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Spiral Traveling Wave Solutions of Nonlinear Diffusion Equations Related to a Model of Spiral Crystal Growth

By

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Abstract

This paper is concerned with nonlinear diffusion equations related to a model of the motion of screw dislocations on crystal surfaces. We prove the existence, uniqueness and asymptotic stability of a rotating and growing solution with a time-independent profile, which we call a spiral traveling wave solution.

§1. Introduction

In this paper we shall investigate a nonlinear diffusion equation on a two-dimensional annulus $\Omega = \{x \in \mathbb{R}^2 \mid a < |x| < b\}$:

$$(1) \qquad \begin{cases} u_t = \Delta u + f(u - \sigma\theta), & x \in \Omega,\ t > 0, \\ u_r = 0, & x \in \partial\Omega,\ t > 0, \end{cases}$$

where σ is a positive integer, (r, θ) denotes the polar coordinates of $x \in \overline{\Omega}$ and f is a periodic function.

Problem (1) is related to a model of spiral crystal growth. Spiral ledges are observed on the surface of many kinds of crystals such as silicon carbide (SiC),

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calcogen, paraffin and polyethylene ([17]). Frank [4] originally proposed the screw dislocation mechanism for crystal growth. Screw dislocation is a kind of lattice defect and produces a line step on the crystal surface. The step provides a preferred site for atoms to bond and moves normal to itself as the atoms attach to it. Since the velocity of the line step can be assumed to be the same at any point, the angular velocity at the center is larger than that at the edge. Thus, the dislocation proceeds in a spiral shape.

Kobayashi [8] has proposed the following reaction-diffusion equation as a model of the motion of screw dislocations:

$$\begin{cases} \tau u_t = \varepsilon^2 \Delta u - \sin(u - \sigma\theta) + \gamma, & x \in \Omega,\ t > 0, \\ u_r = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{2}$$

where $\tau, \varepsilon > 0$ are small parameters and γ is a constant. This equation has a gradient structure

$$\tau \frac{\partial u}{\partial t} = -\frac{\delta H}{\delta u}$$

with the "free energy" functional H defined by

$$H = \int_\Omega \left\{ \frac{\varepsilon^2}{2} |\nabla u|^2 - \cos(u - \sigma\theta) - \gamma u \right\} dx.$$

Here the unknown function $u(x, t)$ represents the local height of the crystal surface and is normalized in order that 2π denotes the size of a unit molecule. In this model, we assume that there exists only one dislocation on the surface with fixed core region $\{x \in \mathbb{R}^2 \mid |x| < a\}$ and that the initial height is given approximately by $\sigma\theta$. Actually, spiral growth with a hollow core at the center can be observed on the surface of SiC crystal ([17]). Kobayashi has also proposed a model of spiral crystal growth for the case where there exist finitely many dislocations on the crystal surface ([8]). Some numerical experiments imply that equation (2) has a rotating and growing solution with a spiral shape (See Figures 1 and 2).

The purpose of the present paper is to show the existence, uniqueness and stability of such a solution, which we call a *spiral traveling wave solution*. More precisely, as we will see later, equation (1) (and (2)) has a solution of the form

$$U(x, t) = \varphi\left(r, \theta - \frac{\omega}{\sigma} t\right) + \omega t, \quad x \in \Omega,\ t > 0,$$

where $\omega \in \mathbb{R}$ and $\varphi(x) = \varphi(r, \theta)$ is $2\pi/\sigma$-periodic in θ.

