Uhlenbeck Spaces for $\mathbb{A}^2$ and Affine Lie Algebra $\widehat{\mathfrak{sl}}_n$

Dedicated to Israel M. Gelfand on the occasion of his 90th birthday

By

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Abstract

We introduce an Uhlenbeck closure of the space of based maps from projective line to the Kashiwara flag scheme of an untwisted affine Lie algebra. For the algebra $\widehat{sl}_n$ this space of based maps is isomorphic to the moduli space of locally free parabolic sheaves on $P^1 \times P^1$ trivialized at infinity. The Uhlenbeck closure admits a resolution of singularities: the moduli space of torsion free parabolic sheaves on $P^1 \times P^1$ trivialized at infinity. We compute the Intersection Cohomology sheaf of the Uhlenbeck space using this resolution of singularities. The moduli spaces of parabolic sheaves of various degrees are connected by certain Hecke correspondences. We prove that these correspondences define an action of $\widehat{sl}_n$ in the cohomology of the above moduli spaces.

§1. Introduction

1.1. For a symmetrizable Cartan matrix A, and the corresponding Kac-Moody algebra $\mathfrak{g}(A)$, M. Kashiwara has introduced a remarkable *flag scheme* $\mathcal{B}(A)$ [13]. It shares many properties of the usual flag varieties of semisimple Lie algebras. For one thing, if $\mathbf{C}$ is a smooth projective curve of genus 0, and $\mathbf{c} \in \mathbf{C}$ a

marked point, the space $\overset{\circ}{\mathbf{M}}{}^{\alpha}(A)$ of *based maps* from $(\mathbf{C}, \mathbf{c})$ to $(\mathcal{B}(A), B_0)$ of degree α turns out surprisingly to be a smooth finite-dimensional quasiaffine variety, though $\mathcal{B}(A)$ itself is of infinite type in general.

In case $\mathfrak{g}(A)$ is semisimple, V. Drinfeld has introduced a remarkable affine closure $\mathbf{M}^{\alpha}(A) \supset \overset{\circ}{\mathbf{M}}{}^{\alpha}(A)$ (the space of *based quasimaps*, alias *Zastava* space) which has found applications in the study of quantum groups at roots of unity and geometric Eisenstein series. In fact, Drinfeld's definition works for arbitrary symmetrizable A, but $\mathbf{M}^{\alpha}(A)$ turns out to be of infinite type in general.

However, if $\mathfrak{g}(A)$ is an *untwisted affine* Lie algebra, it appears that $\mathbf{M}^{\alpha}(A)$ possesses a partial resolution $\mathfrak{M}^{\alpha}(A) \to \mathbf{M}^{\alpha}(A)$ with rather favorable properties (for one thing, $\mathfrak{M}^{\alpha}(A)$ is of finite type). The construction of $\mathfrak{M}^{\alpha}(A)$ we propose at the moment is quite cumbersome, in a sense it occupies the bulk of this paper.

1.2. In case $A = \widetilde{\mathbf{A}}_{n-1}$ we have $\mathfrak{g}(A) = \widehat{\mathfrak{sl}}_n$, and $\mathfrak{M}^{\alpha} := \mathfrak{M}^{\alpha}(\widetilde{\mathbf{A}}_{n-1})$ admits a semismall resolution of singularities $\varpi_{\alpha} : \mathcal{M}^{\alpha} \to \mathfrak{M}^{\alpha}$. The quasiprojective variety $\mathcal{M}^{\alpha}$ is not new; it is just the moduli space of torsion free *parabolic sheaves* of degree α on the surface $\mathbf{C} \times \mathbb{P}^1$ trivialized at infinity. For a torsion free parabolic sheaf $\mathcal{F}_{\bullet} \in \mathcal{M}^{\alpha}$ one can define its *saturation* $\mathcal{N}(\mathcal{F})_{\bullet}$ which is a *locally free* parabolic sheaf containing $\mathcal{F}_{\bullet}$, and *defect* $\mathrm{def}(\mathcal{F}_{\bullet})$ which is roughly speaking a colored zero-cycle on $\mathbf{C} \times \mathbb{P}^1$ measuring the quotient $\mathcal{N}(\mathcal{F})_{\bullet}/\mathcal{F}_{\bullet}$. The proper map $\varpi_{\alpha} : \mathcal{M}^{\alpha} \to \mathfrak{M}^{\alpha}$ glues together various parabolic sheaves with the same saturation and defect.

Thus, the closure $\mathfrak{M}^{\alpha} \supset \overset{\circ}{\mathbf{M}}{}^{\alpha}$ may be obtained in two steps. First we view $\overset{\circ}{\mathbf{M}}{}^{\alpha}$ as the moduli space of locally free parabolic sheaves of degree α on the surface $\mathbf{C} \times \mathbb{P}^1$ trivialized at infinity, and put it inside the moduli space of torsion free parabolic sheaves $\mathcal{M}^{\alpha}$. Second, we glue together certain torsion free parabolic sheaves. This idea is not new: for the moduli spaces of vector bundles on surfaces it gives rise to *Uhlenbeck* compactifications. This is why we call $\mathfrak{M}^{\alpha}$ an *Uhlenbeck flag space* for $\mathbb{A}^2$ (though we work with the surface $\mathbf{C} \times \mathbb{P}^1$, the trivialization at infinity essentially leaves us with $\mathbb{A}^2 \subset \mathbf{C} \times \mathbb{P}^1$).

In the framework of moduli spaces of instantons on Riemannian 4-manifolds the Uhlenbeck compactification was introduced by S. Donaldson, following the pioneering work [23]. In the framework of algebraic geometry, the Uhlenbeck compactification of moduli space of vector bundles on a surface was constructed by J. Li [17]. However, in the special case of the surface $\mathbf{C} \times \mathbb{P}^1$, and vector bundles trivialized at infinity (or equivalently, vector bundles on $\mathbb{P}^2$ trivialized at infinity), the algebraic geometric version of Uhlenbeck space was defined by S. Donaldson, via the ADHM description of

instantons on S^4, see chapter 3 of [7]. The algebraic geometry of Donaldson's Uhlenbeck space was thoroughly studied by H. Nakajima in his remarkable works on Quiver varieties.

So roughly speaking we cook up our $\mathfrak{M}^\alpha$ from Drinfeld's Zastava space $\mathbf{M}^\alpha$ and Donaldson's Uhlenbeck space for $\mathbb{A}^2$. Though the exposition in the main body of the paper concerns the case $\mathfrak{g}(A) = \widehat{\mathfrak{sl}}_n$, we spell it out in such a way that the construction carries out without changes for an arbitrary untwisted affine Lie algebra $\mathfrak{g}(A)$, cf. 6.8.

1.3. The particularity of the case $\mathfrak{g}(A) = \widehat{\mathfrak{sl}}_n$ lies in the existence of a semismall resolution of singularities $\varpi_\alpha : \mathcal{M}^\alpha \to \mathfrak{M}^\alpha$. This is similar to the existence of a *small resolution*, due to G. Laumon, of the Zastava space $\mathbf{M}^\alpha(\mathfrak{sl}_n)$. We apply the resolution ϖ_α to compute the Intersection Cohomology sheaf $\mathcal{IC}(\mathfrak{M}^\alpha)$, similarly to [15], where Laumon's resolution was used to compute $\mathcal{IC}(\mathbf{M}^\alpha(\mathfrak{sl}_n))$. The necessary information about the fibers of ϖ_α was already obtained in [9], [21], so in a sense, all the hard work was already done a long time ago. The generating function of the $\mathcal{IC}$-stalks is governed by the product of Kostant partition function for $\widehat{\mathfrak{sl}}_n$, and another partition function, arising from the invariants of a principal nilpotent element of $\mathfrak{sl}_n$ in the nilpotent radical of the maximal parabolic subalgebra of $\widehat{\mathfrak{sl}}_n$.

For an arbitrary untwisted affine $\mathfrak{g}(A)$ we propose a conjectural answer for the stalks of $\mathcal{IC}(\mathfrak{M}^\alpha(A))$ in 6.7.

1.4. We also study another moduli space $\mathcal{M}^\alpha_{\mathbf{gt}} \supset \mathcal{M}^\alpha$ of parabolic torsion free sheaves of degree α on $\mathbf{C} \times \mathbb{P}^1$, where we relax the condition of triviality at infinity, and impose only a condition that a torsion free sheaf $\mathcal{F}_0$ is *generically trivial*, that is trivial on some line $c \times \mathbb{P}^1$. For any α, γ there is a closed subvariety of middle dimension (*Hecke correspondence*) $\mathfrak{E}^\gamma_\alpha \subset \mathcal{M}^\alpha_{\mathbf{gt}} \times \mathcal{M}^{\alpha+\gamma}_{\mathbf{gt}}$. It is formed by pairs of parabolic sheaves such that the second one is a subsheaf of the first one. The top-dimensional irreducible components of $\mathfrak{E}^\gamma_\alpha$ are naturally numbered by the isomorphism classes $\kappa \in \mathfrak{K}(\gamma)$ of γ-dimensional nilpotent representations of the cyclic quiver $\widetilde{\mathbf{A}}_{n-1}$, independently of α. For $\kappa \in \mathfrak{K}(\gamma)$ the corresponding irreducible component $\mathfrak{E}^\kappa_\alpha$, viewed as a correspondence between $\mathcal{M}^\alpha_{\mathbf{gt}}$ and $\mathcal{M}^{\alpha+\gamma}_{\mathbf{gt}}$, defines two operators:

$$e_\kappa : \ H^\bullet(\mathcal{M}^\alpha_{\mathbf{gt}}) \rightleftharpoons H^\bullet(\mathcal{M}^{\alpha+\gamma}_{\mathbf{gt}}) \ : f_\kappa$$

Let $\mathbf{H}$ denote the generic Hall algebra of nilpotent representations of the cyclic quiver $\widetilde{\mathbf{A}}_{n-1}$ at $q = 1$. It turns out that the linear span of operators e_κ is closed under composition; the algebra they form is naturally isomorphic to $\mathbf{H}$, and the isomorphism takes e_κ to the element of $\mathbf{H}$ corresponding to the isomorphism

class κ. Moreover, for the isomorphism classes of simple representations $\kappa = \{\{i\}\}$, $i \in \mathbb{Z}/n\mathbb{Z}$, the corresponding operators e_i, f_i define the action of the Chevalley generators of $\mathfrak{g}(\widetilde{\mathbf{A}}_{n-1}) = \widehat{\mathfrak{sl}}_n$ on $\bigoplus_\alpha H^\bullet(\mathcal{M}^\alpha_{\mathbf{gt}})$. This $\widehat{\mathfrak{sl}}_n$-action has central charge 2. This is a partial realization of the programme outlined in [9] 1.3.

1.5.   Let us say a few words about the structure of the paper. In §2 we recall the well known facts about the Kashiwara flag scheme for $\widehat{\mathfrak{sl}}_n$, and various realizations thereof. In §3 we introduce the moduli space of torsion free parabolic sheaves $\mathcal{M}^\alpha$, and construct a family of regular functions on it, which will be used in the definition of the resolution $\varpi_\alpha : \mathcal{M}^\alpha \to \mathfrak{M}^\alpha$. In §4 we recall the Drinfeld's spaces of based maps and quasimaps $\overset{\circ}{\mathbf{M}}{}^\alpha \subset \mathbf{M}^\alpha$, and define the Uhlenbeck flag space $\mathfrak{M}^\alpha$ as a closure of $\overset{\circ}{\mathbf{M}}{}^\alpha$ in some quasiaffine embedding (like Schubert varieties are closures of Schubert cells in the usual flag varieties). In §5 we construct the resolution ϖ_α, and in §6 we compute $\mathcal{IC}(\mathfrak{M}^\alpha)$. Note that while the generating function of the stalks of $\varpi_{\alpha*}\mathcal{IC}(\mathcal{M}^\alpha)$ involves the Kostant partition function of $\widehat{\mathfrak{gl}}_n$, the generating function of the $\mathcal{IC}$-stalks of $\mathfrak{M}^\alpha$ involves the Kostant partition function of $\widehat{\mathfrak{sl}}_n$: the *semi*smallness of ϖ_α kills the extra imaginary roots. In §7 we study the Hecke correspondences; among other things, they are used in the proof of connectedness of $\mathcal{M}^\alpha$.

§2.   Kashiwara Flag Scheme for $\widehat{\mathfrak{sl}}_n$

2.1.   Recall that the affine Lie algebra $\widehat{\mathfrak{sl}}_n$ is the canonical central extension

$$0 \to \mathbb{C} \to \widehat{\mathfrak{sl}}_n \to \mathfrak{sl}_n \otimes \mathbb{C}((t^{-1})) \to 0$$

Let us fix an n-dimensional vector space V with a basis $v_1, \dots, v_n$, and identify $\mathfrak{sl}_n$ with $\mathfrak{sl}(V)$. For $1 \le i \ne j \le n$ we denote by $E_{ij} \in \mathfrak{sl}_n$ the operator taking v_j to v_i, and annihilating other base vectors. Then the Chevalley generators of $\widetilde{\mathfrak{sl}}_n$ are as follows: $e_0 = t^{-1}E_{n1}$, $f_0 = tE_{1n}$, $h_0 = [e_0, f_0]$; for $1 \le i \le n-1$ we set $e_i = E_{i,i+1}$, $f_i = E_{i+1,i}$, $h_i = [e_i, f_i]$. Thus the simple positive coroots are naturally numbered by $0 \le i \le n-1$. We will identify this set with $I := \mathbb{Z}/n\mathbb{Z}$. We will denote by Y the coroot lattice $\mathbb{Z}[I]$, and by X the dual weight lattice. We denote the perfect pairing $X \times Y \to \mathbb{Z}$ by $\langle, \rangle$, and the basis of X dual to I consists of fundamental weights ω_i, $i \in I$. Thus $\langle \omega_j, i \rangle = \delta_{ij}$. A simple root dual to a simple coroot $i \in I$ will be denoted by $i' \in X$.

For a dominant weight $X^+ \ni \lambda = \sum_I l_i \omega_i$, $l_i \in \mathbb{N}$, we denote by V_λ the corresponding highest weight integrable $\widehat{\mathfrak{sl}}_n$-module (its highest vector is annihilated by e_i, $i \in I$). We denote by V_λ^* the dual (pro-finite dimensional) vector space.

2.2.   **Fundamental representations.** The general reference for this subsection is [4]. Recall the semi-infinite wedge construction of the fundamental representations V_{ω_i}. Let $\mathbf{V}$ denote the Tate vector space $V \otimes \mathbb{C}((t^{-1}))$. Then the Tate vector space $\mathbf{W} := \mathbf{V} \oplus \mathbf{V}^*$ has a natural symmetric bilinear form which gives rise to the Clifford algebra $\mathsf{Cliff}(\mathbf{W})$. We choose a compact lattice $L_V = V \otimes \mathbb{C}[[t^{-1}]] \subset \mathbf{V}$, and consider a compact lattice $L_W = L_V \oplus L_V^\perp \subset \mathbf{W}$. Then L_W is an isotropic subspace of $\mathbf{W}$, and its exterior algebra embeds naturally into the Clifford algebra: $\Lambda^\bullet(L_W) \subset \mathsf{Cliff}(\mathbf{W})$.

We define the Clifford module $\mathcal{Q}$ as $\mathrm{Ind}_{\Lambda^\bullet(L_W)}^{\mathsf{Cliff}(\mathbf{W})}$. In fact, its isomorphism class is independent of the choice of compact lattice $L_V \subset \mathbf{V}$.

Consider an arbitrary compact lattice $L_1 \subset \mathbf{V}$ and another compact lattice $L_2 \subset L_1^\perp \subset \mathbf{V}^*$. We set $L_{1,2} := L_1 \oplus L_2 \subset \mathbf{W}$. Then the invariants $\mathcal{Q}^{L_{1,2}}$ form a finite dimensional vector subspace canonically isomorphic to $\Lambda^*(L_2^\perp/L_1) \otimes \det(L_2^\perp)$. Here $\Lambda^*(?)$ is a vector space dual to the exterior algebra $\Lambda^\bullet(?)$, and $\det(L_2^\perp)$ is the relative determinant of the lattice $L_2^\perp \subset \mathbf{V}$ with respect to L_V. Clearly, $\mathcal{Q}$ is a union of $\mathcal{Q}^{L_{1,2}}$ as L_1, L_2 shrink.

The algebra $\widehat{\mathfrak{gl}}_n \supset \widehat{\mathfrak{sl}}_n$ acts naturally on $\mathcal{Q}$. It is well known that for any $i \in I$ there is a canonical embedding $s_i : V_{\omega_i} \to \mathcal{Q}$. In fact, $\mathcal{Q}$ is the direct sum of fundamental representations of $\widehat{\mathfrak{gl}}_n$.

2.3. Plücker equations. Kashiwara [13] defines the flag scheme $\mathcal{B}$ for $\widehat{\mathfrak{sl}}_n$ as the (infinite type) subscheme of $\prod_{\lambda \in X^+} \mathbb{P}(V_\lambda^*)$ cut out by *Plücker equations*:

A collection of lines $(\ell_\lambda \subset V_\lambda^*)_{\lambda \in X^+}$ satisfies Plücker equations if

(a) For any nonzero $\widehat{\mathfrak{sl}}_n$-morphism $\varphi : V_\lambda^* \hat{\otimes} V_\mu^* \to V_{\lambda+\mu}^*$ we have $\varphi(\ell_\lambda \otimes \ell_\mu) = \ell_{\lambda+\mu}$;

(b) For any $\widehat{\mathfrak{sl}}_n$-morphism $\varphi : V_\lambda^* \hat{\otimes} V_\mu^* \to V_\nu^*$ such that $\nu < \lambda + \mu$ we have $\varphi(\ell_\lambda \otimes \ell_\mu) = 0$.

The inverse image of the line bundle $\mathcal{O}(1)$ on $\mathbb{P}(V_\lambda^*)$ is the line bundle on $\mathcal{B}$ denoted by $\mathcal{L}_\lambda$. We have $\Gamma(\mathcal{B}, \mathcal{L}_\lambda) = V_\lambda$.

Note that the Plücker equation (a) above implies that $\mathcal{B}$ embeds as a closed subscheme into $\prod_{i \in I} \mathbb{P}(V_{\omega_i}^*)$.

2.4. Discrete lattices. The above definition works in the generality of an arbitrary symmetrizable Kac-Moody algebra. In the particular case of $\widehat{\mathfrak{sl}}_n$ there is another well known definition of $\mathcal{B}$ in terms of periodic flags in the Tate vector space $\mathbf{V}$. Namely, $\mathcal{B}$ is a scheme (of infinite type) parametrizing collections of discrete lattices $(F_k \subset \mathbf{V})_{k \in \mathbb{Z}}$ such that

(a) The kernel and cokernel of the natural map $F_0 \oplus V \otimes \mathbb{C}[[t^{-1}]] \to \mathbf{V}$ have the same dimension.

(b) $F_k \subset F_{k+1}$, and $\dim(F_{k+1}/F_k) = 1$ for any k;

(c) $F_{k+n} = t^{-1} F_k$ for any k;

Let us construct an isomorphism from the second definition of $\mathcal{B}$ to the first one. To this end let us temporarily denote $\mathcal{B}$ in the first (resp. second) definition by $\mathcal{B}_1$ (resp. $\mathcal{B}_2$). Given a flag (F_k) and $0 \leq i \leq n-1$, we consider a discrete lattice $F_{W,i} = F_i \oplus F_i^\perp \subset \mathbf{W}$. It is well known that the coinvariants of the Clifford module $\mathcal{Q}_{F_{W,i}}$ are one-dimensional. Composing with the canonical embedding $s_i : V_{\omega_i} \to \mathcal{Q}$ (see 2.2) we obtain a projection $p_i : V_{\omega_i} \to \mathcal{Q}_{F_{W,i}}$.

Thus we have constructed a line bundle $\mathcal{L}_i$ over $\mathcal{B}_2$ (with a fiber over $(F_k)_{k \in \mathbb{Z}}$ equal to $\mathcal{Q}_{F_{W,i}}$) together with a surjection $p_i : V_{\omega_i} \otimes \mathcal{O}_{\mathcal{B}_2} \twoheadrightarrow \mathcal{L}_i$. It defines a map $\mathcal{B}_2 \to \mathbb{P}(V_{\omega_i}^*)$, and taking the product over $i \in I$ we obtain an embedding $\mathcal{B}_2 \hookrightarrow \prod_{i \in I} \mathbb{P}(V_{\omega_i}^*)$ which identifies it with the image of Plücker embedding of $\mathcal{B}_1$. This way $\mathcal{L}_i$ on $\mathcal{B}_2$ gets identified with $\mathcal{L}_{\omega_i}$ on $\mathcal{B}_1$.

2.5. Parabolic vector bundles. The second definition of the flag scheme $\mathcal{B}$ translates immediately into the language of vector bundles on $\mathbb{P}^1$. Namely, let $\mathbf{X}$

be a smooth projective curve of genus 0. We choose two distinct points $\mathbf{y}, \mathbf{x} \in \mathbf{X}$ and a global rational coordinate $t: \mathbf{X} \to \mathbb{P}^1$ such that $t(\mathbf{y}) = 0$, $t(\mathbf{x}) = \infty$.

Then $\mathcal{B}_2$ is isomorphic to the moduli space $\mathcal{B}_3$ of *parabolic vector bundles* on $\mathbf{X}$ with a trivialization in the formal neighbourhood of $\mathbf{x} \in \mathbf{X}$. More precisely, we consider the moduli space of the collections $(\mathfrak{F}_k, \tau)_{k \in \mathbb{Z}}$ where

(a) $\mathfrak{F}_k$ is a vector bundle on $\mathbf{X}$ of degree k and rank n;

(b) $\mathfrak{F}_k \subset \mathfrak{F}_{k+1}$, and $\mathfrak{F}_{k+1}/\mathfrak{F}_k$ is supported at $\mathbf{y} \in \mathbf{X}$ for any k;

(c) $\mathfrak{F}_{k+n} = \mathfrak{F}_k(\mathbf{y})$ for any k;

(d) τ is a trivialization of $\mathfrak{F}_0$ restricted to the formal neighbourhood $\mathbf{X}_{\widehat{\mathbf{x}}}$ of $\mathbf{x} \in \mathbf{X}$ (and hence τ is a trivialization of any $\mathfrak{F}_k$ in $\mathbf{X}_{\widehat{\mathbf{x}}}$).

Let us recall the isomorphism from $\mathcal{B}_3$ to $\mathcal{B}_2$. Given $(\mathfrak{F}_k, \tau)_{k \in \mathbb{Z}}$ we define the flag of discrete lattices $(F_k)_{k \in \mathbb{Z}}$ as follows. Our coordinate $t: \mathbf{X} \to \mathbb{P}^1$ identifies $\mathcal{O}_{\mathbf{X}_{\widehat{\mathbf{x}}}}$ with $\mathbb{C}[[t^{-1}]]$. Hence τ identifies $\mathfrak{F}_k|_{\mathbf{X}_{\widehat{\mathbf{x}}}}$ with $V \otimes \mathbb{C}[[t^{-1}]]$. Now the space of global sections $\Gamma(\mathbf{X} - \mathbf{x}, \mathfrak{F}_k)$ embeds as a discrete lattice F_k into $\Gamma(\mathbf{X}_{\widehat{\mathbf{x}}} - \mathbf{x}, \mathfrak{F}_k) = V \otimes \mathbb{C}((t^{-1}))$. One checks easily that the conditions (a–c) above imply the conditions 2.4 (a–c).

Under this isomorphism, the fiber of the line bundle $\mathcal{L}_i$ at a point $(F_k)_{k \in \mathbb{Z}}$ gets identified with the determinant of cohomology $\det R\Gamma(\mathbf{X}, \mathfrak{F}_i)$.

2.6. **Schubert divisors.** Recall that V is a vector space with a basis $v_1, \ldots, v_n$. We define a complete flag of vector subspaces $0 = V_0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V_n = V$ where $V_i = \langle v_1, \ldots, v_i \rangle$. We define a transversal flag $V = V^0 \supset V^{-1} \supset \ldots \supset V^{1-n} \supset V^{-n} = 0$ where $V^{-j} = \langle v_n, \ldots, v_{j+1} \rangle$. We denote by B the flag variety of $\mathfrak{sl}_n$. So we have two distinguished points $V^\bullet, V_\bullet \in \mathsf{B}$. Let $\mathfrak{b} \subset \mathfrak{sl}_n$ be a Borel subalgebra formed by all the operators preserving our flag $V_\bullet$. Let $\mathbf{B} \subset SL_n$ be the corresponding Borel subgroup. Let $\mathfrak{n} \subset \mathfrak{b}$ be the nilpotent radical.

We define a subalgebra $\hat{\mathfrak{b}} \subset \widehat{\mathfrak{sl}}_n$ as a full preimage in the central extension of a subalgebra $\mathfrak{b} \oplus \mathfrak{sl}_n \otimes t^{-1}\mathbb{C}[[t^{-1}]] \subset \mathfrak{sl}_n \otimes \mathbb{C}((t^{-1}))$. Let $\hat{\mathbf{B}}$ be the corresponding proalgebraic group. According to [13], $\hat{\mathbf{B}}$ acts on $\mathcal{B}$ with a unique open orbit $\mathcal{U} \subset \mathcal{B}$. The complement $\mathcal{B} - \mathcal{U}$ is a union of $n + 1$ irreducible Cartier divisors naturally numbered by $I: \mathcal{B} - \mathcal{U} = \bigcup_{i \in I} \Delta_i$. We have $\mathcal{L}_i = \mathcal{L}_{\omega_i} = \mathcal{O}(\Delta_i)$, see [14].

Finally, recall that Δ_0 is cut out by the condition that $\mathcal{F}_0$ is a nontrivial vector bundle on $\mathbf{X}$.

2.7. **Base point.** We choose a base point $B_0 \in \mathcal{U} \subset \mathcal{B}$ as follows. In the setup of 2.3 we set $B_0 = (\ell_\lambda^0)_{\lambda \in X^+}$ where ℓ_λ^0 is the unique line in V_λ^* killed by all f_i, $i \in I$. Equivalently, in the setup of 2.4 we have $B_0 = (F_k)_{k \in \mathbb{Z}}$ where for $-n \leq k \leq 0$ we have $F_k = V \otimes t\mathbb{C}[t] \oplus V^k$. Equivalently, in the setup of 2.5 we have $B_0 = (\mathfrak{F}_k, \tau)_{k \in \mathbb{Z}}$ where $\mathfrak{F}_0 = V \otimes \mathcal{O}_\mathbf{X}$, τ is the tautological trivialization, and for $-n \leq k \leq 0$ the local sections of $\mathfrak{F}_k$ are those sections of $\mathfrak{F}_0 = V \otimes \mathcal{O}_\mathbf{X}$ which take value in V^k at $\mathbf{y} \in \mathbf{X}$.

