 1) of Theorem 3.1.*

We divide the proof into a series of lemmas.

Lemma 3.3. *Assume that $\sigma(\theta\partial_\theta) = \theta\partial_\theta$. Then $\sigma(\theta) = bt\partial_\theta, \sigma(\partial_\theta) = b^{-1}t^{-1}\theta$ for some nonzero $b \in \mathbb{C}$.*

Proof. We may assume that

$$\begin{aligned}\sigma(\theta) &= \theta h_-(D) + \partial_\theta t h_+(D), & \text{for some } h_\pm(w) \in \mathbb{C}[w], \\ \sigma(\partial_\theta) &= \theta t^{-1} j_-(D) + \partial_\theta j_+(D), & \text{for some } j_\pm(w) \in \mathbb{C}[w].\end{aligned}$$

Since $[\theta, \theta \partial_\theta] = -\theta$, we have

$$(3.5) \quad [\sigma(\theta \partial_\theta), \sigma(\theta)] = -\sigma(\theta) = -\theta h_-(D) - \partial_\theta t h_+(D).$$

On the other hand we have

$$(3.6) \quad [\sigma(\theta \partial_\theta), \sigma(\theta)] = [\theta \partial_\theta, \theta h_-(D) + \partial_\theta t h_+(D)] = \theta h_-(D) - \partial_\theta t h_+(D).$$

It follows by comparing (3.5) and (3.6) that $h_-(w) = 0$. Similar calculation by using $[\sigma(\theta \partial_\theta), \sigma(\partial_\theta)] = \sigma(\partial_\theta)$ shows that $j_+(w) = 0$. It follows from the following identity

$$\begin{aligned}\theta \partial_\theta &= \sigma(\theta \partial_\theta) \\ &= -\sigma(\partial_\theta) \sigma(\theta) \\ &= -(\theta t^{-1} j_-(D)) (\partial_\theta t h_+(D)) \\ &= -\theta \partial_\theta j_-(D+1) h_+(D)\end{aligned}$$

that $j_-(D+1)h_+(D) = -1$. Since both $j_-(w)$ and $h_+(w)$ are polynomials, $h_+(w)$ is some nonzero constant b while $j_-(w)$ is $-b^{-1}$. \square

Lemma 3.4. *Assume that $\sigma(\theta \partial_\theta) = \theta \partial_\theta$. Then $\sigma(t) = -t$.*

Proof. Assume that

$$\begin{aligned}\sigma(t) &= \partial_\theta \theta t f_0(D) + \theta \partial_\theta t f_1(D), & \text{for some } f_0(w), f_1(w) \in \mathbb{C}[w], \\ \sigma(D) &= \partial_\theta \theta g_0(D) + \theta \partial_\theta g_1(D), & \text{for some } g_0(w), g_1(w) \in \mathbb{C}[w].\end{aligned}$$

Then we have

$$\begin{aligned}(3.7) \quad t &= \sigma^2(t) = \sigma(\partial_\theta \theta t f_0(D) + \theta \partial_\theta t f_1(D)) \\ &= f_0(\partial_\theta \theta g_0(D) + \theta \partial_\theta g_1(D)) (\partial_\theta \theta t f_0(D) + \theta \partial_\theta t f_1(D)) \partial_\theta \theta \\ &\quad + f_1(\partial_\theta \theta g_0(D) + \theta \partial_\theta g_1(D)) (\partial_\theta \theta t f_0(D) + \theta \partial_\theta t f_1(D)) \theta \partial_\theta \\ &= \partial_\theta \theta f_0(g_0(D)) (t f_0(D)) + \theta \partial_\theta f_1(g_1(D)) (t f_1(D)) \\ &= \partial_\theta \theta t f_0(g_0(D+1)) (f_0(D)) + \theta \partial_\theta t f_1(g_1(D+1)) (f_1(D)).\end{aligned}$$

As we may rewrite $t = \partial_\theta \theta t + \theta \partial_\theta t$, it follows from (3.7) that

$$f_0(g_0(w+1)) (f_0(w)) = f_1(g_1(w+1)) (f_1(w)) = 1.$$

This implies that $(f_0(w), f_1(w)) = (1, 1), (-1, -1), (1, -1)$ or $(-1, 1)$. In the first two cases we get $\sigma(t) = \pm t$. The last two solutions for $(f_0(w), f_1(w))$ give rise to $\sigma(t) = \pm(\partial_\theta \theta t - \theta \partial_\theta t)$. We first claim that these last two cases cannot happen. For example let us take $\sigma(t) = \theta \partial_\theta t - \partial_\theta \theta t$ (the case $\sigma(t) = -(\theta \partial_\theta t - \partial_\theta \theta t)$ is similar). On one hand by Lemma 3.3 we have

$$\begin{aligned}\sigma(t)\sigma(\theta) &= (\theta \partial_\theta t - \partial_\theta \theta t)(bt \partial_\theta) = -bt^2 \partial_\theta, \\ \sigma(\theta)\sigma(t) &= (bt \partial_\theta)(\theta \partial_\theta t - \partial_\theta \theta t) = bt^2 \partial_\theta.\end{aligned}$$

So $\sigma(t)\sigma(\theta) = -\sigma(\theta)\sigma(t)$. On the other hand, $\sigma(t)\sigma(\theta) = \sigma(\theta)\sigma(t)$ since $\theta t = t\theta$. This is a contradiction.

We claim that $\sigma(t) = t$ cannot occur either. Indeed if $\sigma(t) = t$ then by Lemma 3.3

$$\theta = \sigma^2(\theta) = \sigma(bt \partial_\theta) = b\sigma(\partial_\theta)\sigma(t) = b(-b^{-1}t^{-1}\theta)t = -\theta,$$

which is a contradiction. \square

Lemma 3.5. *Assume that $\sigma(\theta \partial_\theta) = \theta \partial_\theta$. Then $\sigma(D) = -D + a + \partial_\theta \theta$ for some $a \in \mathbb{C}$.*

Proof. Assume that $\sigma(D) = \partial_\theta \theta g_0(D) + \theta \partial_\theta g_1(D)$ for some $g_0(w), g_1(w) \in \mathbb{C}[w]$. Then we have

$$\begin{aligned}D &= \sigma^2(D) \\ &= \sigma(\partial_\theta \theta g_0(D) + \theta \partial_\theta g_1(D)) \\ &= g_0(\partial_\theta \theta g_0(D) + \theta \partial_\theta g_1(D)) \partial_\theta \theta + g_1(\partial_\theta \theta g_0(D) + \theta \partial_\theta g_1(D)) \theta \partial_\theta \\ &= \partial_\theta \theta g_0(g_0(D)) + \theta \partial_\theta g_1(g_1(D)) \\ &= g_0(g_0(D)) + \theta \partial_\theta (g_1(g_1(D)) - g_0(g_0(D))).\end{aligned}$$

Thus $g_0(g_0(w)) = g_1(g_1(w)) = w$. This implies $g_1(w) = -w + a$ for some $a \in \mathbb{C}$ or $g_1(w) = w$.

We have $\sigma(\theta)\sigma(D) = \sigma(D)\sigma(\theta)$ as $D\theta = \theta D$. On the other hand, we can compute directly that

$$\begin{aligned}\sigma(\theta)\sigma(D) &= (bt \partial_\theta)(\partial_\theta \theta g_0(D) + \theta \partial_\theta g_1(D)) = b \partial_\theta t g_1(D), \\ \sigma(D)\sigma(\theta) &= (\partial_\theta \theta g_0(D) + \theta \partial_\theta g_1(D))(bt \partial_\theta) = b \partial_\theta t g_0(D + 1).\end{aligned}$$

It follows that $g_0(w + 1) = g_1(w)$. In the case that $g_1(w) = -w + a$, $g_0(w) = g_1(w - 1) = -w + a + 1$. Thus $\sigma(D) = -D + a + \partial_\theta \theta$. We claim that $g_1(w) = w$

cannot occur. Indeed if $g_1(w) = w$, then $g_0(w) = w - 1$ and $\sigma(D) = D - \partial_\theta \theta$. Together with Lemma 3.4 and the fact that $[t, D] = -t$ this implies that

$$[D, -t] = [\sigma(D), \sigma(t)] = -\sigma(t) = t.$$

This contradicts the fact that $[t, D] = -t$. \square

Proposition 3.1 now follows from Lemmas 3.3, 3.4 and 3.5.

§3.2. The case when $\sigma(\theta\partial_\theta) = \partial_\theta\theta$

This subsection is devoted to the proof of the following.

Proposition 3.2. *Assume that an anti-involution σ of \mathcal{SD} preserving the principal gradation satisfies $\sigma(\theta\partial_\theta) = \partial_\theta\theta$. Then σ is either σ_{++a} , σ_{+-a} , σ_{-+a} or σ_{--a} in Theorem 3.1.*

We divide the proof into a series of lemmas.

Lemma 3.6. *Assume that $\sigma(\theta\partial_\theta) = \partial_\theta\theta$. Then $\sigma(\theta) = \pm\theta$ and $\sigma(\partial_\theta) = \mp\partial_\theta$.*

Proof. Assume that

$$\begin{aligned}\sigma(\theta) &= \theta h_-(D) + \partial_\theta \theta h_+(D), & \text{for some } h_+(w), h_-(w) \in \mathbb{C}[w], \\ \sigma(\partial_\theta) &= \theta t^{-1} j_-(D) + \partial_\theta j_+(D), & \text{for some } j_+(w), j_-(w) \in \mathbb{C}[w].\end{aligned}$$

Since $[\theta, \theta\partial_\theta] = -\theta$, we have

$$(3.8) \quad [\sigma(\theta\partial_\theta), \sigma(\theta)] = -\sigma(\theta) = -\theta h_-(D) - \partial_\theta \theta h_+(D).$$

On the other hand, we also have

$$(3.9) \quad [\sigma(\theta\partial_\theta), \sigma(\theta)] = [\partial_\theta \theta, \theta h_-(D) + \partial_\theta \theta h_+(D)] = -\theta h_-(D) + \partial_\theta \theta h_+(D).$$

It follows by comparing (3.8) and (3.9) that $h_+(w) = 0$. Similar calculation by using $[\sigma(\theta\partial_\theta), \sigma(\partial_\theta)] = \sigma(\partial_\theta)$ shows that $j_-(w) = 0$. It follows from

$$\partial_\theta \theta = \sigma(\theta\partial_\theta) = -\sigma(\partial_\theta)\sigma(\theta) = -(\partial_\theta j_+(D))(\theta h_-(D)) = -\partial_\theta \theta j_+(D) h_-(D)$$

that $j_+(w)h_-(w) = -1$. Since both $j_+(w)$ and $h_-(w)$ are polynomials, $h_-(w)$ is some nonzero constant b while $j_+(w)$ is $-b^{-1}$. Noting that $\theta = \sigma^2(\theta) = b^2\theta$, we have $b = \pm 1$. \square

Lemma 3.7. *Assume that $\sigma(\theta\partial_\theta) = \partial_\theta\theta$. Then $\sigma(t) = \pm t$.*

Proof. Assume that

$$\begin{aligned}\sigma(t) &= \partial_\theta\theta t f_0(D) + \theta\partial_\theta t f_1(D), & \text{for some } f_0(w), f_1(w) \in \mathbb{C}[w], \\ \sigma(D) &= \partial_\theta\theta t g_0(D) + \theta\partial_\theta g_1(D), & \text{for some } g_0(w), g_1(w) \in \mathbb{C}[w].\end{aligned}$$

We calculate

$$\begin{aligned}t &= \sigma^2(t) \\ &= \sigma(\partial_\theta\theta t f_0(D) + \theta\partial_\theta t f_1(D)) \\ &= f_0(\partial_\theta\theta g_0(D) + \theta\partial_\theta g_1(D))(\partial_\theta\theta t f_0(D) + \theta\partial_\theta t f_1(D))\theta\partial_\theta \\ &\quad + f_1(\partial_\theta\theta g_0(D) + \theta\partial_\theta g_1(D))(\partial_\theta\theta t f_0(D) + \theta\partial_\theta t f_1(D))\partial_\theta\theta \\ &= \theta\partial_\theta f_0(g_1(D))(t f_1(D)) + \partial_\theta\theta f_1(g_0(D))(t f_0(D)) \\ &= \theta\partial_\theta t f_0(g_1(D+1))(f_1(D)) + \partial_\theta\theta t f_1(g_0(D+1))(f_0(D)).\end{aligned}$$

As we may rewrite $t = \partial_\theta\theta t + \theta\partial_\theta t$, it follows that

$$f_0(g_1(w+1))f_1(w) = f_1(g_0(w+1))f_0(w) = 1.$$

This implies that $f_0(w)$ and $f_1(w)$ are constants. Let us put $f_0 = c$ ($c \in \mathbb{C}$) then $f_1 = c^{-1}$ and so $\sigma(t) = c\theta\partial_\theta t + c^{-1}\partial_\theta\theta t$.

It follows from $t\theta = \theta t$ that $\sigma(t)\sigma(\theta) = \sigma(\theta)\sigma(t)$. On the other hand we have by Lemma 3.6

$$\begin{aligned}\sigma(t)\sigma(\theta) &= (c\theta\partial_\theta + c^{-1}\partial_\theta\theta)(\pm\theta) = \pm c\theta, \\ \sigma(\theta)\sigma(t) &= (\pm\theta)(c\theta\partial_\theta + c^{-1}\partial_\theta\theta) = \pm c^{-1}\theta.\end{aligned}$$

This implies $c = c^{-1}$ i.e. $c = \pm 1$. Thus $\sigma(t) = c\theta\partial_\theta t + c^{-1}\partial_\theta\theta t = \pm t$. \square

Lemma 3.8. *Assume that $\sigma(\theta\partial_\theta) = \partial_\theta\theta$. Then $\sigma(D) = -D + a$ for some $a \in \mathbb{C}$.*

Proof. Assume that $\sigma(D) = \partial_\theta\theta g_0(D) + \theta\partial_\theta g_1(D)$ for some $g_0(w), g_1(w) \in \mathbb{C}[w]$. We have

$$\begin{aligned}D &= \sigma^2(D) \\ &= \sigma(\partial_\theta\theta g_0(D) + \theta\partial_\theta g_1(D)) \\ &= g_0(\partial_\theta\theta g_0(D) + \theta\partial_\theta g_1(D))\theta\partial_\theta + g_1(\partial_\theta\theta g_0(D) + \theta\partial_\theta g_1(D))\partial_\theta\theta\end{aligned}$$

$$\begin{aligned} &= \theta\partial_\theta g_0(g_1(D)) + \partial_\theta\theta g_1(g_0(D)) \\ &= g_0(g_1(D)) + \partial_\theta\theta(g_1(g_0(D)) - g_0(g_1(D))). \end{aligned}$$

It follows that

$$g_0(g_1(w)) = g_1(g_0(w)) = w.$$

This implies that $g_0(g_0(w)) = w$ and thus $g_0(w) = -w + a$ for some $a \in \mathbb{C}$ or $g_0(w) = w$.

From $D\theta = \theta D$ we conclude that $\sigma(\theta)\sigma(D) = \sigma(D)\sigma(\theta)$. We calculate

$$\begin{aligned} \sigma(\theta)\sigma(D) &= (\pm\theta)(\theta\partial_\theta g_0(D) + \partial_\theta\theta g_1(D)) = \pm\theta g_1(D), \\ \sigma(D)\sigma(\theta) &= (\theta\partial_\theta g_0(D) + \partial_\theta\theta g_1(D))(\pm\theta) = \pm\theta g_0(D). \end{aligned}$$

Thus $g_0(w) = g_1(w)$. We claim that $g_0(w) = w$ is impossible. Indeed if $g_0(w) = w$, then $\sigma(D) = D$. So by Lemma 3.7 (to be definite we choose $\sigma(t) = t$, the case $\sigma(t) = -t$ is the same) it follows that

$$-t = -\sigma(t) = [\sigma(D), \sigma(t)] = [D, t] = t,$$

which is a contradiction. Thus $g_0(w) = g_1(w) = -w + a$ and so $\sigma(w) = -w + a$. □

Now Proposition 3.2 follows from Lemmas 3.6, 3.7 and 3.8.

It is straightforward to check that the σ 's given in Theorem 3.1 are indeed anti-involutions of \mathcal{SD}_{as} . Combining Lemma 3.2 with Propositions 3.1 and 3.2, we have proved Theorem 3.1.

§4. Subalgebras of \mathcal{SD} Fixed by Anti-involutions

Given an anti-involution σ of \mathcal{SD} , one can check easily that

$$\mathcal{SD}^\sigma \equiv \{e \in \mathcal{SD} \mid \sigma(e) = -e\}$$

is a Lie subalgebra of \mathcal{SD} . If σ preserves the principal $\frac{1}{2}\mathbb{Z}$ -gradation of \mathcal{SD} , then \mathcal{SD}^σ inherits the principal $\frac{1}{2}\mathbb{Z}$ -gradation from \mathcal{SD} :

$$\mathcal{SD}^\sigma = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}} \mathcal{SD}_n^\sigma.$$

Denote by $\mathbb{C}[w]^{(1)}$ the space of all odd polynomials in $\mathbb{C}[w]$, and by $\mathbb{C}[w]^{(0)}$ the space of all even polynomials in $\mathbb{C}[w]$. Let $\bar{k} = 0$ if k is an odd integer and $\bar{k} = 1$ if k is even. The purpose of this section is to give an explicit description for the Lie superalgebra \mathcal{SD}^σ , where σ is an involution given in Theorem 3.1. We also provide a set of linear basis elements for these superalgebras, which are canonical in connections to vertex algebras and free field realizations.

Proposition 4.1. *We have the following description for the graded subspaces $\mathcal{SD}_j^{\sigma_{a,b}}$ and $\mathcal{SD}_{j-1/2}^{\sigma_{a,b}}$ ($j \in \mathbb{Z}$) of $\mathcal{SD}^{\sigma_{a,b}}$:*

$$\mathcal{SD}_j^{\sigma_{a,b}} = \{ \partial_\theta \theta t^j f(D + (j - a - 1)/2) + \theta \partial_\theta t^j g(D + (j - a)/2) \mid f, g \in \mathbb{C}^{(\bar{j})}[w] \},$$

$$\mathcal{SD}_{j-1/2}^{\sigma_{a,b}} = \{ \theta t^{j-1} g_0(D) + \partial_\theta (-t)^j b g_0(-D - j + a + 1), g_0 \in \mathbb{C}[w] \}.$$

Proof. Assume that $A = \partial_\theta \theta t^j f_0(D) + \theta \partial_\theta t^j f_1(D)$ and $B = \theta t^{j-1} g_0(D) + \partial_\theta t^j g_1(D)$ belong to $\mathcal{SD}^{\sigma_{a,b}}$.

$$\begin{aligned} \sigma_{a,b}(A) &= f_0(-D + a + \partial_\theta \theta) (-t)^j \partial_\theta \theta + f_1(-D + a + \partial_\theta \theta) (-t)^j \theta \partial_\theta \\ &= (-t)^j f_0(-D - j + a + \partial_\theta \theta) \partial_\theta \theta + (-t)^j f_1(-D - j + a + \partial_\theta \theta) \theta \partial_\theta \\ &= (-t)^j f_0(-D - j + a + 1) \partial_\theta \theta + (-t)^j f_1(-D - j + a) \theta \partial_\theta. \end{aligned}$$

It follows by comparing with $\sigma_{a,b}(A) = -A$ that

$$\begin{aligned} f_0(w) &= (-1)^{j-1} f_0(-w - j + a + 1), \\ f_1(w) &= (-1)^{j-1} f_1(-w - j + a). \end{aligned}$$

Define $f(w) = f_0(w - (j - a - 1)/2)$ and $g(w) = f_1(w - (j - a)/2)$. We see that $f(w), g(w) \in \mathbb{C}^{(\bar{j})}[w]$. On the other hand, we calculate that

$$\begin{aligned} \sigma_{a,b}(B) &= g_0(-D + a + \partial_\theta \theta) (-t)^{j-1} (bt \partial_\theta) + g_1(-D + a + \partial_\theta \theta) (-t)^j (-b^{-1} t^{-1} \theta) \\ &= (-1)^{j-1} b g_0(-D + a + 1) t^j \partial_\theta + (-1)^{j-1} b^{-1} g_1(-D + a) t^{j-1} \theta \\ &= (-1)^{j-1} b t^j g_0(-D - j + a + 1) \partial_\theta \\ &\quad + (-1)^{j-1} b^{-1} t^{j-1} g_1(-D - j + a + 1) \theta. \end{aligned}$$

By comparing with $\sigma_{a,b}(B) = -B$ we have

$$\begin{aligned} g_0(D) &= (-1)^j b^{-1} g_1(-D - j + a + 1), \\ g_1(D) &= (-1)^j b g_0(-D - j + a + 1). \end{aligned}$$

We observe that these two equations are equivalent to each other. \square

Letting $t \mapsto t, \partial_t \mapsto \partial_t, \theta \mapsto \alpha \theta$ and $\partial_\theta \mapsto \alpha \partial_\theta$ defines an automorphism \sharp_α ($\alpha \in \mathbb{C}^\times$) of \mathcal{SD} . It is easy to verify that

$$\sigma_{a,b} \sharp_\alpha = \sharp_{\alpha^{-1}} \sigma_{a,b} = \sigma_{a,b\alpha}.$$

Denote by Θ_s the automorphism of \mathcal{SD} defined by letting $t \mapsto t, D \mapsto D + s, \theta \mapsto \theta$ and $\partial_\theta \mapsto \partial_\theta$. Clearly

$$\sigma_{a,b} \cdot \Theta_s = \sigma_{a+s,b} = \Theta_{-s} \cdot \sigma_{a,b}.$$

In this way, we may regard the $\sigma_{a,b}$ for different a, b as related to each other by some spectral flow (cf. e.g. [AFMO1]). It follows that the Lie superalgebra $\mathcal{SD}^{\sigma_{a,b}}$ is isomorphic to $\mathcal{SD}^{\sigma_{a',b'}}$ for all $a, a' \in \mathbb{C}, b, b' \in \mathbb{C}^\times$. It turns out that a convenient choice is putting $a = -1, b = 1$, which we will fix from now on and will denote $\mathcal{SD}^{\sigma_{-1,1}}$ by ${}^0\mathcal{SD}$. Also the 2-cocycle (2.5) of \mathcal{SD} restricted to ${}^0\mathcal{SD}$ gives rise to a central extension of ${}^0\mathcal{SD}$, which we will denote by ${}^0\widehat{\mathcal{SD}}$.