Our paper is organized as follows: In Section 2 we introduce basic notation and state our main results (Theorems 2.1, 2.2 and 2.3). We prove Theorems 2.1

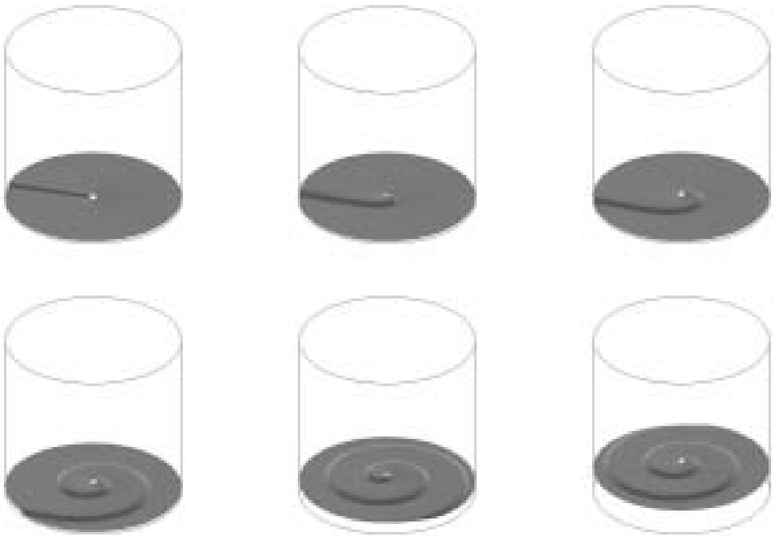

Figure 1. Time evolution of $u(x,t)$ of (2) when $\sigma = 1$, $\varepsilon = \tau = 0.10$, $\gamma = \sqrt{3}/2$.

and 2.2 in Section 3 and Theorem 2.3 in Section 4. In Appendix we recall a monotonicity result in order-preserving dynamical systems in the presence of symmetry obtained by Ogiwara and Matano [14, Proposition B1]. This result plays a crucial role in the proof of the monotonicity of spiral traveling wave solutions.

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§2. Main Results

Throughout this paper, we assume that the nonlinearity f satisfies the following condition:

(F) f is of class C^1 and is 2π-periodic.

We denote by $\langle f \rangle$ the average of f, namely,

$$\langle f \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(v)dv. \tag{3}$$

It is known that, for any $u_0 \in C(\overline{\Omega})$, a solution $u(x,t)$ of (1) with initial data $u(\cdot, 0) = u_0$ exists globally in time, since f is a bounded function (see [6], [12]).

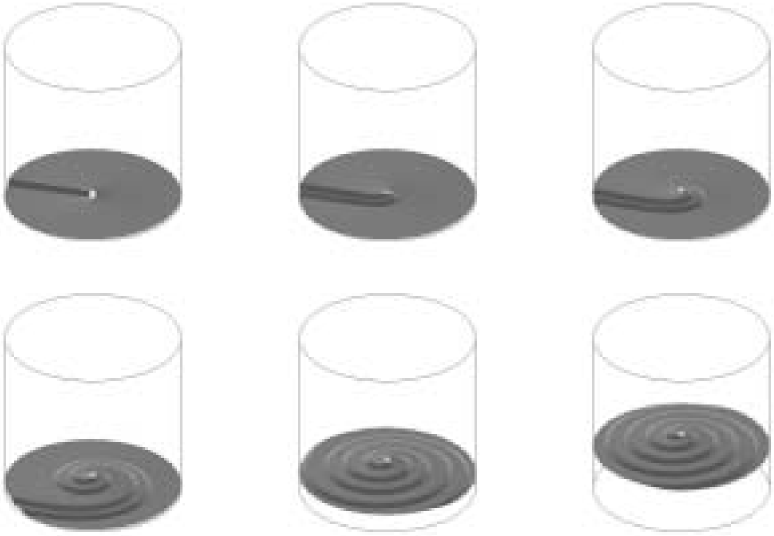

Figure 2. Time evolution of $u(x,t)$ of (2) when $\sigma = 2$ and other parameters are the same as in Figure 1.