2.8. **Beilinson-Drinfeld-Kottwitz flags.** We recall the construction [11] of an ind-scheme of ind-finite type "approximating" the infinite type scheme $\mathcal{B}$. For a positive integer a let $\mathbf{X}^{(a)}$ denote the a-th symmetric power of $\mathbf{X}$. For a test scheme S, and an S-point y of $\mathbf{X}^{(a)}$, we may view the graph Γ_y of y as a subscheme of $S \times \mathbf{X}$ (finite over S).

Following [11], we define the ind-scheme $\mathfrak{B}^a$ representing the functor associating to a test scheme S the set of quadruples $(y, \mathcal{V}, \varsigma, \mathcal{V}_\mathbf{y}^\bullet)$ where

y is an S-point of $(\mathbf{X} - \mathbf{x})^{(a)}$;

$\mathcal{V}$ is an SL_n-bundle on $S \times \mathbf{X}$;

ς is a trivialization $\mathcal{V}|_{S \times \mathbf{X} - \Gamma_y} \to V \otimes \mathcal{O}_{S \times \mathbf{X} - \Gamma_y}$;

$\mathcal{V}_\mathbf{y}^\bullet$ is a reduction of $\mathcal{V}|_{S \times \mathbf{y}}$ to $\mathbf{B} \subset SL_n$.

$\mathfrak{B}^a$ is equipped with an evident projection $p_a : \mathfrak{B}^a \to (\mathbf{X} - \mathbf{x})^{(a)}$, and with a section $s_a : (\mathbf{X} - \mathbf{x})^{(a)} \to \mathfrak{B}^a$ defined as follows. For $s_a(y) \in \mathfrak{B}^a$ we have: $\mathcal{V} = V \otimes \mathcal{O}_{S \times \mathbf{X}}$ is a trivial SL_n-bundle; $\varsigma = \mathrm{Id}$ is the tautological trivialization; $\mathcal{V}_\mathbf{y}^\bullet$ is given by a constant flag $V^\bullet \otimes \mathcal{O}_{S \times \mathbf{y}}$ in $\mathcal{V}_{S \times \mathbf{y}} = V \otimes \mathcal{O}_{S \times \mathbf{y}}$.

We have an evident morphism $m_a : \mathfrak{B}^a \to \mathcal{B}$ restricting a rational trivialization ς to the formal neighbourhood $\mathbf{X}_{\widehat{\mathbf{x}}}$ of $\mathbf{x}$ in $\mathbf{X}$. Note that m_a contracts the section $s_a((\mathbf{X} - \mathbf{x})^{(a)})$ to the base point $B_0 \in \mathcal{B}$.

2.9. **Kashiwara Grassmannian.** Kashiwara scheme $\mathcal{B}$ has an important parabolic version $\mathcal{G}$ which we presently recall. In the setup of 2.3, for $i = 0$, the line bundle $\mathcal{L}_{\omega_0}$ on $\mathcal{B}$ defines a morphism from $\mathcal{B}$ to $\mathbb{P}(V_{\omega_0}^*)$, and $\mathcal{G}$ is the image of this morphism. We have a fiber bundle $\mathcal{B} \to \mathcal{G}$ with the typical fiber $\mathbf{B}$. Thus, the line bundle $\mathcal{L}_{\omega_0}$ on $\mathcal{B}$ descends to the ample *determinant line bundle* $\mathcal{L}_0$ on $\mathcal{G}$.

Equivalently, in the setup of 2.4, $\mathcal{G}$ is the moduli scheme of discrete lattices $F \subset \mathbf{V}$ satisfying the condition (a) of *loc. cit.*, such that $F \subset t^{-1}F$.

Equivalently, in the setup of 2.5, $\mathcal{G}$ is the moduli scheme of pairs $(\mathfrak{F}, \tau)$ where $\mathfrak{F}$ is an SL_n-bundle on $\mathbf{X}$, and τ is a trivialization of $\mathfrak{F}$ in the formal neighbourhood of $\mathbf{x} \in \mathbf{X}$.

We have a divisor $\Delta_0 \subset \mathcal{G}$ cut out by the condition that $\mathcal{F}_0$ is a nontrivial vector bundle on $\mathbf{X}$, and $\mathcal{L}_0 = \mathcal{O}(\Delta_0)$. Also, we have a base point $G_0 \in \mathcal{G}$ which is the image of $B_0 \in \mathcal{B}$. Finally, in the setup of 2.8, for $a \in \mathbb{N}$ we have

the ind-scheme $\mathfrak{G}^a$ (*Beilinson-Drinfeld Grassmannian*) representing the functor associating to a test scheme S the set of triples $(y, \mathcal{V}, \varsigma)$ as in *loc. cit.* We have an evident morphism $m_a : \mathfrak{G}^a \to \mathcal{G}$.

§3.   Parabolic Sheaves on $\mathbb{A}^2$

3.1.   Let $\mathbf{C}$ be a smooth projective curve of genus 0. We choose two distinct points $\mathbf{b}, \mathbf{c} \in \mathbf{C}$ and a global rational coordinate $z : \mathbf{C} \to \mathbb{P}^1$ such that $z(\mathbf{b}) = 0$, $z(\mathbf{c}) = \infty$.

We consider a smooth projective surface $\mathcal{S}' := \mathbf{C} \times \mathbf{X}$ with a normal crossing divisor $\mathbf{D}' := \mathbf{C} \times \mathbf{x} \bigcup \mathbf{c} \times \mathbf{X}$. Note that $\mathcal{S}' - \mathbf{D}'$ is the affine plane $\mathbb{A}^2$ with coordinates z, t.

Blowing up the point $\mathbf{c} \times \mathbf{x} \in \mathcal{S}'$ we obtain a surface $\mathcal{S}$ with a projection $p : \mathcal{S} \to \mathcal{S}'$. It is well known that one can blow down the proper transform of $\mathbf{D}'$ in $\mathcal{S}$ to obtain $q : \mathcal{S} \to \mathcal{S}''$. The surfaces $\mathcal{S}', \mathcal{S}''$ have a common open subscheme $\mathcal{S}' \supset \mathbb{A}^2 \subset \mathcal{S}''$, and the complement $\mathcal{S}'' - \mathbb{A}^2$ is a smooth divisor $\mathbf{D}'' \subset \mathcal{S}''$. In fact, $\mathcal{S}''$ is isomorphic to $\mathbb{P}^2$, and $\mathbf{D}''$ is a projective line.

Finally, we introduce a divisor $\mathbf{D}_0 := \mathbf{C} \times \mathbf{y} \subset \mathcal{S}'$. Note that $\mathbf{D}_0 \cap \mathbb{A}^2$ is cut out by the equation $t = 0$.

3.2. **Torsion free sheaves.** For a positive integer a let $\mathcal{A}^a \supset \overset{\circ}{\mathcal{A}}{}^a$ denote the fine moduli space of torsion free (resp. locally free) coherent sheaves $\mathcal{F}$ on $\mathcal{S}'$ of rank n, and second Chern class a, equipped with a trivialization at $\mathbf{D}' : \mathcal{F}|_{\mathbf{D}'} = V \otimes \mathcal{O}_{\mathbf{D}'}$. Its existence is proved in [12], and its smoothness is well known. For the reader's convenience let us recall the argument.

Lemma 3.1. $\mathcal{A}^a$ *is smooth.*

Proof. Let $\mathcal{F} \in \mathcal{A}^a$ be a torsion free sheaf. The obstruction to smoothness of $\mathcal{A}^a$ at $\mathcal{F}$ lies in $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F}(-\mathbf{D}'))$ which by Serre duality is a vector space dual to $\mathrm{Hom}(\mathcal{F}, \mathcal{F}(\mathbf{D}') \otimes \Omega^2_{\mathcal{S}'}) \simeq \mathrm{Hom}(\mathcal{F}, \mathcal{F}(-\mathbf{D}'))$. We claim that the latter vector space is zero, which at the same time proves that $\mathcal{F}$ has no infinitesimal automorphisms. In effect, since $\mathcal{F}|_{\mathbf{C} \times \mathbf{x}}$ is a trivial vector bundle on $\mathbf{C} = \mathbb{P}^1$, for a general $x \in \mathbf{X}$ the restriction $\mathcal{F}_x := \mathcal{F}|_{\mathbf{C} \times x}$ is also a trivial vector bundle on $\mathbf{C}$. But then $\mathrm{Hom}(\mathcal{F}_x, \mathcal{F}_x(-\mathbf{c})) = 0$ for a general $x \in \mathbf{X}$, and hence already $\mathrm{Hom}(\mathcal{F}, \mathcal{F}(-\mathbf{c} \times \mathbf{X})) = 0$. $\qquad\square$

We will use an equivalent definition of $\mathcal{A}^a$ going back to [1]. Namely, let $\mathcal{A}^a_1$ denote a fine moduli space of torsion free coherent sheaves $\mathcal{E}$ on $\mathcal{S}''$ of rank n, and second Chern class a, equipped with a trivialization at $\mathbf{D}'' : \mathcal{E}|_{\mathbf{D}''} = V \otimes \mathcal{O}_{\mathbf{D}''}$. Its existence is proved in [12].

Following [1], we construct an isomorphism ξ_a from $\mathcal{A}^a$ to $\mathcal{A}_1^a$ sending $\mathcal{F}$ to $\mathcal{E} := q_* p^* \mathcal{F}$ (notations of 3.1); the inverse isomorphism from $\mathcal{A}_1^a$ to $\mathcal{A}^a$ sends $\mathcal{E}$ to $\mathcal{F} := p_* q^* \mathcal{E}$.

Given a torsion free sheaf $\mathcal{F} \in \mathcal{A}^a$ and a point $s \in \mathbb{A}^2 \subset \mathcal{S}'$ we define a *saturation* at s: $\mathcal{N}_s(\mathcal{F}) := \mathbf{j}_{s*}\mathbf{j}_s^* \mathcal{F} \supset \mathcal{F}$ where $\mathbf{j}_s : \mathcal{S}' - s \hookrightarrow \mathcal{S}'$ is an open embedding. It is well known that $\mathcal{N}_s(\mathcal{F})$ is a torsion free sheaf locally free at s. We define a *defect* at s: $\mathrm{def}_s(\mathcal{F})$ as the length of the torsion sheaf $\mathcal{N}_s(\mathcal{F})/\mathcal{F}$. Finally, $\mathcal{N}(\mathcal{F})$ denotes the total *saturation* of $\mathcal{F}$, that is, $\mathcal{N}(\mathcal{F}) := \mathbf{j}_{S*}\mathbf{j}_S^* \mathcal{F}$ where $S \subset \mathbb{A}^2 \subset \mathcal{S}'$ is a finite subset such that $\mathcal{F}$ is locally free off S, and $\mathbf{j}_S : \mathcal{S}' - S \hookrightarrow \mathcal{S}'$ is an open embedding. The sum $\sum_{s \in S} \mathrm{def}_s(\mathcal{F}) \cdot s \in \mathrm{Sym}^d(\mathbb{A}^2)$ is the total *defect* $\mathrm{def}(\mathcal{F})$.

3.3. **Quiver description.** Nakajima ([20], Theorem 2.1) gives another equivalent definition of $\mathcal{A}(n,a) = \mathcal{A}_1^a$ as a certain quiver variety. Recall that $\mathcal{A}(n,a)$ is a moduli space of certain linear algebra data $(B_1, B_2, \imath, \jmath)$, see *loc. cit.* Here $B_1, B_2 \in \mathrm{End}(W)$ where $W = \mathbb{C}^a$, $\imath \in \mathrm{Hom}(V, W)$, $\jmath \in \mathrm{Hom}(W, V)$ satisfy a condition $[B_1, B_2] + \imath\jmath = 0$. Nakajima defines $\mathcal{F}$ as the middle cohomology of a certain monad on $\mathcal{S}''$ constructed from these linear algebra data.

Recall that z is our coordinate on $\mathbf{C}$ which identifies $\mathbf{C} - \mathbf{c}$ with $\mathbb{A}^1$. Also, t is our coordinate on $\mathbf{X}$ which identifies $\mathbf{X} - \mathbf{x}$ with $\mathbb{A}^1$. The restriction of this monad to $\mathbb{A}^2 \subset \mathcal{S}''$ looks as follows:

$$(3.2) \qquad 0 \to W \otimes \mathcal{O}_{\mathbb{A}^2} \xrightarrow{\mathfrak{a}} (W \oplus W \oplus V) \otimes \mathcal{O}_{\mathbb{A}^2} \xrightarrow{\mathfrak{b}} W \otimes \mathcal{O}_{\mathbb{A}^2} \to 0$$

where $\mathfrak{a}$ sends $w \in W \otimes \mathcal{O}_{\mathbb{A}^2}$ to $((B_1 - z)w, (B_2 - t)w, \jmath w)$, and $\mathfrak{b}$ sends a triple (w_1, w_2, v) to the sum $-(B_2 - t)w_1 + (B_1 - z)w_2 + \imath v$.

3.4. **Parabolic sheaves.** Let $\alpha = \sum_{i \in I} a_i i \in \mathbb{N}[I] \subset Y$ be a positive coroot combination. A *parabolic sheaf* $\mathcal{F}_\bullet$ of degree α on $\mathbb{A}^2$ is an infinite flag of torsion free coherent sheaves of rank n on $\mathcal{S}'$: $\ldots \subset \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots$ such that:

(a) $\mathcal{F}_{k+n} = \mathcal{F}_k(\mathbf{D}_0)$ for any k;

(b) $ch_1(\mathcal{F}_k) = k[\mathbf{D}_0]$ for any k: the first Chern classes are proportional to the fundamental class of $\mathbf{D}_0$;

(c) $ch_2(\mathcal{F}_k) = a_i$ for $i \equiv k \pmod{n}$;

(d) $\mathcal{F}_0$ is locally free at $\mathbf{D}'$ and trivialized at $\mathbf{D}'$: $\mathcal{F}_0|_{\mathbf{D}'} = V \otimes \mathcal{O}_{\mathbf{D}'}$;

(e) For $-n \leq k \leq 0$ the sheaf $\mathcal{F}_k$ is locally free at $\mathbf{D}'$, and the quotient sheaves $\mathcal{F}_k/\mathcal{F}_{-n}$, $\mathcal{F}_0/\mathcal{F}_k$ (both supported at $\mathbf{D}_0 = \mathbf{C} \times \mathbf{y} \subset \mathcal{S}'$) are both locally free at the point $\mathbf{c} \times \mathbf{y}$; moreover, the local sections of $\mathcal{F}_k|_{\mathbf{c} \times \mathbf{X}}$ are those sections of $\mathcal{F}_0|_{\mathbf{c} \times \mathbf{X}} = V \otimes \mathcal{O}_{\mathbf{X}}$ which take value in V^k at $\mathbf{y} \in \mathbf{X}$.

We say that a parabolic sheaf $\mathcal{F}_\bullet$ is *locally free* if $\mathcal{F}_k$ is locally free for any k. Note that this condition implies that for any $k \leq l \leq k + n$ the quotient sheaf $\mathcal{F}_l/\mathcal{F}_k$ is a locally free sheaf on $\mathbf{D}_0$, since $\mathbf{D}_0$ is smooth. Indeed, $\mathcal{F}_l/\mathcal{F}_k$ is a subsheaf in the sheaf $\mathcal{F}_{k+n}/\mathcal{F}_k = \mathcal{F}_k(\mathbf{D}_0)/\mathcal{F}_k = \mathcal{F}_k(\mathbf{D}_0)|_{\mathbf{D}_0}$ which is locally free.

3.5.   According to [12], [24], there exists a fine moduli scheme $\mathcal{M}^\alpha$ of parabolic sheaves of degree α on $\mathbb{A}^2$, and its open subscheme $\overset{\circ}{\mathcal{M}}{}^\alpha$ which is a fine moduli space of locally free parabolic sheaves. We have a natural forgetting morphism $\pi_\alpha : \mathcal{M}^\alpha \to \mathcal{A}^{a_0}$, $(\mathcal{F}_k)_{k \in \mathbb{Z}} \mapsto \mathcal{F}_0$.

Lemma 3.3. $\mathcal{M}^\alpha$ *is smooth.*

Proof. Let Coh denote the moduli stack of coherent sheaves on $\mathbf{C}$ of generic rank n, equipped with a trivialization at $\mathbf{c} \in \mathbf{C}$. Let FCoh denote the moduli stack of flags $0 = \mathfrak{F}_0 \subset \mathfrak{F}_1 \subset \ldots \subset \mathfrak{F}_n$ of coherent sheaves on $\mathbf{C}$ with the successive quotients being of generic rank 1, equipped with a trivialization at $\mathbf{c} \in \mathbf{C}$ compatible with the flag. Both Coh and FCoh are smooth, see [16].

We have morphisms $\mathsf{r} : \mathcal{A}^a \to \mathsf{Coh}$, $\mathcal{F} \mapsto \mathcal{F}|_{\mathbf{D}_0} = \mathcal{F}/\mathcal{F}(-\mathbf{D}_0)$, and $\mathfrak{r} : \mathcal{M}^\alpha \to \mathsf{FCoh}$, $\mathcal{F}_\bullet \mapsto (\mathfrak{F}_k := \mathcal{F}_{k-n}/\mathcal{F}_0(-\mathbf{D}_0))_{0 \leq k \leq n}$. Evidently, $\mathcal{M}^\alpha$ is the cartesian product of $\mathcal{A}^{a_0}$ and FCoh over Coh.

So it remains only to check that $\mathsf{r} : \mathcal{A}^a \to \mathsf{Coh}$ is smooth. Both stacks in question being smooth it suffices to show the surjectivity of the corresponding tangent map. The obstruction to r being a submersion at a point $\mathcal{F} \in \mathcal{A}^a$ lies in $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F}(-\mathbf{D}' - \mathbf{D}_0))$. As in the proof of Lemma 3.1, it is enough to check $\mathrm{Hom}(\mathcal{F}, \mathcal{F}(-\mathbf{c} \times \mathbf{X})) = 0$. But exactly this was done in the above cited proof. $\square$

Remark 3.4.   We will see in Proposition 7.1, Corollary 7.3 that $\mathcal{M}^\alpha$ is connected of dimension $\dim \mathcal{M}^\alpha = 2|\alpha| := 2 \sum_I a_i$.

3.6.   For a future use we construct a family of regular functions on $\mathcal{M}^\alpha$ factoring through the projection $\pi_\alpha : \mathcal{M}^\alpha \to \mathcal{A}^{a_0}$.

Let O denote the algebraic variety formed by pairs of lines (P_1, P_2) in the projective plane $\mathcal{S}''$ such that all the three lines $P_1, P_2, \mathbf{D}''$ are distinct. Note that O is an affine algebraic variety. We have a fiber bundle P_1 (resp. P_2) over O whose fiber over (P_1, P_2) is P_1 (resp. P_2). We have a fiber bundle $\mathsf{p} : \widetilde{\mathsf{O}} \to \mathsf{O}$ whose fiber over (P_1, P_2) is $P_2^\circ := P_2 - P_2 \cap \mathbf{D}''$ (isomorphic to $\mathbb{A}^1$). We denote the a-th symmetric power of $\widetilde{\mathsf{O}}$ relative over O by $\widetilde{\mathsf{O}}^{(a)}$.

The relative surface $\mathcal{S}''_\mathsf{O} := \mathcal{S}'' \times \mathsf{O}$ over O has two sections $p_1 := P_1 \cap \mathbf{D}''$, $p_2 := P_2 \cap \mathbf{D}''$, and a relative line $\mathbf{D}''_\mathsf{O}$. Blowing up p_1, p_2 and blowing down

the proper transform of $\mathbf{D}_{\mathsf{O}}''$ we obtain the relative surface $\mathcal{S}_{\mathsf{O}}'$. Its exceptional divisor $\mathbf{D}_{\mathsf{O}}'$ is a union of two $\mathbb{P}^1$-bundles over O; in fact, $\mathcal{S}_{\mathsf{O}}' \simeq \mathsf{P}_1 \times_{\mathsf{O}} \mathsf{P}_2$, and $\mathbf{D}_{\mathsf{O}}' = \mathsf{P}_1 \times_{\mathsf{O}} p_2 \bigcup p_1 \times_{\mathsf{O}} \mathsf{P}_2$.

Given a torsion free sheaf $\mathcal{F} \in \mathcal{A}^a$ on $\mathcal{S}''$ trivialized at $\mathbf{D}''$ we lift it to $\mathcal{S}_{\mathsf{O}}''$, and then apply the relative version of Atiyah's trick 3.2 to get a torsion free sheaf $\mathcal{F}'$ on $\mathcal{S}_{\mathsf{O}}'$ trivialized at $\mathbf{D}_{\mathsf{O}}'$. The sheaf $\mathcal{F}'$ is flat over P_1, and for a point $f = (P_1, P_2, c \in P_2) \in \mathsf{P}_2$ its restriction to the fiber $\mathbb{P}^1_f \simeq P_1$ of $\mathcal{S}'$ over f is a coherent sheaf on a projective line. If $c = P_2 \cap \mathbf{D}''$ then the restriction of $\mathcal{F}'$ to $\mathbb{P}^1_f$ is trivialized by construction: $\mathcal{F}'|_{\mathbb{P}^1_f} = V \otimes \mathcal{O}_{\mathbb{P}^1_f}$. Since the condition of triviality is an open condition in the moduli stack of coherent sheaves on P_1, we get a finite subset $D \subset P_2^\circ$ such that for $c \notin D$ the restriction $\mathcal{F}'|_{\mathbb{P}^1_f}$ is trivial.

In fact, D is not just a finite subset of P_2° but it carries a structure of an effective Cartier divisor in P_2°. Indeed, the restriction of $\mathcal{F}'$ to the fiber $P_1 \times P_2$ of $\mathcal{S}_{\mathsf{O}}'$ over $(P_1, P_2) \in \mathsf{O}$ defines a morphism from P_2 to the moduli stack of coherent sheaves on P_1. This stack has a canonical Cartier divisor Δ_0 of nontrivial coherent sheaves. We define D as the inverse image of Δ_0. It is easy to see that $\deg D = a$, and as (P_1, P_2) vary in O, these effective divisors form a section $\mathsf{D}(\mathcal{F})$ of $\widetilde{\mathsf{O}}^{(a)}$.

Let $'\mathcal{P}^a$ denote the ind-scheme of sections of $\widetilde{\mathsf{O}}^{(a)}$ over O. In fact, it is just an infinite-dimensional vector space. The above construction defines a morphism $\theta_a : \mathcal{A}^a \to {}'\mathcal{P}^a$.

3.7.   Using Nakajima's construction of $\mathcal{A}^a$ as a quiver variety, it is possible to write down an explicit formula for the map θ_a above. Recall that z is our coordinate on $\mathbf{C}$ which identifies $\mathbf{C} - \mathbf{c}$ with $\mathbb{A}^1$. Also, t is our coordinate on $\mathbf{X}$ which identifies $\mathbf{X} - \mathbf{x}$ with $\mathbb{A}^1$. Thus, $\mathbb{A}^2 = \mathcal{S}'' - \mathbf{D}''$ is equipped with coordinates (t, z). The variety O is the variety of pairs of nonparallel lines in $\mathbb{A}^2$. If $(P_1^\circ, P_2^\circ) = (\{t = 0\}, \{z = 0\})$, and $\mathcal{F} \in \mathcal{A}^\alpha$ is represented by a quadruple (B_1, B_2, i, j), then the value of the section $\theta_a(\mathcal{F})$ at the point (P_1, P_2) lies in $(\mathbf{X} - \mathbf{x})^{(a)} = \mathbb{A}^{(a)}$. We will prove that $\mathsf{ev}_{(P_1, P_2)}(\theta_a(\mathcal{F})) = \mathrm{Spec}(B_2)$ (that is the effective divisor in $\mathbb{A}^1$ cut out by the equation $\det(B_2 - t\mathrm{Id})$), and the morphism θ_a is equivariant under the natural action of the group of affine linear transformations of $\mathbb{A}^2$.

More precisely, given $(P_1, P_2) \in \mathsf{O}$ we choose an affine linear transformation g of $\mathbb{A}^2$ such that $g(\{t = 0\}) = P_1^\circ$, $g(\{z = 0\}) = P_2^\circ$. It also identifies P_2° with $\mathbf{X} - \mathbf{x}$ with coordinate t. Let us write g as a linear transformation (with matrix entries $g_{11}, g_{12}, g_{21}, g_{22}$) followed by a translation by $(g_1, g_2) \in \mathbb{A}^2$. The natural action of affine linear transformations of $\mathbb{A}^2$ on $\mathcal{A}^a$ in terms of Nakajima's

quadruples looks like

$$g(B_1, B_2, \imath, \jmath) = (g_{11}B_1 + g_{12}B_2 + g_1\mathrm{Id}, g_{21}B_1$$
$$+ g_{22}B_2 + g_2\mathrm{Id}, (g_{11}g_{22} - g_{12}g_{21})\imath, \jmath)$$

For an operator $B \in \mathrm{End}(W)$ we define $\mathrm{Spec}_g(B) \in (P_2^\circ)^{(a)}$ as follows. First we consider an effective divisor on $\mathbb{A}^2$ cut out by an equation $\det(B - (g_{21}z + g_{22}t - g_2)\mathrm{Id})$. Then we intersect it with P_2°.

Now given $\mathcal{F} \in \mathcal{A}^a$ we compute $\theta_a(\mathcal{F}, P_1, P_2) \in (P_2^\circ)^{(a)}$ in terms of Nakajima's quiver data $(B_1, B_2, \imath, \jmath)$ for $\mathcal{F}$.

The following lemma is borrowed from [3].

Lemma 3.5. $\theta_a(B_1, B_2, \imath, \jmath, P_1, P_2) = \mathrm{Spec}_g(g_{21}B_1 + g_{22}B_2)$.