We introduce a canonical spanning set of ${}^0\mathcal{SD}$ that will be used later. Define

$$\begin{aligned} {}^0W_{0,k}^n &= t^k ([D]_n + (-1)^{k+1}[-D - k]_n) \partial_\theta \theta, \\ {}^0W_{1,k}^n &= t^k ([D]_n + (-1)^{k+1}[-D - k - 1]_n) \theta \partial_\theta, \\ {}^0W_{\times,k}^n &= \theta t^k [D]_n + (-1)^{k+1} \partial_\theta t^{k+1} [-D - k - 1]_n. \end{aligned}$$

Then ${}^0W_{a,k}^n$ ($k \in \mathbb{Z}, n \in \mathbb{N}, a = 0, 1, \times$) span ${}^0\mathcal{SD}$.

Proposition 4.2. *We have the following description for the graded subspaces $\mathcal{SD}_j^{\sigma_{++},a}$ and $\mathcal{SD}_{j-1/2}^{\sigma_{++},a}$ ($j \in \mathbb{Z}$) of $\mathcal{SD}^{\sigma_{++},a}$:*

$$\begin{aligned} \mathcal{SD}_j^{\sigma_{++},a} &= \{ \partial_\theta \theta t^j f(D) - \theta \partial_\theta t^j f(-D - j + a) \mid f(w) \in \mathbb{C}[w] \}, \\ \mathcal{SD}_{j-1/2}^{\sigma_{++},a} &= \{ \theta t^{j-1} f(D + (j - a - 1)/2) + \partial_\theta t^j g(D + (j - a)/2) \\ &\quad \mid f(w) \in \mathbb{C}^{(1)}[w], g(w) \in \mathbb{C}^{(0)}[w] \}. \end{aligned}$$

Proof. Assume that $A = \partial_\theta \theta t^j f(D) + \theta \partial_\theta t^j f_1(D)$ and $B = \theta t^{j-1} g_-(D) + \partial_\theta t^j g_+(D)$ belong to $\mathcal{SD}^{\sigma_{++},a}$.

$$\begin{aligned} \sigma_{++},a(A) &= \theta \partial_\theta f(-D + a) t^j + \partial_\theta \theta f_1(-D + a) t^j \\ &= \theta \partial_\theta t^j f(-D - j + a) + \partial_\theta \theta t^j f_1(-D - j + a). \end{aligned}$$

We obtain by comparing with $\sigma_{++},a(A) = -A$ that

$$-f(D) = f_1(-D - j + a), \quad -f_1(D) = f(-D - j + a).$$

The two equations are clearly equivalent.

On the other hand, we have

$$\begin{aligned} \sigma_{++},a(B) &= g_-(-D + a) t^{j-1} \theta - g_+(-D + a) t^j \partial_\theta \\ &= \theta t^{j-1} g_-(-D - j + 1 + a) + \partial_\theta t^j g_+(-D - j + a). \end{aligned}$$

We obtain by comparing with $\sigma_{++,a}(B) = -B$ that

$$\begin{aligned} -g_-(D) &= g_-(-D - j + 1 + a), \\ g_+(D) &= g_+(-D - j + a). \end{aligned}$$

If we let $f(w) = g_-(w + (a + 1 - j)/2)$ and $g(w) = g_+(w + (a - j)/2)$, we see that $f(w) \in \mathbb{C}^{(1)}[w], g(w) \in \mathbb{C}^{(0)}[w]$. □

In a similar way we determine $\mathcal{SD}^{\sigma_{+-,a}}$ as follows. We omit the arguments, as they are parallel to the ones given above.

Proposition 4.3. *We have the following description for the graded subspaces $\mathcal{SD}_j^{\sigma_{+-,a}}$ and $\mathcal{SD}_{j-1/2}^{\sigma_{+-,a}}$ ($j \in \mathbb{Z}$) of $\mathcal{SD}^{\sigma_{+-,a}}$:*

$$\begin{aligned} \mathcal{SD}_j^{\sigma_{+-,a}} &= \{ \partial_\theta \theta t^j f(D) - \theta \partial_\theta t^j f(-D - j + a) \mid f(w) \in \mathbb{C}[w] \}, \\ \mathcal{SD}_{j-1/2}^{\sigma_{+-,a}} &= \{ \theta t^{j-1} f(D + (j - a - 1)/2) + \partial_\theta t^j g(D + (j - a)/2) \\ &\quad \mid f(w) \in \mathbb{C}^{(0)}[w], g(w) \in \mathbb{C}^{(1)}[w] \}. \end{aligned}$$

Proposition 4.4. *We have the following description for the graded subspaces $\mathcal{SD}_j^{\sigma_{-+,a}}$ and $\mathcal{SD}_{j-1/2}^{\sigma_{-+,a}}$ ($j \in \mathbb{Z}$) of $\mathcal{SD}^{\sigma_{-+,a}}$:*

$$\begin{aligned} \mathcal{SD}_j^{\sigma_{-+,a}} &= \{ \partial_\theta \theta t^j f(D) - (-1)^j \theta \partial_\theta t^j f(-D - j + a) \mid f(w) \in \mathbb{C}[w] \}, \\ \mathcal{SD}_{j-1/2}^{\sigma_{-+,a}} &= \{ \theta t^{j-1} f(D + (j - a - 1)/2) + \partial_\theta t^j g(D + (j - a)/2) \\ &\quad \mid f(w), g(w) \in \mathbb{C}^{(\overline{j+1})}[w] \}. \end{aligned}$$

Proof. Assume that $A = \partial_\theta \theta t^j f(D) + \theta \partial_\theta t^j f_1(D)$ and $B = \theta t^{j-1} g_-(D) + \partial_\theta t^j g_+(D)$ belong to $\mathcal{SD}^{\sigma_{-+,a}}$. We have

$$\begin{aligned} \sigma_{-+,a}(A) &= \theta \partial_\theta f(-D + a)(-t)^j + \partial_\theta \theta f_1(-D + a)(-t)^j \\ &= \theta \partial_\theta (-t)^j f(-D - j + a) + \partial_\theta \theta (-t)^j f_1(-D - j + a). \end{aligned}$$

We obtain by comparing with $\sigma_{-+,a}(A) = -A$ that

$$-f(D) = (-1)^j f_1(-D - j + a), \quad -f_1(D) = (-1)^j f(-D - j + a).$$

These two equations are equivalent.

On the other hand, we have

$$\begin{aligned} \sigma_{-+,a}(B) &= g_-(-D + a)(-t)^{j-1} \theta - g_+(-D + a)(-t)^j \partial_\theta \\ &= \theta (-t)^{j-1} g_-(-D - j + 1 + a) + \partial_\theta (-t)^j g_+(-D - j + a) \end{aligned}$$

We obtain by comparing with $\sigma_{-,a}(B) = -B$ that

$$\begin{aligned} -g_-(D) &= (-1)^{j-1}g_-(-D - j + 1 + a), \\ g_+(D) &= (-1)^jg_+(-D - j + a). \end{aligned}$$

If we let $f(w) = g_-(w + (a + 1 - j)/2)$ and $g(w) = g_+(w + (a - j)/2)$, we see that $f(w), g(w) \in \mathbb{C}^{(j+1)}[w]$. □

In a similar way one determines $\mathcal{SD}^{\sigma--,a}$ as follows.

Proposition 4.5. *We have the following description for the graded subspaces $\mathcal{SD}_j^{\sigma--,a}$ and $\mathcal{SD}_{j-1/2}^{\sigma--,a}$ ($j \in \mathbb{Z}$) of $\mathcal{SD}^{\sigma--,a}$:*

$$\begin{aligned} \mathcal{SD}_j^{\sigma--,a} &= \{ \partial_\theta \theta t^j f(D) - (-1)^j \theta \partial_\theta t^j f(-D - j + a) \mid f(w) \in \mathbb{C}[w] \}, \\ \mathcal{SD}_{j-1/2}^{\sigma--,a} &= \{ \theta t^{j-1} f(D + (j - a - 1)/2) + \partial_\theta t^j g(D + (j - a)/2) \\ &\quad \mid f(w), g(w) \in \mathbb{C}^{(\bar{j})}[w] \}. \end{aligned}$$

Observe that the $\sigma_{\pm\pm,a}$'s for different a are transformed into each other by some spectral flow. More explicitly we have $\sigma_{\pm\pm,a} \cdot \Theta_s = \sigma_{\pm\pm,a+s} = \Theta_{-s} \cdot \sigma_{\pm\pm,a}$. Thus the Lie superalgebra $\mathcal{SD}^{\sigma\pm\pm,a}$ is isomorphic to $\mathcal{SD}^{\sigma\pm\pm,a'}$ for $a, a' \in \mathbb{C}$. As we shall see, it is most convenient to choose $a = -1$ which we will fix from now on. Also we will write ${}^{\pm\pm}\mathcal{SD}$ for $\mathcal{SD}^{\sigma\pm\pm,-1}$. Similarly the corresponding subalgebras of $\widehat{\mathcal{SD}}$, induced by the 2-cocycle (2.5), will be denoted by ${}^{\pm\pm}\widehat{\mathcal{SD}}$.

We conclude this section by displaying a distinguished spanning set for ${}^{\pm\pm}\mathcal{SD}$, that will be of use later on. Let

$$\begin{aligned} {}^{+\pm}W_k^n &= t^k ([D]_n \partial_\theta \theta - [-D - k - 1]_n \theta \partial_\theta), \\ {}^{\pm\pm}W_{+,k}^n &= t^k ([D]_n \mp [-D - k - 1]_n) \theta, \\ {}^{\pm\pm}W_{-,k}^n &= t^k ([D]_n \pm [-D - k - 1]_n) \partial_\theta. \end{aligned}$$

Then ${}^{+\pm}W_k^n, {}^{\pm\pm}W_{+,k}^n$ and ${}^{\pm\pm}W_{-,k}^n$ ($k \in \mathbb{Z}, n \in \mathbb{Z}_+$) span ${}^{+\pm}\mathcal{SD}$. Similarly we set

$$\begin{aligned} -{}^{\pm\pm}W_k^n &= t^k ([D]_n \partial_\theta \theta + (-1)^{k+1} [-D - k - 1]_n \theta \partial_\theta), \\ -{}^{\pm\pm}W_{+,k}^n &= t^k ([D]_n \pm (-1)^{k+1} [-D - k - 1]_n) \theta, \\ -{}^{\pm\pm}W_{-,k}^n &= t^k ([D]_n \pm (-1)^k [-D - k - 1]_n) \partial_\theta. \end{aligned}$$

Then $-{}^{\pm\pm}W_k^n, -{}^{\pm\pm}W_{+,k}^n$ and $-{}^{\pm\pm}W_{-,k}^n$ ($k \in \mathbb{Z}, n \in \mathbb{Z}_+$) span $-{}^{\pm\pm}\mathcal{SD}$.

§5. Criterion for Quasifiniteness of Modules over ${}^i\widehat{SD}$

Given a $\frac{1}{2}\mathbb{Z}$ -graded Lie superalgebra $\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$ (possibly $\dim \mathfrak{g}_j = \infty$) with $\bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ as the even part and $\bigoplus_{j \in \frac{1}{2} + \mathbb{Z}} \mathfrak{g}_j$ as the odd part. A \mathfrak{g} -module $M = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} M_j$ is *graded* if $\mathfrak{g}_i M_j \subset M_{i+j}$. M is called *quasifinite* if $\dim M_j < \infty$, following the terminology of Kac and Radul [KR1]. We assume further that \mathfrak{g}_0 is abelian and regard it as the Cartan subalgebra of \mathfrak{g} .

Given a highest weight ξ in the restricted dual \mathfrak{g}_0^* of \mathfrak{g}_0 , we denote by $L(\mathfrak{g}; \xi)$ the irreducible representation of \mathfrak{g} with highest weight ξ and a highest weight vector v_ξ annihilated by $\mathfrak{g}_+ = \bigoplus_{j \in \frac{1}{2}\mathbb{N}} \mathfrak{g}_j$. That is, it is the unique irreducible quotient of the \mathfrak{g} -module obtained by inducing from the one-dimensional $(\bigoplus_{j \geq 0} \mathfrak{g}_j)$ -module determined by ξ .

From now on \mathfrak{g} will always be one of the Lie superalgebras ${}^0\widehat{SD}$ or ${}^{\pm\pm}\widehat{SD}$. Note that any of these Lie superalgebras has infinite-dimensional graded subspaces. Our problem is to determine for which ξ the corresponding irreducible highest weight module $L(\mathfrak{g}, \xi)$ is quasifinite. Note that the Verma module is never quasifinite. We remark that the $\mathcal{W}_{1+\infty}$ case was first determined in [KR1] and subsequently the \widehat{SD} case was determined in [AFMO1]. Our analysis below has a somewhat different flavor.

An obvious necessary condition for $L(\mathfrak{g}, \xi)$ to be quasifinite is that $\mathfrak{g}_{-j}v_\xi$ ($j > 0$) is finite-dimensional, since $\mathfrak{g}_{-j}v_\xi \subset L(\mathfrak{g}, \xi)_{-j}$. We will be particularly interested in the case $j = \frac{1}{2}$. We will show eventually that the condition that $\mathfrak{g}_{-1/2}v_\xi$ is finite dimensional is also sufficient for $L(\mathfrak{g}; \xi)$ to be quasifinite.

For $j \in \frac{1}{2}\mathbb{Z}$ we define $P_j = \{X \in \mathfrak{g}_j \mid Xv_\xi = 0\}$. Obviously $P = \bigoplus_j P_j$ is a parabolic subalgebra of \mathfrak{g} .

Proposition 5.1. *Let $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ be a \mathbb{Z} -graded Lie superalgebra such that \mathfrak{g}_0 is abelian. Let $\xi \in \mathfrak{g}_0^*$ and $L(\mathfrak{g}, \xi)$ be the irreducible highest weight module of highest weight ξ and v_ξ a corresponding highest weight vector. Suppose that $\mathfrak{g}_j v_\xi$ is finite-dimensional for all j . Then $L(\mathfrak{g}, \xi)$ is quasifinite.*

Proof. Of course if $L(\mathfrak{g}, \xi)$ is quasifinite, then $\mathfrak{g}_j v_\xi$ is necessarily finite-dimensional for all j .

Conversely suppose that $\mathfrak{g}_j v_\xi$ is finite-dimensional for all j . We need to show that for any positive integers $i_1, i_2, \dots, i_m \in \mathbb{N}$ the space $\mathfrak{g}_{-i_1} \mathfrak{g}_{-i_2} \cdots \mathfrak{g}_{-i_m} v_\xi$ is finite-dimensional. We will do this by induction on m , with $m = 1$ being the hypothesis of the proposition.

Now $\mathfrak{g}_{-i_m} v_\xi$ is a finite-dimensional vector space, hence there exist finitely many elements $x_1, x_2, \dots, x_n \in \mathfrak{g}_{-i_m}$ such that \mathfrak{g}_{-i_m} is spanned by $x_1 v_\xi,$

$x_2v_\xi, \dots, x_nv_\xi$. This implies that $\mathfrak{g}_{-i_1}\mathfrak{g}_{-i_2}\cdots\mathfrak{g}_{-i_m}v_\xi$ lies in the span of $\mathfrak{g}_{-i_1}\mathfrak{g}_{-i_2}\cdots\mathfrak{g}_{-i_{m-1}}x_iv_\xi$, for $1 \leq i \leq n$. Thus it is enough to prove that, for each i , each of these $\mathfrak{g}_{-i_1}\mathfrak{g}_{-i_2}\cdots\mathfrak{g}_{-i_{m-1}}x_iv_\xi$ is finite-dimensional.

Now we have

$$\mathfrak{g}_{-i_1}\mathfrak{g}_{-i_2}\cdots\mathfrak{g}_{-i_{m-1}}x_iv_\xi \subseteq x_i\mathfrak{g}_{-i_1}\mathfrak{g}_{-i_2}\cdots\mathfrak{g}_{-i_{m-1}}v_\xi + [x_i, \mathfrak{g}_{-i_1}\mathfrak{g}_{-i_2}\cdots\mathfrak{g}_{-i_{m-1}}]v_\xi.$$

By induction $\mathfrak{g}_{-i_1}\mathfrak{g}_{-i_2}\cdots\mathfrak{g}_{-i_{m-1}}v_\xi$ is a finite-dimensional vector space and hence the space $x_i\mathfrak{g}_{-i_1}\mathfrak{g}_{-i_2}\cdots\mathfrak{g}_{-i_{m-1}}v_\xi$ is finite-dimensional. Thus it is enough to prove that $[x_i, \mathfrak{g}_{-i_1}\mathfrak{g}_{-i_2}\cdots\mathfrak{g}_{-i_{m-1}}]v_\xi$ is finite-dimensional as well. But

$$\begin{aligned} [x_i, \mathfrak{g}_{-i_1}\mathfrak{g}_{-i_2}\cdots\mathfrak{g}_{-i_{m-1}}]v_\xi &\subseteq [x_i, \mathfrak{g}_{-i_1}]\mathfrak{g}_{-i_2}\cdots\mathfrak{g}_{-i_{m-1}}v_\xi \\ &\quad + \mathfrak{g}_{-i_1}[x_i, \mathfrak{g}_{-i_2}]\cdots\mathfrak{g}_{-i_{m-1}}v_\xi + \cdots \\ &\quad + \mathfrak{g}_{-i_1}\mathfrak{g}_{-i_2}\cdots[x_i, \mathfrak{g}_{-i_{m-1}}]v_\xi. \end{aligned}$$

However each of these spaces on the right-hand side lies in some finite-dimensional vector space by induction hypothesis. Thus $[x_i, \mathfrak{g}_{-i_1}\mathfrak{g}_{-i_2}\cdots\mathfrak{g}_{-i_{m-1}}]v_\xi$ is also finite-dimensional. □

Proposition 5.1 implies that quasifiniteness is equivalent to the spaces \mathfrak{g}_j/P_j being finite-dimensional, for all $j < 0$. Hence our problem at hand reduces to showing that the subspaces P_j have finite co-dimension in \mathfrak{g}_j for all $j < 0$.

For a given ξ , we introduce the following generating functions in a variable x for the highest weights:

$$\begin{aligned} {}^0\Delta_0(x) &= -\xi(\partial_\theta\theta \sinh(xD)), \\ {}^0\Delta_1(x) &= -\xi(\theta\partial_\theta \sinh(x(D + 1/2))), \\ {}^{\pm\pm}\Delta_0(x) &= -\xi(\sinh(x(D + 1/2))), \\ {}^{\pm\pm}\Delta_1(x) &= -\xi((\theta\partial_\theta - \partial_\theta\theta) \cosh(x(D + 1/2))), \end{aligned}$$

where we recall that $\sinh(z) = (e^z - e^{-z})/2$ and $\cosh(z) = (e^z + e^{-z})/2$.

§5.1. The case $\mathfrak{g} = {}^0\widehat{SD}$

We will work out this case in detail. The other cases can be worked out in a similar and straightforward fashion and we will omit the arguments for the sake of preserving space.