For u_1, $u_2 \in C(\overline{\Omega})$ we write

$$
\begin{aligned}
&u_1 \le u_2 \quad &&\text{if } u_1(x) \le u_2(x) \text{ for all } x \in \overline{\Omega}, \\
&u_1 < u_2 \quad &&\text{if } u_1(x) \le u_2(x) \text{ for all } x \in \overline{\Omega} \text{ and } u_1 \not\equiv u_2, \\
&u_1 \ll u_2 \quad &&\text{if } u_1(x) < u_2(x) \text{ for all } x \in \overline{\Omega}.
\end{aligned} \tag{4}
$$

Let $\{\Phi_t\}_{t\in[0,\infty)}$ be the semiflow on $C(\overline{\Omega})$ generated by (1). In other words, the map Φ_t on $C(\overline{\Omega})$ is defined by $\Phi_t(u_0) = u(\cdot,t)$ for each $t \ge 0$, where $u(x,t)$ is the solution of (1) with initial data $u(\cdot,0) = u_0$. The strong maximum principle ([16]) shows that Φ_t is strongly order-preserving ([10]), that is, $u_1 < u_2$ implies $\Phi_t(u_1) \ll \Phi_t(u_2)$ for each $t > 0$. Further the standard parabolic estimate ([12]) shows that Φ_t is a compact map on $C(\overline{\Omega})$ for each $t > 0$.

For $\alpha \in \mathbb{R}$ we define a map g_α on $C(\overline{\Omega})$ by

$$(g_\alpha w)(x) = (g_\alpha w)(r,\theta) = w(r,\theta-\alpha) + \sigma\alpha.$$

Here and in the sequel, $x \in \overline{\Omega}$ is often identified with (r,θ), the polar coordinates of x. Note that the map g_α is commutative with Φ_t for all $\alpha \in \mathbb{R}$ and $t \ge 0$.

Since f is 2π-periodic, the semiflow $\{\Phi_t\}_{t\in[0,\infty)}$ also satisfies

$$\Phi_t(w + 2k\pi) = \Phi_t(w) + 2k\pi, \quad t \geq 0 \tag{5}$$

for all $w \in C(\overline{\Omega})$ and $k \in \mathbb{Z}$.

Definition 2.1. A solution $U(x,t)$ of (1) is called a *spiral traveling wave solution* if it is written in the form

$$U(x,t) = \varphi(r, \theta - \omega_1 t) + \omega_2 t, \quad x \in \Omega,\ \ t > 0 \tag{6}$$

for some function $\varphi(r,\theta)$ and some constants ω_1, ω_2. We call the function φ the *profile* and the constant ω_2 the *growth speed* of the spiral traveling wave solution U.

Concerning the existence, uniqueness and stability of spiral traveling wave solutions, we obtain the following results:

Theorem 2.1.

(i) *If $\langle f \rangle \neq 0$, then there exists a spiral traveling wave solution of* (1) *and it is unique up to time shift. More precisely, there exist a function $\varphi = \varphi(r,\theta) \in C(\overline{\Omega})$ and a constant ω with $\omega\langle f \rangle > 0$ such that*

$$U(x,t) = \varphi\left(r, \theta - \frac{\omega}{\sigma} t\right) + \omega t \tag{7}$$

is a solution of (1), *and if $\widetilde{U}$ is a spiral traveling wave solution of* (1) *then there exists some $t_0 \in \mathbb{R}$ such that $\widetilde{U}(\cdot,t) = U(\cdot, t + t_0)$ for all $t \geq 0$. Furthermore, the profile φ is $2\pi/\sigma$-periodic in θ.*

(ii) *If $\langle f \rangle = 0$, then there exists an equilibrium solution $v(x)$ of* (1) *and the set of equilibria of* (1) *coincides with the set $\{g_\alpha v \mid \alpha \in \mathbb{R}\}$. Furthermore, $v = v(r,\theta)$ is $2\pi/\sigma$-periodic in θ.*

Theorem 2.2.