Proof. Recall that $\mathcal{F}|_{\mathbb{A}^2}$ is the middle cohomology of the monad (3.2). We define the map

$$K_g : \ \mathrm{Ker}(\mathfrak{b}) \to V \otimes \det{}^{-1}(g_{21}B_1 + g_{22}B_2 - (g_{21}z + g_{22}t - g_2)\mathrm{Id})\,\mathcal{O}_{\mathbb{A}^2}$$

as follows. It sends $(w_1, w_2, v) \in \mathrm{Ker}(\mathfrak{b})$ to

$$v - \jmath\,(g_{21}(B_1 - z) + g_{22}(B_2 - t) + g_2)^{-1}(g_{21}w_1 + g_{22}w_2)$$

Since $K_g(\mathrm{Im}(\mathfrak{a})) = 0$ we get a well defined map

$$L_g : \ \mathcal{F}|_{\mathbb{A}^2} \to V \otimes \det{}^{-1}(g_{21}B_1 + g_{22}B_2 - (g_{21}z + g_{22}t - g_2)\mathrm{Id})\,\mathcal{O}_{\mathbb{A}^2}$$

It is easy to see that L_g is injective, and its image contains

$$V \otimes \det(g_{21}B_1 + g_{22}B_2 - (g_{21}z + g_{22}t - g_2)\mathrm{Id})\,\mathcal{O}_{\mathbb{A}^2} \qquad \square$$

In particular, in the coordinates g, the degree of section $\theta_a(\mathcal{F})$ as a function of g is less than or equal to a. Let $\mathcal{P}^a \subset {}'\mathcal{P}^a$ denote the space of sections of $\widetilde{\mathsf{O}}^{(a)}$ of degree less than or equal to a. Then our map $\theta_a : \ \mathcal{A}^a \to {}'\mathcal{P}^a$ actually lands into the finite dimensional subspace $\mathcal{P}^a$. Note that the morphism θ_a from $\mathcal{A}^a$ to the ind-scheme ${}'\mathcal{P}^a$ *a priori* lands into a (finite type) subscheme of ${}'\mathcal{P}^a$, and $\mathcal{P}^a$ is just an explicit estimate of such a subscheme.

Composing θ_{a_0} with the projection $\pi_\alpha : \ \mathcal{M}^\alpha \to \mathcal{A}^{a_0}$ we get the desired map $\vartheta_\alpha : \ \mathcal{M}^\alpha \to \mathcal{P}^{a_0}$.

When $P_1^\circ = \{t = 0\}$, and $P_2^\circ = \{z = 0\}$, the fiber of $\widetilde{\mathsf{O}}^{(a)}$ over (P_1, P_2) canonically identifies with $(\mathbf{X} - \mathbf{x})^{(a)}$. Composing θ_a with evaluation at (P_1, P_2)

we get the map $\eta_a : \mathcal{A}^a \to (\mathbf{X} - \mathbf{x})^{(a)}$. Composing η_{a_0} with the projection $\pi_\alpha : \mathcal{M}^\alpha \to \mathcal{A}^{a_0}$ we get the map $\eta_\alpha : \mathcal{M}^\alpha \to (\mathbf{X} - \mathbf{x})^{(a_0)}$.

3.8.   Let $\mathcal{F} \in \mathcal{A}^a$ be a torsion free sheaf, let $\mathrm{def}(\mathcal{F}) = \underline{s} \in \mathrm{Sym}^d(\mathbb{A}^2)$ (notations of 3.2). Then $\mathcal{N}(\mathcal{F}) \in \mathcal{A}^{a-d}$, and we will compute $\theta_a(\mathcal{F})$ in terms of $\theta_{a-d}(\mathcal{N}(\mathcal{F})), \underline{s}$.

To this end we define a morphism $\mho^d$ from the symmetric power $\mathrm{Sym}^d(\mathbb{A}^2)$ to $\mathcal{P}^d$. Namely, for $\underline{s} \in \mathrm{Sym}^d(\mathbb{A}^2)$, and $(P_1, P_2) \in \mathsf{O}$ we can project $\mathbb{A}^2$ onto P_2° along P_1°, and the projection of $\underline{s}$ will be an effective degree d divisor in P_2°. As (P_1, P_2) vary in O we get the desired map $\mho^d : \mathrm{Sym}^d(\mathbb{A}^2) \to \mathcal{P}^d$.

Also note that we have a natural addition $\widetilde{\mathsf{O}}^{(k)} \times_\mathsf{O} \widetilde{\mathsf{O}}^{(l)} \to \widetilde{\mathsf{O}}^{(k+l)}$, $(D_1, D_2) \mapsto D_1 + D_2$, which gives rise to the addition map $\mathcal{P}^k \times \mathcal{P}^l \to \mathcal{P}^{k+l}$. Now we can formulate the following corollary of Lemma 3.5 due to Nakajima. A proof of a similar statement for more general quiver varieties can be found in [19], 3.27; see also [18], 2.30.

Corollary 3.6. $\theta_a(\mathcal{F}) = \mho^d(\underline{s}) + \theta_{a-d}(\mathcal{N}(\mathcal{F}))$.

$\square$

§4.   Quasimaps into the Kashiwara Flag Scheme and Uhlenbeck Spaces

4.1.   **Based maps and quasimaps.** We return to the setup of 2.3. According to [14], $H^2(\mathcal{B}, \mathbb{Z})$ is canonically isomorphic to the weight lattice $X : \lambda \mapsto c_1(\mathcal{L}_\lambda)$. The dual lattice $H_2(\mathcal{B}, \mathbb{Z})$ is canonically isomorphic to the coroot lattice Y. We say that a regular map $\phi : \mathbf{C} \to \mathcal{B}$ has degree $\alpha \in Y$ if the fundamental class of $\mathbf{C}$ in the second homology of $\mathcal{B}$ equals $\alpha : \phi_*[\mathbf{C}] = \alpha$. Equivalently, $\deg(\phi) = \alpha$ iff for any $\lambda \in X$ we have $\deg(\phi^*\mathcal{L}_\lambda) = \langle \lambda, \alpha \rangle$. Then necessarily $\alpha \in Y^+ = \mathbb{N}[I]$.

We say that ϕ is *based* if $\phi(\mathbf{c}) = B_0$ (notations of 2.7).

According to [6], for any $\alpha \in Y^+$ there exists a fine moduli space $\overset{\circ}{\mathbf{M}}{}^\alpha$ of based maps of degree α from $(\mathbf{C}, \mathbf{c})$ to $(\mathcal{B}, B_0)$. Moreover, it is a smooth connected quasiaffine scheme of dimension $2|\alpha|$. Let us recall its quasiaffine embedding. Recall that we have a canonical surjection of vector bundles on $\mathcal{B} : V_\lambda \otimes \mathcal{O}_\mathcal{B} \twoheadrightarrow \mathcal{L}_\lambda$ for any $\lambda \in X^+$. Dually, we have an embedding of vector bundles: $\mathcal{L}_{-\lambda} \hookrightarrow V_\lambda^* \otimes \mathcal{O}_\mathcal{B}$. Thus, $\phi \in \overset{\circ}{\mathbf{M}}{}^\alpha$ gives rise to a collection of line subbundles $(\phi^*\mathcal{L}_{-\lambda} \hookrightarrow V_\lambda^* \otimes \mathcal{O}_\mathbf{C})_{\lambda \in X^+}$ such that

(a) the fiber of $\phi^*\mathcal{L}_{-\lambda}$ at $\mathbf{c} \in \mathbf{C}$ equals $\ell_\lambda^0 \subset V_\lambda^*$ (notations of 2.7);

(b) This collection of line subbundles satisfies fiberwise Plücker equations.

Note that $\phi^*\mathcal{L}_{-\lambda} \simeq \mathcal{O}_{\mathbf{C}}(\langle-\lambda,\alpha\rangle)$. Hence the datum of $\phi^*\mathcal{L}_{-\lambda} \hookrightarrow V_\lambda^* \otimes \mathcal{O}_{\mathbf{C}} \Leftrightarrow \phi^*\mathcal{L}_{-\lambda}(\langle\lambda,\alpha\rangle) \hookrightarrow V_\lambda^*\otimes\mathcal{O}_{\mathbf{C}}(\langle\lambda,\alpha\rangle)$ is equivalent to the datum of *nowhere vanishing* section $s_\lambda \in \Gamma(\mathbf{C}, V_\lambda^* \otimes \mathcal{O}_{\mathbf{C}}(\langle\lambda,\alpha\rangle))$ up to scalar multiplication.

Recall that we have chosen a coordinate z on $\mathbf{C} - \mathbf{c}$. Thus s_λ is just a polynomial in z of degree $\langle\lambda,\alpha\rangle$ with values in V_λ^*. The condition (a) above means that the scalar product of s_λ with a highest vector of V_λ is a (scalar) polynomial of degree exactly $\langle\lambda,\alpha\rangle$, and the scalar product of s_λ with any other weight vector of V_λ is a (scalar) polynomial of degree strictly less than $\langle\lambda,\alpha\rangle$. Now we may scale a constant multiple indeterminacy in the choice of s_λ by requirement that the scalar product of s_λ with *the* highest vector $v_\lambda \in V_\lambda$ is a *monic* polynomial of degree $\langle\lambda,\alpha\rangle$.

All in all, $\overset{\circ}{\mathbf{M}}{}^\alpha$ is formed by collections of such V_λ^*-valued *nowhere vanishing* polynomials s_λ satisfying Plücker equations. If we drop the nowhere vanishing condition, we obtain an affine closure $\mathbf{M}^\alpha \supset \overset{\circ}{\mathbf{M}}{}^\alpha$.

Equivalently, $\mathbf{M}^\alpha$ is formed by collections of *invertible subsheaves* $(\mathfrak{L}_{-\lambda} \subset V_\lambda^* \otimes \mathcal{O}_{\mathbf{C}})_{\lambda\in X^+}$ such that

(a) $\mathfrak{L}_{-\lambda}$ is a line subbundle at $\mathbf{c} \in \mathbf{C}$, and its fiber equals $\ell_\lambda^0 \subset V_\lambda^*$ (notations of 2.7);

(b) This collection of invertible subsheaves satisfies fiberwise Plücker equations;

(c) $\deg(\mathfrak{L}_{-\lambda}) = -\langle\lambda,\alpha\rangle$.

The points of $\mathbf{M}^\alpha$ will be called *based quasimaps of degree α*.

4.2. **Relative based quasimaps.** We will need a slight generalization of the above construction. Let Q be a scheme, let K be a set of indices, and for $k \in K$ let W^k be a (pro)finite dimensional vector bundle over Q. Let $\mathbb{P}(W^k)$ be the corresponding projective scheme over Q. Let $R \subset \prod_{k\in K} \mathbb{P}(W^k)$ (cartesian product over Q) be a closed subscheme. Let $s: Q \to R$, $q \mapsto (w^k(q))$ be a section.

A *relative based quasimap* ϕ from $(\mathbf{C}, \mathbf{c})$ to (R, s) is the following collection of data:

(a) a point $q \in Q$;

(b) an invertible subsheaf $\mathfrak{L}_k \subset W_q^k \otimes \mathcal{O}_{\mathbf{C}}$ for any $k \in K$ satisfying the following conditions:

(i) for an open subset $U \subset \mathbf{C}$ the invertible subsheaves $\mathfrak{L}_k \subset W_q^k \otimes \mathcal{O}_U$ are *line subbundles*, so they give rise to a map $\phi: U \to \prod_{k\in K} \mathbb{P}(W_q^k)$, and its image is required to lie in R;

(ii) we have $\mathbf{c} \in U$, and $\phi(\mathbf{c}) = s(q)$.

The arguments of the previous subsection show the existence of the classifying scheme $M(\mathbf{C}, \mathbf{c}; R, s)$ for the relative based quasimaps. Simultaneously we obtain the open subscheme $\overset{\circ}{M}(\mathbf{C}, \mathbf{c}; R, s) \subset M(\mathbf{C}, \mathbf{c}; R, s)$ classifying the relative based maps (i.e. when U above equals $\mathbf{C}$).

Note that if $R' \hookrightarrow R$ is a closed subscheme, and the section s factors through $s : Q \to R' \hookrightarrow R$, then $M(\mathbf{C}, \mathbf{c}; R', s)$ is a closed subscheme of $M(\mathbf{C}, \mathbf{c}; R, s)$.

4.3. Quasimaps into Grassmannian.
Recall the setup of 2.9. We have a closed subscheme $\mathcal{G} \subset \mathbb{P}(V_{\omega_0}^*)$ with the base point $G_0 \subset \mathcal{G}$. It is well known that $H^2(\mathcal{G}, \mathbb{Z}) = \mathbb{Z}$ is generated by $ch_1(\mathcal{L}_0)$. Thus for a positive integer a we have the classifying scheme $\mathbf{A}^a \supset \overset{\circ}{\mathbf{A}}{}^a$ of based quasimaps (resp. maps) of degree a from $(\mathbf{C}, \mathbf{c})$ to $(\mathcal{G}, G_0)$.

4.4.
We will construct an identification $\overset{\circ}{\mathcal{M}}{}^\alpha \equiv \overset{\circ}{\mathbf{M}}{}^\alpha$. Let $\mathcal{F}_\bullet \in \mathcal{M}^\alpha$ be a parabolic sheaf of degree α on $\mathbb{A}^2$ (notations of 3.4). Then $\mathcal{F}_0$ is trivialized at $\mathbf{D}'$, and in particular, at $\mathbf{C} \times \mathbf{x} \subset \mathbf{D}' : \mathcal{F}_0|_{\mathbf{C} \times \mathbf{x}} \overset{\sigma}{\longrightarrow} V \otimes \mathcal{O}_{\mathbf{C}}$.

Lemma 4.1. *The trivialization σ extends canonically to a trivialization in the formal neighbourhood of $\mathbf{C} \times \mathbf{x}$ in $\mathcal{S}' : \mathcal{F}_0|_{\mathcal{S}'_{\widehat{\mathbf{C} \times \mathbf{x}}}} \overset{\varsigma}{\longrightarrow} V \otimes \mathcal{O}_{\mathcal{S}'_{\widehat{\mathbf{C} \times \mathbf{x}}}}$.*

Proof. We essentially repeat the arguments of 3.6. Let us denote by $\varrho_{\mathbf{X}} : \mathcal{S}' = \mathbf{C} \times \mathbf{X} \to \mathbf{X}$ the canonical projection from $\mathcal{S}'$ to $\mathbf{X}$. For any point $y \in \mathbf{X}$ the fiber $\varrho_y := \varrho_{\mathbf{X}}^{-1}(y)$ is identified with $\mathbf{C}$. The sheaf $\mathcal{F}_0$ is flat over $\mathbf{X}$, and the restrictions $\mathcal{F}_0|_{\varrho_y}$ are coherent sheaves on $\mathbf{C}$. Thus we obtain a morphism from $\mathbf{X}$ to the moduli stack of coherent sheaves on $\mathbf{C}$. By our assumption this morphism sends $\mathbf{x} \in \mathbf{X}$ to the class of trivial vector bundle on $\mathbf{C}$. Let $D \subset \mathbf{X} - \mathbf{x}$ be the inverse image of the Cartier divisor of nontrivial coherent sheaves in this moduli stack. Then our trivialization σ extends to a trivialization on $\mathbf{C} \times (\mathbf{X} - D) : \mathcal{F}_0|_{\mathbf{C} \times (\mathbf{X} - D)} \simeq V \otimes \mathcal{O}_{\mathbf{C} \times (\mathbf{X} - D)}$. There is a unique choice of such an extension ς such that $\varsigma|_{\mathbf{c} \times (\mathbf{X} - D)}$ coincides with the given trivialization of $\mathcal{F}_0|_{\mathbf{c} \times (\mathbf{X} - D)}$ (as $\mathbf{c} \times (\mathbf{X} - D) \subset \mathbf{D}'$). Finally, we just restrict our canonical rational trivialization ς to the formal neighbourhood of $\mathbf{C} \times \mathbf{x}$ in $\mathcal{S}'$. $\qquad\square$

Given a locally free parabolic sheaf $\mathcal{F}_\bullet \in \overset{\circ}{\mathcal{M}}{}^\alpha$, we equip it with the canonical trivialization ς in the formal neighbourhood of $\mathbf{C} \times \mathbf{x}$ in $\mathcal{S}'$. Consider the canonical projection $\varrho_{\mathbf{C}}$ from $\mathcal{S}' = \mathbf{C} \times \mathbf{X}$ to $\mathbf{C}$. For a point $c \in \mathbf{C}$, restricting $(\mathcal{F}_\bullet, \varsigma)$ to the fiber $\varrho_c := \varrho_{\mathbf{C}}^{-1}(c) = \mathbf{X}$ we obtain a parabolic vector bundle

$(\mathfrak{F}_\bullet(c), \tau(c))$ on $\mathbf{X}$ (notations of 2.5). For $q = \mathbf{c}$ the corresponding parabolic vector bundle $(\mathfrak{F}_\bullet(\mathbf{c}), \tau(\mathbf{c}))$ gives the point $B_0 \in \mathcal{B}$. Thus, starting from $\mathcal{F}_\bullet \in \overset{\circ}{\mathcal{M}}{}^\alpha$, we have constructed a based map ϕ from $(\mathbf{C}, \mathbf{c})$ to $(\mathcal{B}, B_0)$. It is easy to see that $\deg(\phi) = \alpha$, so we have constructed a morphism $\zeta : \overset{\circ}{\mathcal{M}}{}^\alpha \to \overset{\circ}{\mathbf{M}}{}^\alpha$.

Conversely, the data of a based map $\phi \in \overset{\circ}{\mathbf{M}}{}^\alpha$, by the very definition, consists of a family of parabolic vector bundles over $\mathbf{C}$, that is, a locally free parabolic sheaf $\mathcal{F}_\bullet$ on $\mathbb{A}^2$ along with a trivialization of $\mathcal{F}_0$ in the formal neighbourhood $\mathcal{S}'_{\widehat{\mathbf{C} \times \mathbf{x}}}$, compatible with a given trivialization of $\mathcal{F}_0|_{\mathbf{D}'}$. To define $\zeta^{-1}(\phi)$ we just forget the formal trivialization. Thus we have constructed the inverse isomorphism $\zeta^{-1} : \overset{\circ}{\mathbf{M}}{}^\alpha \to \overset{\circ}{\mathcal{M}}{}^\alpha$.

The same argument establishes an identification $\overset{\circ}{\mathcal{A}}{}^a \equiv \overset{\circ}{\mathbf{A}}{}^a$.

4.5. **Uhlenbeck space.** Recall that Nakajima (following Donaldson) defines $\mathcal{A}^a$ as the moduli space of stable quadruples $(B_1, B_2, \imath, \jmath)$, see 3.3. He also defines the *Uhlenbeck space* $\mathfrak{N}^a$ as the GIT quotient of the space of quadruples with respect to the natural $\mathrm{GL}(W)$-action. There is a natural proper morphism $\Upsilon_a : \mathcal{A}^a \to \mathfrak{N}^a$ which is a semismall resolution of singularities, see [19]. In fact, $\mathfrak{N}^a$ is the affinization of $\mathcal{A}^a$, that is the spectrum of the algebra of regular functions on $\mathcal{A}^a$.

We propose two more definitions of the Uhlenbeck space; we conjecture that they are both equivalent to Donaldson's definition, see 5.4. From now on we will identify $\overset{\circ}{\mathbf{A}}{}^a$ with $\overset{\circ}{\mathcal{A}}{}^a$. In particular, we have an open embedding $\mathbf{j} : \overset{\circ}{\mathcal{A}}{}^a \hookrightarrow \mathbf{A}^a$. Recall also the morphism $\theta_a : \overset{\circ}{\mathcal{A}}{}^\alpha \to \mathcal{P}^a$ defined in 3.7.

We define[1] the Uhlenbeck space $\mathfrak{A}^a$ as the closure of $\overset{\circ}{\mathcal{A}}{}^a$ in $\mathbf{A}^a \times \mathcal{P}^a$ (with respect to the locally closed embedding $(\mathbf{j}, \theta_a)$).

4.6. The virtue of our second construction of the Uhlenbeck space is that it carries a natural action of the group of affine linear transformations of $\mathbb{A}^2$. Recall the setup of 3.6. For a projective line P_2 with a point $p_2 = P_2 \cap \mathbf{D}''$ we consider the moduli space $\mathcal{G}_{P_2, p_2}$ of SL_n-bundles on P_2 trivialized in the formal neighbourhood of $p_2 \in P_2$. It carries an ample line bundle $\mathcal{L}_{P_2, p_2}$ whose fiber at $(\mathfrak{F}, \tau)$ is $\det R\Gamma(P_2, \mathfrak{F})$ (where we view $\mathfrak{F}$ as a vector bundle of rank n on P_2). Also, we have a base point $G_{P_2, p_2} \in \mathcal{G}_{P_2, p_2}$, namely, $G_{P_2, p_2} = (\mathfrak{F}, \tau)$ where $\mathfrak{F} = V \otimes \mathcal{O}_{P_2}$, and τ is the tautological trivialization. Thus we have a relative scheme $\mathcal{G}_{\mathsf{O}}$ over O together with a section G_{O}.

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[1] It was V.Drinfeld who noticed that the closure $'\mathfrak{A}^a$ of $\overset{\circ}{\mathcal{A}}{}^a$ in $\mathbf{A}^a \times (\mathbf{X} - \mathbf{x})^{(a)}$ is a wrong candidate for the Uhlenbeck space; in particular, $'\mathfrak{A}^a$ is not normal in general.

Following 4.2 we define a *relative based quasimap of degree a* from (P_1, p_1) to $(\mathcal{G}_O, G_O)$ as the following collection of data:

(a) a point $(P_1, P_2) \in O$;

(b) an invertible subsheaf $\mathfrak{L} \subset \Gamma^*(\mathcal{G}_{P_2, p_2}, \mathcal{L}_{P_2, p_2}) \otimes \mathcal{O}_{P_1}$ of degree $-a$, satisfying the conditions (i),(ii) of *loc. cit.*

The arguments of *loc. cit.* establish the existence of the classifying scheme $M^a(P_1, p_1; \mathcal{G}_O, G_O)$ for the relative based quasimaps. We have an evident projection $M^a(P_1, p_1; \mathcal{G}_O, G_O) \to O$ with a typical fiber isomorphic to $\mathbf{A}^a$. All the schemes in question are affine. Let M^a be the ind-scheme of sections of $M^a(P_1, p_1; \mathcal{G}_O, G_O)$ over O.

Given a locally free sheaf $\mathcal{F} \in \overset{\circ}{\mathcal{A}}{}^a$ we get a locally free sheaf $\mathcal{F}'$ on $\mathcal{S}'_O$ trivialized at $\mathbf{D}'_O$ as in 3.6. Applying the relative version of the arguments in 4.4 to $\mathcal{F}'$ we produce from it a section of $M^a(P_1, p_1; \mathcal{G}_O, G_O)$ over O. Thus we construct a morphism $\mathsf{j} : \overset{\circ}{\mathcal{A}}{}^a \to \mathsf{M}^a$.

Finally, we define our second version of the Uhlenbeck space A^a as the closure of $\overset{\circ}{\mathcal{A}}{}^a$ in $\mathsf{M}^a \times \mathcal{P}^a$ with respect to the locally closed embedding (j, θ_a).

Evaluating the sections in M^a at the point $(P_1, P_2) = (\{t = 0\}, \{z = 0\})$ we obtain the morphisms $\mathsf{M}^a \to \mathbf{A}^a$ and $\Xi_a : \mathsf{A}^a \to \mathfrak{A}^a$.

4.7. Uhlenbeck flag space. From now on we will identify $\overset{\circ}{\mathbf{M}}{}^\alpha$ with $\overset{\circ}{\mathcal{M}}{}^\alpha$. In particular, we have an open embedding $\mathbf{j} : \overset{\circ}{\mathcal{M}}{}^\alpha \hookrightarrow \mathbf{M}^\alpha$. Recall also the morphism $\vartheta_\alpha : \overset{\circ}{\mathcal{M}}{}^\alpha \to \mathcal{P}^{a_0}$ defined in 3.7. We are finally able to introduce our main character.

We define the *Uhlenbeck flag space* $\mathfrak{M}^\alpha$ as the closure of $\overset{\circ}{\mathcal{M}}{}^\alpha$ in $\mathbf{M}^\alpha \times \mathcal{P}^{a_0}$ (with respect to the locally closed embedding $(\mathbf{j}, \vartheta_\alpha)$).

Proposition 4.2. $\mathfrak{M}^\alpha$ *is an irreducible affine scheme of finite type, of dimension* $\dim(\mathfrak{M}^\alpha) = 2|\alpha|$.

Proof. According to [6] (alternatively, see Proposition 7.1, Corollary 7.3), $\overset{\circ}{\mathcal{M}}{}^\alpha$ is smooth, connected of dimension $2|\alpha|$. Hence it only remains to prove that $\mathfrak{M}^\alpha$ is of finite type, the problem being that $\mathbf{M}^\alpha$ is not of finite type, see *loc. cit.* Recall the morphism $\eta_\alpha : \overset{\circ}{\mathcal{M}}{}^\alpha \to (\mathbf{X} - \mathbf{x})^{(a_0)}$ defined in 3.7. Let us consider the closure $'\mathfrak{M}^\alpha$ of $\overset{\circ}{\mathcal{M}}{}^\alpha$ in $\mathbf{M}^\alpha \times (\mathbf{X} - \mathbf{x})^{(a_0)}$ (with respect to the locally closed embedding $(\mathbf{j}, \eta_\alpha)$). Since η_α factors through ϑ_α, we may equivalently define $\mathfrak{M}^\alpha$ as the closure of $\overset{\circ}{\mathcal{M}}{}^\alpha$ in $'\mathfrak{M}^\alpha \times \mathcal{P}^{a_0}$. Thus it suffices to prove that $'\mathfrak{M}^\alpha$ is of finite type.

Let us denote a_0 by a till the end of the proof. Recall the Beilinson-Drinfeld-Kottwitz ind-scheme $\mathfrak{B}^a$. We have a closed embedding[2] $(m_a, p_a) : \mathfrak{B}^a \hookrightarrow \mathcal{B} \times (\mathbf{X} - \mathbf{x})^{(a)}$ (notations of 2.8). Recall the notion of relative based (quasi)maps, see 4.2. We apply it to the case $Q = (\mathbf{X} - \mathbf{x})^{(a)}, R = \mathcal{B} \times (\mathbf{X} - \mathbf{x})^{(a)}$ with the (relative) Plücker embedding, and the evident section s. Then evidently $M^\alpha(\mathbf{C}, \mathbf{c}; R, s) = \mathbf{M}^\alpha \times (\mathbf{X} - \mathbf{x})^{(a)}$, and $\overset{\circ}{M}{}^\alpha(\mathbf{C}, \mathbf{c}; R, s) = \overset{\circ}{\mathbf{M}}{}^\alpha \times (\mathbf{X} - \mathbf{x})^{(a)}$. We will use an embedding $(\mathrm{Id}, \eta_\alpha) : \overset{\circ}{\mathbf{M}}{}^\alpha \hookrightarrow \overset{\circ}{M}{}^\alpha(\mathbf{C}, \mathbf{c}; R, s)$.