Let us first recall that for $j \in \mathbb{Z}$

$$\begin{aligned} \mathfrak{g}_j &= {}^0\widehat{\mathcal{SD}}_j = \{\partial_\theta \theta t^j f(D + j/2) + \theta \partial_\theta t^j g(D + (j + 1)/2) \mid f, g \in \mathbb{C}^{(\bar{j})}[w]\}, \\ \mathfrak{g}_{j-\frac{1}{2}} &= {}^0\widehat{\mathcal{SD}}_{j-\frac{1}{2}} = \{\theta t^{j-1} g_0(D) + \partial_\theta (-t)^j g_0(-D - j), g_0 \in \mathbb{C}[w]\}. \end{aligned}$$

For $j \in \mathbb{Z}$, we define I_j^0 (respectively I_j^1) to be the subspace of $\mathbb{C}^{(\bar{j})}[w]$ consisting of polynomials $f(w)$ such that $\partial_\theta \theta t^j f(D + j/2) \in P_j$ (respectively $\theta \partial_\theta t^j f(D + (j + 1)/2) \in P_j$). Similarly we let $I_{j-\frac{1}{2}}$ be the subspace of $\mathbb{C}[w]$ consisting of polynomials $f(w)$ such that $\theta t^{j-1} f(D) + \partial_\theta (-t)^j f(-D - j) \in P_{j-\frac{1}{2}}$.

Lemma 5.1. *For $j \in \mathbb{Z}$, $I_{j-\frac{1}{2}}$ is an ideal of $\mathbb{C}[w]$, while I_j^0 and I_j^1 are $\mathbb{C}^{(0)}[w]$ -submodules of $\mathbb{C}^{(\bar{j})}[w]$. Thus there exist monomials $\alpha_{j-\frac{1}{2}}(w) \in \mathbb{C}[w]$ and $\beta_j^0(w), \beta_j^1(w) \in \mathbb{C}^{(\bar{j})}[w]$ such that $I_{j-\frac{1}{2}} = \mathbb{C}[w]\alpha_{j-\frac{1}{2}}(w)$, $I_j^0 = \mathbb{C}^{(0)}[w]\beta_j^0(w)$ and $I_j^1 = \mathbb{C}^{(0)}[w]\beta_j^1(w)$. Furthermore*

$$P_j = \left\{ \partial_\theta \theta t^j f \left(D + \frac{j}{2} \right) + \theta \partial_\theta t^j g \left(D + \frac{j+1}{2} \right) \mid f \in I_j^0, g \in I_j^1 \right\}.$$

Proof. Consider $f(w) \in \mathbb{C}^{(1)}[w]$ so that $\partial_\theta \theta f(D) \in \mathfrak{g}_0$. Take any $\theta t^{j-1} g(D) + \partial_\theta (-t)^j g(-D - j) \in P_{j-\frac{1}{2}}$, for some $g(w) \in I_{j-\frac{1}{2}}$. We compute their bracket

$$\begin{aligned} & [\partial_\theta \theta f(D), \theta t^{j-1} g(D) + \partial_\theta (-t)^j g(-D - j)] \\ &= -(\theta t^{j-1} g(D) f(D) + \partial_\theta (-t)^j f(-D - j) g(-D - j)) \in P_{j-\frac{1}{2}}. \end{aligned}$$

Thus $f(w)g(w) \in I_{j-\frac{1}{2}}$, for any $f(w) \in \mathbb{C}^{(1)}[w]$, that is, $I_{j-\frac{1}{2}}$ is invariant under the multiplication of any odd polynomial. But then it follows that $I_{j-\frac{1}{2}}$ is invariant under the multiplication of any polynomial, and hence is an ideal of $\mathbb{C}[w]$.

Again take any $f(w) \in \mathbb{C}^{(1)}[w]$ so that $\partial_\theta \theta f(D) \in \mathfrak{g}_0$, and any element

$$\partial_\theta \theta t^j g \left(D + \frac{j}{2} \right) + \theta \partial_\theta t^j \bar{g} \left(D + \frac{j+1}{2} \right) \in P_j$$

for some $g(w) \in I_j^0$ and $\bar{g}(w) \in I_j^1$. We compute

$$\left[\partial_\theta \theta f(D), \partial_\theta \theta t^j g \left(D + \frac{j}{2} \right) + \theta \partial_\theta t^j \bar{g} \left(D + \frac{j+1}{2} \right) \right] = \partial_\theta \theta g \left(D + \frac{j}{2} \right) h \left(D + \frac{j}{2} \right),$$

where $h(D + \frac{j}{2}) = f(D + j) - f(D)$. As $f(w)$ ranges over all odd polynomials, $h(w)$ ranges over all even polynomials. From this it follows that I_j^0 is a module over $\mathbb{C}^{(0)}[w]$.

Replacing $\partial_\theta \theta f(D)$ with $\theta \partial_\theta f(D)$ in the above calculation one shows similarly that I_j^1 is a module over $\mathbb{C}^{(0)}[w]$. \square

It follows from Lemma 5.1 that $L(\mathfrak{g}, \xi)$ is quasifinite if and only if I_j^0, I_j^1 , and $I_{j+\frac{1}{2}}$ are all nonzero for $j \in -\mathbb{N}$.

Lemma 5.2. *Suppose that $I_{-\frac{1}{2}} \neq 0$ and $j \in \mathbb{N}$. We have:*

- (i) $I_j^a \neq 0$ if and only if $I_{j-\frac{1}{2}} \neq 0$, for $a = 0, 1$.
- (ii) $I_{j+\frac{1}{2}} \neq 0$ if and only if $I_j^a \neq 0$, for $a = 0, 1$.

Proof. We will first prove (ii). Suppose that $0 \neq f(w) \in I_j^0$. Let $g(w) \in \mathbb{C}[w]$ so that $\theta g(D) - \partial_\theta t g(-D - 1) \in \mathfrak{g}_{\frac{1}{2}}$. We compute

$$\begin{aligned} & \left[\partial_\theta \theta t^j f \left(D + \frac{j}{2} \right), \theta g(D) - \partial_\theta t g(-D - 1) \right] \\ &= \theta t^j h(D) + \partial_\theta (-t)^{j+1} h(-D - j - 1) \in P_{j+\frac{1}{2}}, \end{aligned}$$

where $h(D) = g(D + j)f(D + \frac{j}{2})$. Since $f(w) \neq 0$ and $g(w)$ can be taken to be an arbitrary non-zero element of $\mathbb{C}[w]$, it follows that $I_{j+\frac{1}{2}} \neq 0$. The claim that $I_j^1 \neq 0$ implies that $I_{j+\frac{1}{2}} \neq 0$ is proved similarly and thus omitted.

Conversely suppose that $I_{j+\frac{1}{2}} \neq 0$. Let $0 \neq f(w) \in I_{j+\frac{1}{2}}$ and $0 \neq g(w) \in I_{-\frac{1}{2}}$ and compute

$$\begin{aligned} & [\theta t^{-1} g(D) + \partial_\theta g(-D), \theta t^j f(D) + \partial_\theta (-t)^{j+1} f(-D - j - 1)] \\ &= \partial_\theta \theta t^j h \left(D + \frac{j}{2} \right) + \theta \partial_\theta t^j k \left(D + \frac{j+1}{2} \right), \end{aligned}$$

where $h(D + \frac{j}{2}) = f(D)g(-D - j) + (-1)^{j+1} f(-D - j)g(D)$ and $k(D + \frac{j+1}{2}) = f(D)g(-D) + (-1)^{j+1} f(-D - j - 1)g(D + j + 1)$.

Now $h(w) = 0$ if and only if $f(w - \frac{j}{2})g(-w - \frac{j}{2}) \in \mathbb{C}^{\overline{(j+1)}}[w]$. Now since $I_{j+\frac{1}{2}}$ is an ideal of $\mathbb{C}[w]$, we may replace $f(w) \in I_{j+\frac{1}{2}}$ above by $0 \neq (w + \frac{j}{2})f(w) \in I_{j+\frac{1}{2}}$. This implies then that $wf(w - \frac{j}{2})g(-w - \frac{j}{2}) \in \mathbb{C}^{\overline{(j+1)}}[w]$ as well. But this is impossible unless $f(w - \frac{j}{2})g(-w - \frac{j}{2}) = 0$, which is a contradiction. Hence $I_j^0 \neq 0$. Similarly $I_j^1 \neq 0$. This proves (ii).

Next we prove (i). For this suppose $0 \neq f(w) \in I_j^0$ and $0 \neq g(w) \in I_{-\frac{1}{2}}$. Then

$$\left[\partial_\theta \theta t^j f \left(D + \frac{j}{2} \right), \theta t^{-1} g(D) + \partial_\theta g(-D) \right]$$

$$= (-1)^j \left(\theta t^{j-1} g(D+j) f \left(-D - \frac{j}{2} \right) + (-t)^j \partial_\theta f \left(D + \frac{j}{2} \right) g(-D) \right) \in P_{j-\frac{1}{2}},$$

and hence $I_{j-\frac{1}{2}} \neq 0$. Similarly $I_j^1 \neq 0$ implies $I_{j-\frac{1}{2}} \neq 0$.

Finally suppose that $0 \neq g(w) \in I_{\frac{1}{2}} = \mathbb{C}[w]$ and $0 \neq f(w) \in I_{j-\frac{1}{2}}$ and consider

$$\begin{aligned} & [\theta g(D) - \partial_\theta t g(-D-1), \theta t^{j-1} f(D) + \partial_\theta (-t)^j f(-D-j)] \\ &= \partial_\theta \theta t^j h \left(D + \frac{j}{2} \right) + \theta \partial_\theta t^j k \left(D + \frac{j+1}{2} \right), \end{aligned}$$

where $h(D + \frac{j}{2}) = (-1)^j f(-D-j)g(D) - f(D)g(-D-j)$ and $k(D + \frac{j+1}{2}) = (-1)^j g(D+j)f(-D-j) - f(D+1)g(-D-1)$. Now the same argument proving $I_{j+\frac{1}{2}} \neq 0$ implying $I_j^a \neq 0$ can be used to derive that $I_j^a \neq 0$, for $a = 0, 1$. \square

Corollary 5.1. *The irreducible highest weight module $L(\mathfrak{g}, \xi)$ is quasifinite if and only if $I_{-\frac{1}{2}} \neq 0$.*

In particular setting $\alpha(w) = \alpha_{-\frac{1}{2}}(w)$ in Lemma 5.1 we obtain the following.

Corollary 5.2. *There exists a monomial $\alpha(w) \in \mathbb{C}[w]$ such that any element in $({}^0\widehat{\mathcal{SD}})_{-\frac{1}{2}}$ annihilating v_ξ is of the form $\theta t^{-1}g(D)\alpha(D) + \partial_\theta g(-D)\alpha(-D)$, for some $g(w) \in \mathbb{C}[w]$.*

Take this $\theta t^{-1}\alpha(D) + \partial_\theta \alpha(-D) \in \mathfrak{g}_{-1/2}$ and any element $\theta f(D) - \partial_\theta t f(-D-1) \in \mathfrak{g}_{1/2}$. One computes

$$\begin{aligned} & [\theta f(D) - \partial_\theta t f(-D-1), \theta t^{-1}\alpha(D) + \partial_\theta \alpha(-D)] \\ &= \partial_\theta \theta (\alpha(-D)f(D) - \alpha(D)f(-D)) \\ &+ \theta \partial_\theta (\alpha(-D)f(D) - \alpha(D+1)f(-D-1)) - \alpha(0)f(0)C. \end{aligned}$$

Since $[\theta f(D) - \partial_\theta t f(-D-1), \theta t^{-1}\alpha(D) + \partial_\theta \alpha(-D)]v_\xi = 0$, we obtain

$$\begin{aligned} & \xi \{ \partial_\theta \theta (\alpha(-D)f(D) - \alpha(D)f(-D)) \\ &+ \theta \partial_\theta (\alpha(-D)f(D) - \alpha(D+1)f(-D-1)) - \alpha(0)f(0)c \} = 0. \end{aligned}$$

Putting $f(D) = e^{-x D}$ this is equivalent to

$$(5.1) \quad \alpha \left(\frac{d}{dx} \right) \left({}^0\Delta_0(x) + {}^0\Delta_1(x)e^{x/2} - \frac{c}{2} \right) = 0.$$

Combining the results of this section we obtain the following characterization of quasifinite ${}^0\widehat{\mathcal{SD}}$ -modules.

Theorem 5.1. *The irreducible highest weight module $L({}^0\widehat{\mathcal{SD}}, \xi)$ is quasifinite if and only if ${}^0\Delta_0(x)$ and ${}^0\Delta_1(x)$ satisfy the differential equation (5.1).*

§5.2. The case $\mathfrak{g} = {}^{\pm\pm}\widehat{\mathcal{SD}}$

First we let $\mathfrak{g} = {}^{++}\widehat{\mathcal{SD}}$. The arguments leading to the following proposition is similar to the ones we have given in the previous section leading to Corollaries 5.1 and 5.2 and so we will omit the details.

Proposition 5.2. *There exist $\alpha(w) \in \mathbb{C}^{(1)}[w]$ and $\beta(w) \in \mathbb{C}^{(0)}[w]$ such that any element in $({}^{++}\widehat{\mathcal{SD}})_{-\frac{1}{2}}$ annihilating v_ξ is of the form $\theta t^{-1}f(D)\alpha(D) + \partial_\theta g(D)\beta(D + 1/2)$, where $f(w), g(w) \in \mathbb{C}^{(0)}[w]$. Furthermore the irreducible highest weight module $L(\mathfrak{g}, \xi)$ is quasifinite if and only if $\alpha(w) \neq 0$ and $\beta(w) \neq 0$.*

Take $\theta f(D + 1/2) + \partial_\theta t g(D + 1) \in \mathfrak{g}_{1/2}$ and $\theta t^{-1}\alpha(D) + \partial_\theta \beta(D + 1/2) \in P_{-1/2}$, where $\alpha, f \in \mathbb{C}^{(1)}[w]$ and $\beta, g \in \mathbb{C}^{(0)}[w]$. One computes

$$\begin{aligned} & [\theta f(D + 1/2) + \partial_\theta t g(D + 1), \theta t^{-1}\alpha(D) + \partial_\theta \beta(D + 1/2)] \\ &= \beta(D + 1/2)f(D + 1/2) + \frac{1}{2}(\alpha(D + 1)g(D + 1) + \alpha(D)g(D)) \\ & \quad + \frac{1}{2}(\theta\partial_\theta - \partial_\theta\theta)(\alpha(D + 1)g(D + 1) - \alpha(D)g(D)) + \alpha(0)g(0)c. \end{aligned}$$

This implies that

$$\begin{aligned} (5.2) \quad & \xi \left(\beta(D + 1/2)f(D + 1/2) + \frac{1}{2}(\alpha(D + 1)g(D + 1) + \alpha(D)g(D)) \right. \\ & \left. + \frac{1}{2}(\theta\partial_\theta - \partial_\theta\theta)(\alpha(D + 1)g(D + 1) - \alpha(D)g(D)) + \alpha(0)g(0)c \right) = 0. \end{aligned}$$

By putting $g(D) = 0$ and $f(D + 1/2) = \frac{1}{2}(e^{x(D+1/2)} - e^{-x(D+1/2)})$ we have

$$(5.3) \quad \beta \left(\frac{d}{dx} \right) ({}^{++}\Delta_0(x)) = 0.$$

Now putting $f(D) = 0$ and $g(D) = \frac{1}{2}(e^{xD} + e^{-xD})$ we obtain

$$(5.4) \quad \alpha \left(\frac{d}{dx} \right) \left((e^{x/2} + e^{-x/2}) \cdot {}^{++}\Delta_0(x) + (e^{x/2} - e^{-x/2}) \cdot {}^{++}\Delta_1(x) - 2c \right) = 0.$$

Next, we consider $\mathfrak{g} = {}^{+-}\widehat{\mathcal{SD}}$. In this case we have similarly the following proposition.

Proposition 5.3. *There exist $\alpha(w) \in \mathbb{C}^{(0)}[w]$ and $\beta(w) \in \mathbb{C}^{(1)}[w]$ such that any element in $({}^{+-}\widehat{\mathcal{SD}})_{-\frac{1}{2}}$ annihilating v_ξ is of the form $\theta t^{-1}f(D)\alpha(D) + \partial_\theta g(D)\beta(D+1/2)$, where $f(w), g(w) \in \mathbb{C}^{(0)}[w]$. Furthermore the irreducible highest weight module $L(\mathfrak{g}, \xi)$ is quasifinite if and only if $\alpha(w) \neq 0$ and $\beta(w) \neq 0$.*

Take $\theta f(D+1/2) + \partial_\theta t g(D+1) \in \mathfrak{g}_{1/2}$ and $\theta t^{-1}\alpha(D) + \partial_\theta \beta(D+1/2) \in \mathfrak{g}_{-1/2}$, where $\alpha, f \in \mathbb{C}^{(0)}[w]$ and $\beta, g \in \mathbb{C}^{(1)}[w]$. As before we obtain identity (5.2). Now using the pairs $g(D) = 0$ together with $f(D+1/2) = \frac{1}{2}(e^{x(D+1/2)} + e^{-x(D+1/2)})$ and $f(D) = 0$ together with $g(D) = \frac{1}{2}(e^{xD} + e^{-xD})$ we obtain again the equations (5.3) and (5.4), where $\Delta_i^{++}(x)$ is replaced by $\Delta_i^{+-}(x)$.

Next, we consider $\mathfrak{g} = {}^{-+}\widehat{\mathcal{SD}}$. In this case we have similarly the following proposition.

Proposition 5.4. *There exist $\alpha(w), \beta(w) \in \mathbb{C}^{(0)}[w]$ such that any element in $({}^{+-}\widehat{\mathcal{SD}})_{-\frac{1}{2}}$ annihilating v_ξ is of the form $\theta t^{-1}f(D)\alpha(D) + \partial_\theta g(D)\beta(D+1/2)$, where $f(w), g(w) \in \mathbb{C}^{(0)}[w]$. Furthermore the irreducible highest weight module $L(\mathfrak{g}, \xi)$ is quasifinite if and only if $\alpha(w) \neq 0$ and $\beta(w) \neq 0$.*

Take $\theta f(D+1/2) + \partial_\theta t g(D+1) \in \mathfrak{g}_{1/2}$ and $\theta t^{-1}\alpha(D) + \partial_\theta \beta(D+1/2) \in \mathfrak{g}_{-1/2}$, where $\alpha, f \in \mathbb{C}^{(0)}[w]$ and $\beta, g \in \mathbb{C}^{(1)}[w]$, and taking their bracket gives identity (5.2). Now using the pairs $g(D) = 0$ together with $f(D+1/2) = \frac{1}{2}(e^{x(D+1/2)} - e^{-x(D+1/2)})$ and $f(D) = 0$ together with $g(D) = \frac{1}{2}(e^{xD} - e^{-xD})$ we obtain again (5.3) and (5.4), where $\Delta_i^{++}(x)$ is replaced by $\Delta_i^{-+}(x)$.

Finally, we consider $\mathfrak{g} = {}^{--}\widehat{\mathcal{SD}}$. In this case we have similarly the following proposition.

Proposition 5.5. *There exist $\alpha(w), \beta(w) \in \mathbb{C}^{(1)}[w]$ such that any element in $({}^{+-}\widehat{\mathcal{SD}})_{-\frac{1}{2}}$ annihilating v_ξ is of the form $\theta t^{-1}f(D)\alpha(D) + \partial_\theta g(D)\beta(D+1/2)$, where $f(w), g(w) \in \mathbb{C}^{(0)}[w]$. Furthermore the irreducible highest weight module $L(\mathfrak{g}, \xi)$ is quasifinite if and only if $\alpha(w) \neq 0$ and $\beta(w) \neq 0$.*

As before we obtain identity (5.2). Now we use the pairs $g(D) = 0$ together with $f(D+1/2) = \frac{1}{2}(e^{x(D+1/2)} + e^{-x(D+1/2)})$ and $f(D) = 0$ together with $g(D) = \frac{1}{2}(e^{xD} + e^{-xD})$ to obtain (5.3) and (5.4), where $\Delta_i^{++}(x)$ is replaced by $\Delta_i^{--}(x)$.

We summarize our results in the following.

Theorem 5.2. *The irreducible highest weight module $L(\pm\pm\widehat{\mathcal{SD}}, \xi)$ is quasifinite if and only if $\pm\pm\Delta_0(x)$ and $\pm\pm\Delta_1(x)$ satisfy the following differential equations:*

$$\begin{aligned} \beta\left(\frac{d}{dx}\right)(\pm\pm\Delta_0(x)) &= 0, \\ \alpha\left(\frac{d}{dx}\right)\left((e^{x/2} + e^{-x/2}) \cdot \pm\pm\Delta_0(x)\right) \\ + \alpha\left(\frac{d}{dx}\right)\left((e^{x/2} - e^{-x/2}) \cdot \pm\pm\Delta_1(x) - 2c\right) &= 0. \end{aligned}$$

§6. Subalgebras of Lie Superalgebras $\widehat{\mathfrak{gl}}_{\infty|\infty}$ and $\widehat{\mathcal{SD}}$

In this section we will describe Lie subalgebras of $\mathfrak{gl}_{\infty|\infty}$ of types *OSP* and *P*. We study the embeddings of the Lie subalgebras of \mathcal{SD} introduced in Section 4 into (the subalgebras of) $\mathfrak{gl}_{\infty|\infty}$. We then study the relations among the central extensions of these superalgebras. Also cf. [AFMO1] for the embedding of $\widehat{\mathcal{SD}}$ into $\widehat{\mathfrak{gl}}_{\infty|\infty}$.