(i) *Suppose $\langle f \rangle \neq 0$ and let U be as in Theorem* 2.1(i). *Then U is stable in the sense of Lyapunov and is strictly monotone in t, that is,*

$$U_t(x,t) \begin{cases} > 0, & \text{if } \langle f \rangle > 0, \\ < 0, & \text{if } \langle f \rangle < 0, \end{cases}$$

for $x \in \overline{\Omega}$ and $t > 0$.

(ii) *Suppose* $\langle f\rangle = 0$ *and let* v *be as in Theorem* 2.1(ii)*. Then* $g_\alpha v$ *is strictly monotone increasing in* α*, that is,*

$$\frac{\partial}{\partial\alpha}(g_\alpha v)(x) > 0, \quad x \in \overline{\Omega}$$

for all $\alpha \in \mathbb{R}$*.*

Theorem 2.3. *Let* U *and* v *be as in Theorem* 2.1*. Then there exists a positive constant* μ *such that for any* $u_0 \in C(\overline{\Omega})$ *the solution* $u(x,t)$ *of* (1) *with initial value* u_0 *satisfies*

$$\begin{aligned} \|u(\cdot,t) - U(\cdot,t+\tau_0)\|_{C(\overline{\Omega})} &\le M_0 e^{-\mu t}, \quad \text{if } \langle f\rangle \neq 0, \\ \|u(\cdot,t) - g_{\tau_0} v\|_{C(\overline{\Omega})} &\le M_0 e^{-\mu t}, \quad \text{if } \langle f\rangle = 0, \end{aligned} \tag{8}$$

for all $t \ge 0$*, where* $\tau_0 \in \mathbb{R}$ *and* $M_0 > 0$ *are constants depending on* u_0*.*

Remark 2.1. By (7), the spiral traveling wave solution U satisfies

$$U(x,t+T) = U(x,t) + 2\pi, \qquad x \in \overline{\Omega},\ t > 0, \tag{9}$$

where $T = 2\pi/\omega$. Solutions with property (9) have been studied for other equations such as systems of ordinary differential equations ([11], [3], [1]) and parabolic equations in the whole space $\mathbb{R}^N$ ([13]). For our problem (1), as we will see in Lemma 3.3, if a solution U satisfies (9) for some $T \neq 0$ then it is a spiral traveling wave solution with growth speed $\omega = 2\pi/T$.

§3. Existence, Uniqueness and Monotonicity

In this section, we show the existence, uniqueness and monotonicity of spiral traveling wave solutions. First we show the uniqueness of solutions of the form (6).

Proposition 3.1. *Let* $U = \varphi(r, \theta - \omega_1 t) + \omega_2 t$ *and* $\widetilde{U} = \widetilde{\varphi}(r, \theta - \widetilde{\omega}_1 t) + \widetilde{\omega}_2 t$ *be spiral traveling wave solutions of* (1)*. Then we have* $\widetilde{\omega_j} = \omega_j$ $(j = 1, 2)$ *and* $\widetilde{\varphi} = g_{\alpha_0}\varphi$ *for some* $\alpha_0 \in \mathbb{R}$*.*

Proof. By the comparison theorem, we can easily see $\widetilde{\omega}_2 = \omega_2$. We define $\alpha_0 = \inf\{\alpha \in \mathbb{R} \mid \widetilde{\varphi} \le g_\alpha \varphi\}$. Then $\widetilde{\varphi} \le g_{\alpha_0}\varphi$ and $\widetilde{\varphi}(r_0,\theta_0) = (g_{\alpha_0}\varphi)(r_0,\theta_0)$ for some $(r_0,\theta_0) \in \overline{\Omega}$.