Lemma 4.3. *There exists a closed subscheme of finite type $\mathfrak{B}_+^\alpha \subset \mathfrak{B}^a$ such that for any based map $\phi \in \overset{\circ}{\mathbf{M}}{}^\alpha$ the relative based map $\phi' := (\mathrm{Id}, \eta_\alpha)(\phi)$ factors through $\mathfrak{B}_+^\alpha$.*

Proof. For a fixed ϕ, the proof of Lemma 4.1 shows that ϕ' factors through $\mathfrak{B}^a \hookrightarrow \mathcal{B} \times (\mathbf{X} - \mathbf{x})^{(a)}$, and hence through its closed subscheme of finite type. We have to choose such a subscheme uniformly, as ϕ varies. Recall that $\overset{\circ}{\mathbf{M}}{}^\alpha = \overset{\circ}{\mathcal{M}}{}^\alpha$ is a scheme of finite type, and we have a natural evaluation morphism $\mathsf{ev} : \overset{\circ}{\mathbf{M}}{}^\alpha \times \mathbf{C} \to \mathfrak{B}^a$, $(\phi, c) \mapsto \phi'(c)$. By definition, such a morphism into the ind-scheme $\mathfrak{B}^a$ factors through a finite type scheme $\mathfrak{B}_+^\alpha$. $\square$

Remark 4.4. Let us give a more concrete description of $\mathfrak{B}_+^\alpha$. Let $(y_1, \ldots, y_a) \in (\mathbf{X} - \mathbf{x} - \mathbf{y})^{(a)} - \Delta$ be a collection of *distinct* points. Then the fiber of $\mathfrak{B}^a$ over $(y_1, \ldots, y_a)$ equals the product of B (the flag variety of $\mathfrak{sl}_n$) and of a copies of the affine Grassmannian Gr of $\mathfrak{sl}_n$, see [11]. Let $\mathsf{Gr}_{\gamma_0} \subset \mathsf{Gr}$ be the closure of $SL_n(\mathcal{O})$-orbit numbered by the highest (co)root γ_0 of $\mathfrak{sl}_n$. Let $\mathfrak{B}_1^a \subset p_a^{-1}((\mathbf{X} - \mathbf{x} - \mathbf{y})^{(a)} - \Delta)$ be a closed subscheme of finite type whose fiber over $(y_1, \ldots, y_a) \in (\mathbf{X} - \mathbf{x} - \mathbf{y})^{(a)} - \Delta$ equals $\mathsf{B} \times \prod \mathsf{Gr}_{\gamma_0} \subset \mathsf{B} \times \prod \mathsf{Gr}$. Let $\mathfrak{B}_2^a \subset \mathfrak{B}^a$ be the closure of $\mathfrak{B}_1^a$.

The proof of Lemma 3.5 actually shows that if $\eta_\alpha(\phi) \in (\mathbf{X} - \mathbf{x} - \mathbf{y})^{(a)}$ then ϕ' factors through $\mathfrak{B}_2^a$.

According to [11], the fiber of $\mathfrak{B}^a$ over $a \cdot \mathbf{y} \in (\mathbf{X} - \mathbf{x})^{(a)}$ equals the affine flag variety Fl of $\mathfrak{sl}_n$. We define a closed subscheme of finite type $\mathsf{Fl}_a \subset \mathsf{Fl}$ as the fiber of $\mathfrak{B}_2^a$ over $a \cdot \mathbf{y} \in (\mathbf{X} - \mathbf{x})^{(a)}$.

We have an embedding $(\mathbf{X} - \mathbf{x} - \mathbf{y})^{(a-1)} - \Delta \hookrightarrow (\mathbf{X} - \mathbf{x})^{(a)} - \Delta$, $(y_1, \ldots, y_{a-1}) \mapsto (\mathbf{y}, y_1, \ldots, y_{a-1})$. According to [11], the fiber of $\mathfrak{B}^a$ over $(\mathbf{y}, y_1, \ldots, y_{a-1})$ equals the product of Fl and $a - 1$ copies of Gr. We define $b := \max(a_i)_{i \in I}$. Let $\mathfrak{B}_3^a \subset p_a^{-1}((\mathbf{X} - \mathbf{x} - \mathbf{y})^{(a-1)} - \Delta)$ be a closed subscheme

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[2] We just mean that the restriction of (m_a, p_a) to a finite type closed subscheme of $\mathfrak{B}^a$ is a closed embedding.

of finite type whose fiber over $(\mathbf{y}, y_1, \ldots, y_{a-1})$ equals $\mathsf{Fl}_b \times \prod \mathsf{Gr}_{\gamma_0}$. Let $\mathfrak{B}_4^a \subset \mathfrak{B}^a$ be the closure of $\mathfrak{B}_3^a$.

Finally, we define $\mathfrak{B}_+^\alpha := \mathfrak{B}_2^a \bigcup \mathfrak{B}_4^a$. Clearly, it is a closed subscheme of finite type solving our problem.

4.8.   We return to the proof of Proposition 4.2. In the setup of 4.2 we set $R' := \mathfrak{B}_+^a \overset{(m_a, p_a)}{\hookrightarrow} \mathcal{B} \times (\mathbf{X} - \mathbf{x})^{(a)} =: R$. Thus we have a closed embedding $M(\mathbf{C}, \mathbf{c}; R', s) \hookrightarrow M(\mathbf{C}, \mathbf{c}; R, s)$. Since R' is a scheme of finite type, $M(\mathbf{C}, \mathbf{c}; R', s)$ is a scheme of finite type as well. By Lemma 4.3, $\overset{\circ}{M}(\mathbf{C}, \mathbf{c}; R', s)$ coincides with the image of $\overset{\circ}{\mathbf{M}}{}^\alpha$ under the embedding $(\mathrm{Id}, \eta_\alpha)$. Thus, the closure $'\mathfrak{M}^\alpha$ of $(\mathrm{Id}, \eta_\alpha)(\overset{\circ}{\mathbf{M}}{}^\alpha)$ in $M(\mathbf{C}, \mathbf{c}; R, s)$ is a closed subscheme in a scheme of finite type $M(\mathbf{C}, \mathbf{c}; R', s)$. Hence $'\mathfrak{M}^\alpha$ is a scheme of finite type itself. This completes the proof of Proposition 4.2.   □

4.9.   The same proof as above (using the Beilinson-Drinfeld ind-scheme $\mathfrak{G}^a$, see 2.9, instead of $\mathfrak{B}^\alpha$) shows that both A^a and $\mathfrak{A}^a$ are irreducible affine schemes of finite type.

§5.   Resolution of Singularities $\varpi_\alpha : \mathcal{M}^\alpha \to \mathfrak{M}^\alpha$

5.1. **Determinant line bundles.** We start with a construction of a morphism $\omega^\alpha : \mathcal{M}^\alpha \to \mathbf{M}^\alpha$. So let $\mathcal{F}_\bullet$ be an S-point of $\mathcal{M}^\alpha$, that is, a parabolic sheaf on $S \times \mathcal{S}'$ flat over S along with a trivialization σ of $\mathcal{F}_0$ at $S \times \mathbf{D}'$. We have to construct the invertible subsheaves $\mathfrak{L}_{-\omega_i} \subset V_{\omega_i}^* \otimes \mathcal{O}_{S \times \mathbf{C}}$ satisfying the Plücker equations. Equivalently, we have to construct the generically surjective maps $V_{\omega_i} \otimes \mathcal{O}_{S \times \mathbf{C}} \to \mathfrak{L}_{\omega_i} =: \mathfrak{L}_i$ defined up to scalar multiplication. Recall that the fundamental representation V_{ω_i} is canonically embedded into the semi-infinite wedge power $\mathcal{Q}$ (see 2.2). Hence it suffices to construct the generically surjective maps $p_i : \mathcal{Q} \otimes \mathcal{O}_{S \times \mathbf{C}} \to \mathfrak{L}_i$ defined up to scalar multiplication.

According to Lemma 4.1, the trivialization σ restricted to $S \times \mathbf{C} \times \mathbf{x}$ canonically extends to a trivialization ς of $\mathcal{F}_0$ (and hence $\mathcal{F}_k$, $k \in \mathbb{Z}$) in the formal neighbourhood of $S \times \mathbf{C} \times \mathbf{x}$ in $S \times \mathcal{S}'$.

Let us denote by $\mathcal{U} \subset S \times \mathcal{S}'$ the open subset $S \times \mathcal{S}' - S \times \mathbf{C} \times \mathbf{x}$. Let us denote by $\mathbf{p} : S \times \mathcal{S}' \to S \times \mathbf{C}$ the natural projection, and by $\overset{\circ}{\mathbf{p}} : \mathcal{U} \to S \times \mathbf{C}$ its restriction to $\mathcal{U}$. Let us denote by $\overset{\bullet}{\mathcal{U}}$ the intersection of $\mathcal{U}$ with the formal neighbourhood of $S \times \mathbf{C} \times \mathbf{x}$ in $S \times \mathcal{S}'$ (the "pointed formal neighbourhood"), and by $\overset{\bullet}{\mathbf{p}} : \overset{\bullet}{\mathcal{U}} \to \mathbf{C}$ the natural projection. Then for any $k \in \mathbb{Z}$ the trivialization ς identifies $\overset{\bullet}{\mathbf{p}}_*(\mathcal{F}_k|_{\overset{\bullet}{\mathcal{U}}})$ with $\mathbf{V} \otimes \mathcal{O}_{S \times \mathbf{C}}$ (notations of 2.2), and $\mathsf{F}_k := \overset{\circ}{\mathbf{p}}_*(\mathcal{F}_k|_{\mathcal{U}})$ is

naturally a discrete lattice in $\overset{\bullet}{\mathbf{p}}_*(\mathcal{F}_k|_{\overset{\bullet}{\mathcal{U}}}) = \mathbf{V} \otimes \mathcal{O}_{S \times \mathbf{C}}$. Recall that $L_V \otimes \mathcal{O}_{S \times \mathbf{C}} := V \otimes \mathbb{C}[[t^{-1}]] \otimes \mathcal{O}_{S \times \mathbf{C}}$ is a compact lattice in $\mathbf{V} \otimes \mathcal{O}_{S \times \mathbf{C}}$.

We define $\mathfrak{L}_i$ as the dual of the determinant line bundle of a natural *Fredholm operator* $(L_V \otimes \mathcal{O}_{S \times \mathbf{C}}) \oplus \mathsf{F}_i \to \mathbf{V} \otimes \mathcal{O}_{S \times \mathbf{C}}$ (notations of [4]).

5.2.   We still have to construct the generically surjective maps $p_i : \mathcal{Q} \otimes \mathcal{O}_{S \times \mathbf{C}} \to \mathfrak{L}_i$. Recall that $\mathcal{Q}$ is a union of finite dimensional subspaces $\mathcal{Q}^{L_{1,2}}$ (see 2.2). It suffices to construct a compatible system of maps $p_i^{L_{1,2}} : \mathcal{Q}^{L_{1,2}} \otimes \mathcal{O}_{S \times \mathbf{C}} \to \mathfrak{L}_i$.

For small enough compact lattices L_1, L_2 (such that $L_2 \subset L_1^\perp \subset \mathbf{V}^*$) we have $L_1 \bigcap \mathsf{F}_i = 0$, and $L_2^\perp + \mathsf{F}_i = \mathbf{V} \otimes \mathcal{O}_{S \times \mathbf{C}}$ for any $0 \le i \le n-1$. In effect, by Čech calculation, this is equivalent to $R^0 \mathbf{p}_* \mathcal{F}_i(-N) = 0$ and $R^1 \mathbf{p}_* \mathcal{F}_i(N) = 0$ for $N \gg 0$. We define $\mathsf{F}_i^{L_2} := \mathrm{Ker}(L_2^\perp \oplus \mathsf{F}_i \to \mathbf{V} \otimes \mathcal{O}_{S \times \mathbf{C}})$. This is a coherent sheaf flat over $S \times \mathbf{C}$ equipped with a canonical embedding into a vector bundle $L_2^\perp / L_1 \otimes \mathcal{O}_{S \times \mathbf{C}}$. Note that $\det(\mathsf{F}_i^{L_2}) = \mathfrak{L}_i^* \otimes \det(L_2^\perp)$, thus $\det^*(\mathsf{F}_i^{L_2}) = \mathfrak{L}_i \otimes \det^*(L_2^\perp)$.

The embedding $\mathsf{F}_i^{L_2} \hookrightarrow L_2^\perp / L_1 \otimes \mathcal{O}_{S \times \mathbf{C}}$ gives rise to an invertible subsheaf $\det(\mathsf{F}_i^{L_2}) \subset \Lambda^\bullet(L_2^\perp / L_1) \otimes \mathcal{O}_{S \times \mathbf{C}}$. Dually, we have a generically surjective morphism $\Lambda^*(L_2^\perp / L_1) \otimes \mathcal{O}_{S \times \mathbf{C}} \to \det^*(\mathsf{F}_i^{L_2})$, or equivalently, $\Lambda^*(L_2^\perp / L_1) \otimes \det(L_2^\perp) \otimes \mathcal{O}_{S \times \mathbf{C}} \to \det^*(\mathsf{F}_i^{L_2}) \otimes \det(L_2^\perp) = \mathfrak{L}_i$. Now recall that $\Lambda^*(L_2^\perp / L_1) \otimes \det(L_2^\perp)$ is canonically isomorphic to $\mathcal{Q}^{L_{1,2}}$ (see 2.2), hence we have obtained the desired morphism $p_i^{L_{1,2}} : \mathcal{Q}^{L_{1,2}} \otimes \mathcal{O}_{S \times \mathbf{C}} \to \mathfrak{L}_i$.

5.3.   Over an open subset $U \subset S \times \mathbf{C}$ such that $\mathcal{F}_\bullet|_{\mathbf{p}^{-1}(U)}$ is a parabolic vector bundle, the above construction reduces to $\mathcal{Q} \otimes \mathcal{O}_U \twoheadrightarrow \mathfrak{L}_i := (\mathcal{Q} \otimes \mathcal{O}_U)_{\mathsf{F}_{W,i}}$ (notations of 2.4). Hence $\mathfrak{L}_{-\omega_i}|_U \subset V_{\omega_i}^* \otimes \mathcal{O}_U$ satisfy Plücker relations, hence $\mathfrak{L}_{-\omega_i} \subset V_{\omega_i}^* \otimes \mathcal{O}_{S \times \mathbf{C}}$ satisfy Plücker relations. Evidently, $U \supset S \times \mathbf{c}$, and the fibers of $\mathfrak{L}_{-\omega_i}$ at $S \times \mathbf{c}$ are as prescribed. So all in all we have constructed the desired morphism $\omega^\alpha : \mathcal{M}^\alpha \to \mathbf{M}^\alpha$. By the same token, we have constructed the morphism $\upsilon^a : \mathcal{A}^a \to \mathbf{A}^a$.

Recall the morphism $\vartheta_\alpha : \mathcal{M}^\alpha \to \mathcal{P}^{a_0}$ constructed in 3.7.

We define the morphism $\varpi_\alpha := (\omega^\alpha, \vartheta_\alpha) : \mathcal{M}^\alpha \to \mathbf{M}^\alpha \times \mathcal{P}^{a_0}$. Since $\overset{\circ}{\mathcal{M}}{}^\alpha$ is dense in $\mathcal{M}^\alpha$, the morphism ϖ_α factors through $\mathcal{M}^\alpha \to \mathfrak{M}^\alpha \hookrightarrow \mathbf{M}^\alpha \times \mathcal{P}^{a_0}$. Thus we obtain the same named morphism $\varpi_\alpha : \mathcal{M}^\alpha \to \mathfrak{M}^\alpha$.

Proposition 5.1. ϖ_α *is a proper morphism.*

Proof. Recall that $'\mathfrak{M}^\alpha$ is the closure of $\overset{\circ}{\mathcal{M}}{}^\alpha$ in $\mathbf{M}^\alpha \times (\mathbf{X} - \mathbf{x})^{(a_0)}$ (with respect to the locally closed embedding $(\mathbf{j}, \eta_\alpha)$). The morphism $'\varpi_\alpha := (\omega^\alpha, \eta_\alpha) : \mathcal{M}^\alpha \to \mathbf{M}^\alpha \times (\mathbf{X} - \mathbf{x})^{(a_0)}$ factors through the morphism $'\varpi_\alpha : \mathcal{M}^\alpha \to {}'\mathfrak{M}^\alpha$,

and $'\varpi_\alpha$ factors through $\mathcal{M}^\alpha \xrightarrow{\varpi_\alpha} \mathfrak{M}^\alpha \to {}'\mathfrak{M}^\alpha$, so it suffices to check that $'\varpi_\alpha$ is projective.

Let us consider the moduli scheme $\mathbb{M}^\alpha \supset \mathbf{M}^\alpha$ of quasimaps from $\mathbf{C}$ to $\mathcal{B}$ removing the based condition in the definition of $\mathbf{M}^\alpha$. Recall that $\varrho_\mathbf{X} : \mathcal{S}' = \mathbf{C} \times \mathbf{X} \to \mathbf{X}$ is the canonical projection. For an effective divisor $D \in (\mathbf{X} - \mathbf{x})^{(a_0)}$ we denote by $\mathbf{D}_D$ the effective divisor $\varrho_\mathbf{X}^{-1}(D)$ in $\mathcal{S}'$. Let $\widetilde{\mathbb{M}}^\alpha$ be the moduli ind-scheme of the following data:

(a) a divisor $D \in (\mathbf{X} - \mathbf{x})^{(a_0)}$;

(b) a parabolic sheaf $\mathcal{F}_\bullet$ of degree α on $\mathcal{S}'$ such that $V \otimes \mathcal{O}_{\mathcal{S}'}(-\infty \cdot \mathbf{D}_D) \subset \mathcal{F}_0 \subset V \otimes \mathcal{O}_{\mathcal{S}'}(\infty \cdot \mathbf{D}_D)$.

Note in particular that $\mathcal{F}_0$ (and hence all the $\mathcal{F}_k$, $k \in \mathbb{Z}$) are trivialized in a Zariski (and hence in the formal) neighbourhood of $\mathbf{C} \times \mathbf{x} \subset \mathcal{S}'$. Now the construction of 5.1-5.2 defines a morphism $\Omega^\alpha : \widetilde{\mathbb{M}}^\alpha \to \mathbb{M}^\alpha$. The construction of 3.6 defines a locally closed embedding $\mathcal{M}^\alpha \to \widetilde{\mathbb{M}}^\alpha$, and we have a cartesian diagram

$$
\begin{array}{ccc}
\mathcal{M}^\alpha & \longrightarrow & \widetilde{\mathbb{M}}^\alpha \\
\downarrow & & \downarrow \\
\mathbf{M}^\alpha & \longrightarrow & \mathbb{M}^\alpha
\end{array}
$$

We denote by $\eta' : \widetilde{\mathbb{M}}^\alpha \to (\mathbf{X} - \mathbf{x})^{(a_0)}$ the tautological projection. Note that η_α factors through $\mathcal{M}^\alpha \to \widetilde{\mathbb{M}}^\alpha \xrightarrow{\eta'} (\mathbf{X} - \mathbf{x})^{(a_0)}$.

It suffices to prove that $(\Omega^\alpha, \eta') : \widetilde{\mathbb{M}}^\alpha \to \mathbb{M}^\alpha \times (\mathbf{X} - \mathbf{x})^{(a_0)}$ is ind-projective. Moreover, it is enough to prove that $\eta' : \widetilde{\mathbb{M}}^\alpha \to (\mathbf{X} - \mathbf{x})^{(a_0)}$ is ind-projective. Let us view a "universal divisor" $\mathbf{D}_D$ as a closed subscheme of $(\mathbf{X} - \mathbf{x})^{(a_0)} \times \mathcal{S}'$, projective over $(\mathbf{X} - \mathbf{x})^{(a_0)}$. Then $\widetilde{\mathbb{M}}^\alpha$ is a closed ind-subscheme of a product of certain inductive limits of Quot-schemes over $\mathbf{D}_{k \cdot D + \mathbf{y}}$, $k \to \infty$, cf. [9], p. 164. These Quot-schemes being projective, η' is ind-projective. This completes the proof of the Proposition. $\qquad \square$

Remark 5.2.   Lemma 3.5 shows that instead of the ind-scheme $\widetilde{\mathbb{M}}^\alpha$ in the above argument, one could use the scheme $'\widetilde{\mathbb{M}}^\alpha$ defined as $\widetilde{\mathbb{M}}^\alpha$ with the condition (b) being replaced by

('b) $V \otimes \mathcal{O}_{\mathcal{S}'}(-\mathbf{D}_D) \subset \mathcal{F}_0 \subset V \otimes \mathcal{O}_{\mathcal{S}'}(\mathbf{D}_D)$.

5.4.   Recall the setup of 4.5, 4.6. The relative version over O of the construction 5.2–5.3 defines the proper morphisms $\epsilon_a : \mathcal{A}^a \to \mathsf{A}^a$, $\varepsilon_a : \mathcal{A}^a \to \mathfrak{A}^a$

such that $\varepsilon_a = \Xi_a \circ \epsilon_a$. Since A^a (resp. $\mathfrak{A}^a$) is affine, ϵ_a (resp. ε_a) factors through the affinization $\Upsilon_a : \mathcal{A}^a \to \mathfrak{N}^a$, and we get the morphisms $\Psi_a : \mathfrak{N}^a \to \mathsf{A}^a$, $\Phi_a : \mathfrak{N}^a \to \mathfrak{A}^a$ such that $\Phi_a = \Xi_a \circ \Psi_a$.

We conjecture[3] that all the maps Ψ_a, Φ_a, Ξ_a are isomorphisms. This can be checked at the level of $\mathbb{C}$-points. In effect, it is well known that Donaldson's space $\mathfrak{N}^a$ has a decomposition into locally closed pieces $\mathfrak{N}^a = \bigsqcup_{b \leq a} \overset{\circ}{\mathcal{A}}{}^{a-b} \times \mathrm{Sym}^b(\mathbb{A}^2)$. On the other hand, recall that $\mathsf{A}^a = \bigsqcup_{b \leq a} \overset{\circ}{\mathcal{A}}{}^{a-b} \times \mathrm{Sym}^b(\mathbf{C} - \mathbf{c})$ (see e.g. [5]). Also, recall the embedding $\mho^b : \mathrm{Sym}^b(\mathbb{A}^2) \hookrightarrow \mathcal{P}^b$, see 3.8. We have $\mathfrak{A}^a \subset \mathsf{A}^a \times \mathcal{P}^a$, and the arguments of 6.1 below show that $\mathfrak{A}^a$ has a decomposition into locally closed pieces

$$\mathfrak{A}^a = \bigsqcup_{b \leq a} \left(\overset{\circ}{\mathcal{A}}{}^{a-b} \times \mathrm{Sym}^b(\mathbf{C} - \mathbf{c}) \right) \times_{\mathrm{Sym}^b(\mathbf{C}-\mathbf{c})} \mathrm{Sym}^b(\mathbb{A}^2)$$

$$= \bigsqcup_{b \leq a} \overset{\circ}{\mathcal{A}}{}^{a-b} \times \mathrm{Sym}^b(\mathbb{A}^2)$$

Here the embedding $\overset{\circ}{\mathcal{A}}{}^{a-b} \times \mathrm{Sym}^b(\mathbb{A}^2) \hookrightarrow \mathsf{A}^a \times \mathcal{P}^a$ goes as follows: $(\mathcal{F}; \underline{s}) \mapsto (\mathcal{F}, \varrho_{\mathbf{C}}(\underline{s}); \theta_{a-b}(\mathcal{F}) + \mho^b(\underline{s}))$ where we use the natural projection $\varrho_{\mathbf{C}} : \mathrm{Sym}^b(\mathbb{A}^2) \to \mathrm{Sym}^b(\mathbf{C} - \mathbf{c})$.

§6. $\mathcal{IC}$ Stalks

6.1. **Uhlenbeck stratification.** We refine the decomposition of the Uhlenbeck space $\mathfrak{A}^a$ into locally closed pieces described in 5.4. We define the *diagonal stratification* of $\mathrm{Sym}^b(\mathbb{A}^2)$ as follows. For a positive integer b we denote by $\mathfrak{P}(b)$ the set of partitions of b (in the traditional meaning). For $\mathfrak{P} = (b_1 \geq b_2 \geq \ldots \geq b_m > 0) \in \mathfrak{P}(b)$ the corresponding stratum $\mathrm{Sym}^b(\mathbb{A}^2)_{\mathfrak{P}}$ of $\mathrm{Sym}^b(\mathbb{A}^2)$ is formed by configurations which can be subdivided into m groups of points, the r-th group containing b_r points; all the points in one group equal to each other, the different groups being disjoint. We have $\mathrm{Sym}^b(\mathbb{A}^2) = \bigsqcup_{\mathfrak{P} \in \mathfrak{P}(b)} \mathrm{Sym}^b(\mathbb{A}^2)_{\mathfrak{P}}$.

In the setup of 5.4, for $b \leq a$, $\mathfrak{P} \in \mathfrak{P}(b)$, we define a locally closed subscheme $\mathfrak{A}^a_{a-b,\mathfrak{P}} = \overset{\circ}{\mathcal{A}}{}^{a-b} \times \mathrm{Sym}^b(\mathbb{A}^2)_{\mathfrak{P}} \subset \overset{\circ}{\mathcal{A}}{}^{a-b} \times \mathrm{Sym}^b(\mathbb{A}^2) \subset \mathsf{A}^a \times \mathcal{P}^a$.

In order to show $\mathfrak{A}^a = \bigsqcup_{\substack{b \leq a \\ \mathfrak{P} \in \mathfrak{P}(b)}} \mathfrak{A}^a_{a-b,\mathfrak{P}}$ we describe the inverse image $\varepsilon^{-1}(\mathfrak{A}^a_{a-b,\mathfrak{P}}) \subset \mathcal{A}^a$. First we define the *saturation* and *defect* of a based

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[3]This conjecture is proved in [6].

quasimap $\phi = (\mathfrak{L}_0 \subset V_{\omega_0}^* \otimes \mathcal{O}_\mathbf{C}) \in \mathbf{A}^a$. Namely, the saturation $\mathcal{N}(\phi) \in \overset{\circ}{\mathbf{A}}{}^d$ is the based map $(\widetilde{\mathfrak{L}}_0 \subset V_{\omega_0}^* \otimes \mathcal{O}_\mathbf{C})$ where the line subbundle $\widetilde{\mathfrak{L}}_0$ is the saturation of the invertible subsheaf $\mathfrak{L}_0$. The quotient $\widetilde{\mathfrak{L}}_0/\mathfrak{L}_0$ is a torsion sheaf on $\mathbf{C} - \mathbf{c}$ of length b supported at a finite subset S, and we define the defect $\mathrm{def}(\phi) \in (\mathbf{C} - \mathbf{c})^{(b)}$ as $\sum_{s \in S} \mathrm{length}_s(\widetilde{\mathfrak{L}}_0/\mathfrak{L}_0) \cdot s$. Note that necessarily $a = d + b$.