§6.1. Lie superalgebra $\mathfrak{gl}_{\infty|\infty}$

Let M_∞ be the associative algebra consisting of matrices $(a_{ij})_{i,j \in \mathbb{Z}}$ such that $a_{ij} = 0$ for $|i - j| \gg 0$. We denote by \mathfrak{gl}_∞ the Lie algebra obtained from M_∞ by taking the usual commutator. Denote by E_{ij} the elementary matrix with 1 at the (i, j) -th entry and 0 elsewhere. Define the Lie superalgebra $\mathfrak{gl}_{\infty|\infty}$ to be $\mathfrak{gl}_\infty \otimes M(1, 1)$ with the induced \mathbb{Z}_2 -graded structure from $M(1, 1)$ (recall that $M(1, 1)$ is defined earlier in Section 2).

One may have two different ways of looking at $\mathfrak{gl}_{\infty|\infty}$. First we may regard $\mathfrak{gl}_{\infty|\infty} = \bigoplus_{a=0,1,\pm} \mathfrak{gl}_\infty M_a$, that is,

$$\mathfrak{gl}_{\infty|\infty} = \begin{bmatrix} \mathfrak{gl}_\infty & \mathfrak{gl}_\infty \\ \mathfrak{gl}_\infty & \mathfrak{gl}_\infty \end{bmatrix}.$$

Secondly, one may identify $\mathfrak{gl}_{\infty|\infty} = \{(a_{ij})_{i,j \in \mathbb{Z}/2} \mid a_{ij} = 0, \text{ for } |i - j| \gg 0\}$. Under this identification, the \mathbb{Z}_2 -graded structure is given by letting E_{ij} ($i - j \in \mathbb{Z}$) be even while E_{ij} ($i - j \in \mathbb{Z} + 1/2$) be odd. One could easily identify the two presentations of $\mathfrak{gl}_{\infty|\infty}$ as follows ($i, j \in \mathbb{Z}$):

$$\begin{aligned} E_{ij}M_0 &= E_{ij}, & E_{ij}M_1 &= E_{i-1/2, j-1/2}, \\ E_{ij}M_+ &= E_{i, j-1/2}, & E_{ij}M_- &= E_{i-1/2, j}. \end{aligned}$$

The Lie superalgebra $\mathfrak{gl}_{\infty|\infty}$ is equipped with a natural $\frac{1}{2}\mathbb{Z}$ -gradation

$$\mathfrak{gl}_{\infty|\infty} = \bigoplus_{r \in \frac{1}{2}\mathbb{Z}} (\mathfrak{gl}_{\infty|\infty})_r$$

where $(\mathfrak{gl}_{\infty|\infty})_r$ is the completion of the linear span of E_{ij} with $j - i = r$ in the notation of the second realization above. We will call this gradation the *principal gradation* of $\mathfrak{gl}_{\infty|\infty}$. This gradation naturally gives a choice of a Borel subalgebra $B_{\infty|\infty} = \bigoplus_{r \geq 0} (\mathfrak{gl}_{\infty|\infty})_r$ and a choice of a Cartan subalgebra $\mathfrak{h}_{\infty|\infty} = (\mathfrak{gl}_{\infty|\infty})_0$. We obtain thus a triangular decomposition of $\mathfrak{gl}_{\infty|\infty}$

$$\mathfrak{gl}_{\infty|\infty} = (\mathfrak{gl}_{\infty|\infty})_+ \bigoplus (\mathfrak{gl}_{\infty|\infty})_0 \bigoplus (\mathfrak{gl}_{\infty|\infty})_-$$

where $(\mathfrak{gl}_{\infty|\infty})_{\pm} = \bigoplus_{r \in \frac{1}{2}\mathbb{N}} (\mathfrak{gl}_{\infty|\infty})_{\pm r}$.

§6.2. Subalgebras \mathcal{B} and $\mathcal{P}_{\pm\pm}$ of $\mathfrak{gl}_{\infty|\infty}$

Consider the superspace $\mathbb{C}^{\infty|\infty}$ with even basis vectors $\{e_i | i \in \mathbb{Z}\}$ and odd basis vectors $\{e_r | r \in \frac{1}{2} + \mathbb{Z}\}$. The Lie superalgebra $\mathfrak{gl}_{\infty|\infty}$ acts on $\mathbb{C}^{\infty|\infty}$ in the usual way:

$$E_{ij}e_k = \delta_{jk}e_i, \quad i, j, k \in \frac{1}{2}\mathbb{Z}.$$

Consider the *even super-symmetric* non-degenerate bilinear form $(\cdot | \cdot)$ on $\mathbb{C}^{\infty|\infty}$ given by

$$\begin{aligned} (e_i | e_j) &= (-1)^i \delta_{i,-j}, & i, j \in \mathbb{Z}; \\ (e_r, e_s) &= (-1)^{r+\frac{1}{2}} \delta_{r,-s}, & r, s \in \frac{1}{2} + \mathbb{Z}; \\ (e_i | e_r) &= 0, & i \in \mathbb{Z}, r \in \frac{1}{2} + \mathbb{Z}. \end{aligned}$$

Associated to this form we define the Lie superalgebra $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1$ to be the subalgebra of $\mathfrak{gl}_{\infty|\infty}$ preserving this form, where

$$\mathcal{B}_\epsilon = \{T \in (\mathfrak{gl}_{\infty|\infty})_\epsilon | (Tv|w) = -(-1)^{\epsilon|v|}(v, Tw)\}, \quad \epsilon = 0, 1.$$

This is a Lie superalgebra of type *OSP*. It is easy to see that the subalgebra \mathcal{B}^f spanned by the following elements is a dense subalgebra inside \mathcal{B} ($i, j \in \mathbb{Z}$, $r, s \in \frac{1}{2} + \mathbb{Z}$):

$$E_{i,j} - (-1)^{i+j} E_{-j,-i}, \quad E_{i,r} - (-1)^{i+r-\frac{1}{2}} E_{-r,-i}, \quad E_{r,s} - (-1)^{r+s-1} E_{-s,-r}.$$

In our first realization of $\mathfrak{gl}_{\infty|\infty}$ the subalgebra \mathcal{B}^f is the span of elements of the form $(E_{i,j} - (-1)^{i+j} E_{-j,-i})M_0$, $(E_{i,j} - (-1)^{i+j} E_{1-j,1-i})M_1$, and $E_{i,j}M_+ + (-1)^{i+j} E_{1-j,-i}M_-$.

In the remainder of this subsection we will introduce four subalgebras of $\widehat{\mathcal{SD}}$ of type P . Note that the type P Lie superalgebra is one of the two *strange* series of Lie superalgebras, cf. e.g. [K] for the finite-dimensional case.

First, we introduce an *odd symmetric* non-degenerate bilinear form $(\cdot|\cdot)$ on $\mathbb{C}^{\infty|\infty}$ which is uniquely determined by

$$(6.1) \quad (e_i, e_{j-\frac{1}{2}}) = \delta_{i+j-\frac{1}{2}, \frac{1}{2}}, \quad i, j \in \mathbb{Z}.$$

We denote by \mathcal{P}_{++} the subalgebra of $\mathfrak{gl}_{\infty|\infty}$ preserving this form. Then using the notation in the second realization of $\mathfrak{gl}_{\infty|\infty}$ the subalgebra spanned by

$$E_{ij} - E_{-j+\frac{1}{2}, -i+\frac{1}{2}}, \quad E_{i,j-\frac{1}{2}} + E_{-j+1, -i+\frac{1}{2}}, \quad E_{i-\frac{1}{2}, j} - E_{-j+\frac{1}{2}, -i+1}$$

forms a dense subalgebra \mathcal{P}_{++}^f inside \mathcal{P}_{++} . In the first realization of $\mathfrak{gl}_{\infty|\infty}$ the subalgebra \mathcal{P}_{++}^f is the span of elements of the form

$$E_{ij}M_1 - E_{1-j,1-i}M_0, \quad (E_{ij} - E_{1-j,1-i})M_-, \quad (E_{ij} + E_{1-j,1-i})M_+.$$

We may also use (6.1) to (uniquely) determine an *odd skew-symmetric* non-degenerate bilinear form $(\cdot|\cdot)$ on $\mathbb{C}^{\infty|\infty}$. We denote by \mathcal{P}_{+-} the subalgebra of $\mathfrak{gl}_{\infty|\infty}$ preserving this form. Then the subalgebra spanned by

$$E_{ij} - E_{-j+\frac{1}{2}, -i+\frac{1}{2}}, \quad E_{i,j-\frac{1}{2}} - E_{-j+1, -i+\frac{1}{2}}, \quad E_{i-\frac{1}{2}, j} + E_{-j+\frac{1}{2}, -i+1}$$

forms a dense subalgebra \mathcal{P}_{+-}^f inside \mathcal{P}_{+-} . In the first realization \mathcal{P}_{+-}^f corresponds to the subalgebra spanned by elements of the form

$$E_{ij}M_1 - E_{1-j,1-i}M_0, \quad (E_{ij} + E_{1-j,1-i})M_-, \quad (E_{ij} - E_{1-j,1-i})M_+.$$

Next, we consider the *odd symmetric* non-degenerate bilinear form $(\cdot|\cdot)$ on $\mathbb{C}^{\infty|\infty}$ uniquely determined by

$$(6.2) \quad (e_i, e_{j-\frac{1}{2}}) = (-1)^i \delta_{i+j-\frac{1}{2}, \frac{1}{2}}, \quad i, j \in \mathbb{Z}.$$

Let \mathcal{P}_{-+} be the subalgebra of $\mathfrak{gl}_{\infty|\infty}$ preserving this form. Then the subalgebra spanned by $E_{ij} - (-1)^{i+j} E_{-j+\frac{1}{2}, -i+\frac{1}{2}}$, $E_{i,j-\frac{1}{2}} + (-1)^{i+j} E_{-j+1, -i+\frac{1}{2}}$, and $E_{i-\frac{1}{2}, j} -$

$(-1)^{i+j}E_{-j+\frac{1}{2},-i+1}$ forms a dense subalgebra \mathcal{P}_{-+}^f inside \mathcal{P}_{-+} . In the first realization of $\mathfrak{gl}_{\infty|\infty}$ the subalgebra \mathcal{P}_{-+}^f is the span of elements of the form

$$E_{ij}M_1 - (-1)^{i+j}E_{1-j,1-i}M_0, \quad (E_{ij} - (-1)^{i+j}E_{1-j,1-i})M_-,$$

and $(E_{ij} + (-1)^{i+j}E_{1-j,1-i})M_+$.

Similarly we may consider the *odd skew-symmetric* non-degenerate bilinear form $(\cdot|\cdot)$ on $\mathbb{C}^{\infty|\infty}$ determined by (6.2). Let \mathcal{P}_{--} be the subalgebra of $\mathfrak{gl}_{\infty|\infty}$ preserving this form. Then the subalgebra spanned by

$$E_{ij} - (-1)^{i+j}E_{-j+\frac{1}{2},-i+\frac{1}{2}}, \quad E_{i,j-\frac{1}{2}} - (-1)^{i+j}E_{-j+1,-i+\frac{1}{2}},$$

and $E_{i-\frac{1}{2},j} + (-1)^{i+j}E_{-j+\frac{1}{2},-i+1}$ forms a dense subalgebra \mathcal{P}_{--}^f inside \mathcal{P}_{--} . In the first realization \mathcal{P}_{--}^f corresponds to the subalgebra spanned by elements of the form

$$E_{ij}M_1 - (-1)^{i+j}E_{1-j,1-i}M_0, \quad (E_{ij} + (-1)^{i+j}E_{1-j,1-i})M_-,$$

and $(E_{ij} - (-1)^{i+j}E_{1-j,1-i})M_+$. We note the Lie superalgebras \mathcal{B} and $\mathcal{P}_{\pm\pm}$ inherit from $\mathfrak{gl}_{\infty|\infty}$ the principal gradation and the triangular decomposition.

§6.3. Embedding of subalgebras of \mathcal{SD} into $\mathfrak{gl}_{\infty|\infty}$

Take a basis in $\mathbb{C}^{\infty|\infty} = \mathbb{C}^\infty \oplus \mathbb{C}^\infty\theta = t^s\mathbb{C}[t, t^{-1}] \oplus \theta t^s\mathbb{C}[t, t^{-1}]$ as $\{v_i = t^{-i+s}, v_{i-1/2} = t^{-i+s}\theta, i \in \mathbb{Z}\}$, where $s \in \mathbb{C}$. The Lie superalgebra $\mathfrak{gl}_{\infty|\infty}$ acts on $\mathbb{C}^{\infty|\infty}$ via $E_{ij}v_k = \delta_{jk}v_i, i, j, k \in \mathbb{Z}/2$. The Lie superalgebra \mathcal{SD} acts on $\mathbb{C}^{\infty|\infty}$ as differential operators. In this way we obtain a family of embeddings ϕ_s of \mathcal{SD} into $\mathfrak{gl}_{\infty|\infty}$:

$$(6.3) \quad \phi_s(t^k F(D)) = \begin{bmatrix} \phi_s(t^k f_0(D)) & \phi_s(t^k f_+(D)) \\ \phi_s(t^k f_-(D)) & \phi_s(t^k f_1(D)) \end{bmatrix},$$

where

$$\begin{aligned} \phi_s(t^k f_0(D)) &= \sum_{j \in \mathbb{Z}} f_0(-j + s)E_{j-k,j}M_0, \\ \phi_s(t^k f_1(D)) &= \sum_{j \in \mathbb{Z}} f_1(-j + s)E_{j-k,j}M_1, \\ \phi_s(t^k f_+(D)) &= \sum_{j \in \mathbb{Z}} f_+(-j + s)E_{j-k,j}M_+, \end{aligned}$$

$$\phi_s(t^k f_-(D)) = \sum_{j \in \mathbb{Z}} f_-(-j+s) E_{j-k,j} M_-.$$

Note that the principal gradation on $\mathfrak{gl}_{\infty|\infty}$ is compatible with that on \mathcal{SD} under the map ϕ_s .

Denote by \mathcal{O} the algebra of all holomorphic functions on \mathbb{C} with topology of uniform convergence on compact sets. For an integer k we let

$$\begin{aligned} \mathcal{O}^{(2k)} &\equiv \mathcal{O}^{(0)} = \{f \in \mathcal{O} \mid f(w) = f(-w)\}, \\ \mathcal{O}^{(2k+1)} &\equiv \mathcal{O}^{(1)} = \{f \in \mathcal{O} \mid f(w) = -f(-w)\}. \end{aligned}$$

We define a completion $\mathcal{SD}^\mathcal{O}$ of \mathcal{SD} consisting of all differential operators of the form $\sum_{a,j} t^j f_a(D) M_a$, where $f_a \in \mathcal{O}$ and $j \in \mathbb{Z}$. This induces completions ${}^0\mathcal{SD}^\mathcal{O}$ and ${}^{\pm\pm}\mathcal{SD}^\mathcal{O}$ of ${}^0\mathcal{SD}$ and ${}^{\pm\pm}\mathcal{SD}$, respectively. The embedding ϕ_s extends naturally to ${}^0\mathcal{SD}^\mathcal{O}$ and ${}^{\pm\pm}\mathcal{SD}^\mathcal{O}$.

We define

$$I_s := \{f \in \mathcal{O} \mid f(n+s) = 0, \forall n \in \mathbb{Z}\},$$

and, for $j, k \in \mathbb{Z}$,

$$I_{s,k}^{(j)} = \{f \in \mathcal{O}^{(j)} \mid f(n+k/2+s) = 0, \forall n \in \mathbb{Z}\}.$$

Setting $J_s = \bigoplus_{k \in \mathbb{Z}} \{\sum_{a=0,1,\pm} t^k f_a(D) M_a \mid f_a \in I_s\}$ we have the following proposition whose proof is a straightforward calculation (compare [KR1]).

Proposition 6.1. *We have the following exact sequence of Lie superalgebras:*

$$0 \longrightarrow J_s \longrightarrow \mathcal{SD}^\mathcal{O} \xrightarrow{\phi_s} \mathfrak{gl}_{\infty|\infty} \longrightarrow 0.$$

For $k \in \mathbb{Z}$, let ${}^0J_{s,k}$ be the linear span of

$$t^k f_0 \left(D + \frac{k}{2} \right) M_0, t^k f_1 \left(D + \frac{k+1}{2} \right) M_1, (-t)^k g(-D-k) M_+ + t^{k-1} g(D) M_-,$$

where $f_0 \in I_{s,k}^{(k+1)}$, $f_1 \in I_{s,k+1}^{(k+1)}$ and $g \in I_s$. Set ${}^0J_s = \bigoplus_{k \in \mathbb{Z}} {}^0J_{s,k}$ we can easily show the following.

Proposition 6.2. *We have the following exact sequences of Lie superalgebras:*

$$\begin{aligned} 0 \longrightarrow {}^0J_s \longrightarrow {}^0\mathcal{SD}^\mathcal{O} \xrightarrow{\phi_s} \mathfrak{gl}_{\infty|\infty} \longrightarrow 0, & \quad s \notin \frac{1}{2}\mathbb{Z}, \\ 0 \longrightarrow {}^0J_0 \longrightarrow {}^0\mathcal{SD}^\mathcal{O} \xrightarrow{\phi_0} \mathcal{B} \longrightarrow 0. & \end{aligned}$$

For $k \in \mathbb{Z}$ we let ${}^{\pm\pm}J_{s,k}$ be the linear span of $t^k(f(D + \frac{k}{2})M_0 - f(-D - k - 1)M_1)$, $t^{k-1}f_-(D + \frac{k}{2})M_-$, and $t^k f_+(D + \frac{k-1}{2})M_+$, where $f \in I_s$ and $f_- \in I_{s,k}^{(1)}$, $f_+ \in I_{s,k-1}^{(0)}$ in the case of ${}^{++}J_{s,k}$, and $f_- \in I_{s,k}^{(0)}$, $f_+ \in I_{s,k-1}^{(1)}$ in the case of ${}^{+-}J_{s,k}$.

For $k \in \mathbb{Z}$ we let ${}^{-\pm}J_{s,k}$ be the linear span of

$$t^k(f(D + k/2)M_0 - (-1)^k f(-D - k - 1)M_1), \quad t^{k-1}f_-(D + k/2)M_-,$$

and $t^k f_+(D + (k - 1)/2)M_+$, where $f \in I_s$ and $f_- \in I_{s,k}^{(k)}$, $f_+ \in I_{s,k-1}^{(k)}$ in the case of ${}^{-+}J_{s,k}$, and $f_- \in I_{s,k}^{(k+1)}$, $f_+ \in I_{s,k-1}^{(k+1)}$ in the case of ${}^{--}J_{s,k}$.

Similarly, putting ${}^{\pm\pm}J_s = \bigoplus_{k \in \mathbb{Z}} {}^{\pm\pm}J_{s,k}$, we can show the following.

Proposition 6.3. *We have the following short exact sequences of Lie superalgebras:*

$$\begin{aligned} 0 \longrightarrow {}^{\pm\pm}J_s \longrightarrow {}^{\pm\pm}SD^{\mathcal{O}} \xrightarrow{\phi_s} \mathfrak{gl}_{\infty|\infty} \longrightarrow 0, \quad s \notin \frac{1}{2}\mathbb{Z}, \\ 0 \longrightarrow {}^{\pm\pm}J_0 \longrightarrow {}^{\pm\pm}SD^{\mathcal{O}} \xrightarrow{\phi_0} \mathcal{P}_{\pm\pm} \longrightarrow 0. \end{aligned}$$

Remark 6.1. One can also explicitly describe the images of ϕ_s ($s \in \frac{1}{2}\mathbb{Z}$) which are Lie subalgebras of $\widehat{\mathfrak{gl}}_{\infty|\infty}$. For our purpose in this paper, we only need the surjectivity of the homomorphisms ϕ_s in Propositions 6.1 and 6.2.

§6.4. Lifting of the embeddings to the central extensions

Next we extend the above homomorphism ϕ_s to a homomorphism between the central extensions of the corresponding Lie superalgebras.

We begin by describing the central extension of $\mathfrak{gl}_{\infty|\infty}$. The Lie superalgebra $\widehat{\mathfrak{gl}}_{\infty|\infty}$ is the central extension of $\mathfrak{gl}_{\infty|\infty}$ by a 1-dimensional vector space spanned by $\mathbf{1}$. The 2-cocycle giving rise to this central extension is explicitly given by

$$\alpha(A, B) = \text{Str}([J, A]B), \quad A, B \in \mathfrak{gl}_{\infty|\infty},$$

where J denotes the matrix $\sum_{r \leq 0} E_{rr}$, and for a matrix $C = (c_{ij}) \in \mathfrak{gl}_{\infty|\infty}$, $\text{Str}(C)$ stands for the supertrace of the matrix C , which is given by $\sum_{r \in \frac{1}{2}\mathbb{Z}} (-1)^{2r} c_{rr}$. We note that the expression $\alpha(A, B)$ is well-defined for $A, B \in \mathfrak{gl}_{\infty|\infty}$. The restriction of α to the subalgebras \mathcal{B} , $\mathcal{P}_{\pm\pm}$ gives rise to subalgebras of $\widehat{\mathfrak{gl}}_{\infty|\infty}$, which we denote accordingly by $\widehat{\mathcal{B}}$ and $\widehat{\mathcal{P}}_{\pm\pm}$, respectively.