Recall that the saturation and defect of a torsion free sheaf $\mathcal{F} \in \mathcal{A}^a$ were defined in 3.2. Recall the morphism $\varepsilon_a = (v^a, \theta_a) : \mathcal{A}^a \to \mathfrak{A}^a \subset \mathbf{A}^a \times \mathcal{P}^a$, and the natural projection $\varrho_\mathbf{C} : \mathrm{Sym}^b(\mathbb{A}^2) \to (\mathbf{C} - \mathbf{c})^{(b)}$. We have the following

Lemma 6.1. *Suppose the defect* $\mathrm{def}(\mathcal{F})$ *of a torsion free sheaf* $\mathcal{F} \in \mathcal{A}^a$ *has degree* $b \le a$, *so that the saturation* $\mathcal{N}(\mathcal{F})$ *lies in* $\overset{\circ}{\mathcal{A}}{}^{a-b}$. *Then the quasimap* $\phi := v^a(\mathcal{F}) \in \mathbf{A}^a$ *has saturation* $\mathcal{N}(\phi) = v^{a-b}(\mathcal{N}(\mathcal{F}))$, *and defect* $\mathrm{def}(\phi) = \varrho_\mathbf{C}(\mathrm{def}(\mathcal{F})) \in (\mathbf{C} - \mathbf{c})^{(b)}$.

Proof. Let $\phi = (\mathfrak{L}_0 \subset V_{\omega_0}^* \otimes \mathcal{O}_\mathbf{C})$, and $\phi' := v^{a-b}(\mathcal{N}(\mathcal{F})) = (\mathfrak{L}_0' \subset V_{\omega_0}^* \otimes \mathcal{O}_\mathbf{C})$. Since $\mathcal{F} \subset \mathcal{N}(\mathcal{F})$, by the construction of v, we see that $\mathfrak{L}_0 \subset \mathfrak{L}_0'$. Moreover, since $\mathcal{N}(\mathcal{F}) \in \overset{\circ}{\mathcal{A}}{}^{a-b} = \overset{\circ}{\mathbf{A}}{}^{a-b}$, we see that $\mathfrak{L}_0'$ is saturated; hence $\mathfrak{L}_0'$ is the saturation of $\mathfrak{L}_0$. Since $\mathcal{F}$ and $\mathcal{N}(\mathcal{F})$ coincide off the support of $\mathrm{def}(\mathcal{F})$, by the construction of v, $\mathfrak{L}_0$ and $\mathfrak{L}_0'$ coincide off the support of $\varrho_\mathbf{C}(\mathrm{def}(\mathcal{F}))$. Finally, since $\mathrm{length}(\mathfrak{L}_0'/\mathfrak{L}_0) = b$, and the definition of v is local over $\mathbf{C}$, we conclude that $\mathrm{def}(\phi) = \varrho_\mathbf{C}(\mathrm{def}(\mathcal{F}))$. $\square$

It follows immediately from the above Lemma and Corollary 3.6 that $\mathcal{A}_{a-b,\mathfrak{P}}^a := \varepsilon^{-1}(\mathfrak{A}_{a-b,\mathfrak{P}}^a) \subset \mathcal{A}^a$ is formed by all the torsion free sheaves $\mathcal{F} \in \mathcal{A}^a$ such that $\mathcal{N}(\mathcal{F})$ lies in $\overset{\circ}{\mathcal{A}}{}^{a-b}$, and $\mathrm{def}(\mathcal{F})$ lies in $\mathrm{Sym}^b(\mathbb{A}^2)_\mathfrak{P}$. Clearly, $\mathcal{A}^a$ is covered by the above locally closed pieces: $\mathcal{A}^a = \coprod\limits_{\substack{b \le a}}^{\mathfrak{P} \in \mathfrak{P}(b)} \mathcal{A}_{a-b,\mathfrak{P}}^a$. Hence $\mathfrak{A}^a$ being the image of $\mathcal{A}^a$ under the proper morphism ε_a, is covered by the Uhlenbeck strata: $\mathfrak{A}^a = \coprod\limits_{\substack{b \le a}}^{\mathfrak{P} \in \mathfrak{P}(b)} \mathfrak{A}_{a-b,\mathfrak{P}}^a$.

6.2. **Saturation and defect.** Our next goal is to describe a similar stratification of the Uhlenbeck flag space.

Given a parabolic sheaf $\mathcal{F}_\bullet \in \mathcal{M}^\alpha$ we define its *saturation* $\mathcal{N}(\mathcal{F})_\bullet$ as the parabolic sheaf formed by saturations of all components of the parabolic sheaf $\mathcal{F}_\bullet$:

$$\mathcal{N}(\mathcal{F})_k := \mathcal{N}(\mathcal{F}_k).$$

Clearly, $\mathcal{N}(\mathcal{F})_\bullet$ is indeed a parabolic sheaf. Moreover, it is evidently locally free.

By construction, $\mathcal{F}_k \subset \mathcal{N}(\mathcal{F})_k$ for any $k \in \mathbb{Z}$.

Given $\beta = \sum_i b_i i \in \mathbb{N}[I]$ we define $\mathbb{A}^\beta = (\mathbf{C} - \mathbf{c})^\beta$ as the product $\prod_i (\mathbf{C} - \mathbf{c})^{(b_i)}$. This is the moduli space of effective I-colored divisors of degree β on $\mathbb{A}^1 = \mathbf{C} - \mathbf{c}$. We define $\overset{\circ}{\mathbb{A}}{}^2 \subset \mathbb{A}^2$ as $\mathbb{A}^2 - \mathbf{D}_0$. Note that $\mathbf{D}_0 \cap \mathbb{A}^2 = \mathbf{C} - \mathbf{c}$.

For a parabolic sheaf $\mathcal{F}_\bullet \subset \mathcal{M}^\alpha$ we define its *defect* $\mathrm{def}(\mathcal{F}_\bullet) = \mathrm{def}^\dagger(\mathcal{F}_\bullet) + \mathrm{def}^\circ(\mathcal{F}_\bullet) = \sum_{i \in I} \mathrm{def}_i(\mathcal{F}_\bullet) + \mathrm{def}^\circ(\mathcal{F}_\bullet) \in \mathbb{A}^\beta \times \mathrm{Sym}^d(\overset{\circ}{\mathbb{A}}{}^2)$ (for some β, d) as follows. Note that for any $k \in \mathbb{Z}$ the quotient sheaf $\mathcal{N}(\mathcal{F})_k/\mathcal{F}_k|_{\overset{\circ}{\mathbb{A}}{}^2}$ is a torsion sheaf of finite length d on $\overset{\circ}{\mathbb{A}}{}^2$ independent of k. In particular, it is supported on a finite subset $S \subset \overset{\circ}{\mathbb{A}}{}^2$. So $\mathrm{def}^\circ(\mathcal{F}_\bullet) := \sum_{s \in S} \mathrm{length}_s(\mathcal{N}(\mathcal{F})_k/\mathcal{F}_k) \cdot s \in \mathrm{Sym}^d(\overset{\circ}{\mathbb{A}}{}^2)$.

Now for $i \in I$ the quotient $\mathcal{N}(\mathcal{F})_i/\mathcal{F}_i$ is a torsion sheaf of finite length $d + b_i$. In particular, it is supported on a finite subset $S_i \subset \mathbb{A}^2$. So $\mathrm{def}_i(\mathcal{F}_\bullet) := \sum_{s \in S_i \cap \mathbf{D}_0} \mathrm{length}_s(\mathcal{N}(\mathcal{F})_i/\mathcal{F}_i) \cdot s$, and $\mathrm{def}^\dagger(\mathcal{F}_\bullet) := \sum_i \sum_{s \in S_i \cap \mathbf{D}_0} \mathrm{length}_s(\mathcal{N}(\mathcal{F})_i/\mathcal{F}_i) \cdot s \in \prod_i (\mathbf{C} - \mathbf{c})^{(b_i)}$.

Finally, we define $\mathrm{def}^\ddagger(\mathcal{F}_\bullet) := \sum_{s \in S_0} \mathrm{length}_s(\mathcal{N}(\mathcal{F})_0/\mathcal{F}_0) \cdot s \in \mathrm{Sym}^{d+b_0}(\mathbb{A}^2)$.

Recall that the *imaginary coroot* of $\widehat{\mathfrak{sl}}_n$ is $\delta_0 := \sum_i \alpha_i$. Note that if $\mathcal{F}_\bullet \subset \mathcal{M}^\alpha$, and $\mathcal{N}(\mathcal{F})_\bullet \in \overset{\circ}{\mathcal{M}}{}^\gamma$, and $\mathrm{def}(\mathcal{F}_\bullet) \in \mathbb{A}^\beta \times \mathrm{Sym}^d(\overset{\circ}{\mathbb{A}}{}^2)$, then $\alpha = \gamma + \beta + d\delta_0$.

6.3. Partitions.
We refer the reader to [9] 2.2 for a general terminology on partitions. So for $\alpha \in \mathbb{N}[I]$ we have the set of *usual partitions* $\Gamma(\alpha)$. The definition of Kostant partitions $\mathfrak{K}(\alpha)$ of *loc. cit.* makes reference to the set R^+ of ψ-*roots*, or in other words, "positive roots of $\widehat{\mathfrak{gl}}_n$". To avoid a confusion we will denote the set of ψ-roots by $R^+_{\widehat{\mathfrak{gl}}_n}$, and the corresponding set of $\widehat{\mathfrak{gl}}_n$-*Kostant partitions* by $\mathfrak{K}_{\widehat{\mathfrak{gl}}_n}(\alpha)$. We define a subset $R^+_{\widehat{\mathfrak{sl}}_n} \subset R^+_{\widehat{\mathfrak{gl}}_n}$ whose complement consists of ψ-roots $(0, kn)$, $k = 1, 2, \ldots$ (notations of *loc. cit.* 2.1). The set $R^+_{\widehat{\mathfrak{sl}}_n}$ together with the dimension function restricted from $R^+_{\widehat{\mathfrak{gl}}_n}$ gives rise to the set of $\widehat{\mathfrak{sl}}_n$-*Kostant partitions* $\mathfrak{K}_{\widehat{\mathfrak{sl}}_n}(\alpha)$.

The number of summands in a partition ? is denoted by $\mathsf{K}(?)$.

6.4. Configurations.
We define the *diagonal stratification* of $\mathbb{A}^\alpha = (\mathbf{C} - \mathbf{c})^\alpha$.

Recall that the *multisubsets* of a set S are defined as elements of some symmetric power $S^{(m)}$, and we denote the image of $(s_1, \ldots, s_m) \in S^m$ by $\{\{s_1, \ldots, s_m\}\}$. In particular, the set of usual partitions $\Gamma(\alpha)$ is formed by all the multisubsets $\Gamma = \{\{\gamma_1, \ldots, \gamma_m\}\}$ of $\mathbb{N}[I] - \{0\}$ such that $\sum_{r=1}^m \gamma_r = \alpha$.

For $\Gamma \in \Gamma(\alpha)$ the corresponding stratum $\mathbb{A}^\alpha_\Gamma$ is defined as follows. It is formed by configurations which can be subdivided into m groups of points, the r-th group containing γ_r points; all the points in one group equal to each other, the different groups being disjoint. We have $\dim(\mathbb{A}^\alpha_\Gamma) = \mathsf{K}(\Gamma)$. For example, the main diagonal in $\mathbb{A}^\alpha$ is the closed stratum given by partition $\alpha = \alpha$, while the complement to all diagonals in $\mathbb{A}^\alpha$ is the open stratum given by partition $\alpha = \sum_{i \in I}(\underbrace{i + i + \cdots + i}_{a_i \text{ times}})$. Evidently, $\mathbb{A}^\alpha = \bigsqcup_{\Gamma \in \Gamma(\alpha)} \mathbb{A}^\alpha_\Gamma$.

Similarly, we define the *diagonal stratification* of $\mathrm{Sym}^d(\overset{\circ}{\mathbb{A}}{}^2)$. We have $\mathrm{Sym}^d(\overset{\circ}{\mathbb{A}}{}^2) = \bigsqcup_{\mathfrak{P} \in \mathfrak{P}(d)} \mathrm{Sym}^d(\overset{\circ}{\mathbb{A}}{}^2)_{\mathfrak{P}}$. Also, in evident notations, $\mathrm{Sym}^d(\mathbb{A}^2) = \bigsqcup_{\mathfrak{P} \in \mathfrak{P}(d)} \mathrm{Sym}^d(\mathbb{A}^2)_{\mathfrak{P}}$. For a partition $\mathfrak{P} \in \mathfrak{P}(d)$ let $\mathrm{Sym}^d(\mathbb{A}^2)_{\overline{\mathfrak{P}}}$ denote the closure of the stratum $\mathrm{Sym}^d(\mathbb{A}^2)_{\mathfrak{P}}$. If $\mathfrak{P} = \{\{k_1 \cdot d_1, \ldots, k_m \cdot d_m\}\}$ for some $0 < d_1 < \ldots < d_m$, then we have an evident finite morphism $N^{\mathfrak{P}} : \prod_{l=1}^m \mathrm{Sym}^{k_l}(\mathbb{A}^2) \to \mathrm{Sym}^d(\mathbb{A}^2)_{\overline{\mathfrak{P}}}$. The morphism $N^{\mathfrak{P}}$ is generically one-to-one; moreover, $\prod_{l=1}^m \mathrm{Sym}^{k_l}(\mathbb{A}^2)$ is just the normalization $\mathrm{Sym}^d(\mathbb{A}^2)_{\widetilde{\mathfrak{P}}}$ of the stratum closure $\mathrm{Sym}^d(\mathbb{A}^2)_{\overline{\mathfrak{P}}}$. We have $\dim(\mathrm{Sym}^d(\mathbb{A}^2)_{\overline{\mathfrak{P}}}) = \dim(\mathrm{Sym}^d(\mathbb{A}^2)_{\widetilde{\mathfrak{P}}}) = 2\mathsf{K}(\mathfrak{P})$.

6.5. **Defect stratification.** We define the *defect stratification* of $\mathcal{M}^\alpha$. Recall the setup of 6.2. For a decomposition $\alpha = \gamma + \beta + d\delta_0$, and a partition $\Gamma \in \Gamma(\beta)$, and a partition $\mathfrak{P} \in \mathfrak{P}(d)$, the stratum $\mathcal{M}^\alpha_{\gamma,\Gamma,\mathfrak{P}}$ is formed by all $\mathcal{F}_\bullet \in \mathcal{M}^\alpha$ such that $\mathrm{def}^\dagger(\mathcal{F}_\bullet) \in \mathbb{A}^\beta_\Gamma$, and $\mathrm{def}^\circ(\mathcal{F}_\bullet) \in \mathrm{Sym}^d(\overset{\circ}{\mathbb{A}}{}^2)_{\mathfrak{P}}$. We have $\mathcal{M}^\alpha = \bigsqcup_{\substack{\Gamma \in \Gamma(\beta), \mathfrak{P} \in \mathfrak{P}(d) \\ \alpha = \gamma + \beta + d\delta_0}} \mathcal{M}^\alpha_{\gamma,\Gamma,\mathfrak{P}}$.

We have an evident projection $\mathcal{N} : \mathcal{M}^\alpha_{\gamma,\Gamma,\mathfrak{P}} \to \overset{\circ}{\mathcal{M}}{}^\gamma$. Also we have a morphism $\mathrm{def}^\ddagger : \mathcal{M}^\alpha_{\gamma,\Gamma,\mathfrak{P}} \to \mathrm{Sym}^{d+b_0}(\mathbb{A}^2)$.

We define the *defect stratification* of $\mathbf{M}^\alpha$ as follows. For a based quasimap $\mathbf{M}^\alpha \ni \phi = (\mathfrak{L}_{-\omega_i} \subset V^*_{\omega_i} \otimes \mathcal{O}_{\mathbf{C}})_{i \in I}$ we define its *saturation* $\mathcal{N}(\phi) \in \overset{\circ}{\mathbf{M}}{}^\gamma$ as the collection $(\widetilde{\mathfrak{L}}_{-\omega_i} \subset V^*_{\omega_i} \otimes \mathcal{O}_{\mathbf{C}})_{i \in I}$ where the line subbundle $\widetilde{\mathfrak{L}}_{-\omega_i}$ is the saturation of the invertible subsheaf $\mathfrak{L}_{-\omega_i}$. The quotient $\widetilde{\mathfrak{L}}_{-\omega_i}/\mathfrak{L}_{-\omega_i}$ is a torsion sheaf on $\mathbf{C} - \mathbf{c}$ of length b_i supported at a finite subset S_i, and we define the *defect* $\mathrm{def}(\phi) = \sum_{i \in I} \mathrm{def}_i(\phi) \in \mathbb{A}^\beta$ as $\sum_{i \in I} \sum_{s \in S_i} \mathrm{length}_s(\widetilde{\mathfrak{L}}_{-\omega_i}/\mathfrak{L}_{-\omega_i}) \cdot s$. Note that necessarily $\alpha = \gamma + \beta$.

Finally, for a decomposition $\alpha = \gamma + \beta$, and a partition $\Gamma \in \Gamma(\beta)$, the stratum $\mathbf{M}^\alpha_{\gamma,\Gamma}$ is formed by all $\phi \in \mathbf{M}^\alpha$ such that $\mathrm{def}(\phi) \in \mathbb{A}^\beta_\Gamma$. Note that $\mathbf{M}^\alpha_{\gamma,\Gamma} \simeq \overset{\circ}{\mathbf{M}}{}^\gamma \times \mathbb{A}^\beta_\Gamma$, and $\mathbf{M}^a = \bigsqcup_{\substack{\alpha=\gamma+\beta}}^{\Gamma \in \Gamma(\beta)} \mathbf{M}^\alpha_{\gamma,\Gamma}$ (cf. [6]).

Recall that $\varrho_{\mathbf{C}} : \mathcal{S}' = \mathbf{C} \times \mathbf{X} \to \mathbf{C}$ is the canonical projection from $\mathcal{S}'$ to $\mathbf{C}$, and $\varrho_{\mathbf{X}} : \mathcal{S}' = \mathbf{C} \times \mathbf{X} \to \mathbf{X}$ is the canonical projection from $\mathcal{S}'$ to $\mathbf{X}$. Recall also the setup of 3.7, 3.8.

Lemma 6.2. *For a parabolic sheaf $\mathcal{F}_\bullet \in \mathcal{M}^\alpha_{\gamma,\Gamma,\mathfrak{P}}$ the quasimap $\phi := \omega^\alpha(\mathcal{F}_\bullet) \in \mathbf{M}^\alpha$ has saturation $\mathcal{N}(\phi) = \omega^\gamma(\mathcal{N}(\mathcal{F})_\bullet)$, and defect $\mathrm{def}(\phi) = \mathrm{def}^\dagger(\mathcal{F}_\bullet) + \delta_0 \cdot \varrho_{\mathbf{C}}(\mathrm{def}^\circ(\mathcal{F}_\bullet))$. Furthermore, $\vartheta_\alpha(\mathcal{F}_\bullet) = \mathfrak{V}^{d+b_0}(\mathrm{def}^\ddagger(\mathcal{F}_\bullet)) + \vartheta_\gamma(\mathcal{N}(\mathcal{F})_\bullet)$.*

Proof. Clear from definitions and Corollary 3.6. $\qquad\square$

6.6. **Uhlenbeck flag stratification.** For a decomposition $\alpha = \gamma + \beta + d\delta_0$, and a partition $\Gamma \in \Gamma(\beta)$, and a partition $\mathfrak{P} \in \mathfrak{P}(d)$, we define a constructible subset $\mathfrak{M}^\alpha_{\gamma,\Gamma,\mathfrak{P}} \subset \mathfrak{M}^\alpha$ as $\varpi_\alpha(\mathcal{M}^\alpha_{\gamma,\Gamma,\mathfrak{P}})$. Lemma 6.2 implies that $\mathcal{M}^\alpha_{\gamma,\Gamma,\mathfrak{P}} = \varpi_\alpha^{-1}(\mathfrak{M}^\alpha_{\gamma,\Gamma,\mathfrak{P}})$. It follows that $\mathfrak{M}^\alpha_{\gamma,\Gamma,\mathfrak{P}}$ is a locally closed subscheme of $\mathfrak{M}^\alpha$, and $\varpi_\alpha : \mathcal{M}^\alpha_{\gamma,\Gamma,\mathfrak{P}} \to \mathfrak{M}^\alpha_{\gamma,\Gamma,\mathfrak{P}}$ is proper. Moreover, one can see easily that $\mathfrak{M}^\alpha_{\gamma,\Gamma,\mathfrak{P}}$ is smooth.

Thus we have the Uhlenbeck flag stratification $\mathfrak{M}^\alpha = \bigsqcup_{\substack{\alpha=\gamma+\beta+d\delta_0}}^{\Gamma \in \Gamma(\beta), \mathfrak{P} \in \mathfrak{P}(d)} \mathfrak{M}^\alpha_{\gamma,\Gamma,\mathfrak{P}}$.

Lemma 6.2 implies that $\mathfrak{M}^\alpha_{\gamma,\Gamma,\mathfrak{P}} \simeq \overset{\circ}{\mathcal{M}}{}^\gamma \times \mathbb{A}^\beta_\Gamma \times \mathrm{Sym}^d(\overset{\circ}{\mathbb{A}}{}^2)_{\mathfrak{P}}$. Let $\overline{\mathfrak{M}^\alpha_{\gamma,\Gamma,\mathfrak{P}}}$ denote the closure of the stratum $\mathfrak{M}^\alpha_{\gamma,\Gamma,\mathfrak{P}}$. Furthermore, if $\beta = 0$ (hence $\Gamma = \emptyset$), the above isomorphism extends to the finite morphism $\mathfrak{M}^\gamma \times \mathrm{Sym}^d(\mathbb{A}^2)_{\overline{\mathfrak{P}}} \to \overline{\mathfrak{M}^\alpha_{\gamma,\emptyset,\mathfrak{P}}}$. Composing it with the normalization morphism $N^{\mathfrak{P}}$ from 6.4 we obtain the finite morphism $N^{\gamma,\mathfrak{P}} : \mathfrak{M}^\gamma \times \mathrm{Sym}^d(\mathbb{A}^2)_{\widetilde{\mathfrak{P}}} \to \overline{\mathfrak{M}^\alpha_{\gamma,\emptyset,\mathfrak{P}}}$ which is generically one-to-one.

Theorem 6.3. *$\varpi_\alpha : \mathcal{M}^\alpha \to \mathfrak{M}^\alpha$ is semismall. The relevant strata in $\mathfrak{M}^\alpha$ are the ones with $\beta = 0$, $\Gamma = \emptyset$, i.e. $\mathfrak{M}^\alpha_{\gamma,\emptyset,\mathfrak{P}}$ for $\alpha = \gamma + d\delta_0$, $\mathfrak{P} \in \mathfrak{P}(d)$. The fibers of ϖ_α over the relevant strata are irreducible.*

Proof. Recall that $\dim(\mathfrak{M}^\alpha) = \dim(\mathcal{M}^\alpha) = 2|\alpha|$. It follows from the above discussion that $\dim(\mathfrak{M}^\alpha_{\gamma,\Gamma,\mathfrak{P}}) = 2|\gamma| + \mathsf{K}(\Gamma) + 2\mathsf{K}(\mathfrak{P})$. The fibers of $\varpi_\alpha : \mathcal{M}^\alpha_{\gamma,\Gamma,\mathfrak{P}} \to \mathfrak{M}^\alpha_{\gamma,\Gamma,\mathfrak{P}}$ were computed in [2], [9], [21]. Namely, consider a locally free parabolic sheaf $\mathcal{F}_\bullet \in \overset{\circ}{\mathcal{M}}{}^\gamma$, and $\beta' \in \mathbb{N}[I]$. Recall the projective variety $K_{\beta'}(\mathcal{F}_\bullet)$ introduced in [9], 3.1.3. For a point $c \in \mathbf{C} - \mathbf{c}$ let $\mathsf{Fib}_{\beta',c}(\mathcal{F}_\bullet) \subset$

$K_{\beta'}(\mathcal{F}_\bullet)$ be a closed subvariety given by the condition that the quotient sheaf $T_\bullet$ (see *loc. cit.*) is concentrated at $c \in \mathbf{C}$. For a point $s \in \overset{\circ}{\mathbb{A}}{}^2$ and $d' \in \mathbb{N}$ let $\mathsf{Fib}_{d',s}(\mathcal{F}_0)$ be the projective variety classifying all the torsion free subsheaves $\mathcal{F}' \subset \mathcal{F}_0$ with $\mathrm{def}(\mathcal{F}') = d' \cdot s$ (see the Appendix of [2]).

Let $\mathfrak{P} = \{\{k_1 \cdot d_1, \dots, k_m \cdot d_m\}\}$, and $\Gamma = \{\{n_1 \cdot \beta_1, \dots, n_g \cdot \beta_g\}\}$ for distinct $\beta_1, \dots, \beta_g$. Let $\underline{s} = (d_1 \cdot s_1^1, \dots, d_1 \cdot s_1^{k_1}, \dots, d_m \cdot s_m^1, \dots, d_m \cdot s_m^{k_m}) \in \mathrm{Sym}^d(\overset{\circ}{\mathbb{A}}{}^2)_\Gamma$, and $\underline{c} = ((\beta_1 \cdot c_1^1, \dots, \beta_1 \cdot c_1^{n_1}, \dots, \beta_g \cdot c_g^1, \dots, \beta_g \cdot c_g^{n_g}) \in \mathbb{A}_\Gamma^\beta$, and $\phi \in \overset{\circ}{\mathbf{M}}{}^\gamma$ correspond to $\mathcal{F}_\bullet$ in $\overset{\circ}{\mathcal{M}}{}^\gamma$. Then the reduced fiber $\varpi_\alpha^{-1}(\phi, \underline{c}, \underline{s})$ is isomorphic to

$$\prod_{l=1}^m (\mathsf{Fib}_{\beta_l, c_l^1}(\mathcal{F}_\bullet) \times \cdots \times \mathsf{Fib}_{\beta_l, c_l^{n_l}}(\mathcal{F}_\bullet)) \times \prod_{l=1}^g (\mathsf{Fib}_{d_l, s_l^1}(\mathcal{F}_0) \times \cdots \times \mathsf{Fib}_{d_l, s_l^{k_l}}(\mathcal{F}_0))$$

Now according to the Appendix of [2], the variety $\mathsf{Fib}_{d',s}(\mathcal{F}_0)$ is irreducible of dimension $nd' - 1$. And according to [9], Theorem 1, $\mathsf{Fib}_{\beta',c}(\mathcal{F}_\bullet)$ is a union of various irreducible components of dimension smaller than or equal to $|\beta'| - 1$.