The linear map σ sending D to $D + 1$ induces an associative algebra automorphism of \mathcal{SD} . Furthermore $(1 - \sigma)$ maps \mathcal{SD} onto \mathcal{SD} , thus the map $(1 - \sigma) : \mathcal{SD}/\ker(1 - \sigma) \rightarrow \mathcal{SD}$ is a linear isomorphism. Now let $\text{Str}_s : \mathcal{SD} \rightarrow \mathbb{C}$, $s \in \mathbb{C}$, be the linear map defined by

$$\text{Str}_s(t^r \begin{pmatrix} f^0(D) & f^+(D) \\ f^-(D) & f^1(D) \end{pmatrix}) = \delta_{r,0}(f^0(s) - f^0(0) - f^1(s) + f^1(0)).$$

We note that Str_s vanishes on $\ker(1 - \sigma)$ and hence the linear map $\text{Str}_s \circ (1 - \sigma)^{-1} : \mathcal{SD} \rightarrow \mathbb{C}$ is well-defined. The following lemma is a consequence of a straightforward computation (cf. [KR1]).

Lemma 6.1. *For $t^r F(D), t^s G(D) \in \mathcal{SD}$, $r, s \in \mathbb{Z}$, we have*

$$\begin{aligned} \Psi(t^r F(D), t^s G(D)) + \text{Str}_s \circ (1 - \sigma)^{-1}([t^r F(D), t^s G(D)]) \\ = \alpha(\phi_s(t^r F(D)), \phi_s(t^s G(D))). \end{aligned}$$

It follows from Lemma 6.1 that the map $\widehat{\phi}_s : \widehat{\mathcal{SD}} \rightarrow \widehat{\mathfrak{gl}}_{\infty|\infty}$ defined by

$$(6.4) \quad \widehat{\phi}_s(t^r F(D)) = \phi_s(t^r F(D)) + \text{Str}_s \circ (1 - \sigma)^{-1}(t^r F(D))$$

is a homomorphism of Lie superalgebras.

Let x be a formal variable and define the generating functions $e^{xD\theta\partial_\theta}$ and $e^{xD\partial_\theta\theta}$ in the usual way by $\sum_{n \geq 0} \frac{(xD\theta\partial_\theta)^n}{n!}$ and $\sum_{n \geq 0} \frac{(xD\partial_\theta\theta)^n}{n!}$, respectively. Since $(1 - \sigma)^{-1}(e^{xw}) = -\frac{e^{xw}-1}{e^x-1}$ we have the following description of the map $\widehat{\phi}_s$. A somewhat different proof can be given parallel to the proof of Lemma 5.1, p. 87, [KWY].

Proposition 6.4. *The \mathbb{C} -linear map $\widehat{\phi}_s : \widehat{\mathcal{SD}} \rightarrow \widehat{\mathfrak{gl}}_{\infty|\infty}$ defined by*

$$\begin{aligned} \widehat{\phi}_s|_{\widehat{\mathcal{SD}}_j} &= \phi_s|_{\mathcal{SD}_j}, \quad j \neq 0, \\ \widehat{\phi}_s(e^{xD\partial_\theta\theta}) &= \phi_s(e^{xD\partial_\theta\theta}) - \frac{e^{sx} - 1}{e^x - 1} \cdot \mathbf{1}, \\ \widehat{\phi}_s(e^{xD\theta\partial_\theta}) &= \phi_s(e^{xD\theta\partial_\theta}) + \frac{e^{sx} - 1}{e^x - 1} \cdot \mathbf{1}, \\ \widehat{\phi}_s(C) &= \mathbf{1}, \end{aligned}$$

is a homomorphism of Lie superalgebras.

We note that it follows from Proposition 4.1 that ${}^0\widehat{\mathcal{SD}}_0$ is spanned by the basis elements $\{D^n \partial_\theta, (D + \frac{1}{2})^n \theta \partial_\theta | n \text{ odd}\}$. We thus obtain the following description for the embedding of ${}^0\widehat{\mathcal{SD}}$ into $\widehat{\mathfrak{gl}}_{\infty|\infty}$ by restricting the map $\widehat{\phi}_s$. (Compare Proposition 5.2, p. 87, [KWY]).

Proposition 6.5. *The \mathbb{C} -linear map $\widehat{\phi}_s : {}^0\widehat{\mathcal{SD}} \rightarrow \widehat{\mathfrak{gl}}_{\infty|\infty}$ defined below is a homomorphism of Lie superalgebras:*

$$\begin{aligned}\widehat{\phi}_s|_{{}^0\widehat{\mathcal{SD}}_j} &= \phi_s|_{{}^0\widehat{\mathcal{SD}}_j}, \quad j \neq 0, \\ \widehat{\phi}_s(\sinh(xD\partial_\theta\theta)) &= \phi_s(\sinh(xD\partial_\theta\theta)) - \frac{\cosh((s-\frac{1}{2})x) - \cosh(\frac{x}{2})}{\sinh(\frac{x}{2})} \cdot \mathbf{1}, \\ \widehat{\phi}_s\left(\sinh\left(x\left(D + \frac{1}{2}\right)\theta\partial_\theta\right)\right) &= \phi_s\left(\sinh\left(x\left(D + \frac{1}{2}\right)\theta\partial_\theta\right)\right) + \frac{\cosh(sx) - 1}{\sinh(\frac{x}{2})} \cdot \mathbf{1}, \\ \widehat{\phi}_s(C) &= \mathbf{1}.\end{aligned}$$

In particular, $\widehat{\phi}_0$ is a homomorphism from ${}^0\widehat{\mathcal{SD}}$ to $\widehat{\mathcal{B}}$.

It follows from Propositions 4.2, 4.3, 4.4 and 4.5 that the Cartan subalgebra $\pm\pm\widehat{\mathcal{SD}}_0$ is spanned by the basis elements $\{(D + \frac{1}{2})^n \partial_\theta\theta - (-1)^n (D + \frac{1}{2})^n \theta\partial_\theta | n \in \mathbb{Z}_+\}$ which is the same as $\{(D + \frac{1}{2})^{2k+1}, (D + \frac{1}{2})^{2k}(\partial_\theta\theta - \theta\partial_\theta) | k \in \mathbb{Z}_+\}$. We may consider the following generating functions:

$$\sinh((D + 1/2)x), \quad \cosh((D + 1/2)x)(\partial_\theta\theta - \theta\partial_\theta).$$

Thus restricting $\widehat{\psi}_s$ we obtain the following description for the embedding of $\pm\pm\widehat{\mathcal{SD}}$ into $\widehat{\mathfrak{gl}}_{\infty|\infty}$.

Proposition 6.6. *The \mathbb{C} -linear map $\widehat{\phi}_s : \pm\pm\widehat{\mathcal{SD}} \rightarrow \widehat{\mathfrak{gl}}_{\infty|\infty}$ defined below is a homomorphism of Lie superalgebras:*

$$\begin{aligned}\widehat{\phi}_s|_{\pm\pm\widehat{\mathcal{SD}}_j} &= \phi_s|_{\pm\pm\widehat{\mathcal{SD}}_j}, \quad j \neq 0, \\ \widehat{\phi}_s\left(\sinh\left(\left(D + \frac{1}{2}\right)x\right)\right) &= \phi_s\left(\sinh\left(\left(D + \frac{1}{2}\right)x\right)\right), \\ \widehat{\phi}_s\left(\cosh\left(\left(D + \frac{1}{2}\right)x\right)(\partial_\theta\theta - \theta\partial_\theta)\right) &= \phi_s\left(\cosh\left(\left(D + \frac{1}{2}\right)x\right)(\partial_\theta\theta - \theta\partial_\theta)\right) \\ &\quad - 2\frac{\cosh((s + \frac{1}{2})x) - \cosh(\frac{x}{2})}{e^x - 1} \cdot \mathbf{1}, \\ \widehat{\phi}_s(C) &= \mathbf{1}.\end{aligned}$$

In particular, $\widehat{\phi}_0$ is a homomorphism from $\pm\pm\widehat{\mathcal{SD}}$ to $\widehat{\mathcal{P}}_{\pm\pm}$.

We conclude this section with the following proposition whose proof is analogous to the one of Proposition 4.3 in [KR1].

Proposition 6.7. *Let \mathfrak{g} be either $\widehat{\mathcal{SD}}$ or ${}^i\widehat{\mathcal{SD}}$, where $i = 0, \pm\pm$. Let $\mathfrak{g}^\mathcal{O}$ denote the corresponding holomorphic completion of \mathfrak{g} and V a quasifinite \mathfrak{g} -module. The action of \mathfrak{g} extends naturally to an action of at least $\mathfrak{g}_k^\mathcal{O}$, for all $k \neq 0$.*

§7. Finite-dimensional Howe Duality

Howe duality [H1], [H2] was generalized in [CW1], [CW2] in a systematic way (also cf. [OP], [Naz], [Se] for other independent approaches¹) to finite-dimensional Lie superalgebras. However in order to apply those results to our infinite dimensional setting, we need to use Borel subalgebras different from the ones used in [CW1]. In this section we will obtain explicit formulas of the joint highest weight vectors for a dual pair of Lie superalgebras acting on a supersymmetric module with respect to the new Borel subalgebras, and thus obtain an explicit isotypic decomposition of the supersymmetric module with respect to the action of dual pairs. A variation of these formulas will be applied to the study of Howe duality involving infinite-dimensional Lie superalgebras in the next section.

Let $\mathfrak{gl}(m|n)$ denote the general linear Lie superalgebra of linear transformations on the $(m|n)$ -dimensional complex superspace $\mathbb{C}^{m|n}$. Let the subspace $\sum_{i=1}^{m+n} \mathbb{C}E_{ii}$ of diagonal matrices be our choice of a Cartan subalgebra \mathfrak{h} and B any Borel subalgebra containing \mathfrak{h} . A finite-dimensional irreducible representation $V_{m|n}^\lambda$ is determined by a unique (up to a scalar) non-zero highest weight vector $v \in V_{m|n}^\lambda$, where $\lambda \in \mathfrak{h}^*$, which in turn is uniquely determined by the properties

$$\begin{aligned} hv &= \lambda(h)v, & h \in \mathfrak{h}, \\ bv &= 0, & b \in [B, B]. \end{aligned}$$

Consider the natural action of the Lie superalgebra $\mathfrak{gl}(n|n)$ on the $\mathbb{C}^{n|n}$ and the natural action of the Lie algebra $\mathfrak{gl}(l)$ on the space \mathbb{C}^l . We obtain an action of $\mathfrak{gl}(n|n) \times \mathfrak{gl}(l)$ on the space $\mathbb{C}^{n|n} \otimes \mathbb{C}^l$.

We will **assume for the rest of this section that $n \geq l$** , which is all we need for the application in the next section.

We consider the induced action of $\mathfrak{gl}(n|n) \times \mathfrak{gl}(l)$ on the k -th skew-symmetric tensor $\Lambda^k(\mathbb{C}^{n|n} \otimes \mathbb{C}^l)$. As a $\mathfrak{gl}(n|n) \times \mathfrak{gl}(l)$ -module $\Lambda^k(\mathbb{C}^{n|n} \otimes \mathbb{C}^l)$ is completely reducible and we have

$$(7.1) \quad \Lambda^k(\mathbb{C}^{n|n} \otimes \mathbb{C}^l) \cong \sum_{\lambda} V_{n|n}^{\lambda'} \otimes V_l^{\lambda},$$

¹We thank M. Nazarov for bringing the reference [Naz], [OP] to our attention.

where $\lambda = (\lambda_1, \lambda_2, \dots)$ is summed over all partitions of size $|\lambda| = k$ and length $l(\lambda) \leq l$. As usual λ' denotes the transpose of λ . Here the partition λ is viewed as highest weights of $\mathfrak{gl}(n|n)$ and $\mathfrak{gl}(l)$ with respect to their standard Borel subalgebras of upper triangular matrices as follows. As a $\mathfrak{gl}(l)$ -highest weight we have $\lambda(E_{ii}) = \lambda_i$, for $i = 1, \dots, l$, while as a $\mathfrak{gl}(n|n)$ -highest weight we have $\lambda(E_{ii}) = \lambda'_i$, for $i = 1, \dots, n$, and $\lambda(E_{j+n, j+n}) = \langle \lambda_j - n \rangle$, for $j = 1, \dots, n$ (cf. [CW1], [OP], [Se]). The symbol $\langle m \rangle$, $m \in \mathbb{Z}$, stands for m , if $m \in \mathbb{Z}_+$, and stands for zero otherwise.

For our application in next section, we will use the technique developed in [CW1] to derive explicit formulas for the joint highest weight vectors with respect to certain non-standard Borel subalgebras (compare [OP] where less explicit formulas are also presented).

We note that while all Borel subalgebras of $\mathfrak{gl}(l)$ are conjugate to each other, this is no longer true for Lie superalgebras [K]. Therefore it follows that a different choice of Borel subalgebra for $\mathfrak{gl}(n|n)$ above may result in different description of $\mathfrak{gl}(n|n)$ -highest weights in the decomposition (7.1).

Let $\{e^1, \dots, e^n, e^{n+1}, \dots, e^{n+n}\}$ be the standard basis of $\mathbb{C}^{n|n}$. In particular $\{e^i | 1 \leq i \leq n\}$ are even basis vectors, while $\{e^i | n+1 \leq i \leq 2n\}$ are odd basis vectors. Of course any ordering of this basis gives rise, via taking the upper triangular matrices with respect to this ordering, to a Borel subalgebra of $\mathfrak{gl}(n|n)$ containing the Cartan subalgebra of diagonal matrices. We consider two orderings that are relevant for our discussion. The first one is the ordering $\{e^1, e^{n+1}, e^2, e^{n+2}, \dots, e^n, e^{n+n}\}$ and we denote by B_1 the Borel subalgebra corresponding to this ordering. The second ordering is $\{e^{n+1}, e^1, e^{n+2}, e^2, \dots, e^{n+n}, e^n\}$ and we denote the corresponding Borel subalgebra by B_2 . Our present goal is to find the $\mathfrak{gl}(n|n) \times \mathfrak{gl}(l)$ -joint highest weight vectors inside $\Lambda^k(\mathbb{C}^{n|n} \otimes \mathbb{C}^l)$ with respect to the Borel subalgebra $B_i \times B'$, $i = 1, 2$, where B' is the Borel subalgebra of $\mathfrak{gl}(l)$ consisting of upper triangular $l \times l$ matrices.

Denote by e_j , $j = 1, \dots, l$ the standard basis for \mathbb{C}^l . Then $\{e^i \otimes e_j | 1 \leq i \leq 2n, 1 \leq j \leq l\}$ is a basis for $\mathbb{C}^{n|n} \otimes \mathbb{C}^l$. We set for $1 \leq i \leq n$ and $1 \leq j \leq l$:

$$\begin{aligned} \xi_j^i &= e^i \otimes e_j, \\ x_j^i &= e^{n+i} \otimes e_j. \end{aligned}$$

Then the skew-symmetric algebra $\Lambda^\bullet(\mathbb{C}^{n|n} \otimes \mathbb{C}^l) = \sum_{k=0}^\infty \Lambda^k(\mathbb{C}^{n|n} \otimes \mathbb{C}^l)$ may be identified with the superalgebra $\mathbb{C}[x_j^i] \otimes \Lambda(\xi_j^i)$, the tensor algebra of the polynomial algebra in the variables x_j^i and the Grassmann superalgebra in the variables ξ_j^i , for $1 \leq i \leq n$ and $1 \leq j \leq l$.

Under this identification the action of $\mathfrak{gl}(n|n)$ on $\mathbb{C}[x_j^i] \otimes \Lambda(\xi_j^i)$ may be realized as first order linear differential operators

$$\sum_{j=1}^l \xi_j^i \frac{\partial}{\partial \xi_j^k}, \quad \sum_{j=1}^l x_j^i \frac{\partial}{\partial \xi_j^k}, \quad \sum_{j=1}^l x_j^i \frac{\partial}{\partial x_j^k}, \quad \sum_{j=1}^l \xi_j^i \frac{\partial}{\partial x_j^k}, \quad 1 \leq i, k \leq n,$$

while that of $\mathfrak{gl}(l)$ may be realized as

$$\sum_{i=1}^n \xi_j^i \frac{\partial}{\partial \xi_k^i} + \sum_{i=1}^n x_j^i \frac{\partial}{\partial x_k^i}, \quad 1 \leq j, k \leq l.$$

The Cartan subalgebra of $\mathfrak{gl}(n|n)$ is spanned by the basis vectors $\sum_{j=1}^l \xi_j^i \frac{\partial}{\partial \xi_j^i}$ and $\sum_{j=1}^l x_j^i \frac{\partial}{\partial x_j^i}$, for $1 \leq i \leq n$, while the nilpotent radical of the Borel subalgebra B_1 is generated by the odd simple root vectors

$$(7.2) \quad \sum_{j=1}^l \xi_j^i \frac{\partial}{\partial x_j^i}, \quad 1 \leq i \leq n;$$

$$(7.3) \quad \sum_{j=1}^l x_j^i \frac{\partial}{\partial \xi_j^{i+1}}, \quad 1 \leq i \leq n - 1.$$

The nilpotent radical of the Borel subalgebra B_2 is generated by the odd simple root vectors

$$(7.4) \quad \sum_{j=1}^l x_j^i \frac{\partial}{\partial \xi_j^i}, \quad 1 \leq i \leq n;$$

$$(7.5) \quad \sum_{j=1}^l \xi_j^i \frac{\partial}{\partial x_j^{i+1}}, \quad i \leq i \leq n - 1.$$

The Cartan subalgebra of $\mathfrak{gl}(l)$ on the other hand is spanned by $\sum_{i=1}^n \xi_j^i \frac{\partial}{\partial \xi_j^i} + \sum_{i=1}^n x_j^i \frac{\partial}{\partial x_j^i}$, for $1 \leq j \leq l$, while the nilpotent radical of the Borel subalgebra is generated by the simple root vectors

$$(7.6) \quad \sum_{i=1}^n \xi_j^i \frac{\partial}{\partial \xi_{j+1}^i} + \sum_{i=1}^n x_j^i \frac{\partial}{\partial x_{j+1}^i}, \quad 1 \leq j \leq l - 1.$$

Consider the following $l \times l$ matrices

$$X^1 = \begin{pmatrix} \xi_1^1 & \xi_2^1 & \cdots & \xi_l^1 \\ \xi_1^1 & \xi_2^1 & \cdots & \xi_l^1 \\ \vdots & \vdots & \cdots & \vdots \\ \xi_1^1 & \xi_2^1 & \cdots & \xi_l^1 \end{pmatrix},$$

$$\begin{aligned}
 X^2 &= \begin{pmatrix} x_1^1 & x_2^1 & \cdots & x_l^1 \\ \xi_1^2 & \xi_2^2 & \cdots & \xi_l^2 \\ \vdots & \vdots & \cdots & \vdots \\ \xi_1^2 & \xi_2^2 & \cdots & \xi_l^2 \end{pmatrix}, \\
 X^3 &= \begin{pmatrix} x_1^1 & x_2^1 & \cdots & x_l^1 \\ x_1^2 & x_2^2 & \cdots & x_l^2 \\ \xi_1^3 & \xi_2^3 & \cdots & \xi_l^3 \\ \vdots & \vdots & \cdots & \vdots \\ \xi_1^3 & \xi_2^3 & \cdots & \xi_l^3 \end{pmatrix}, \\
 &\quad \vdots \\
 X^k \equiv X^{l-1} &= \begin{pmatrix} x_1^1 & x_2^1 & \cdots & x_l^1 \\ x_1^2 & x_2^2 & \cdots & x_l^2 \\ x_1^3 & x_2^3 & \cdots & x_l^3 \\ \vdots & \vdots & \cdots & \vdots \\ x_1^l & x_2^l & \cdots & x_l^l \end{pmatrix}, \quad k \geq l.
 \end{aligned}$$

For $0 \leq r \leq l$ we let X_r^i denote the first $r \times r$ minor of the matrix X^i .

Remark 7.1. By the determinant of an $r \times r$ matrix $A = (a_i^j)$, denoted by $\det A$, we will always mean the expression $\sum_{\sigma \in S_r} \text{sgn}(\sigma) a_{\sigma_1}^1 a_{\sigma_2}^2 \cdots a_{\sigma_r}^r$. It is necessary to specify the order of product as some of the a_i^j 's might be odd variables.

The following lemma, which is the Corollary 4.1 in [CW1], plays an important role.

Lemma 7.1. *Let x_i^j be even variables for $i = 1, \dots, r$ and $j = 1, \dots, q$ with $r \geq t > q$. Let ξ_i^q and ξ_i^s be odd variables for $i = 1, \dots, l$. Then*

$$\det \begin{pmatrix} x_1^1 & x_2^1 & \cdots & x_r^1 \\ x_1^2 & x_2^2 & \cdots & x_r^2 \\ \vdots & \vdots & \cdots & \vdots \\ x_1^q & x_2^q & \cdots & x_r^q \\ \xi_1^q & \xi_2^q & \cdots & \xi_r^q \\ \xi_1^q & \xi_2^q & \cdots & \xi_r^q \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ \xi_1^q & \xi_2^q & \cdots & \xi_r^q \end{pmatrix} \cdot \det \begin{pmatrix} x_1^1 & x_2^1 & \cdots & x_t^1 \\ x_1^2 & x_2^2 & \cdots & x_t^2 \\ \vdots & \vdots & \cdots & \vdots \\ x_1^q & x_2^q & \cdots & x_t^q \\ \xi_1^q & \xi_2^q & \cdots & \xi_t^q \\ \xi_1^s & \xi_2^s & \cdots & \xi_t^s \\ \vdots & \vdots & \cdots & \vdots \\ \xi_1^s & \xi_2^s & \cdots & \xi_t^s \end{pmatrix} = 0.$$

Corollary 7.1. *Let x_i^j be even variables for $i = 1, \dots, r$ and $j = 1, \dots, q + m$ with $r \geq t > q$. Let ξ_i^q and ξ_i^s be odd variables for $i = 1, \dots, l$. Then*

$$\det \begin{pmatrix} x_1^1 & x_2^1 & \cdots & x_r^1 \\ x_1^2 & x_2^2 & \cdots & x_r^2 \\ \vdots & \vdots & \cdots & \vdots \\ x_1^q & x_2^q & \cdots & x_r^q \\ \xi_1^q & \xi_2^q & \cdots & \xi_r^q \\ \xi_1^q & \xi_2^q & \cdots & \xi_r^q \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ \xi_1^q & \xi_2^q & \cdots & \xi_r^q \end{pmatrix} \cdot \det \begin{pmatrix} x_1^1 & x_2^1 & \cdots & x_t^1 \\ \vdots & \vdots & \cdots & \vdots \\ x_1^q & x_2^q & \cdots & x_t^q \\ \xi_1^q & \xi_2^q & \cdots & \xi_t^q \\ x_1^{q+2} & x_2^{q+2} & \cdots & x_t^{q+2} \\ \vdots & \vdots & \cdots & \vdots \\ x_1^{q+m} & x_2^{q+m} & \cdots & x_t^{q+m} \\ \xi_1^s & \xi_2^s & \cdots & \xi_t^s \\ \vdots & \vdots & \cdots & \vdots \\ \xi_1^s & \xi_2^s & \cdots & \xi_t^s \end{pmatrix} = 0.$$

Proof. The identity follows by applying successively the differential operators of the form $\sum_{i=1}^t x_i^{q+2} \frac{\partial}{\partial \xi_i^s}$, $\sum_{i=1}^t x_i^{q+3} \frac{\partial}{\partial \xi_i^s}, \dots, \sum_{i=1}^t x_i^{q+m} \frac{\partial}{\partial \xi_i^s}$ to the identity in Lemma 7.1. \square

Theorem 7.1. *Let λ be a partition of length $l(\lambda) \leq l$. Then the expression*

$$\prod_{i=1}^{\lambda_1} \det X_{\lambda_i}^i$$

is annihilated by (7.2), (7.3) and (7.6), and hence is a $\mathfrak{gl}(n|n) \times \mathfrak{gl}(l)$ highest weight vector with respect to the Borel subalgebra $B_1 \times B'$.

Proof. From the definition of the determinant it is easy to see that the simple root vectors in (7.6) kill each $\det X_{\lambda_i}^i$. Hence they kill $\prod_{i=1}^{\lambda_1} \det X_{\lambda_i}^i$.

Also it is clear that $\prod_{i=1}^{\lambda_1} \det X_{\lambda_i}^i$ is annihilated by (7.3), since such an expression acts on $\det X_r^{l+1}$ by replacing the row $(\xi_1^{l+1}, \xi_2^{l+1}, \dots, \xi_r^{l+1})$ by $(x_1^l, x_2^l, \dots, x_r^l)$, where $l \geq 1$. As the latter row already appears in X_r^{l+1} , the resulting determinant is zero.

Hence it remains to show that (7.2) also annihilates $\prod_{i=1}^{\lambda_1} \det X_{\lambda_i}^i$. But this now can be easily seen using Corollary 7.1. \square

Let λ be a partition of size k and length $l(\lambda) \leq l$. Define $\phi_1(\lambda)$ to be the $n|n$ -tuple of integers

$$\phi_1(\lambda) := (\lambda'_1, \langle \lambda'_2 - 1 \rangle, \dots, \langle \lambda'_n - n + 1 \rangle; \langle \lambda_1 - 1 \rangle, \langle \lambda_2 - 2 \rangle, \dots, \langle \lambda_n - n \rangle).$$

The following corollary follows from an easy computation of the weight of vector in Theorem 7.1.

Corollary 7.2. *As a $\mathfrak{gl}(n|n) \times \mathfrak{gl}(l)$ -module we have*

$$\Lambda^k(\mathbb{C}^{n|n} \otimes \mathbb{C}^l) \cong \sum_{\lambda} V_{n|n}^{\phi_1(\lambda)} \otimes V_l^\lambda,$$

where $\phi_1(\lambda)$ is the highest weight of $\mathfrak{gl}(n|n)$ with respect to the Borel subalgebra B_1 and λ above is summed over all partitions with $l(\lambda) \leq l$ and $|\lambda| = k$.

Theorem 7.2. *Let λ be a partition of length $l(\lambda) \leq l$. Then the expression*

$$\prod_{j=1}^{\lambda_1} \det X_{\lambda'_j}^{j+1}$$

is annihilated by (7.4), (7.5) and (7.6), and hence is a $\mathfrak{gl}(n|n) \times \mathfrak{gl}(l)$ highest weight vector with respect to the Borel subalgebra $B_2 \times B'$.

Proof. The proof is similar to that of Theorem 7.1. Again it is easy to show that the expression is annihilated by operators of the form (7.6) and (7.4). The fact that it is annihilated by operators of the form (7.5) is again a consequence of Corollary 7.1. □

Similarly for a partition λ with $|\lambda| = k$ and $l(\lambda) \leq l$ we define an $n|n$ -tuple of integers

$$\phi_2(\lambda) := (\langle \lambda'_1 - 1 \rangle, \langle \lambda'_2 - 2 \rangle, \dots, \langle \lambda'_n - n \rangle; \lambda_1, \langle \lambda_2 - 1 \rangle, \dots, \langle \lambda_n - n + 1 \rangle).$$

Corollary 7.3. *As a $\mathfrak{gl}(n|n) \times \mathfrak{gl}(l)$ -module we have*

$$\Lambda^k(\mathbb{C}^{n|n} \otimes \mathbb{C}^l) \cong \sum_{\lambda} V_{n|n}^{\phi_2(\lambda)} \otimes V_l^\lambda,$$

where $\phi_2(\lambda)$ is the highest weight of $\mathfrak{gl}(n|n)$ with respect to the Borel subalgebra B_2 and λ above is summed over all partitions with $l(\lambda) \leq l$ and $|\lambda| = k$.

Remark 7.2. The arguments and results in this section remain valid for the dual pair $\mathfrak{gl}(m|n) \times \mathfrak{gl}(l)$, as long as $m, n \geq l$. In this more general setting we need to replace B_1 and B_2 with appropriate Borel subalgebras. Let's denote the standard basis of $\mathbb{C}^{m|n}$ by $\{e^1, \dots, e^m; e^{m+1}, \dots, e^{m+n}\}$. We may replace B_1 with any Borel subalgebra corresponding to any ordering of the basis according to which $e^{m+1}, e^1, e^{m+2}, e^2, \dots, e^{m+l}, e^l$ appear as the first $2l$ members. Similarly we may replace B_2 by any Borel subalgebra whose corresponding order of basis has $e^1, e^{m+1}, e^2, e^{m+2}, \dots, e^l, e^{m+l}$ as its first $2l$ members.

Remark 7.3. One can modify some results in the next section to obtain a finite dimensional $(\text{Pin}(2l), \text{osp}(2n + 1, 2m))$ duality $(n, m \geq 1)$ which seems to be new as well.

§8. Free Field Realizations and Duality

Take a pair of fermionic fields (*bc* fields)

$$\psi^+(z) = \sum_{n \in \mathbb{Z}} \psi_n^+ z^{-n-1}, \quad \psi^-(z) = \sum_{n \in \mathbb{Z}} \psi_n^- z^{-n},$$

with the following anti-commutation relations

$$[\psi_m^+, \psi_n^-] = \delta_{m+n,0}, \quad [\psi_m^\pm, \psi_n^\pm] = 0, \quad m, n \in \mathbb{Z}.$$

Also we take a pair of bosonic ghost fields ($\beta\gamma$ fields)

$$\gamma^\pm(z) = \sum_{r \in \frac{1}{2} + \mathbb{Z}} \gamma_r^\pm z^{-r-\frac{1}{2}},$$

with the following commutation relations

$$[\gamma_r^+, \gamma_s^-] = \delta_{r+s,0}, \quad [\gamma_r^\pm, \gamma_s^\pm] = 0, \quad r, s \in \frac{1}{2} + \mathbb{Z}.$$

Denote by \mathfrak{F} the Fock space of $\psi^\pm(z)$ and $\gamma^\pm(z)$ generated by a vacuum vector $|0\rangle$ which is annihilated by $\psi_m^+, \psi_{m+1}^- (m \in \mathbb{Z}_+), \gamma_r^\pm (r \in \frac{1}{2} + \mathbb{Z}_+)$.

More generally we take l (independent) copies of *bcβγ* fields $\psi^{\pm,k}(z), \gamma^{\pm,k}(z) (k = 1, \dots, l)$ and consider the corresponding Fock space $\mathfrak{F}^{\otimes l}$.

§8.1. The case of $\widehat{\mathfrak{gl}}_{\infty|\infty}$ and $\widehat{\mathcal{SD}}$

In this subsection, we will realize $\widehat{\mathfrak{gl}}_{\infty|\infty}$ and $\widehat{\mathcal{SD}}$ in the Fock space $\mathfrak{F}^{\otimes l}$, also cf. [AFMO1]. We then establish a $(\text{GL}(l), \mathfrak{gl}_{\infty|\infty})$ duality in $\mathfrak{F}^{\otimes l}$. A $(\text{GL}(l), \widehat{\mathcal{SD}})$ duality follows from this via the embeddings ϕ_s .

Introduce the following generating functions for $\widehat{\mathfrak{gl}}_{\infty|\infty}$:

$$\begin{aligned} E_0(z, w) &= \sum_{i,j \in \mathbb{Z}} E_{ij} z^{i-1} w^{-j}, \\ E_1(z, w) &= \sum_{r,s \in \frac{1}{2} + \mathbb{Z}} E_{rs} z^{r-\frac{1}{2}} w^{-s-\frac{1}{2}}, \\ E_+(z, w) &= \sum_{i \in \mathbb{Z}, s \in \frac{1}{2} + \mathbb{Z}} E_{is} z^{i-1} w^{-s-\frac{1}{2}}, \end{aligned}$$

$$E_-(z, w) = \sum_{r \in \frac{1}{2} + \mathbb{Z}, j \in \mathbb{Z}} E_{rj} z^{r-\frac{1}{2}} w^{-j}.$$

We have the following free field realization for $\widehat{\mathcal{SD}}$.

Proposition 8.1. *Let*

$$\begin{aligned} E_0(z, w) &= \sum_{k=1}^l : \psi^{+,k}(z) \psi^{-,k}(w) :, \\ E_1(z, w) &= - \sum_{k=1}^l : \gamma^{+,k}(z) \gamma^{-,k}(w) :, \\ E_+(z, w) &= \sum_{k=1}^l : \psi^{+,k}(z) \gamma^{-,k}(w) :, \\ E_-(z, w) &= - \sum_{k=1}^l : \gamma^{+,k}(z) \psi^{-,k}(w) :. \end{aligned}$$

This defines a representation of $\widehat{\mathfrak{gl}}_{\infty|\infty}$ on $\mathfrak{F}^{\otimes l}$ of central charge l .

Componentwise, we can write down the above representation of $\widehat{\mathfrak{gl}}_{\infty|\infty}$ on $\mathfrak{F}^{\otimes l}$ as follows:

$$\begin{aligned} E_{ij} &= \sum_{k=1}^l : \psi_{-i}^{+,k} \psi_j^{-,k} :, & E_{rs} &= - \sum_{k=1}^l : \gamma_{-r}^{+,k} \gamma_s^{-,k} :, \\ E_{is} &= \sum_{k=1}^l : \psi_{-i}^{+,k} \gamma_s^{-,k} :, & E_{rj} &= - \sum_{k=1}^l : \gamma_{-r}^{+,k} \psi_j^{-,k} :, \end{aligned}$$

where $i, j \in \mathbb{Z}$ and $r, s \in \frac{1}{2} + \mathbb{Z}$. The normal ordering $::$ is defined by moving the annihilation operators to the right.

It is well known that the fields $:\psi^{+,i}(z)\psi^{j,-}(z):$ define a representation of the affine algebra $\widehat{\mathfrak{gl}}(l)$ on the Fock space $\mathfrak{F}^{\otimes l}$ of central charge l . On the other hand the components of fields $-\ : \gamma^{+,i}(z)\gamma^{-,j}(z) :$ define a representation of $\widehat{\mathfrak{gl}}(l)$ on $\mathfrak{F}^{\otimes l}$ of central charge $-l$. Hence

$$:\psi^{+,i}(z)\psi^{j,-}(z) : - : \gamma^{+,i}(z)\gamma^{-,j}(z) :, \quad 1 \leq i, j \leq l,$$

give a representation of $\widehat{\mathfrak{gl}}(l)$ of central charge zero on $\mathfrak{F}^{\otimes l}$. The horizontal subalgebra is $\mathfrak{gl}(l)$ and its action on $\mathfrak{F}^{\otimes l}$ lifts to a rational action of the group $\text{GL}(l)$. In particular as a $\text{GL}(l)$ -module $\mathfrak{F}^{\otimes l}$ is completely reducible. The following proposition is a variant in our infinite-dimensional setting of the description of $\text{GL}(l)$ -invariants in $\text{End}(\mathfrak{F}^{\otimes l})$ [H1].

Proposition 8.2. $GL(l)$ and $\widehat{\mathfrak{gl}}_{\infty|\infty}$ are mutual centralizers in $\text{End}(\mathfrak{F}^{\otimes l})$. In particular $\mathfrak{F}^{\otimes l}$ is a multiplicity-free direct sum of irreducible $\widehat{\mathfrak{gl}}_{\infty|\infty} \times GL(l)$ -modules.

Our next task is to determine the decomposition of $\mathfrak{F}^{\otimes l}$ with respect to the joint action of $\widehat{\mathfrak{gl}}_{\infty|\infty} \times GL(l)$.

We will do this by explicitly displaying all joint $\widehat{\mathfrak{gl}}_{\infty|\infty} \times GL(l)$ highest weight vectors with respect to the (joint) Borel subalgebra $\widehat{B}_{\infty|\infty} \times B'$, where $\widehat{B}_{\infty|\infty} = \bigoplus_{r \geq 0} (\widehat{\mathfrak{gl}}_{\infty|\infty})_r$.

For this purpose we need some notation. Define for $j \in \mathbb{Z}_+$ the matrices X^{-j} as follows:

$$\begin{aligned}
 X^0 &= \begin{pmatrix} \psi_0^{-,l} & \psi_0^{-,l-1} & \cdots & \psi_0^{-,1} \\ \psi_0^{-,l} & \psi_0^{-,l-1} & \cdots & \psi_0^{-,1} \\ \vdots & \vdots & \cdots & \vdots \\ \psi_0^{-,l} & \psi_0^{-,l-1} & \cdots & \psi_0^{-,1} \end{pmatrix}, \\
 X^{-1} &= \begin{pmatrix} \gamma_{-\frac{1}{2}}^{-,l} & \gamma_{-\frac{1}{2}}^{-,l-1} & \cdots & \gamma_{-\frac{1}{2}}^{-,1} \\ \psi_{-1}^{-,l} & \psi_{-1}^{-,l-1} & \cdots & \psi_{-1}^{-,1} \\ \vdots & \vdots & \cdots & \vdots \\ \psi_{-1}^{-,l} & \psi_{-1}^{-,l-1} & \cdots & \psi_{-1}^{-,1} \end{pmatrix}, \\
 X^{-2} &= \begin{pmatrix} \gamma_{-\frac{1}{2}}^{-,l} & \gamma_{-\frac{1}{2}}^{-,l-1} & \cdots & \gamma_{-\frac{1}{2}}^{-,1} \\ \gamma_{-\frac{3}{2}}^{-,l} & \gamma_{-\frac{3}{2}}^{-,l-1} & \cdots & \gamma_{-\frac{3}{2}}^{-,1} \\ \psi_{-2}^{-,l} & \psi_{-2}^{-,l-1} & \cdots & \psi_{-2}^{-,1} \\ \vdots & \vdots & \cdots & \vdots \\ \psi_{-2}^{-,l} & \psi_{-2}^{-,l-1} & \cdots & \psi_{-2}^{-,1} \end{pmatrix}, \\
 &\quad \vdots \\
 &\quad \vdots \\
 X^{-k} \equiv -X^{-l} &= \begin{pmatrix} \gamma_{-\frac{1}{2}}^{-,l} & \gamma_{-\frac{1}{2}}^{-,l-1} & \cdots & \gamma_{-\frac{1}{2}}^{-,1} \\ \gamma_{-\frac{3}{2}}^{-,l} & \gamma_{-\frac{3}{2}}^{-,l-1} & \cdots & \gamma_{-\frac{3}{2}}^{-,1} \\ \vdots & \vdots & \cdots & \vdots \\ \gamma_{-l+\frac{1}{2}}^{-,l} & \gamma_{-l+\frac{1}{2}}^{-,l-1} & \cdots & \gamma_{-l+\frac{1}{2}}^{-,1} \\ \gamma_{-l-\frac{1}{2}}^{-,l} & \gamma_{-l-\frac{1}{2}}^{-,l-1} & \cdots & \gamma_{-l-\frac{1}{2}}^{-,1} \end{pmatrix}, \quad k \geq l.
 \end{aligned}$$

The matrices X^j , for $j \in \mathbb{N}$, are defined in a similar fashion. Namely, X^j is obtained from X^{-j} by replacing $\psi_i^{-,k}$ by $\psi_i^{+,l-k+1}$ and $\gamma_r^{-,k}$ by $\gamma_r^{+,l-k+1}$. For example,

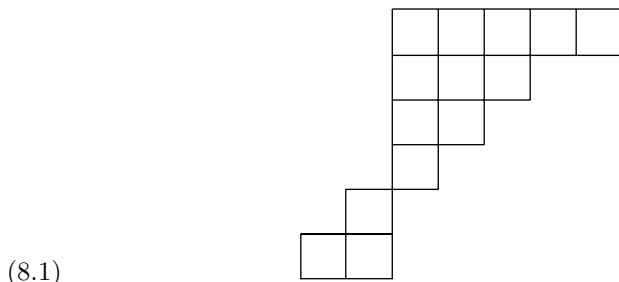
$$X^1 = \begin{pmatrix} \gamma_{-\frac{1}{2}}^{+,1} & \gamma_{-\frac{1}{2}}^{+,2} & \cdots & \gamma_{-\frac{1}{2}}^{+,l} \\ \psi_{-1}^{+,1} & \psi_{-1}^{+,2} & \cdots & \psi_{-1}^{+,l} \\ \vdots & \vdots & \cdots & \vdots \\ \psi_{-1}^{+,1} & \psi_{-1}^{+,2} & \cdots & \psi_{-1}^{+,l} \end{pmatrix}$$

For $0 \leq r \leq l$, we let $X_r^i (i \geq 0)$ denote the first $r \times r$ minor of the matrix X^i and let $X_{-r}^i (i < 0)$ denote the first $r \times r$ minor of the matrix X^i .

Consider a generalized Young diagram $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p)$ of length l with

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_i > \lambda_{i+1} = 0 = \cdots = \lambda_{j-1} > \lambda_j \geq \cdots \geq \lambda_l.$$

It is well-known that the irreducible rational representations of $GL(l)$ are parameterized by generalized Young diagrams, hence they may be interpreted as highest weights of irreducible representations of $GL(l)$. As usual we denote by λ'_j the length of the j -th column of λ . We use the convention that the first column of λ is the first column of the Young diagram $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_i$. The column to the right is the second column of λ , while the column to the left of it is the zeroth column and the column to the left of the zeroth column is the -1 -st column. We also use the convention that a non-positive column has negative length. As an example consider $\lambda = (5, 3, 2, 1, -1, -2)$ with $l(\lambda) = 6$. We have $\lambda'_{-1} = -1, \lambda'_0 = -2, \lambda'_1 = 4$ etc. (see (8.1)).