Now a routine arithmetical check completes the proof of the Theorem. □

Corollary 6.4. $\displaystyle \varpi_{\alpha*}\underline{\mathbb{C}}[2|\alpha|] = \bigoplus_{\substack{\alpha = \gamma + d\delta_0}}^{\mathfrak{P} \in \mathfrak{P}(d)} \mathcal{IC}(\mathfrak{M}_{\gamma, \emptyset, \mathfrak{P}}^\alpha).$

□

6.7. **Symmetric algebras.** We compute the stalks of $\varpi_{\alpha*}\underline{\mathbb{C}}[2|\alpha|]$. To this end we have to know the cohomology of fibers of ϖ_α. A cellular decomposition of $\mathsf{Fib}_{\beta',c}(\mathcal{F}_\bullet)$ is constructed, and the dimensions of the cells are computed in [9], p. 165. The cohomology of $\mathsf{Fib}_{d',s}(\mathcal{F}_0)$ (equal to $H^\bullet(\mathcal{A}(n, d'))$) are computed in [21], Theorem 2.3. To arrange the cited information into a neat form we need some linear algebraic preliminaries.

Recall the *Hall algebra* $\mathbf{H}$ of the category of nilpotent representations of the cyclic quiver $\widetilde{\mathbf{A}}_{n-1}$, see e.g. [9], 1.3. It is naturally $\mathbb{N}[I]$-graded by the dimension of representation $\mathbf{H} = \oplus_{\beta \in \mathbb{N}[I]} \mathbf{H}_\beta$, see *loc. cit.* It has also a natural filtration $F^0\mathbf{H} \subset F^1\mathbf{H} \subset \dots$, namely, we say that a class $[M]$ of a nilpotent representation M lies in $F^k\mathbf{H}$ if M is a direct sum of k indecomposable representations. In particular, $F^1\mathbf{H}/F^0\mathbf{H} = \mathbb{C}[R_{\widehat{\mathfrak{sl}}_n}^+]$ (notations of 6.3). We denote by $\mathbf{H}^\bullet$ the associated graded algebra $\mathrm{gr}_F \mathbf{H}$. It is canonically isomorphic to $\mathrm{Sym}^\bullet(\mathbb{C}[R_{\widehat{\mathfrak{sl}}_n}^+])$. We have $\mathbf{H}^\bullet = \oplus_{\beta \in \mathbb{N}[I]} \mathbf{H}_\beta^\bullet$.

Let $\widehat{\mathfrak{n}}_+$ be a subalgebra of $\widehat{\mathfrak{sl}}_n$ generated by e_i, $i \in I$. Choosing a root basis[4] we identify it with $\mathbb{C}[R_{\widehat{\mathfrak{sl}}_n}^+]$.   Thus we have $\mathrm{Sym}^\bullet(\widehat{\mathfrak{n}}_+)$ =

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[4]To avoid a confusion between roots and coroots, we should have worked with the *Langlands dual* Lie algebra $\widehat{\mathfrak{sl}}_n^\vee$. We prefer to use a coincidence $\widehat{\mathfrak{sl}}_n \simeq \widehat{\mathfrak{sl}}_n^\vee$ to save notations at this moment. To clear up things, the interested reader is referred to 6.8.

$\mathrm{Sym}^\bullet(\mathbb{C}[R^+_{\widehat{\mathfrak{sl}}_n}]) \subset \mathrm{Sym}^\bullet(\mathbb{C}[R^+_{\widehat{\mathfrak{gl}}_n}]) = \mathbf{H}^\bullet$. Also, we have a natural grading $\mathrm{Sym}^\bullet(\widehat{\mathfrak{n}}_+) = \oplus_{\beta \in \mathbb{N}[I]}\mathrm{Sym}^\bullet(\widehat{\mathfrak{n}}_+)_\beta$.

We also define a bigraded space $\mathfrak{u}(\widehat{\mathfrak{gl}}_n) = \bigoplus\limits_{\substack{1 \le k \le n \\ d=1,2,\dots}} \mathfrak{u}(\widehat{\mathfrak{gl}}_n)^k_d$ where $\mathfrak{u}(\widehat{\mathfrak{gl}}_n)^k_d$ is 1-dimensional $\mathbb{C}$-vector space. We define $\mathfrak{u}(\widehat{\mathfrak{sl}}_n) = \bigoplus\limits_{\substack{2 \le k \le n \\ d=1,2,\dots}} \mathfrak{u}(\widehat{\mathfrak{gl}}_n)^k_d \subset \mathfrak{u}(\widehat{\mathfrak{gl}}_n)$. Thus, the symmetric algebras $\mathrm{Sym}(\mathfrak{u}(\widehat{\mathfrak{gl}}_n))$, $\mathrm{Sym}(\mathfrak{u}(\widehat{\mathfrak{sl}}_n))$ are also bigraded: $\mathrm{Sym}(\mathfrak{u}(\widehat{\mathfrak{gl}}_n)) = \bigoplus\limits_{\substack{k \in \mathbb{N} \\ d \in \mathbb{N}}} \mathrm{Sym}(\mathfrak{u}(\widehat{\mathfrak{gl}}_n))^k_d$, $\mathrm{Sym}(\mathfrak{u}(\widehat{\mathfrak{sl}}_n)) = \bigoplus\limits_{\substack{k \in \mathbb{N} \\ d \in \mathbb{N}}} \mathrm{Sym}(\mathfrak{u}(\widehat{\mathfrak{sl}}_n))^k_d$.

Recall the notations of the proof of Theorem 6.3.

Proposition 6.5. *The stalk of $\varpi_{\alpha*}\underline{\mathbb{C}}[2|\alpha|]$ at a point $(\phi,\underline{c},\underline{s}) \in \mathfrak{M}^\alpha_{\gamma,\Gamma,\mathfrak{P}}$ is isomorphic to*

$$\bigotimes_{l=1}^{m} \left(\oplus_{r \in \mathbb{N}}\mathbf{H}^r_{\beta_l}[2r]\right)^{\otimes n_l} \otimes \bigotimes_{l=1}^{g} \left(\oplus_{r \in \mathbb{N}}\mathrm{Sym}(\mathfrak{u}(\widehat{\mathfrak{gl}}_n))^r_{d_l}[2r]\right)^{\otimes k_l}[2|\gamma|]$$

Proof. Follows immediately from the proof of Theorem 6.3, and the above cited results of [21], [9]. $\qquad\square$

Theorem 6.6. *The stalk of $\mathcal{IC}(\mathfrak{M}^\alpha)$ at a point $(\phi,\underline{c},\underline{s}) \in \mathfrak{M}^\alpha_{\gamma,\Gamma,\mathfrak{P}}$ is isomorphic to*

$$\bigotimes_{l=1}^{m} (\oplus_{r \in \mathbb{N}}\mathrm{Sym}^r(\widehat{\mathfrak{n}}_+)_{\beta_l}[2r])^{\otimes n_l} \otimes \bigotimes_{l=1}^{g} \left(\oplus_{r \in \mathbb{N}}\mathrm{Sym}(\mathfrak{u}(\widehat{\mathfrak{sl}}_n))^r_{d_l}[2r]\right)^{\otimes k_l}[2|\gamma|]$$

Proof. Recall the finite, generically one-to-one morphism $N^{\gamma,\mathfrak{P}} : \mathfrak{M}^\gamma \times \mathrm{Sym}^d(\mathbb{A}^2)_{\widetilde{\mathfrak{P}}} \to \mathfrak{M}^\alpha_{\gamma,\emptyset,\mathfrak{P}}$ introduced in 6.6. It is well known that $\mathrm{Sym}^d(\mathbb{A}^2)_{\widetilde{\mathfrak{P}}}$ is rationally smooth; hence $\mathcal{IC}(\mathrm{Sym}^d(\mathbb{A}^2)_{\widetilde{\mathfrak{P}}}) = \underline{\mathbb{C}}[2\mathsf{K}(\mathfrak{P})]$, and $\mathcal{IC}(\mathfrak{M}^\alpha_{\gamma,\emptyset,\mathfrak{P}}) = N^{\gamma,\mathfrak{P}}_*(\mathcal{IC}(\mathfrak{M}^\gamma) \boxtimes \underline{\mathbb{C}}[2\mathsf{K}(\mathfrak{P})])$.

Now we use Corollary 6.4, Proposition 6.5, and induction in α, d (cf. the argument in [2], 3.7). $\qquad\square$

6.8. Uhlenbeck flag spaces for untwisted affine Lie algebras. Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra, and let $\widehat{\mathfrak{g}}$ be the corresponding untwisted affine Lie algebra with the coroot lattice $Y = \mathbb{Z}[I]$, and the dual lattice of weights X. Let $\widehat{\mathfrak{g}}^\vee$ be the *Langlands dual* affine Lie algebra, with the roles of X and Y interchanged (note that if $\mathfrak{g}$ is not simply laced, then $\widehat{\mathfrak{g}}^\vee$ is twisted). Let $\widehat{\mathfrak{n}}^\vee_+ \subset \widehat{\mathfrak{g}}^\vee$ be the standard maximal nilpotent subalgebra graded by $\mathbb{N}[I]$, and $\widehat{\mathfrak{n}}^\vee(\mathfrak{p}) \subset \widehat{\mathfrak{n}}^\vee_+$ the nilpotent radical of the standard maximal parabolic subalgebra

$\mathfrak{p} \subset \widehat{\mathfrak{g}}^\vee$. Let $\delta_0 \in \mathbb{N}[I]$ be the minimal imaginary root of $\widehat{\mathfrak{n}}_+^\vee$. Then $\widehat{\mathfrak{n}}^\vee(\mathfrak{p})$ is naturally graded by $\mathbb{N}[\delta_0]$: $\widehat{\mathfrak{n}}^\vee(\mathfrak{p}) = \oplus_{d \in \mathbb{N}} \widehat{\mathfrak{n}}^\vee(\mathfrak{p})_{d\delta_0}$. Also, $\widehat{\mathfrak{n}}^\vee(\mathfrak{p})$ carries a natural integrable action of the Langlands dual algebra $\mathfrak{g}^\vee$. Let $f \in \mathfrak{g}^\vee$ be a principal nilpotent element. Let $W_\bullet \widehat{\mathfrak{n}}^\vee(\mathfrak{p})$ be the monodromy filtration associated to the action of f. Then the invariants $(\widehat{\mathfrak{n}}^\vee(\mathfrak{p}))^f$ project injectively into $\mathrm{gr}_{W_\bullet} \widehat{\mathfrak{n}}^\vee(\mathfrak{p})$, and hence we obtain a grading on the space $(\widehat{\mathfrak{n}}^\vee(\mathfrak{p}))^f =: \mathfrak{u}(\widehat{\mathfrak{g}}^\vee) = \oplus_{k \in \mathbb{N}} \mathfrak{u}(\widehat{\mathfrak{g}}^\vee)^k$. Recall that we also have another grading on $\mathfrak{u}(\widehat{\mathfrak{g}}^\vee)$, so that it is actually bigraded: $\mathfrak{u}(\widehat{\mathfrak{g}}^\vee) = \bigoplus_{d \in \mathbb{N}}^{k \in \mathbb{N}} \mathfrak{u}(\widehat{\mathfrak{g}}^\vee)_{d\delta_0}^k$. Thus, the symmetric algebra $\mathrm{Sym}(\mathfrak{u}(\widehat{\mathfrak{g}}^\vee))$ is also bigraded: $\mathrm{Sym}(\mathfrak{u}(\widehat{\mathfrak{g}}^\vee)) = \bigoplus_{d \in \mathbb{N}}^{k \in \mathbb{N}} \mathrm{Sym}(\mathfrak{u}(\widehat{\mathfrak{g}}^\vee))_{d\delta_0}^k$.

The Kashiwara definition of the flag scheme $\mathcal{B}$, and the Drinfeld definition of the based quasimaps' scheme $\mathbf{M}^\alpha$ works for the affine Lie algebra $\widehat{\mathfrak{g}}$ as well. Repeating the constructions of 3.6, 4.7 we define the Uhlenbeck flag space $\mathfrak{M}^\alpha$. It is stratified as in 6.6. On the base of [5], 7.3, we propose the following generalization[5] of Theorem 6.6 to arbitrary $\widehat{\mathfrak{g}}$.

Conjecture 6.7.   *The stalk of* $\mathcal{IC}(\mathfrak{M}^\alpha)$ *at a point* $(\phi, \underline{c}, \underline{s}) \in \mathfrak{M}^\alpha_{\gamma, \Gamma, \mathfrak{P}}$ *is isomorphic to*

$$\bigotimes_{l=1}^m \left(\oplus_{r \in \mathbb{N}} \mathrm{Sym}^r(\widehat{\mathfrak{n}}_+^\vee)_{\beta_l}[2r] \right)^{\otimes n_l} \otimes \bigotimes_{l=1}^g \left(\oplus_{r \in \mathbb{N}} \mathrm{Sym}(\mathfrak{u}(\widehat{\mathfrak{g}}^\vee))_{d_l}^r[2r] \right)^{\otimes k_l} [2|\gamma|]$$

§7.   Hecke Correspondences

7.1. **Boundary.** We define an open subvariety $\mathcal{M}^\alpha \supset \overset{\bullet}{\mathcal{M}}{}^\alpha \supset \overset{\circ}{\mathcal{M}}{}^\alpha$ formed by the parabolic sheaves $\mathcal{F}_\bullet$ which are locally free parabolic sheaves in some Zariski open neighbourhood of $\mathbf{D}_0 \subset \mathcal{S}'$. The complementary closed subvariety $\mathcal{M}^\alpha - \overset{\bullet}{\mathcal{M}}{}^\alpha$ is denoted by $\overset{\bullet}{\mathcal{M}}{}_\bullet^\alpha$. For $\alpha, \gamma \in \mathbb{N}[I]$ we consider the *Hecke correspondence* $\mathfrak{E}_\alpha^\gamma \subset \mathcal{M}^\alpha \times \mathcal{M}^{\alpha+\gamma}$ formed by the pairs $(\mathcal{F}_\bullet, \mathcal{F}_\bullet')$ of parabolic sheaves such that $\mathcal{F}_\bullet' \subset \mathcal{F}_\bullet$. The first projection $\mathfrak{E}_\alpha^\gamma \to \mathcal{M}^\alpha$ will be denoted by $\mathbf{p}$, and the second projection $\mathfrak{E}_\alpha^\gamma \to \mathcal{M}^{\alpha+\gamma}$ will be denoted by $\mathbf{q}$. Note that $\mathbf{q}$ is proper while $\mathbf{p}$ is not.

Proposition 7.1. $\mathcal{M}^\alpha$ *is connected.*

Proof. Assume for a moment that $\overset{\bullet}{\mathcal{M}}{}^\alpha$ is connected for any α. Then we just have to prove that the boundary $\overset{\bullet}{\mathcal{M}}{}_\bullet^\alpha$ is connected. By induction in α we

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[5]This conjecture is proved in [6].

may assume that $\mathcal{M}^\beta$ is connected for any $\beta < \alpha$. Recall that $\alpha = \sum_{i \in I} a_i i$, and for $i \in I$ such that $a_i > 0$ we set $\alpha_i = \alpha - i$. Then, evidently, $\overset{\bullet}{\mathcal{M}}{}^\alpha = \bigcup_i \mathbf{q}(\mathfrak{E}^i_{\alpha_i})$, and any two nonempty pieces of the boundary intersect nontrivially: $\mathbf{q}(\mathfrak{E}^i_{\alpha_i}) \bigcap \mathbf{q}(\mathfrak{E}^j_{\alpha_j}) \neq \emptyset$. So we only have to prove that $\mathbf{q}(\mathfrak{E}^i_{\alpha_i})$ is connected. But the fibers of projection $\mathfrak{E}^i_{\alpha_i} \to \mathcal{M}^{\alpha_i}$ were computed in [9], 4.3.5. It follows in particular that these fibers are connected. Since $\mathcal{M}^{\alpha_i}$ is connected by induction assumption, $\mathfrak{E}^i_{\alpha_i}$ is connected itself, hence $\mathbf{q}(\mathfrak{E}^i_{\alpha_i})$ is also connected. Thus it remains to prove that $\overset{\bullet}{\mathcal{M}}{}^\alpha$ is connected. This is the subject of the following Lemma.

Lemma 7.2. $\overset{\bullet}{\mathcal{M}}{}^\alpha$ *is connected.*

Proof. We introduce a still bigger open subvariety $\mathcal{M}^\alpha \supset \check{\mathcal{M}}^\alpha \supset \overset{\bullet}{\mathcal{M}}{}^\alpha$ formed by the parabolic sheaves $\mathcal{F}_\bullet$ such that $\mathcal{F}_0$ is locally free in some Zariski open neighbourhood of $\mathbf{D}_0 \subset \mathcal{S}'$, and for $-n \leq k \leq 0$ the quotient sheaves $\mathcal{F}_k/\mathcal{F}_{-n}$ on $\mathbf{D}_0$ are locally free. For such a parabolic sheaf $\mathcal{F}_\bullet$ the quotients $\mathcal{F}_k/\mathcal{F}_{-n}$ are locally free subsheaves of the vector bundle $\mathcal{F}_0|_{\mathbf{D}_0}$ on $\mathbf{C}$, and the parabolic sheaf $\mathcal{F}_\bullet$ can be uniquely reconstructed from $\mathcal{F}_0$, and the flag of locally free subsheaves $\mathcal{F}_k/\mathcal{F}_{-n}$ of $\mathcal{F}_0|_{\mathbf{D}_0}$.

Thus we have a cartesian diagram

$$
\begin{array}{ccc}
\check{\mathcal{M}}^\alpha & \overset{\mathfrak{r}}{\longrightarrow} & \mathsf{Q}^{\overline{\alpha}} \\
{\scriptstyle r}\downarrow & & \downarrow{\scriptstyle \rho} \\
\check{\mathcal{A}}^{a_0} & \overset{\mathfrak{r}}{\longrightarrow} & \mathsf{Bun}
\end{array}
$$

Here $\check{\mathcal{A}}^{a_0} \subset \mathcal{A}^{a_0}$ is an open subvariety formed by torsion free sheaves which are locally free in a Zariski open neighbourhood of $\mathbf{D}_0 \subset \mathcal{S}'$. Furthermore, Bun is the moduli stack of SL_n-bundles on $\mathbf{C}$, and $\mathfrak{r}$ sends $\check{\mathcal{A}}^{a_0} \ni \mathcal{F}_0$ to $\mathcal{F}_0|_{\mathbf{D}_0}$. Also, r sends $\mathcal{F}_\bullet$ to $\mathcal{F}_0$. Finally, $\mathsf{Q}^{\overline{\alpha}} \overset{\rho}{\to} \mathsf{Bun}$ is Laumon's stack of quasiflags of degree $\overline{\alpha}$, see [16]. Here $\overline{\alpha}$ is an element of coroot lattice $\mathbb{Z}[I - 0] \subset \mathbb{Z}[I] = Y$ of $\mathfrak{sl}_n \subset \widehat{\mathfrak{sl}}_n$. It is given by the formula $\overline{\alpha} = \sum_{0 \neq i \in I}(a_i - a_0)i$.

Now $\check{\mathcal{A}}^{a_0}$ is connected being an open subvariety in $\mathcal{A}^{a_0}$ which is connected by Nakajima's quiver realization [20], 2.1, and cohomology computation [21], 2.3. Moreover, the fibers of ρ are connected since $\mathsf{Q}^{\overline{\alpha}}$ is connected, and Bun is normal. Hence $\check{\mathcal{M}}^\alpha$ is connected. This completes the proof of the Lemma along with Proposition 7.1. $\square$

Corollary 7.3. $\dim \mathcal{M}^{\alpha} = 2|\alpha|$.

Proof. We use the cartesian diagram in the proof of the above Lemma, together with the known formulas for the dimensions of $\mathsf{Bun}, \mathsf{Q}^{\overline{\alpha}}$ (see [16]) and of $\check{\mathcal{A}}^{a_0}$ (see [20]).   □

7.2. **Generically trivial parabolic sheaves.** Our next goal is to study the action of Hecke correspondences $\mathfrak{E}^i$ on the cohomology of $\mathcal{M}^{\alpha}$. Recall that the first projection $\mathbf{p}:\ \mathfrak{E}^i_{\alpha} \to \mathcal{M}^{\alpha}$ is not proper, and this causes a difficulty in the definition of the desired action. To get around this difficulty we will introduce another version of moduli spaces $\mathcal{M}^{\alpha}_{\mathbf{gt}}$ and Hecke correspondences between them which have proper projections. Recall that $\mathcal{M}^{\alpha}$ is the moduli space of parabolic sheaves which are trivialized at $\mathbf{C} \times \mathbf{x} \bigcup \mathbf{c} \times \mathbf{X}$. In the definition of $\mathcal{M}^{\alpha}_{\mathbf{gt}}$ we replace the condition of triviality at $\mathbf{c} \times \mathbf{X}$ by the condition of triviality of $\mathcal{F}_0$ at *some* line $c \times \mathbf{X}$, i.e. the condition of *generic triviality* of $\mathcal{F}_0$. To give a rigorous definition we need some preparations.

For any collection of points $c_1, \ldots, c_m$ of the curve $\mathbf{C}$ we consider the moduli scheme $\widetilde{\mathcal{M}}^{\alpha}(c_1, \ldots, c_m)$ of all infinite flags $\cdots \subset \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots$ of torsion free coherent sheaves of rank n on $\mathcal{S}'$ such that

(a) $\mathcal{F}_{k+n} = \mathcal{F}_k(\mathbf{D}_0)$ for any k;

(b) $ch_1(\mathcal{F}_k) = k[\mathbf{D}_0]$ for any k: the first Chern classes are proportional to the fundamental class of $\mathbf{D}_0$;

(c) $ch_2(\mathcal{F}_k) = a_i$ for $i \equiv k \pmod{n}$;

(d) For all $k \in \mathbb{Z}$ the sheaf $\mathcal{F}_k$ is locally free at the lines $c_1 \times \mathbf{X}, \ldots, c_m \times \mathbf{X} \subset \mathcal{S}'$;

(e) The sheaf $\mathcal{F}_0$ is trivialized at the line $\mathbf{C} \times \mathbf{x} \subset \mathcal{S}'$ and trivial at the lines $c_1 \times \mathbf{X}, \ldots, c_m \times \mathbf{X} \subset \mathcal{S}'$.

It is instructive to compare this definition with 3.4. The difference is the following. First, we replace one fixed line $\mathbf{c} \times \mathbf{X} \subset \mathcal{S}'$ with an m-tuple of lines $c_1 \times \mathbf{X}, \ldots, c_m \times \mathbf{X} \subset \mathcal{S}'$; and secondly, we drop the condition (e) of 3.4, imposing the behavior of the restrictions $\mathcal{F}_{k|\mathbf{c} \times \mathbf{X}}$ and replace it by a much weaker condition of locally freeness.

Proposition 7.4.   *For all $m > 0$ and all $c_1, \ldots, c_m \in C$ the moduli scheme $\widetilde{\mathcal{M}}^{\alpha}(c_1, \ldots, c_m)$ exists. It is smooth connected scheme of dimension $\dim \mathsf{B} + |2\alpha|$. For any permutation σ of the set $\{1, \ldots, m\}$ the schemes $\widetilde{\mathcal{M}}^{\alpha}(c_1, \ldots, c_m)$ and $\widetilde{\mathcal{M}}^{\alpha}(c_{\sigma(1)}, \ldots, c_{\sigma(m)})$ are canonically isomorphic.*

Proof. The first part follows from [12] and [24] as in 3.5, and the third part is evident. Thus it remains to check the smoothness and the connectedness and to compute the dimension.

First, consider the moduli scheme $\widetilde{\mathcal{M}}^\alpha(\mathbf{c})$. Consider the evaluation map

$$\mathsf{ev}_{(\mathbf{c},\mathbf{y})} : \widetilde{\mathcal{M}}^\alpha(\mathbf{c}) \to \mathsf{B}, \quad (\mathcal{F}_k) \mapsto \big(\mathsf{Im}(\mathcal{F}_{k|(\mathbf{c},\mathbf{y})} \to \mathcal{F}_{0|(\mathbf{c},\mathbf{y})}) \big)_{-n \leq k \leq 0}.$$

The locally freeness condition 7.2 (d) implies that $\mathcal{F}_{k|(\mathbf{c},\mathbf{y})}$ is an n-dimensional vector space, while the conditions (a) and (b) imply that the map $\mathcal{F}_{k|(\mathbf{c},\mathbf{y})} \to \mathcal{F}_{0|(\mathbf{c},\mathbf{y})}$ has rank $k + n$. Hence $\mathsf{ev}_{(\mathbf{c},\mathbf{y})}(\mathcal{F}_k)$ is a flag in the vector space $\mathcal{F}_{0(\mathbf{c},\mathbf{y})}$ which is canonically isomorphic to V, thus a point of the flag variety B. Now note that the map $\mathsf{ev}_{(\mathbf{c},\mathbf{y})}$ is evidently a locally trivial fibration, and the fiber of $\mathsf{ev}_{(\mathbf{c},\mathbf{y})}$ is canonically isomorphic to the variety $\mathcal{M}^\alpha$. Hence Lemma 3.3 and Remark 3.4 imply that $\widetilde{\mathcal{M}}^\alpha(\mathbf{c})$ is a smooth connected variety of dimension $\dim \mathsf{B} + 2|\alpha|$.

Further, for any point $c_1 \in \mathbf{C}$ choose an automorphism g of the curve $\mathbf{C}$ such that $g(c_1) = \mathbf{c}$. Then g identifies the moduli schemes $\widetilde{\mathcal{M}}^\alpha(c_1)$ and $\widetilde{\mathcal{M}}^\alpha(\mathbf{c})$, hence $\widetilde{\mathcal{M}}^\alpha(c_1)$ is a smooth connected variety of dimension $\dim \mathsf{B} + 2|\alpha|$ for all $c_1 \in \mathbf{C}$.

Finally, it is clear that $\widetilde{\mathcal{M}}^\alpha(c_1, \ldots, c_m)$ is an open subscheme of $\widetilde{\mathcal{M}}^\alpha(c_1)$, hence $\widetilde{\mathcal{M}}^\alpha(c_1, \ldots, c_m)$ is a smooth connected variety of dimension $\dim \mathsf{B} + 2|\alpha|$ for all $c_1, \ldots, c_m \in \mathbf{C}$. $\square$

Definition 7.5. Let $\mathcal{M}^\alpha_{\mathbf{gt}}$ denote the gluing of schemes $\widetilde{\mathcal{M}}^\alpha(c_1)$ for all $c_1 \in \mathbf{C}$ with respect to the open subsets $\widetilde{\mathcal{M}}^\alpha(c_1) \supset \widetilde{\mathcal{M}}^\alpha(c_1, c_2) \subset \widetilde{\mathcal{M}}^\alpha(c_2)$.