Let $\Lambda \in (\widehat{\mathfrak{gl}}_{\infty|\infty})_0^*$. Then Λ may be interpreted as a highest weight for a highest weight irreducible representation of $\widehat{\mathfrak{gl}}_{\infty|\infty}$. We let $\lambda_a = \Lambda(E_{aa})$, for $a \in \frac{1}{2}\mathbb{Z}$. Given a generalized Young diagram λ with $l(\lambda) \leq l$, we define $\Lambda(\lambda)$ of

$(\widehat{\mathfrak{gl}}_{\infty|\infty})_0^*$ to be the weight whose components are given by:

$$\begin{aligned} \lambda_i &= \langle \lambda'_i - i \rangle, & i \in \mathbb{N}, \\ \lambda_j &= -\langle -\lambda'_j + j \rangle, & j \in -\mathbb{Z}_+, \\ \lambda_r &= \langle \lambda_{r+1/2} - (r - 1/2) \rangle, & r \in \frac{1}{2} + \mathbb{Z}_+, \\ \lambda_s &= -\langle -\lambda_{p+(s+1/2)} + (s - 1/2) \rangle, & s \in -\frac{1}{2} - \mathbb{Z}_+. \end{aligned}$$

Theorem 8.1. (1) *As a $\mathrm{GL}(l) \times \widehat{\mathfrak{gl}}_{\infty|\infty}$ -module, $\mathfrak{F}^{\otimes l}$ is completely reducible and decomposes into isotypic components as follows:*

$$\mathfrak{F}^{\otimes l} \cong \bigoplus_{\lambda} V_l^{\lambda} \otimes L(\widehat{\mathfrak{gl}}_{\infty|\infty}, \Lambda(\lambda)),$$

where the summation is summed over all generalized Young diagrams λ with $l(\lambda) \leq l$.

(2) *The $\mathrm{GL}(l) \times \widehat{\mathfrak{gl}}_{\infty|\infty}$ -highest weight vector with respect to the Borel subalgebra $\widehat{B}_{\infty|\infty} \times B'$ in the λ -isotypic component of $\mathfrak{F}^{\otimes l}$ is*

$$(8.2) \quad \det X_{\lambda'_{l-1}}^{\lambda_l+1} \cdots \det X_{\lambda'_{-1}}^{-1} \cdot \det X_{\lambda'_0}^0 \cdot \det X_{\lambda'_1}^1 \cdot \det X_{\lambda'_2}^2 \cdots \det X_{\lambda'_{\lambda_1}}^{\lambda_1} |0\rangle.$$

Proof. The second part is a variation in our infinite-dimensional setting of Theorem 7.1. Since $\mathfrak{F}^{\otimes l}$ is a rational representation of $\mathrm{GL}(l)$, it follows from the second part that all irreducible representations of $\mathrm{GL}(l)$ appears in the decomposition of $\mathfrak{F}^{\otimes l}$. The multiplicity-freeness follows from Proposition 8.2. □

The irreducible representation $L(\widehat{\mathfrak{gl}}_{\infty|\infty}, \Lambda(\lambda))$ pulls back via $\hat{\phi}_s$ to an irreducible representation of $\widehat{\mathcal{SD}}$. For $\xi \in (\widehat{\mathcal{SD}})_0^*$, we denote as usual by $L(\widehat{\mathcal{SD}}, \xi)$ the irreducible highest weight representation of $\widehat{\mathcal{SD}}$. Introduce the generating functions

$$\begin{aligned} \Delta_0(x) &= -\xi(e^{x D \partial_{\theta}}), \\ \Delta_1(x) &= -\xi(e^{x D \theta \partial_{\theta}}). \end{aligned}$$

Proposition 8.3. *Let $L(\widehat{\mathfrak{gl}}_{\infty|\infty}, \Lambda)$ be the irreducible highest weight representation of highest weight Λ and central charge c such that $\Lambda(E_{aa}) = \lambda_a$.*

Then via $\hat{\phi}_s$ the module $L(\widehat{\mathfrak{gl}}_{\infty|\infty}, \Lambda)$ pulls back to the irreducible highest weight module $L(\widehat{\mathcal{SD}}, \lambda_{\widehat{\mathcal{SD}}})$, where $\lambda_{\widehat{\mathcal{SD}}}$ is specified by the generating functions

$$\Delta_0(x) = \frac{\sum_{i \in \mathbb{Z}} (\lambda_{i,i} - \lambda_{i+1,i+1} + \delta_{i,0}c) e^{(-i+s)x} + c}{e^x - 1},$$

$$\Delta_1(x) = \frac{\sum_{i \in \mathbb{Z}} (\lambda_{i-\frac{1}{2},i-\frac{1}{2}} - \lambda_{i+\frac{1}{2},i+\frac{1}{2}} - \delta_{i,0}c) e^{(-i+s)x} + c}{e^x - 1}.$$

Proof. The generating functions above can be directly calculated by using the embedding $\hat{\phi}_s$ given in (6.3) and Proposition 6.4. The irreducibility follows from Propositions 6.1 and 6.7. \square

Remark 8.1. It follows that $\widehat{\mathcal{SD}}$ and $GL(l)$ form a dual pair in the sense of Howe on $\mathfrak{F}^{\otimes l}$. The decomposition of $\mathfrak{F}^{\otimes l}$ with respect to the joint action $\widehat{\mathcal{SD}} \times GL(l)$ follows from Theorem 8.1 and Proposition 8.3.

Indeed we can write down the action of $\widehat{\mathcal{SD}}$ on $\mathfrak{F}^{\otimes l}$ in an explicit manner. Note that $J_n^{a,k} \equiv J_n^k M_a$ ($a = 0, 1, \pm$) together with C span $\widehat{\mathcal{SD}}$. Define

$$J^{a,n}(z) = \sum_{k \in \mathbb{Z}} J_k^{a,n} z^{-n-k-1}.$$

Proposition 8.4. *The action of $\widehat{\mathcal{SD}}$ on $\mathfrak{F}^{\otimes l}$ of central charge l is given in terms of the following generating functions:*

$$J^{0,n}(z) = - \sum_{p=1}^l : \psi^{+p}(z) \partial^n \psi^{-p}(z) :,$$

$$J^{1,n}(z) = \sum_{p=1}^l : \gamma^{+p}(z) \partial^n \gamma^{-p}(z) :,$$

$$J^{+,n}(z) = - \sum_{p=1}^l : \psi^{+p}(z) \partial^n \gamma^{-p}(z) :,$$

$$J^{-,n}(z) = \sum_{p=1}^l : \gamma^{+p}(z) \partial^n \psi^{-p}(z) :.$$

Here and further $\partial^n(\cdot)$ denotes the n -th derivative with respect to z .

Proof. We will prove the last identity only and the proof of the others is similar. Using the embedding ϕ_0 we calculate

$$\begin{aligned} J^{-,n}(z) &= \sum_{k \in \mathbb{Z}} (-t^k [D]_n M_-) z^{-n-k-1} \\ &= \sum_{k \in \mathbb{Z}} -[-j]_n E_{j-k,j} M_- z^{-k-n-1} \\ &= \sum_{k \in \mathbb{Z}} [-j]_n \sum_{p=1}^l : \gamma_{k-j}^{+p} \psi_j^{-p} : z^{-k-n-1} \\ &= \sum_{p=1}^l : \gamma^{+p}(z) \partial^n \psi^{-p}(z) : . \end{aligned}$$

□

Let $\mathfrak{P} = \{J_k^{a,n} \mid n+k \geq 0, k \in \mathbb{Z}, n \in \mathbb{Z}_+, a = 0, 1, \pm\}$ and let $\widehat{\mathfrak{P}} = \mathfrak{P} \oplus \mathbb{C}C$. One can check that $\widehat{\mathfrak{P}}$ is a parabolic subalgebra of $\widehat{\mathcal{SD}}$ (the central extension when restricted to \mathfrak{P} is trivial). Geometrically, \mathfrak{P} consists of those differential operators in \mathcal{SD} which extend to the interior of the circle. Denote by $M_c(\widehat{\mathcal{SD}})$ ($c \in \mathbb{C}$) the vacuum module (which is a generalized Verma module)

$$M(\widehat{\mathcal{SD}}, \widehat{\mathfrak{P}}, \Lambda) = \mathcal{U}(\widehat{\mathcal{SD}}) \otimes_{\mathcal{U}(\widehat{\mathfrak{P}})} \mathbb{C}_c,$$

where $\mathcal{U}(\cdot)$ denotes the universal enveloping algebra, \mathbb{C}_c is the one-dimensional representation of $\widehat{\mathfrak{P}}$ by letting $C = c\text{Id}$ and $\widehat{\mathfrak{P}} \cdot \mathbb{C}_l = 0$. Denote by $V_c(\widehat{\mathcal{SD}})$ the irreducible quotient of the $\widehat{\mathcal{SD}}$ -module $M_c(\widehat{\mathcal{SD}})$. Then $M_c(\widehat{\mathcal{SD}})$ and $V_c(\widehat{\mathcal{SD}})$ carry vertex superalgebra structures. For example, a proof of this can be given based on the above free field realization given in Proposition 8.4 parallel to the proof of Theorem 14.1, p. 132, [KWY].

Now the duality in Theorem 8.1 (cf. Remark 8.1) can be interpreted as a duality between $GL(l)$ and the vertex superalgebra $V_l(\widehat{\mathcal{SD}})$ on the Fock space $\mathfrak{F}^{\otimes l}$. The irreducible $\widehat{\mathfrak{gl}}_{\infty|\infty}$ -module appearing in $\mathfrak{F}^{\otimes l}$ becomes irreducible module over the vertex superalgebra $V_l(\widehat{\mathcal{SD}})$.

§8.2. The case of $\widehat{\mathcal{B}}$ and ${}^0\widehat{\mathcal{SD}}$

In this subsection we will construct actions of $\widehat{\mathcal{B}}$ and ${}^0\widehat{\mathcal{SD}}$ on $\mathfrak{F}^{\otimes l}$, and establish a $(\widehat{\mathcal{B}}, \text{Pin}(2l))$ duality and then a $({}^0\widehat{\mathcal{SD}}, \text{Pin}(2l))$ duality.

Let

$$\begin{aligned}
 & E_0(z, w) + z^{-1}wE_0(-w, -z) \\
 &= \sum_{i,j \in \mathbb{Z}} (E_{ij} - (-1)^{i+j}E_{-j,-i})M_0 z^{i-1}w^{-j} \\
 &= \sum_{k=1}^l (: \psi^{+,k}(z)\psi^{-,k}(w) : -z^{-1}w : \psi^{+,k}(-w)\psi^{-,k}(-z) :), \\
 & E_1(z, w) + E_1(-w, -z) \\
 &= \sum_{i,j \in \mathbb{Z}} (E_{ij} - (-1)^{i+j}E_{1-j,1-i})M_1 z^{i-1}w^{-j} \\
 &= - \sum_{k=1}^l (: \gamma^{+,k}(z)\gamma^{-,k}(w) : + : \gamma^{+,k}(-, w)\gamma^{-,k}(-z) :), \\
 & zE_+(z, w) + E_-(-w, -z) \\
 &= \sum_{i,j \in \mathbb{Z}} (E_{ij}M_+ + (-1)^{i+j}E_{1-j,-i}M_-)z^i w^{-j} \\
 &= \sum_{k=1}^l (z : \psi^{+,k}(z)\gamma^{-,k}(w) : - : \gamma^{+,k}(-, w)\psi^{-,k}(-z) :).
 \end{aligned}$$

The last equation above is equivalent to

$$\begin{aligned}
 & E_-(z, w) - wE_+(-w, -z) \\
 &= - \sum_{k=1}^l (: \gamma^{+,k}(z)\psi^{-,k}(w) : + w : \psi^{+,k}(-w)\gamma^{-,k}(-z) :).
 \end{aligned}$$

Proposition 8.5. *The above equations define a representation of $\widehat{\mathfrak{B}}$ on $\mathfrak{F}^{\otimes l}$ of central charge l .*

According to Feingold and Frenkel [FF], the Fourier components of the generating functions

$$(8.3) \quad : \psi^{+,p}(z)\psi^{+,q}(-z) :, : \psi^{-,p}(z)\psi^{-,q}(-z) :, : \psi^{+,p}(z)\psi^{-,q}(z) : + \frac{1}{2}\delta_{p,q}z^{-1}$$

generate a representation of the twisted affine algebra $\mathfrak{gl}^{(2)}(2l)$ of type $A_{2l-1}^{(2)}$ on $\mathfrak{F}^{\otimes l}$ with central charge 1. On the other hand, the Fourier components of the following generating functions

$$(8.4) \quad - : \gamma^{+,p}(z)\gamma^{+,q}(-z) :, \quad - : \gamma^{-,p}(z)\gamma^{-,q}(-z) :, \quad - : \gamma^{+,p}(z)\gamma^{-,q}(z) :$$

generate a representation of the twisted affine algebra $\mathfrak{gl}^{(2)}(2l)$ of type $A_{2l-1}^{(2)}$ on $\mathfrak{F}^{\otimes -l}$ with central charge -1 . We observe that the Fourier components of the generating functions in (8.3) and (8.4) correspond to the same generators in $\mathfrak{gl}^{(2)}(2l)$ with the appropriate coefficients as in (8.3) and (8.4) (cf. Eqs. (3.55–3.57), p. 139–140 in [FF]; note that our convention here is a little different). Then the diagonal action on $\mathfrak{F}^{\otimes l}$ given by

$$\begin{aligned} e_+^{pq}(z) &\equiv \sum_{n \in \mathbb{Z}} e_+^{pq}(n)z^{-n-1} =: \psi^{+,p}(z)\psi^{+,q}(-z) : - : \gamma^{+,p}(z)\gamma^{+,q}(-z) : \\ e_-^{pq}(z) &\equiv \sum_{n \in \mathbb{Z}} e_-^{pq}(n)z^{-n-1} =: \psi^{-,p}(z)\psi^{-,q}(-z) : - : \gamma^{-,p}(z)\gamma^{-,q}(-z) : \\ e^{pq}(z) &\equiv \sum_{n \in \mathbb{Z}} e^{pq}(n)z^{-n-1} =: \psi^{+,p}(z)\psi^{-,q}(z) : - : \gamma^{+,p}(z)\gamma^{-,q}(z) : + \frac{1}{2}\delta_{p,q}z^{-1} \end{aligned}$$

$(p, q = 1, \dots, l)$ is that of an affine algebra $\mathfrak{gl}^{(2)}(2l)$ of type $A_{2l-1}^{(2)}$ of central charge zero. The horizontal subalgebra of the affine algebra $\mathfrak{gl}^{(2)}(2l)$ spanned by $e_+^{pq}(0), e_-^{pq}(0), e^{pq}(0)$ ($p, q = 1, \dots, l$) is isomorphic to the Lie algebra $\mathfrak{so}(2l)$.

We claim that the action of the horizontal subalgebra $\mathfrak{so}(2l)$ can be lifted to an action of the Lie group $\text{Pin}(2l)$, also cf. [W1]. First let us take a digression to recall the group $\text{Pin}(2l)$.

The Pin group $\text{Pin}(n)$ is the double covering group of $O(n)$, namely we have an exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \text{Pin}(n) \longrightarrow O(n) \longrightarrow 1.$$

We then define the Spin group $\text{Spin}(n)$ to be the inverse image of $SO(n)$ under the projection from $\text{Pin}(n)$ to $O(n)$. Then we have the following exact sequence of Lie groups:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \text{Spin}(n) \longrightarrow SO(n) \longrightarrow 1.$$

Set $n = 2l$. Denote $\mathbf{1}_l = (1, 1, \dots, 1, 1) \in \mathbb{Z}^l$ and $\bar{\mathbf{1}}_l = (1, 1, \dots, 1, -1) \in \mathbb{Z}^l$. The irreducible representations of $\text{Spin}(2l)$ that do not factor to $SO(2l)$ are irreducible representations of $\mathfrak{so}(2l)$ of highest weights parameterized by

$$(8.5) \quad \lambda = \frac{1}{2}\mathbf{1}_l + (m_1, m_2, \dots, m_l)$$

and

$$(8.6) \quad \lambda = \frac{1}{2}\bar{\mathbf{1}}_l + (m_1, m_2, \dots, -m_l)$$

where $m_1 \geq \dots \geq m_l \geq 0, m_i \in \mathbb{Z}$. We are interested in irreducible representations of $\text{Pin}(2l)$ induced from irreducible representations of $\text{Spin}(2l)$ with

highest weight of (8.5) or (8.6). When restricted to $\text{Spin}(2l)$, it will decompose into a sum of the two irreducible representations of highest weights (8.5) and (8.6). We will use $V(\text{Pin}(2l), \lambda)$, where $\lambda = \frac{1}{2}|\mathbf{1}_l| + (m_1, m_2, \dots, \overline{m}_l), m_i \geq 0$, to denote this irreducible representation of $\text{Pin}(2l)$. Denote by

$$\Sigma(\text{Pin}) = \left\{ \frac{1}{2}|\mathbf{1}_l| + (m_1, m_2, \dots, \overline{m}_l), m_1 \geq \dots \geq m_l \geq 0, m_i \in \mathbb{Z} \right\}.$$

The proof of the following lemma is straightforward.

Lemma 8.1. *The action of $\mathfrak{so}(2l)$ commutes with that of $\widehat{\mathcal{B}}$ on $\mathfrak{F}^{\otimes l}$.*

Remark 8.2. As a representation of $\mathfrak{so}(2l)$, the Fock space $\mathfrak{F}^{\otimes l}$ is isomorphic to $\wedge(\mathbb{C}^l) \wedge \wedge(\mathbb{C}^{2l} \otimes \mathbb{C}^{\mathbb{N}}) \otimes S(\mathbb{C}^{2l} \otimes \mathbb{C}^{\mathbb{N}})$, where $\wedge(\mathbb{C}^l)$ is the sum of two half-spin representations and $\mathfrak{so}(2l)$ acts on $\mathbb{C}^{2l} \otimes \mathbb{C}^{\mathbb{N}}$ naturally by the left action on \mathbb{C}^{2l} . The action of $\mathfrak{so}(2l)$ can be lifted to $\text{Spin}(2l)$ which extends naturally to $\text{Pin}(2l)$. It follows that any irreducible representation of $\text{Spin}(2l)$ appearing in $\mathfrak{F}^{\otimes l}$ cannot factor to $\text{SO}(2l)$. A similar argument to the classical dual pair case [H2] shows that $\text{Pin}(2l)$ and $\widehat{\mathcal{B}}$ form a dual pair on $\mathfrak{F}^{\otimes l}$.

The Cartan subalgebra $\widehat{\mathcal{B}}_0$ of $\widehat{\mathcal{B}}$ is spanned by the basis elements $E_{aa} - E_{-a,-a}$, $a \in \frac{1}{2}\mathbb{N}$, and the central element $\mathbf{1}$. Let $\Lambda \in (\widehat{\mathcal{B}}_0)^*$, the restricted dual of $\widehat{\mathcal{B}}_0$. We denote its components $\Lambda(E_{aa} - E_{-a,-a})$ by λ_a .

Given a Young diagram $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ of length at most l with

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{l-1} \geq \lambda_l \geq 0,$$

we may define a highest weight $\Lambda(\lambda) \in (\widehat{\mathcal{B}}_0)^*$ as follows:

$$\begin{aligned} \lambda_j &= \langle \lambda'_j - j \rangle, & j \in \mathbb{N}, \\ \lambda_r &= \langle \lambda_{r+1/2} - (r - 1/2) \rangle, & r \in \frac{1}{2} + \mathbb{Z}_+. \end{aligned}$$

For each $j \in \mathbb{N}$ define the matrix \widetilde{X}^j to be the matrix obtained from X^j by replacing its last column $(\gamma_{-\frac{1}{2}}^{+,l}, \dots, \gamma_{-j+\frac{1}{2}}^{+,l}, \psi_{-j}^{+,l}, \dots, \psi_{-j}^{+,l})$ by

$$\left(-\gamma_{-\frac{1}{2}}^{+,l}, \gamma_{-\frac{3}{2}}^{+,l}, -\gamma_{-\frac{5}{2}}^{+,l}, \dots, (-1)^j \gamma_{-j+\frac{1}{2}}^{+,l}, (-1)^{j+1} \psi_{-j}^{+,l}, \dots, (-1)^{j+1} \psi_{-j}^{+,l} \right).$$

For $0 \leq r \leq l$ let us, as usual, denote by \widetilde{X}_r^j the first $r \times r$ minor of \widetilde{X}^j .