Theorem 7.6. *The scheme $\mathcal{M}^\alpha_{\mathbf{gt}}$ is a smooth connected variety of dimension $\dim \mathsf{B} + 2|\alpha|$. The moduli schemes $\widetilde{\mathcal{M}}^\alpha(c_1)$ form an open covering of $\mathcal{M}^\alpha_{\mathbf{gt}}$ and $\widetilde{\mathcal{M}}^\alpha(c_1) \cap \widetilde{\mathcal{M}}^\alpha(c_2) = \widetilde{\mathcal{M}}^\alpha(c_1, c_2)$.*

Proof. The only thing we need to check is that $\mathcal{M}^\alpha_{\mathbf{gt}}$ is a scheme of finite type. Then all the rest follows from Proposition 7.4. On the other hand, if we want to check that $\mathcal{M}^\alpha_{\mathbf{gt}}$ is of finite type, it suffices to show that there exists an integer s such that for any collection of distinct points $c_1, c_2^1, \ldots, c_2^s \in \mathbf{C}$ we have

$$\widetilde{\mathcal{M}}^\alpha(c_1) = \bigcup_{p=1}^{s} \widetilde{\mathcal{M}}^\alpha(c_1, c_2^p).$$

Then it would follow that $\mathcal{M}^\alpha_{\mathbf{gt}}$ is in fact a gluing of $(s + 1)$ moduli schemes $\widetilde{\mathcal{M}}^\alpha(c_1^i)$ for an arbitrary collection of $(s + 1)$ distinct points $c_1^1, \ldots, c_1^{s+1} \in \mathbf{C}$.

Let us show that $s = |\alpha| + 1$ is big enough. Since the group of automorphisms of $\mathbf{C}$ acts transitively, it suffices to consider only the case $c_1 = \mathbf{c}$. So let $\mathcal{F}_k$ be a point of $\widetilde{\mathcal{M}}^\alpha(\mathbf{c})$. Let $\mathcal{N}(\mathcal{F})_k$ be its saturation and denote $\beta = \sum b_i i$, where $b_k = ch_2(\mathcal{N}(\mathcal{F})_k)$. Then $\mathcal{N}(\mathcal{F})_0 \in \mathcal{A}^{a_0 - b_0}$ and for any $k \in \mathbb{Z}$ the sheaf $\mathcal{N}(\mathcal{F})_k/\mathcal{F}_k$ is a sheaf on $\mathcal{S}'$ of length b_k. Recall that according to Lemma 3.5 the subset $D \subset (\mathbf{C} - \mathbf{c})$ formed by the points $x \in \mathbf{C}$ such that the restriction $\mathcal{N}(\mathcal{F})_0 | x \times \mathbf{X}$ is nontrivial, has cardinality at most $a_0 - b_0$. Therefore

$$D \cup \left(\bigcup_{k=0}^{n-1} \varrho_{\mathbf{C}}(\mathrm{supp}(\mathcal{N}(\mathcal{F})_k/\mathcal{F}_k)) \right)$$

is a subset in $\mathbf{C} - \mathbf{c}$ of cardinality not greater than

$$(a_0 - b_0) + \sum_{k=0}^{n-1} b_k = (a_0 - b_0) + |\beta|.$$

Now if $s > (a_0 - b_0) + |\beta|$ and $c_2^1, \ldots, c_2^s$ are distinct points of $\mathbf{C} - \mathbf{c}$ then there exists $1 \le p \le s$ such that for all $k \in \mathbb{Z}$ the sheaf $\mathcal{F}_k$ is locally free and the sheaf $\mathcal{F}_0$ is trivial at the line $c_2^p \times \mathbf{X}$. Thus $\mathcal{F}_\bullet$ lies in $\widetilde{\mathcal{M}}^\alpha(\mathbf{c}, c_2^p)$. Thus it remains to check that the integer $(a_0 - b_0) + |\beta|$ is uniformly bounded for all $\mathcal{F}_\bullet$. But this is evident, because we always have $\beta \le \alpha$, hence $(a_0 - b_0) + |\beta| \le |\alpha|$. Thus $s = |\alpha| + 1$ is indeed big enough. $\square$

7.3. Correspondences.
For any $\alpha, \gamma \in \mathbb{N}[I]$ we define the *Hecke correspondence* $\mathfrak{E}_\alpha^\gamma \subset \mathcal{M}_{\mathbf{gt}}^\alpha \times \mathcal{M}_{\mathbf{gt}}^{\alpha+\gamma}$ as a closed subvariety formed by all the pairs $(\mathcal{F}_\bullet, \mathcal{F}_\bullet') \in \mathcal{M}_{\mathbf{gt}}^\alpha \times \mathcal{M}_{\mathbf{gt}}^{\alpha+\gamma}$ such that $\mathcal{F}_\bullet' \subset \mathcal{F}_\bullet$, and the quotient is supported at $\mathbf{D}_0 \subset \mathcal{S}'$.

We have canonical projections

$$\mathbf{p} : \mathfrak{E}_\alpha^\gamma \to \mathcal{M}_{\mathbf{gt}}^\alpha, \quad \mathbf{q} : \mathfrak{E}_\alpha^\gamma \to \mathcal{M}_{\mathbf{gt}}^{\alpha+\gamma}, \quad \text{and} \quad \mathbf{r} : \mathfrak{E}_\alpha^\gamma \to \mathbf{C}^\gamma.$$

Here $\mathbf{p}$ and $\mathbf{q}$ are induced by projections of the product $\mathcal{M}_{\mathbf{gt}}^\alpha \times \mathcal{M}_{\mathbf{gt}}^{\alpha+\gamma}$ to the factors, and $\mathbf{r}$ is defined as follows:

$$\mathbf{r}(\mathcal{F}_\bullet, \mathcal{F}_\bullet') = \mathrm{supp}(\mathcal{F}_\bullet, \mathcal{F}_\bullet') = \sum_i \mathrm{supp}(\mathcal{F}_i/\mathcal{F}_i')i \in \mathbf{D}_0^\gamma = \mathbf{C}^\gamma.$$

Lemma 7.7. *The maps* $\mathbf{p}$ *and* $\mathbf{q}$ *are proper.*

Proof. Evident. $\square$

7.4. Top-dimensional components.
We begin with some notation. Recall that for any $\mathcal{F}_\bullet \subset \mathcal{F}_\bullet'$ the quotient $T_\bullet = \mathcal{F}_\bullet/\mathcal{F}_\bullet'$ can be considered as a representation of the infinite linear quiver $\mathbf{A}_\infty$ in the category of torsion sheaves

supported on $\mathbf{D}_0$ (see e.g. [9]). It is clear that $T_\bullet$ is a nilpotent representation. On the other hand, the periodicity of $\mathcal{F}_\bullet$ and $\mathcal{F}'_\bullet$ imply the periodicity of $T_\bullet$, namely a canonical isomorphism (the triviality of the normal bundle $\mathcal{N}_{\mathbf{D}_0/\mathcal{S}'}$ is used here)

$$T_{k+n} \cong T_k.$$

Thus $T_\bullet$ can (and will) be considered as a nilpotent representation of the cyclic quiver $\widetilde{\mathbf{A}}_{n-1}$. Following [9] we denote by $\mathbf{NR}_n(\mathbf{D}_0)$ the category of nilpotent representations of the cyclic quiver $\widetilde{\mathbf{A}}_{n-1}$ in the category of torsion sheaves supported on $\mathbf{D}_0$. For any object $T_\bullet \in \mathbf{NR}_n(\mathbf{D}_0)$ and a point $x \in \mathbf{D}_0$ we denote by $\Gamma_x(T_\bullet)$ the representation of the cyclic quiver $\widetilde{\mathbf{A}}_{n-1}$ in the category of vector spaces, formed by sections of $T_\bullet$ with support at the point $x \in \mathbf{D}_0$. Recall that the isomorphism classes of nilpotent representations of the cyclic quiver $\widetilde{\mathbf{A}}_{n-1}$ in the category of vector spaces are numbered by Kostant partitions of $\widehat{\mathfrak{gl}}_n$. We denote by $\kappa_x(T_\bullet) \in \mathfrak{K}_{\widehat{\mathfrak{gl}}_n}$ the isomorphism class of $\Gamma_x(T_\bullet)$.

Now we are going to use the results of [9] to describe the top-dimensional components of $\mathfrak{E}_\alpha^\gamma$. Choose a Kostant partition $\kappa = \{\{\theta_1, \ldots, \theta_m\}\} \in \mathfrak{K}_{\widehat{\mathfrak{gl}}_n}(\gamma)$, where $\theta_1, \ldots, \theta_m \in R_{\widehat{\mathfrak{gl}}_n}^+$. Consider a subset $\overset{\circ}{\mathfrak{E}}_\alpha^\kappa \subset \mathfrak{E}_\alpha^\gamma$ consisting of all pairs $(\mathcal{F}_\bullet, \mathcal{F}'_\bullet)$ such that

(1) $\mathbf{r}(\mathcal{F}_\bullet, \mathcal{F}'_\bullet) \in \mathbf{C}_\kappa^\gamma$, that is $\mathrm{supp}(\mathcal{F}_\bullet, \mathcal{F}'_\bullet) = \sum_{r=1}^m |\theta_r| x_r$ with all x_r distinct;

(2) $\mathcal{F}_\bullet$ is locally free at the points $x_1, \ldots, x_m$;

(3) $\kappa_{x_r}(\mathcal{F}_\bullet/\mathcal{F}'_\bullet) = \{\{\theta_r\}\}$ for all $1 \leq r \leq m$.

We define $\mathfrak{E}_\alpha^\kappa$ as the closure of $\overset{\circ}{\mathfrak{E}}_\alpha^\kappa$.

Proposition 7.8. *Dimension of any irreducible component of $\mathfrak{E}_\alpha^\gamma$ is not greater than $\dim \mathsf{B} + 2|\alpha| + |\gamma|$. Any component of this dimension coincides with $\mathfrak{E}_\alpha^\kappa$ for some $\kappa \in \mathfrak{K}_{\widehat{\mathfrak{gl}}_n}(\gamma)$.*

Proof. Consider a stratification of $\mathcal{M}_{\mathbf{gt}}^\alpha \times \mathbf{C}^\gamma$ via the defect of $\mathcal{F}_\bullet$ at the support of $\sum \gamma_r x_r \in \mathbf{C}^\gamma$, namely

$$\mathcal{M}_{\mathbf{gt}}^\alpha \times \mathbf{C}^\gamma = \bigsqcup_{\substack{\Gamma \in \Gamma(\gamma) \\ |\kappa'_1| + \cdots + |\kappa'_m| = \gamma' \leq \alpha}} \mathsf{Z}_\alpha^\Gamma(\kappa'_1, \ldots, \kappa'_m).$$

Here $\mathsf{Z}_\alpha^\Gamma(\kappa'_1, \ldots, \kappa'_m) \subset \mathcal{M}_{\mathbf{gt}}^\alpha \times \mathbf{C}^\gamma$ is the subspace of all pairs $(\mathcal{F}_\bullet, \sum \gamma_r x_r)$ such that

(1) $\{\{\gamma_1, \ldots, \gamma_m\}\} = \Gamma$;

(2) $\kappa_{x_r}(\mathcal{N}(\mathcal{F})_\bullet/\mathcal{F}_\bullet) = \kappa'_r$ for all $1 \le r \le m$;

Consider the partial saturation map

$$\mathsf{Z}^\Gamma_\alpha(\kappa'_1, \ldots, \kappa'_m) \to \mathcal{M}^{\alpha-\gamma'}_{\mathbf{gt}} \times \mathbf{C}^\gamma_\Gamma, \quad \left(\mathcal{F}_\bullet, \sum \gamma_r x_r \right)$$
$$\mapsto \left(\mathcal{N}_{x_1, \ldots, x_m}(\mathcal{F})_\bullet, \sum \gamma_r x_r \right).$$

The fiber of this saturation map was described in [9]. It was denoted there by K_μ, where $\mu = \{\{\kappa'_1, \ldots, \kappa'_m\}\}$ — the corresponding multipartition. It was shown in Lemma 3.1.4 and Theorem 1 of *loc. cit.* that

$$\dim K_\mu = \sum (||\kappa'_r|| - \mathsf{K}(\kappa'_r)).$$

This implies

$$\begin{aligned}
\dim \mathsf{Z}^\Gamma_\alpha(\kappa'_1, \ldots, \kappa'_m) &= \dim \mathsf{B} + 2|\alpha - \gamma'| + m + \sum(||\kappa'_r|| - \mathsf{K}(\kappa'_r)) \\
&= \dim \mathsf{B} + 2|\alpha| - |\gamma'| - \sum \mathsf{K}(\kappa'_r) + m \\
&= \dim \mathsf{B} + 2|\alpha| + |\gamma| + \sum(1 - |\gamma_r + \gamma'_r| - \mathsf{K}(\kappa'_r)),
\end{aligned}$$

where $\gamma'_r = |\kappa'_r|$.

Now consider a stratification of $\mathfrak{E}^\gamma_\alpha$ via the defect of the sheaves $\mathcal{F}_\bullet$ and $\mathcal{F}'_\bullet$ at the support of $\mathcal{F}'_\bullet/\mathcal{F}_\bullet$, namely

$$\mathfrak{E}^\gamma_\alpha = \bigsqcup_{\substack{\Gamma \in \Gamma(\gamma) \\ |\kappa'_1| + \cdots + |\kappa'_m| = \gamma' \le \alpha \\ |\tilde{\kappa}_r| = |\kappa'_r| + \gamma_r}} \mathsf{Z}^\Gamma_\alpha(\kappa'_1, \ldots, \kappa'_m; \tilde{\kappa}_1, \ldots, \tilde{\kappa}_m).$$

Here $\mathsf{Z}^\Gamma_\alpha(\kappa'_1, \ldots, \kappa'_m; \tilde{\kappa}_1, \ldots, \tilde{\kappa}_m) \subset \mathfrak{E}^\gamma_\alpha$ is the subspace of all pairs $(\mathcal{F}_\bullet, \mathcal{F}'_\bullet)$ such that

(1) $\mathbf{r}(\mathcal{F}_\bullet, \mathcal{F}'_\bullet) = \sum \gamma_r x_r \in \mathbf{C}^\gamma_\Gamma$;

(2) $\kappa_{x_r}(\mathcal{N}(\mathcal{F})_\bullet/\mathcal{F}_\bullet) = \kappa'_r$ for all $1 \le r \le m$;

(3) $\kappa_{x_r}(\mathcal{N}(\mathcal{F}')_\bullet/\mathcal{F}'_\bullet) = \tilde{\kappa}_r$ for all $1 \le r \le m$;

Note that $\mathcal{F}'_\bullet \subset \mathcal{F}_\bullet$ implies $\mathcal{N}(\mathcal{F}') = \mathcal{N}(\mathcal{F})$, hence we indeed have $|\tilde{\kappa}_r| = |\kappa'_r| + \gamma_r$.

Now consider the map $\mathbf{p} \times \mathbf{r}$ restricted to the stratum $\mathsf{Z}^\Gamma_\alpha(\kappa'_1, \ldots, \kappa'_m; \tilde{\kappa}_1, \ldots, \tilde{\kappa}_m)$. It is clear that

$$(\mathbf{p} \times \mathbf{r})(\mathsf{Z}^\Gamma_\alpha(\kappa'_1, \ldots, \kappa'_m; \tilde{\kappa}_1, \ldots, \tilde{\kappa}_m)) = \mathsf{Z}^\Gamma_\alpha(\kappa'_1, \ldots, \kappa'_m).$$

Furthermore, it is easy to see that the fiber of this map over a point $(\mathcal{F}_\bullet, \sum \gamma_r x_r)$ consists of all $\mathcal{F}'_\bullet$, such that

(1) $\kappa_{x_r}(\mathcal{N}(\mathcal{F})_\bullet / \mathcal{F}'_\bullet) = \tilde{\kappa}_r$ for all $1 \leq r \leq m$;

(2) $\mathcal{F}'_\bullet \subset \mathcal{F}$.

It follows that this fiber can be embedded into the variety $K_{\tilde{\mu}}$, where $\tilde{\mu} = \{\{\tilde{\kappa}_1, \ldots, \tilde{\kappa}_m\}\}$ as a closed subvariety (the closed condition is the condition 2 above). In particular, the dimension of the fiber is not greater than

$$\sum_{r=1}^m (||\tilde{\kappa}_r|| - \mathsf{K}(\tilde{\kappa}_r)) = \sum_{r=1}^m (|\gamma'_r + \gamma_r| - \mathsf{K}(\tilde{\kappa}_r)).$$

Comparing this with the formula for the dimension of the stratum $\mathsf{Z}_\alpha^\Gamma(\kappa'_1, \ldots, \kappa'_m)$ we see that

$$\dim \mathsf{Z}_\alpha^\Gamma(\kappa'_1, \ldots, \kappa'_m; \tilde{\kappa}_1, \ldots, \tilde{\kappa}_m)$$

$$\leq \dim \mathsf{B} + 2|\alpha| + |\gamma| + \sum (1 - |\gamma_r + \gamma'_r| - \mathsf{K}(\kappa'_r)) + \sum_{r=1}^m (|\gamma'_r + \gamma_r| - \mathsf{K}(\tilde{\kappa}_r))$$

$$= \dim \mathsf{B} + 2|\alpha| + |\gamma| + \sum_{r=1}^m (1 - \mathsf{K}(\kappa'_r) - \mathsf{K}(\tilde{\kappa}_r)).$$

Note that since $\gamma_r > 0$ we have $\mathsf{K}(\tilde{\kappa}_r) \geq 1$, hence the last term is always nonpositive. Therefore,

$$\dim \mathsf{Z}_\alpha^\Gamma(\kappa'_1, \ldots, \kappa'_m; \tilde{\kappa}_1, \ldots, \tilde{\kappa}_m) \leq \dim \mathsf{B} + 2|\alpha| + |\gamma|$$

and the equality is possible only when $\mathsf{K}(\kappa'_r) = 0$, $\mathsf{K}(\tilde{\kappa}_r) = 1$ for all $1 \leq r \leq m$. This means that $\mathcal{F}_\bullet$ is locally free at the points $x_1, \ldots, x_r$ and that $\tilde{\kappa}_r = \{\{\theta_r\}\}$ for some $\theta_r \in R^+_{\mathfrak{sl}_n}$. Moreover, it is easy to see that in the latter case the condition (2) above is void, hence

$$\dim \mathsf{Z}_\alpha^\Gamma(0, \ldots, 0; \{\{\theta_1\}\}, \ldots, \{\{\theta_1\}\}) = \dim \mathsf{B} + 2|\alpha| + |\gamma|.$$

Finally, it remains to note that $\overset{\circ}{\mathfrak{E}}_\alpha^{\{\{\theta_1, \ldots, \theta_m\}\}} = \mathsf{Z}_\alpha^\Gamma(0, \ldots, 0; \{\{\theta_1\}\}, \ldots, \{\{\theta_1\}\})$. $\qquad \square$

7.5. In Proposition 7.8 we gave an explicit description of open parts of the top-dimensional irreducible components of $\mathfrak{E}_\alpha^\gamma$. Below we will need for technical reasons also an explicit description of some closed subset $\widehat{\mathfrak{E}}_\alpha^\kappa \subset \mathfrak{E}_\alpha^\gamma$ such that $\mathfrak{E}_\alpha^\kappa \subset \widehat{\mathfrak{E}}_\alpha^{\kappa'}$ iff $\kappa = \kappa'$.

Now we will define such closed subsets. We begin with some notation. For $T_\bullet \in \mathbf{NR}_n(\mathbf{D}_0)$ we denote

$$\Gamma(T_\bullet) = \Gamma(\mathcal{S}, T_\bullet) = \oplus_{x \in \mathbf{D}_0} \Gamma_x(T_\bullet), \quad \text{and} \quad \kappa(T_\bullet) = \sum_{x \in \mathbf{D}_0} \kappa_x(T_\bullet).$$

Thus $\kappa(T_\bullet)$ is the isomorphism class of $\Gamma(T_\bullet)$. Now, for every $\gamma \in \mathbb{Z}[I]$ let $\mathbf{V}(\gamma)$ denote the representation space of all γ-dimensional nilpotent representations of the cyclic quiver $\widetilde{\mathbf{A}}_{n-1}$ and let $\mathrm{GL}(\gamma)$ denote the group acting on $\mathbf{V}(\gamma)$ by change of bases. Then the $\mathrm{GL}(\gamma)$-orbits on $\mathbf{V}(\gamma)$ are nothing but the isomorphism classes of nilpotent representations of $\widetilde{\mathbf{A}}_{n-1}$. Let $\mathbb{O}_\kappa$ denote the orbit corresponding to a Kostant partition κ. Note that we have a canonical partial order on the set of orbits. It induces a partial order on the set of Kostant partition: for $\kappa, \kappa' \in \mathfrak{K}_{\widehat{\mathfrak{gl}}_n}(\gamma)$ we have

$$\kappa \leq \kappa' \quad \text{iff} \quad \mathbb{O}_\kappa \subset \overline{\mathbb{O}_{\kappa'}} \subset \mathbf{V}(\gamma).$$

Now take an arbitrary Kostant partition $\kappa \in \mathfrak{K}_{\widehat{\mathfrak{gl}}_n}(\gamma)$ and consider the subset $\widehat{\mathfrak{E}}_\alpha^\kappa \subset \mathfrak{E}_\alpha^\gamma$ defined as

$$\widehat{\mathfrak{E}}_\alpha^\kappa = \{(\mathcal{F}_\bullet \supset \mathcal{F}_\bullet') \mid \kappa(\mathcal{F}_\bullet/\mathcal{F}_\bullet') \leq \kappa \text{ and } \mathbf{r}(\mathcal{F}_\bullet, \mathcal{F}_\bullet') \in \overline{\mathbf{C}_\kappa^\gamma}\}$$

Lemma 7.9.   *The subset $\widehat{\mathfrak{E}}_\alpha^\kappa \subset \mathfrak{E}_\alpha^\gamma$ is closed and $\overset{\circ}{\mathfrak{E}}_\alpha^\kappa \subset \widehat{\mathfrak{E}}_\alpha^{\kappa'}$ if and only if $\kappa = \kappa'$.*

Proof. It is clear that both conditions defining $\widehat{\mathfrak{E}}_\alpha^\kappa \subset \mathfrak{E}_\alpha^\gamma$ are closed, hence $\widehat{\mathfrak{E}}_\alpha^\kappa$ is closed. Now assume that $\overset{\circ}{\mathfrak{E}}_\alpha^\kappa \subset \widehat{\mathfrak{E}}_\alpha^{\kappa'}$. It follows than that

$$\kappa \leq \kappa' \qquad \text{and} \qquad \mathbf{C}_\kappa^\gamma \subset \overline{\mathbf{C}_{\kappa'}^\gamma}.$$

Let us check that this is possible only if $\kappa = \kappa'$. It is clear that the partial order given by adjacency of strata $\mathbf{C}_\kappa^\gamma$ can be described as follows. Assume that $\theta_1, \ldots, \theta_r, \theta_1', \ldots, \theta_m' \in R_{\widehat{\mathfrak{gl}}_n}^+$, $m = s_1 + \cdots + s_r$, and for all $1 \leq p \leq r$ we have

$$|\theta_p| = \sum_{q=1}^{s_p} |\theta_{s_1 + \cdots + s_{p-1}+q}'|.$$

Then we have

$$\mathbf{C}_{\{\{\theta_1, \ldots, \theta_r\}\}}^\gamma \subset \overline{\mathbf{C}_{\{\{\theta_1', \ldots, \theta_m'\}\}}^\gamma}$$

and all adjacencies have such form. In particular, if $\overline{\mathbf{C}_\kappa^\gamma} \subset \overline{\mathbf{C}_{\kappa'}^\gamma}$ is a proper inclusion, then the number of entries $\mathsf{K}(\kappa)$ is strictly smaller than $\mathsf{K}(\kappa')$.

On the other hand, Ringel's explicit description of the order $\kappa \leq \kappa'$ in [22], 4.7 implies that we must have $\mathsf{K}(\kappa) \geq \mathsf{K}(\kappa')$ whenever $\kappa \leq \kappa'$. The Lemma follows. $\qquad \square$

7.6. **Action of the Hall algebra.** Let $\widetilde{\mathbf{H}}_n$ denote the generic Hall algebra of the category of nilpotent representation of the cyclic quiver $\widetilde{\mathbf{A}}_{n-1}$. Recall that the generic algebra $\widetilde{\mathbf{H}}_n$ is an algebra over $\mathbb{Z}[q]$ (polynomials in a formal variable q) with a basis S_κ indexed by isomorphism classes of representations, that is by Kostant partitions, and with the following multiplication rule

$$S_{\kappa'} \cdot S_{\kappa''} = \sum_\kappa c^\kappa_{\kappa',\kappa''}(q) S_\kappa,$$

where the structure constants $c^\kappa_{\kappa',\kappa''}(q)$ are defined as follows. Take $\mathbb{F}_q$ for a base field and choose a representation $W_\bullet$ over $\mathbb{F}_q$ in the isomorphism class κ. Then $c^\kappa_{\kappa',\kappa''}(q)$ is defined as the number of subrepresentations $W'_\bullet \subset W_\bullet$, such that the isomorphism class of $W'_\bullet$ equals κ' and the isomorphism class of $W_\bullet/W'_\bullet$ equals κ''. It turns out that $c^\kappa_{\kappa',\kappa''}(q)$ is a polynomial function of q, thus we can consider $\widetilde{\mathbf{H}}_n$ as an algebra over $\mathbb{Z}[q]$.

From now on we consider a specialization of the Hall algebra $\widetilde{\mathbf{H}}_n$ at $q = 1$ and denote it by $\mathbf{H}$. As before, $\mathbf{H}$ is a $\mathbb{Q}$-algebra having S_κ for a basis and $c^\kappa_{\kappa',\kappa''}(1)$ for structure constants.

Consider a graded vector spaces

(7.10) $$\mathfrak{H} = \bigoplus_{\alpha \in \mathbb{Z}[I]} H^{\bullet - |\alpha|}(\mathcal{M}^\alpha_{\mathbf{gt}}, \mathbb{Q})$$

(note the shift of the cohomological grading).