Theorem 8.2. (1) *As a $\text{Pin}(2l) \times \widehat{\mathcal{B}}$ -module, $\mathfrak{F}^{\otimes l}$ is completely reducible and decomposes into isotypic components as follows:*

$$\mathfrak{F}^{\otimes l} \cong \bigoplus_{\lambda \in \Sigma(\text{Pin})} V(\text{Pin}(2l); \lambda) \otimes L(\widehat{\mathcal{B}}, \Lambda(\lambda)).$$

(2) *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ be a Young diagram of length at most l . The following vectors are joint highest weight vectors of $\mathfrak{so}(2l) \times \widehat{\mathcal{B}}$ in $\mathfrak{F}^{\otimes l}$:*

(a) $\det X_{\lambda_1}^1 \det X_{\lambda_2}^2 \cdots \det X_{\lambda_{\lambda_1}}^{\lambda_1}.$

(b) $\psi_0^{-l} \det \tilde{X}_{\lambda_1}^1 \det \tilde{X}_{\lambda_2}^2 \cdots \det \tilde{X}_{\lambda_{\lambda_1}}^{\lambda_1}.$

Furthermore the highest weight with respect to $\widehat{\mathcal{B}}$ of (a) and (b) is $\Lambda(\lambda)$, while their highest weights with respect to $\mathfrak{so}(2l)$ are $\lambda = (\lambda_1 + \frac{1}{2}, \lambda_2 + \frac{1}{2}, \dots, \lambda_l + \frac{1}{2})$ and $\lambda = (\lambda_1 + \frac{1}{2}, \lambda_2 + \frac{1}{2}, \dots, -\lambda_l - \frac{1}{2})$, respectively.

Proof. From the second part we see that all irreducible modules of $\text{Pin}(2l)$ that do not factor through $O(2l)$ appear in the Fock space. The first part follows from this and the fact that $\widehat{\mathcal{B}}$ and $\text{Pin}(2l)$ form a dual pair.

In order to show that (a) is a highest weight vector it is enough to check that it is annihilated by $e_+^{pq}(0)$, for all $p \neq q$, due to Theorem 8.1, which is an easy computation.

The proof of (b) is somewhat more tedious, and we will omit the details, as it is also a rather straightforward calculation. We only remark that the crucial identity used in the computation is again Lemma 7.1.

Finally, once the highest weight vectors are explicitly given, the computation of weights of these vectors is straightforward. □

By composing the homomorphism $\hat{\phi}_0$ with the action of $\widehat{\mathcal{B}}$ on $\mathfrak{F}^{\otimes l}$, we realize a representation of ${}^0\widehat{\mathcal{SD}}$ on $\mathfrak{F}^{\otimes l}$. Let us write down the action of ${}^0\widehat{\mathcal{SD}}$ on $\mathfrak{F}^{\otimes l}$ explicitly. We introduce the following generating functions:

$$\begin{aligned} {}^0W_a^n(z) &= \sum_{k \in \mathbb{Z}} {}^0W_{a,k}^n z^{-k-n-1}, & a = 0, 1, \\ {}^0W_{\pm}^n(z) &= \sum_{k \in \mathbb{Z}} {}^0W_{\pm,k}^n z^{-k-n-1}. \end{aligned}$$

Proposition 8.6. *The action of ${}^0\widehat{\mathcal{SD}}$ on $\mathfrak{F}^{\otimes l}$ is given in terms of the following generating functions:*

$$\begin{aligned} {}^0W_0^n(z) &= \sum_{p=1}^l (: \psi^{+p}(z)\partial^n\psi^{-p}(z) : -z^{-1} : \partial^n(z\psi^{+p}(-z))\psi^{-p}(-z) :), \\ {}^0W_1^n(z) &= -\sum_{p=1}^l (: \partial^n\gamma^{-p}(z)\gamma^{+p}(z) : + : \partial^n(\gamma^{+p}(-z))\gamma^{-p}(-z) :), \\ {}^0W_{\times}^n(z) &= -\sum_{p=1}^l (: \partial^n\psi^{-p}(z)\gamma^{+p}(z) : + : \partial^n(z\psi^{+p}(-z))\gamma^{-p}(-z) :). \end{aligned}$$

Proof. We will only give a proof of the first identity, as proofs for the other two are obtained similarly.

$$\begin{aligned} {}^0W_0^n(z) &= \sum_{k \in \mathbb{Z}} t^k ([D]_n + (-1)^{k+1}[-D - k]_n) M_0 z^{-k-n-1} \\ &= \sum_{k, j \in \mathbb{Z}} ([-j]_n + (-1)^{k+1}[j - k]_n) E_{j-k, j} M_0 z^{-k-n-1} \\ &= \sum_{k, j \in \mathbb{Z}} ([-j]_n E_{j-k, j} + (-1)^{k+1}[-j]_n E_{-j, k-j}) M_0 z^{-k-n-1} \\ &= \sum_{k, j \in \mathbb{Z}} \sum_{p=1}^l \left([-j]_n : \psi_{k-j}^{+p} \psi_j^{-p} : + (-1)^{k+1}[-j]_n : \psi_j^{+p} \psi_{k-j}^{-p} : \right) \\ &\quad \times M_0 z^{-k-n-1} \\ &= \sum_{p=1}^l \left(: \psi^{+p}(z)\partial^n\psi^{-p}(z) : + \frac{1}{z} : \partial^n(z\psi^{+p}(-z))\psi^{-p}(-z) : \right). \end{aligned}$$

□

Remark 8.3. We recall that $E_0(z, w) + z^{-1}wE_0(-w, -z)$, $E_1(z, w) + E_1(-w, -z)$ and $E_-(z, w) - wE_+(-w, -z)$ are the generating functions for generators of $\widehat{\mathcal{B}}$. Indeed, one can also express the field as

$${}^0W_0^n(z) = \partial_w^n (E_0(z, w) + z^{-1}wE_0(-w, -z))|_{w=z}.$$

In other words, we have for $|w| > |z|$

$$E_0(z, w) + z^{-1}wE_0(-w, -z) = \sum_{n=0}^{\infty} \frac{1}{n!} {}^0W_0^n(z)(w - z)^n.$$

Similarly for $|w| > |z|$ we have

$$E_1(z, w) + E_1(-w, -z) = \sum_{n=0}^{\infty} \frac{1}{n!} {}^0W_1^n(z)(w-z)^n,$$

$$E_-(z, w) - wE_+(-w, -z) = \sum_{n=0}^{\infty} \frac{1}{n!} {}^0W_{\times}^n(z)(w-z)^n.$$

Now the irreducible module $L(\widehat{\mathcal{B}}, \Lambda(\lambda))$ pulls back to via $\hat{\phi}_0$ to a module over ${}^0\widehat{\mathcal{SD}}$, which remains irreducible thanks to Propositions 6.2 and 6.7. For $\xi \in ({}^0\widehat{\mathcal{SD}})_0^*$, we denote by $L({}^0\widehat{\mathcal{SD}}, \xi)$ the irreducible highest weight representation of ${}^0\widehat{\mathcal{SD}}$. Recall that the Cartan subalgebra of ${}^0\widehat{\mathcal{SD}}$ is spanned, aside from the central element, by

$$\{D^n \partial_{\theta} \theta | n \text{ odd}\} \cup \left\{ \left(D + \frac{1}{2} \right)^n \theta \partial_{\theta} | n \text{ odd} \right\},$$

so that its generating functions are ${}^0\Delta_0(x) = -\xi(\partial_{\theta} \theta \sinh(xD))$ and ${}^0\Delta_1(x) = -\xi(\theta \partial_{\theta} \sinh(x(D + 1/2)))$.

Proposition 8.7. *Let $L(\widehat{\mathcal{B}}, \Lambda)$ be the irreducible highest weight representation of highest weight Λ such that $\Lambda(E_{aa} - E_{-a,-a}) = \lambda_a$, $a \in \frac{1}{2}\mathbb{N}$. Then via $\hat{\phi}_0$ the module $L(\widehat{\mathfrak{gl}}_{\infty|\infty}, \Lambda)$ pulls back to the irreducible highest weight module $L({}^0\widehat{\mathcal{SD}}, \Lambda_{{}^0\widehat{\mathcal{SD}}})$, where the highest weight $\Lambda_{{}^0\widehat{\mathcal{SD}}}$ is specified by the generating functions*

$${}^0\Delta_0(x) = - \sum_{i \in \frac{1}{2} + \mathbb{Z}_+} \lambda_{rr} \sinh(-rx),$$

$${}^0\Delta_1(x) = - \sum_{i \in \mathbb{N}} \lambda_{ii} \sinh(-ix).$$

Proof. The generating functions for ${}^0\widehat{\mathcal{SD}}$ can be computed using Proposition 6.5. Irreducibility is a consequence of Propositions 6.2 and 6.7. □

Now via $\hat{\phi}_s$, $s \neq \frac{1}{2}\mathbb{Z}$, the irreducible representation $L(\widehat{\mathfrak{gl}}_{\infty|\infty}, \Lambda(\lambda))$ pulls back to an irreducible representation of ${}^0\widehat{\mathcal{SD}}$ by Proposition 6.2. Again using Proposition 6.5 we can show the following.

Proposition 8.8. *Let $L(\widehat{\mathfrak{gl}}_{\infty|\infty}, \Lambda)$ be the irreducible highest weight representation of highest weight Λ and central charge c such that $\Lambda(E_{aa}) = \lambda_a$.*

Then via $\hat{\phi}_s, s \neq \frac{1}{2}\mathbb{Z}$, the module $L(\widehat{\mathfrak{gl}}_{\infty|\infty}, \Lambda)$ pulls back to the irreducible highest weight module $L({}^0\widehat{\mathcal{SD}}, \Lambda_0\widehat{\mathcal{SD}})$, where the highest weight $\Lambda_0\widehat{\mathcal{SD}}$ is specified by the generating functions

$$\begin{aligned}
 {}^0\Delta_0(x) &= - \sum_{r \in \frac{1}{2} + \mathbb{Z}} \sinh(-rx) \lambda_{rr} - \frac{\cosh(sx) - 1}{\sinh(x/2)} c, \\
 {}^0\Delta_1(x) &= - \sum_{i \in \mathbb{Z}} \sinh((-i + s)x) \lambda_{ii} + \frac{\cosh((s - \frac{1}{2})x) - \cosh(x/2)}{\sinh(x/2)} c.
 \end{aligned}$$

Remark 8.4. It follows that ${}^0\widehat{\mathcal{SD}}$ and $\text{Pin}(2l)$ form a dual pair in the sense of Howe on $\mathfrak{F}^{\otimes l}$. The decomposition of $\mathfrak{F}^{\otimes l}$ with respect to the joint action ${}^0\widehat{\mathcal{SD}} \times \text{Pin}(2l)$ is easily obtained from Theorem 8.2 and Proposition 8.7. We further note that Theorem 8.1 together with Proposition 8.8 imply that ${}^0\widehat{\mathcal{SD}}$ and $\text{GL}(l)$ form a dual pair on $\mathfrak{F}^{\otimes l}$ as well.

§8.3. The case of $\widehat{\mathcal{P}}_{\pm\pm}$ and ${}^{\pm\pm}\widehat{\mathcal{SD}}$

We may also compose the homomorphism $\hat{\phi}_0$ with the action of $\widehat{\mathcal{P}}_{\pm\pm}$ on $\mathfrak{F}^{\otimes l}$. This way we obtain a representation of ${}^{\pm\pm}\widehat{\mathcal{SD}}$ on $\mathfrak{F}^{\otimes l}$ which we will describe explicitly. We introduce the following generating functions (we recall that the elements ${}^{\pm\pm}W_k^n, {}^{\pm\pm}W_{+,k}^n$ and ${}^{\pm\pm}W_{-,k}^n$ in ${}^{\pm\pm}\widehat{\mathcal{SD}}$ were introduced at the end of Section 4):

$$\begin{aligned}
 {}^{\pm\pm}W^n(z) &= \sum_{k \in \mathbb{Z}} {}^{\pm\pm}W_k^n z^{-k-n-1}, \\
 {}^{\pm\pm}W_{\pm}^n(z) &= \sum_{k \in \mathbb{Z}} {}^{\pm\pm}W_{\pm,k}^n z^{-k-n-1}, \\
 {}^{\pm\pm}W_{\pm}^n(z) &= \sum_{k \in \mathbb{Z}} {}^{\pm\pm}W_{-,k}^n z^{-k-n-1}.
 \end{aligned}$$

Proposition 8.9. *The action of ${}^{\pm\pm}\widehat{\mathcal{SD}}$ on $\mathfrak{F}^{\otimes l}$ is given in terms of the following generating functions:*

$$\begin{aligned}
 {}^{+\pm}W^n(z) &= \sum_{p=1}^l (: \psi^{+p}(z) \partial^n \psi^{-p}(z) : + : \partial^n \gamma^{+p}(z) \gamma^{-p}(z) :), \\
 {}^{+\pm}W_+^n(z) &= - \sum_{p=1}^l (: \gamma^{+p}(z) \partial^n \psi^{-p}(z) : \mp : \partial^n (\gamma^{+p}(z)) \psi^{-p}(z) :), \\
 {}^{+\pm}W_-^n(z) &= \sum_{p=1}^l (: \psi^{+p}(z) \partial^n \gamma^{-p}(z) : \pm : \partial^n \psi^{+p}(z) \psi^{-p}(z) :).
 \end{aligned}$$

Proof. We will only give a proof of the second identity, as proofs for the other two are obtained similarly.

$$\begin{aligned}
 {}^{\pm\pm}W_+^n(z) &= \sum_{k \in \mathbb{Z}} t^k ([D]_n \mp [-D - k - 1]_n) M_- z^{-k-n-1} \\
 &= \sum_{k, j \in \mathbb{Z}} ([-j]_n \mp [j - k - 1]_n) E_{j-k, j} M_- z^{-k-n-1} \\
 &= \sum_{k, j \in \mathbb{Z}} - ([-j]_n \mp [j - k - 1]_n) \sum_{p=1}^l \gamma_{k-j+1/2}^{+p} \psi_j^{-p} z^{-k-n-1} \\
 &= - \sum_{p=1}^l (: \gamma^{+p}(z) \partial^n \psi^{-p}(z) : \mp : \partial^n (\gamma^{+p}(z)) \psi^{-p}(z) :).
 \end{aligned}$$

□

Remark 8.5. Let $\mathfrak{P}^\pm = \{ {}^{\pm\pm}W_k^n, {}^{\pm\pm}W_{+,k}^n, {}^{\pm\pm}W_{-,k}^n \mid n+k \geq 0, k \in \mathbb{Z}, n \in \mathbb{Z}_+ \}$. One can show that $\widehat{\mathfrak{P}}^\pm = \mathfrak{P}^\pm \oplus \mathbb{C}\mathbb{C}$ is a parabolic subalgebra of ${}^{\pm\pm}\widehat{\mathcal{SD}}$. As in the $\widehat{\mathcal{SD}}$ case, we define the vacuum ${}^{\pm\pm}\widehat{\mathcal{SD}}$ -module $M_c({}^{\pm\pm}\widehat{\mathcal{SD}})$ ($c \in \mathbb{C}$)

$$M_c({}^{\pm\pm}\widehat{\mathcal{SD}}) = \mathcal{U}({}^{\pm\pm}\widehat{\mathcal{SD}}) \underset{\mathcal{U}(\widehat{\mathfrak{P}}^\pm)}{\otimes} \mathbb{C}_c$$

and its irreducible quotient $V_c({}^{\pm\pm}\widehat{\mathcal{SD}})$. Then $M_c({}^{\pm\pm}\widehat{\mathcal{SD}})$ and $V_c({}^{\pm\pm}\widehat{\mathcal{SD}})$ also carry vertex superalgebra structures.

Proposition 8.10. *The action of ${}^{-\pm}\widehat{\mathcal{SD}}$ on $\mathfrak{F}^{\otimes l}$ is given in terms of the following generating functions:*

$$\begin{aligned}
 {}^{-\pm}W^n(z) &= \sum_{p=1}^l (: \psi^{+p}(z) \partial^n \psi^{-p}(z) : - : \partial^n \gamma^{+p}(-z) \gamma^{-p}(-z) :), \\
 {}^{-\pm}W_+^n(z) &= \sum_{p=1}^l (: \gamma^{+p}(z) \partial^n \psi^{-p}(z) : \pm : \partial^n (\gamma^{+p}(-z)) \psi^{-p}(-z) :), \\
 {}^{-\pm}W_-^n(z) &= \sum_{p=1}^l (: \psi^{+p}(z) \partial^n \gamma^{-p}(z) : \mp : \partial^n (\psi^{+p}(z)) \gamma^{-p}(z) :).
 \end{aligned}$$

Proof. We will give a proof of the second identity only.

$$\begin{aligned}
 {}^{\pm}W_+^n(z) &= \sum_{k \in \mathbb{Z}} t^k ([D]_n \pm (-1)^{k+1}[-D - k - 1]_n) M_- z^{-k-n-1} \\
 &= \sum_{k, j \in \mathbb{Z}} ([-j]_n \pm (-1)^{k+1}[j - k - 1]_n) E_{j-k, j} M_- z^{-k-n-1} \\
 &= \sum_{k, j \in \mathbb{Z}} ([-j]_n \pm (-1)^{k+1}[j - k - 1]_n) \sum_{p=1}^l \gamma_{k-j+1/2}^{+p} \psi_j^{-p} z^{-k-n-1} \\
 &= \sum_{p=1}^l (:\gamma^{+p}(z)\partial^n \psi^{-p}(z): \pm : \partial^n(\gamma^{+p}(-z))\psi^{-p}(-z):).
 \end{aligned}$$

□

Remark 8.6. The Fock space $\mathfrak{F}^{\otimes l}$ is not completely reducible with respect to the action of $\widehat{\mathcal{P}}_{\pm\pm}$. Similar remarks as Remark 8.3 hold in the cases of $\pm\pm\widehat{\mathcal{SD}}$.

§9. Conclusion and Discussion

In this paper we have classified all anti-involutions of Lie superalgebra $\widehat{\mathcal{SD}}$ preserving the principal gradation, where $\widehat{\mathcal{SD}}$ is the central extension of the Lie superalgebra of differential operators on the super circle $S^{1|1}$. There are five families of them, while the anti-involutions within a family is related to each other by a spectral flow. We found close relations between the subalgebras ${}^0\widehat{\mathcal{SD}}, \pm\pm\widehat{\mathcal{SD}}$ fixed by the five anti-involutions and subalgebras $\widehat{\mathcal{B}}, \widehat{\mathcal{P}}_{\pm\pm}$ of $\widehat{\mathfrak{gl}}_{\infty|\infty}$. We realize these Lie superalgebras by a free field realization, and further establish dualities between them and certain finite-dimensional classical Lie groups on Fock spaces. Our construction generalizes [KWY].

There are various interesting questions arising from our current work. Below we will list some of them. Some of these questions have been better understood in the $\mathcal{W}_{1+\infty}$ case.

- (1) The Lie superalgebra $\widehat{\mathcal{SD}}$, i.e. the super extension of $\mathcal{W}_{1+\infty}$ was first studied by Manin and Radul [MR] in the context of supersymmetric KP hierarchy. It is natural to ask to construct the hierarchy corresponding to the Lie superalgebras ${}^0\widehat{\mathcal{SD}}$ and $\pm\pm\widehat{\mathcal{SD}}$.
- (2) Calculate the principle q -character formulas for the irreducible modules of ${}^0\widehat{\mathcal{SD}}$ and $\pm\pm\widehat{\mathcal{SD}}$ that appear in the free field realizations. For $\widehat{\mathcal{SD}}$ this problem is solved in [CL].

- (3) Identify the vertex superalgebras $V_l(\widehat{\mathcal{SD}})$ and $V_l(+\pm\widehat{\mathcal{SD}})$ of central charge $l \in \mathbb{N}$ with more familiar \mathcal{W} superalgebras (cf. [BS]). Such a question has been answered for $\mathcal{W}_{1+\infty}$ when $l \in \mathbb{N}$ [FKRW] and when $l = -1$ [W2].
- (4) Classify the irreducible modules of the vertex superalgebras $V_l(\widehat{\mathcal{SD}})$ and $V_l(+\pm\widehat{\mathcal{SD}})$.
- (5) What is the fusion ring of the vertex superalgebras $V_l(\widehat{\mathcal{SD}})$ and $V_l(+\pm\widehat{\mathcal{SD}})$ of central charge $l \in \mathbb{N}$? The duality between $(GL(l), \mathcal{W}_{1+\infty}|_{c=l})$ lead the authors in [FKRW] to conjecture that the fusion ring for $\mathcal{W}_{1+\infty}|_{c=l}$ is isomorphic to the representation ring of $GL(l)$. (Indeed one can argue that the former contains the later). It is natural to conjecture that the fusion ring for $\widehat{\mathcal{SD}}$ of central charge l is also isomorphic to the representation ring of $GL(l)$ based on our duality results.
- (6) We have studied the superalgebra of differential operator on the super circle $S^{1|N}$ with $N = 1$ extended symmetry. One may ask similar questions for the super circle $S^{1|N}$ for a general N .

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