For each Kostant partition $\kappa \in \mathfrak{K}_{\widehat{\mathfrak{sl}}_n}(\gamma)$ we consider an operator on cohomology given by a correspondence $\mathfrak{E}^\kappa_\alpha \subset \mathcal{M}^\alpha_{\mathbf{gt}} \times \mathcal{M}^{\alpha+\gamma}_{\mathbf{gt}}$:

$$e_\kappa = [\mathfrak{E}^\kappa_\alpha] : H^\bullet(\mathcal{M}^\alpha_{\mathbf{gt}}, \mathbb{Q}) \to H^\bullet(\mathcal{M}^{\alpha+\gamma}_{\mathbf{gt}}, \mathbb{Q}).$$

Since $\dim \mathcal{M}^{\alpha+\gamma}_{\mathbf{gt}} = \dim \mathcal{M}^\alpha_{\mathbf{gt}} + 2|\gamma|$ and $\dim \mathfrak{E}^\kappa_\alpha = \dim \mathcal{M}^\alpha_{\mathbf{gt}} + |\gamma|$ it follows that e_κ shifts the cohomological degree by $|\gamma|$. Hence e_κ considered as an operator in the vector space $\mathfrak{H}$ preserves the cohomological degree.

Remark 7.11. Instead of defining e_κ as the operator given by the correspondence $[\mathfrak{E}^\kappa_\alpha]$ we could define e_κ as the component of the operator given by the correspondence $[\widehat{\mathfrak{E}}^\kappa_\alpha]$, which increases the cohomological dimension by $2||\kappa||$. According to Lemma 7.9 these definitions are equivalent.

Theorem 7.12. *The vector space $\mathfrak{H}$ is a module over the Hall algebra $\mathbf{H}$, where S_κ act via e_κ.*

Proof. Let $\kappa' \in \mathfrak{K}_{\widehat{\mathfrak{sl}}_n}(\gamma')$, $\kappa'' \in \mathfrak{K}_{\widehat{\mathfrak{sl}}_n}(\gamma'')$ and put $\gamma = \gamma' + \gamma''$. We have to compute the composition of correspondences $[\mathfrak{E}^{\kappa''}_{\alpha+\gamma'}] \cdot [\mathfrak{E}^{\kappa'}_\alpha]$. Instead, using

Remark 7.11 we can compute the component of the composition of correspondences $[\widehat{\mathfrak{E}}^{\kappa''}_{\alpha+\gamma'}] \cdot [\widehat{\mathfrak{E}}^{\kappa'}_{\alpha}]$ that increase the cohomological dimension by $2|\gamma|$.

Consider the product $\mathcal{M}^{\alpha}_{\mathbf{gt}} \times \mathcal{M}^{\alpha+\gamma'}_{\mathbf{gt}} \times \mathcal{M}^{\alpha+\gamma}_{\mathbf{gt}}$ and let p_{ij} denote the projection to the product of the i-th and j-th factors. Consider the subset $\widehat{\mathfrak{E}}^{\kappa',\kappa''}_{\alpha} \subset \mathcal{M}^{\alpha}_{\mathbf{gt}} \times \mathcal{M}^{\alpha+\gamma'}_{\mathbf{gt}} \times \mathcal{M}^{\alpha+\gamma}_{\mathbf{gt}}$ defined as

$$\widehat{\mathfrak{E}}^{\kappa',\kappa''}_{\alpha} := p^{-1}_{12}(\widehat{\mathfrak{E}}^{\kappa'}_{\alpha}) \cap p^{-1}_{23}(\widehat{\mathfrak{E}}^{\kappa''}_{\alpha+\gamma'})$$
$$= \{(\mathcal{F}_{\bullet} \supset \mathcal{F}'_{\bullet} \supset \mathcal{F}''_{\bullet}) \mid (\mathcal{F}_{\bullet}, \mathcal{F}'_{\bullet}) \in \widehat{\mathfrak{E}}^{\kappa'}_{\alpha} \text{ and } (\mathcal{F}'_{\bullet}, \mathcal{F}''_{\bullet}) \in \widehat{\mathfrak{E}}^{\kappa''}_{\alpha+\gamma'}\}$$

Then $[\widehat{\mathfrak{E}}^{\kappa'}_{\alpha}] \cdot [\widehat{\mathfrak{E}}^{\kappa''}_{\alpha+\gamma'}]$ is given by $p_{13*}[\widehat{\mathfrak{E}}^{\kappa',\kappa''}_{\alpha}]$. But it is clear that $p_{13}(\widehat{\mathfrak{E}}^{\kappa',\kappa''}_{\alpha}) \subset \mathfrak{E}^{\gamma}_{\alpha}$, hence by Proposition 7.8 the component of $[\widehat{\mathfrak{E}}^{\kappa''}_{\alpha+\gamma'}] \cdot [\widehat{\mathfrak{E}}^{\kappa'}_{\alpha}]$ increasing the cohomological dimension by $2|\gamma|$ equals to

$$\sum_{\kappa \in \mathfrak{K}_{\widehat{\mathfrak{gl}}_n}} d^{\kappa}_{\kappa',\kappa''} [\mathfrak{E}^{\kappa}_{\alpha}]$$

for some constants $d^{\kappa}_{\kappa',\kappa''}$ which we have to compute. Further, it is clear that $d^{\kappa}_{\kappa',\kappa''}$ equals the number of points of $\widehat{\mathfrak{E}}^{\kappa',\kappa''}_{\alpha}$ over a generic point of $\mathfrak{E}^{\kappa}_{\alpha}$. Since we are interested in a generic point, we can take a point in $\overset{\circ}{\mathfrak{E}}{}^{\kappa}_{\alpha}$. So let $(\mathcal{F}_{\bullet}, \mathcal{F}''_{\bullet}) \in \overset{\circ}{\mathfrak{E}}{}^{\kappa}_{\alpha}$ and denote $T_{\bullet} = \mathcal{F}_{\bullet}/\mathcal{F}''_{\bullet}$. Then it is clear that $d^{\kappa}_{\kappa',\kappa''}$ equals the number of subobjects $T''_{\bullet} \subset T_{\bullet}$ such that for $T'_{\bullet} = T_{\bullet}/T''_{\bullet}$ the following conditions are satisfied:

(1) $\kappa(T'_{\bullet}) \leq \kappa'$;

(2) $\mathrm{supp}(T'_{\bullet}) \subset \overline{\mathbf{C}^{\gamma'}_{\kappa'}}$;

(3) $\kappa(T''_{\bullet}) \leq \kappa''$;

(4) $\mathrm{supp}(T''_{\bullet}) \subset \overline{\mathbf{C}^{\gamma''}_{\kappa''}}$;

Now assume that $\kappa = \{\{\theta_1, \ldots, \theta_m\}\}$ and that $\mathrm{supp}(T_{\bullet}) = \sum \theta_r x_r$. Then it is clear that $\kappa_{x_r}(T_{\bullet}) = \{\{\theta_r\}\}$ for all $1 \leq r \leq m$. Assume that $T''_{\bullet}$ is a subobject in $T_{\bullet}$. Then $W''_r = \Gamma_{x_r}(T''_{\bullet}) \subset \Gamma_{x_r}(T_{\bullet}) = W_r$ is a subrepresentation. Moreover, $T''_{\bullet}$ is uniquely determined by this collection of subrepresentations. Indeed, it is equal to the image of the natural map

$$\oplus^m_{r=1} W''_r \otimes \mathcal{O}_{\mathcal{S}} \to \oplus^m_{r=1} W_r \otimes \mathcal{O}_{\mathcal{S}} \to T_{\bullet}.$$

Finally, note that the set of all nontrivial subrepresentations $W''_r \subset W_r$ is in a bijection with the set of all θ''_r such that θ''_r ends at the same vertex as θ_r does,

and has smaller length. Put $\theta'_r = \theta_r / \theta''_r$. Then we have $\theta_r = \theta'_r \star \theta''_r$. Note that if $T''_\bullet$ and $T'_\bullet$ is the subobject and the quotient object of $T_\bullet$ corresponding to such collection θ''_r then

$$\kappa(T'_\bullet) = \{\{\theta'_1, \ldots, \theta'_m\}\}, \qquad \operatorname{supp}(T'_\bullet) \in C_{\gamma'}^{\{\{\theta'_1, \ldots, \theta'_m\}\}};$$
$$\kappa(T''_\bullet) = \{\{\theta''_1, \ldots, \theta''_m\}\}, \qquad \operatorname{supp}(T''_\bullet) \in C_{\gamma''}^{\{\{\theta''_1, \ldots, \theta''_m\}\}};$$

Thus $d^\kappa_{\kappa', \kappa''}$ equals the number of collections $(\theta'_r, \theta''_r)_{r=1}^m$ such that

$$\kappa' = \{\{\theta'_1, \ldots, \theta'_m\}\}, \quad \kappa'' = \{\{\theta'_1, \ldots, \theta'_m\}\}, \quad \text{and}$$
$$\theta_r = \theta'_r \star \theta''_r \text{ for all } 1 \le r \le m.$$

It remains to note that this is precisely $c^\kappa_{\kappa', \kappa''}(1)$ (see e.g. [10]). $\qquad \square$

7.7. Action of $\widehat{\mathfrak{sl}}_n$.
In addition to the operators e_κ introduced above, we define operators f_κ as the operators on the cohomology induced by the transposed correspondences:

$$f_\kappa = [(\mathfrak{E}^\kappa_{\alpha-\gamma})^T] : H^\bullet(\mathcal{M}^\alpha_{\mathbf{gt}}, \mathbb{Q}) \to H^\bullet(\mathcal{M}^{\alpha-\gamma}_{\mathbf{gt}}, \mathbb{Q}).$$

To unburden the notation denote the operators $e_{\{\{i\}\}}$ and $f_{\{\{i\}\}}$ by e_i and f_i respectively. Further, define the operator h_i on $H^\bullet(\mathcal{M}^\alpha_{\mathbf{gt}}, \mathbb{Q})$ as a scalar $\langle i', \alpha \rangle + 2$-multiplication.

Proposition 7.13. *We have $[e_i, f_j] = \delta_{ij} h_i$.*

Proof. We have to compare the following compositions of correspondences:

$$e_i f_j = [\mathfrak{E}^{\{\{i\}\}}_\alpha] \cdot [\mathfrak{E}^{\{\{j\}\}}_{\alpha+i-j}]^T \quad \text{and} \quad f_j e_i = [\mathfrak{E}^{\{\{j\}\}}_{\alpha-j}]^T \cdot [\mathfrak{E}^{\{\{i\}\}}_{\alpha-j}].$$

Instead, as in the Proof of Theorem 7.12 we will compare the components of the compositions

$$[\mathfrak{E}^i_\alpha] \cdot [\mathfrak{E}^j_{\alpha+i-j}]^T \quad \text{and} \quad [\mathfrak{E}^j_{\alpha-j}]^T \cdot [\mathfrak{E}^i_{\alpha-j}]$$

preserving the cohomological degree (note that for $\kappa = \{\{i\}\}$ we have $\widehat{\mathfrak{E}}^\kappa_\alpha = \mathfrak{E}^i_\alpha$). To this end we consider the following subspaces

$$\mathfrak{E}\mathfrak{F} = p_{12}^{-1}(\mathfrak{E}^i_\alpha) \cap p_{23}^{-1}((\mathfrak{E}^j_{\alpha+i-j})^T) \subset \mathcal{M}^\alpha_{\mathbf{gt}} \times \mathcal{M}^{\alpha+i}_{\mathbf{gt}} \times \mathcal{M}^{\alpha+i-j}_{\mathbf{gt}},$$
$$\mathfrak{F}\mathfrak{E} = p_{12}^{-1}((\mathfrak{E}^j_{\alpha-j})^T) \cap p_{23}^{-1}(\mathfrak{E}^i_{\alpha-j}) \subset \mathcal{M}^\alpha_{\mathbf{gt}} \times \mathcal{M}^{\alpha-j}_{\mathbf{gt}} \times \mathcal{M}^{\alpha+i-j}_{\mathbf{gt}}.$$

Consider the following open subset

$$U = \begin{cases} \mathcal{M}^\alpha_{\mathbf{gt}} \times \mathcal{M}^{\alpha+i-j}_{\mathbf{gt}}, & \text{if } i \ne j \\ \mathcal{M}^\alpha_{\mathbf{gt}} \times \mathcal{M}^\alpha_{\mathbf{gt}} - \Delta & \text{if } i = j \end{cases}$$

where Δ is the diagonal. Then it is easy to see that

$$\mathfrak{E}\mathfrak{F} \cap p_{13}^{-1}(U) \cong \mathfrak{F}\mathfrak{E} \cap p_{13}^{-1}(U).$$

Indeed, the map

$$\mathfrak{F}\mathfrak{E} \cap p_{13}^{-1}(U) \ni (\mathcal{F}_\bullet \subset \mathcal{F}'_\bullet \supset \mathcal{F}''_\bullet) \mapsto (\mathcal{F}_\bullet \supset (\mathcal{F}_\bullet \cap \mathcal{F}''_\bullet) \subset \mathcal{F}''_\bullet) \in \mathfrak{E}\mathfrak{F} \cap p_{13}^{-1}(U)$$

gives such an isomorphism. Hence we have

$$[e_i, f_j] = 0 \quad \text{for } i \neq j, \quad \text{and} \quad [e_i, f_i] = b_\alpha^i [\Delta] \quad \text{for } i \neq j,$$

and it remains to compute b_α^i.

So, assume that $i = j$. Let us begin with the contribution of $\mathfrak{F}\mathfrak{E}$ into b_α^i. To this end, note that the fiber of $\mathfrak{F}\mathfrak{E}$ over generic point of the diagonal Δ (with respect to the projection p_{13}) is empty. The reason is the fact that for a locally free $\mathcal{F}_\bullet$ there exists no $\mathcal{F}'_\bullet$, such that $\mathcal{F}_\bullet \subset \mathcal{F}'_\bullet$. Thus $\mathfrak{F}\mathfrak{E}$ doesn't contribute into b_α^i.

As for $\mathfrak{E}\mathfrak{F}$, the situation here is quite opposite. For generic point $\xi = (\mathcal{F}_\bullet, \mathcal{F}_\bullet) \in \Delta \subset \mathcal{M}_{\mathbf{gt}}^\alpha \times \mathcal{M}_{\mathbf{gt}}^\alpha$ the fiber of $\mathfrak{E}\mathfrak{F}$ over ξ is isomorphic to $\mathbf{C} \cong \mathbb{P}^1$: it consists of all $\mathcal{F}'_\bullet \subset \mathcal{F}_\bullet$ such that $\kappa(\mathcal{F}_\bullet/\mathcal{F}'_\bullet) = \{\{i\}\}$, and such subobjects are uniquely determined by the point $x = \text{supp}(\mathcal{F}_\bullet/\mathcal{F}'_\bullet) = \mathbf{r}(\mathcal{F}_\bullet, \mathcal{F}'_\bullet)$. Moreover, the intersection $p_{12}^{-1}(\mathfrak{E}_\alpha^i) \cap p_{23}^{-1}((\mathfrak{E}_{\alpha+i-j}^j)^T)$ in this case has dimension greater by 1 than expected, thus we are in the excess intersection situation. It follows that b_α^i equals to the degree of the excess intersection line bundle restricted to the fiber $\mathfrak{E}\mathfrak{F}_\xi$. Further, acting as in [8] 3.6.1 we can show that

$$b_\alpha^i = \deg \mathbf{q}_\xi^* \mathcal{N}_{D_\alpha^i/\mathcal{M}_{\mathbf{gt}}^{\alpha+i}},$$

where $D_\alpha^i = \mathbf{q}(\mathfrak{E}_\alpha^i)$, and $\mathbf{q}_\xi : \mathbf{p}^{-1}(\mathcal{F}_\bullet) \cong \mathbf{C} \subset \mathfrak{E}_\alpha^i \to \mathcal{M}_{\mathbf{gt}}^{\alpha+i}$ is the canonical projection.

Now let us identify the normal bundle $\mathcal{N}_{D_\alpha^i/\mathcal{M}_{\mathbf{gt}}^{\alpha+i}}$. Let $\mathcal{F}_\bullet \in \mathcal{M}_{\mathbf{gt}}^\alpha$ be a locally free parabolic sheaf and assume that $\mathcal{F}'_\bullet \subset \mathcal{F}_\bullet$ is such that $\kappa(\mathcal{F}_\bullet/\mathcal{F}'_\bullet) = \{\{i\}\}$. Let $c = \text{supp}(\mathcal{F}_\bullet/\mathcal{F}'_\bullet) = \mathbf{r}(\mathcal{F}_\bullet, \mathcal{F}'_\bullet)$. Then $\mathcal{F}'_\bullet \in D_\alpha^i$, and we have the following exact sequence

$$(7.14) \qquad 0 \to \mathcal{F}'_\bullet \to \mathcal{F}_\bullet \to (\mathcal{F}_i/\mathcal{F}_{i-1})_c \otimes_\mathbb{C} \mathcal{O}_c[i] \to 0,$$

where the right term is considered as an n-periodic representation of an infinite linear quiver in the category of sheaves on $\mathcal{S}'$ with the sheaves $(\mathcal{F}_i/\mathcal{F}_{i-1})_c \otimes_\mathbb{C} \mathcal{O}_c$ placed at $k \equiv i \pmod{n}$ and with zero at all other places k. Now we want to compute the tangent space (see [24])

$$\mathcal{T}_{\mathcal{F}'_\bullet} \mathcal{M}_{\mathbf{gt}}^{\alpha+i} = \text{Ext}^1(\mathcal{F}'_\bullet, \mathcal{F}'_\bullet(-\mathbf{D}_\infty))$$

using the exact sequence (7.14). Here $\mathbf{D}_\infty$ stands for $\mathbf{C} \times \mathbf{x}$. To this end we have to compute

$$\text{Ext}^*((\mathcal{F}_i/\mathcal{F}_{i-1})_c \otimes_\mathbb{C} \mathcal{O}_c[i], \mathcal{F}_\bullet(-\mathbf{D}_\infty)),$$
$$\text{Ext}^*((\mathcal{F}_i/\mathcal{F}_{i-1})_c \otimes_\mathbb{C} \mathcal{O}_c[i], (\mathcal{F}_i/\mathcal{F}_{i-1})_c \otimes_\mathbb{C} \mathcal{O}_c[i](-\mathbf{D}_\infty)),$$
$$\text{Ext}^*(\mathcal{F}_\bullet, (\mathcal{F}_i/\mathcal{F}_{i-1})_c \otimes_\mathbb{C} \mathcal{O}_c[i](-\mathbf{D}_\infty)).$$

The third Ext is easiest to compute. It is clear that we have

$$(\mathcal{F}_i/\mathcal{F}_{i-1})_c \otimes_\mathbb{C} \mathcal{O}_c[i](-\mathbf{D}_\infty) \cong (\mathcal{F}_i/\mathcal{F}_{i-1})_c \otimes_\mathbb{C} \mathcal{O}_c[i]$$

and

$$\text{Hom}(\mathcal{F}_\bullet, (\mathcal{F}_i/\mathcal{F}_{i-1})_c \otimes_\mathbb{C} \mathcal{O}_c[i]) = (\mathcal{F}_i/\mathcal{F}_{i-1})_c \otimes_\mathbb{C} (\mathcal{F}_i/\mathcal{F}_{i-1})_c^* \cong \mathbb{C}, \quad \text{Ext}^{>0} = 0.$$

To compute the other two Ext-s we use the following locally free resolution of $\mathcal{O}_c[i]$:

$$
\begin{array}{ccccccccc}
& \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
0 \to & \mathcal{O}(-\mathbf{X}_c) & \to & \mathcal{O}(-\mathbf{X}_c) \oplus \mathcal{O} & \to & \mathcal{O} & \to & 0 & \to 0 \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
0 \to \mathcal{O}(-\mathbf{X}_c - \mathbf{D}_0) \to & & \mathcal{O}(-\mathbf{X}_c) \oplus \mathcal{O}(-\mathbf{D}_0) \to & & \mathcal{O} & \to & \mathcal{O}_c & \to 0 \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
0 \to \mathcal{O}(-\mathbf{X}_c - \mathbf{D}_0) \to \mathcal{O}(-\mathbf{X}_c - \mathbf{D}_0) \oplus \mathcal{O}(-\mathbf{D}_0) \to \mathcal{O}(-\mathbf{D}_0) \to & & 0 & \to 0 \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow &
\end{array}
$$

Here $\mathbf{X}_c$ stands for $c \times \mathbf{X}$. The rows of the above diagram are exact sequences of coherent sheaves and the columns are n-quasi-periodic representations of an infinite linear quiver. The quasi-periodicity means that when one shifts to n positions up, the sheaf became twisted by $\mathcal{O}(\mathbf{D}_0)$.

Using this resolution one can easily compute Ext-s:

$$\text{Ext}^p((\mathcal{F}_i/\mathcal{F}_{i-1})_c \otimes_\mathbb{C} \mathcal{O}_c[i], \mathcal{F}_\bullet(-\mathbf{D}_\infty)) =$$
$$\begin{cases} (\mathcal{F}_i/\mathcal{F}_{i-1})_c^* \otimes_\mathbb{C} (\mathcal{F}_{i+1}/\mathcal{F}_i)_c \otimes_\mathbb{C} \mathcal{O}(\mathbf{X}_c)_c, & p = 2 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{Ext}^p((\mathcal{F}_i/\mathcal{F}_{i-1})_c \otimes_\mathbb{C} \mathcal{O}_c[i], (\mathcal{F}_i/\mathcal{F}_{i-1})_c \otimes_\mathbb{C} \mathcal{O}_c[i](-\mathbf{D}_\infty)) = \begin{cases} \mathbb{C}, & p = 0, 1; \\ 0, & \text{otherwise} \end{cases}$$

Now, computing $\text{Ext}^*(\mathcal{F}'_\bullet, \mathcal{F}'_\bullet(-\mathbf{D}_\infty))$ with the help of (7.14) one gets a spectral sequence with the first term as follows:

$$
\begin{array}{ccc}
(\mathcal{F}_i/\mathcal{F}_{i-1})_c^* \otimes_\mathbb{C} (\mathcal{F}_{i+1}/\mathcal{F}_i)_c \otimes_\mathbb{C} \mathcal{O}(\mathbf{X}_c)_c & 0 & 0 \\
0 & \text{Ext}^1(\mathcal{F}_\bullet, \mathcal{F}_\bullet(-\mathbf{D}_\infty)) \oplus \mathbb{C} & 0 \\
0 & \mathbb{C} \xrightarrow{\hspace{3cm}} & \mathbb{C}
\end{array}
$$

Here the map in the bottom row is the map

$$\mathrm{Hom}((\mathcal{F}_i/\mathcal{F}_{i-1})_c \otimes_\mathbb{C} \mathcal{O}_c[i], (\mathcal{F}_i/\mathcal{F}_{i-1})_c \otimes_\mathbb{C} \mathcal{O}_c[i](-\mathbf{D}_\infty)) \to$$
$$\mathrm{Hom}(\mathcal{F}_\bullet, (\mathcal{F}_i/\mathcal{F}_{i-1})_c \otimes_\mathbb{C} \mathcal{O}_c[i](-\mathbf{D}_\infty))$$

induced by the projection $\mathcal{F}_\bullet \to (\mathcal{F}_i/\mathcal{F}_{i-1})_c \otimes_\mathbb{C} \mathcal{O}_c[i]$. It is clear that it takes the identity homomorphism to this projection. Thus this map is not trivial, hence induces an isomorphism in the bottom row of the spectral sequence. It follows that the spectral sequence degenerates in the second term and gives the following exact sequence

$$0 \to \mathrm{Ext}^1(\mathcal{F}_\bullet, \mathcal{F}_\bullet(-\mathbf{D}_\infty)) \oplus \mathbb{C} \to \mathrm{Ext}^1(\mathcal{F}'_\bullet, \mathcal{F}'_\bullet(-\mathbf{D}_\infty)) \to$$
$$(\mathcal{F}_i/\mathcal{F}_{i-1})_c^* \otimes_\mathbb{C} (\mathcal{F}_{i+1}/\mathcal{F}_i)_c \otimes_\mathbb{C} \mathcal{O}(\mathbf{X}_c)_c \to 0.$$

It is clear that the first term in this exact sequence is the tangent space to the divisor D^i_α at the point $\mathcal{F}'_\bullet \in \mathcal{M}^\alpha_{\mathbf{gt}}$. Hence, the fiber of the normal bundle at this point is isomorphic to

$$(7.15) \qquad (\mathcal{F}_i/\mathcal{F}_{i-1})_c^* \otimes_\mathbb{C} (\mathcal{F}_{i+1}/\mathcal{F}_i)_c \otimes_\mathbb{C} \mathcal{O}(\mathbf{X}_c)_c.$$

Now we can compute b^i_α. To this end we should let the point c vary within the curve $\mathbf{C}$ and compute the degree of the line bundle formed by spaces (7.15). The bundle in question is clearly

$$(\mathcal{F}_i/\mathcal{F}_{i-1})^* \otimes (\mathcal{F}_{i+1}/\mathcal{F}_i) \otimes \mathcal{O}(2),$$

the last factor is the restriction of the sheaf $\mathcal{O}(\Delta_\mathbf{C})$ on $\mathbf{C} \times \mathbf{C}$ to the diagonal. Thus

$$b^i_\alpha = \deg((\mathcal{F}_i/\mathcal{F}_{i-1})^* \otimes (\mathcal{F}_{i+1}/\mathcal{F}_i) \otimes \mathcal{O}(2)) = -\deg(\mathcal{F}_i/\mathcal{F}_{i-1}) + \deg(\mathcal{F}_{i+1}/\mathcal{F}_i) + 2.$$

Applying Lemma 3.1.1 from [9] we get

$$b^i_\alpha = \langle i', \alpha \rangle + 2.$$

This completes the proof of the Proposition. □

Recall now that $\widehat{\mathfrak{sl}}_n$ is a Kac-Moody algebra with generators e_i, f_i, h_i, $i \in I$, and standard relations. Theorem 7.12 together with Proposition 7.13 combine into the following

Theorem 7.16. *The operators e_i, f_i, h_i, $i \in I$, generate an action of $\widehat{\mathfrak{sl}}_n$ on $\bigoplus_\alpha H^\bullet(\mathcal{M}^\alpha_{\mathbf{gt}})$. This action has central charge 2.*

Proof. It is well known that the subalgebra $U^+ \subset U(\widehat{\mathfrak{sl}}_n)$ generated by $e_i,\ i \in I$, embeds into the Hall algebra **H**, $e_i \mapsto S_{\{\{i\}\}}$. Thus the Serre relations for $e_i, i \in I$, follow. It only remains to check the Serre relations for the operators $f_i,\ i \in I$. But they are given by correspondences transpose to those of e_i. $\square$

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