Suppose that $\widetilde{\varphi} < g_{\alpha_0}\varphi$. Then we have $\Phi_t(\widetilde{\varphi}) \ll \Phi_t(g_{\alpha_0}\varphi) = g_{\alpha_0}\Phi_t(\varphi)$ for $t > 0$, namely,

$$\widetilde{\varphi}(r, \theta - \widetilde{\omega}_1 t) + \widetilde{\omega}_2 t < (g_{\alpha_0}\varphi)(r, \theta - \omega_1 t) + \omega_2 t, \quad (r,\theta) \in \overline{\Omega},\ \ t > 0.$$

This implies $\widetilde{\varphi}(r_0, \theta_0) < (g_{\alpha_0}\varphi)(r_0, \theta_0 + (\widetilde{\omega}_1 - \omega_1)t)$ for all $t > 0$. This contradiction proves the proposition. ☐

Let $\zeta(x,t)$ be a solution of (1) with initial data $\zeta(\cdot, 0) \equiv 0$ and we define

$$\zeta^* = \limsup_{t\to+\infty} \max_{x\in\overline{\Omega}} \zeta(x,t), \quad \zeta_* = \liminf_{t\to+\infty} \min_{x\in\overline{\Omega}} \zeta(x,t).$$

Lemma 3.1. *There exists a positive constant C independent of t such that*

$$\max\{\zeta(x,t) \mid x \in \overline{\Omega}\} - \min\{\zeta(x,t) \mid x \in \overline{\Omega}\} \le C, \quad t \ge 0.$$

Proof. Let $R = \sup_{u\in\mathbb{R}} |f(u)|$ and let

$$\eta(x,t) = \zeta(x,t) - \frac{1}{|\Omega|}\int_\Omega \zeta(x,t)dx,$$

where $|\Omega|$ denotes the area of Ω. Then η satisfies

$$\begin{cases} \eta_t = L\eta + h(x,t), & x \in \Omega,\ t > 0, \\ \eta_r = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{10}$$

where L is the restriction of Δ on $X_0 = \{u \in C(\overline{\Omega}) \mid \int_\Omega u(x)dx = 0\}$ and $h(x,t)$ is a bounded function defined by

$$h(x,t) = f(\zeta(x,t) - \sigma\theta) - \frac{1}{|\Omega|}\int_\Omega f(\zeta(x,t) - \sigma\theta)dx.$$

We note that L generates an analytic semigroup $\{e^{tL}\}_{t\in(0,\infty]}$ on X_0 and that

$$\eta(\cdot, t) = \int_0^t e^{(t-s)L}h(\cdot, s)ds.$$

See [6] and [12] for details. Let $\lambda_1 > 0$ be the least positive eigenvalue of $-\Delta$ in $C(\overline{\Omega})$ with homogeneous Neumann boundary conditions. Then there exist constants $C_0 > 0$ and $\lambda \in (0, \lambda_1)$ such that $\|e^{tL}u\|_{C(\overline{\Omega})} \le C_0 e^{-\lambda t}\|u\|_{C(\overline{\Omega})}$ for all $t \ge 0$ and $u \in X_0$. Therefore, we have

$$\|\eta(\cdot, t)\|_{C(\overline{\Omega})} \le \int_0^t C_0 e^{-\lambda(t-s)}\|h(\cdot, s)\|_{C(\overline{\Omega})}ds \le \frac{2C_0 R}{\lambda},$$

hence

$$\max_{x\in\overline{\Omega}} \zeta(x,t) - \min_{x\in\overline{\Omega}} \zeta(x,t) = \max_{x\in\overline{\Omega}} \eta(x,t) - \min_{x\in\overline{\Omega}} \eta(x,t) \le \frac{4C_0 R}{\lambda}.$$

The lemma is proved. □

Define

$$\operatorname{sgn}(\nu) = \begin{cases} 1, & \text{if } \nu > 0, \\ 0, & \text{if } \nu = 0, \\ -1, & \text{if } \nu < 0. \end{cases}$$

For $\nu \neq 0$, we put

$$Z_\nu = \{w \in C(\overline{\Omega}) \mid \Phi_{|\nu|}(w) = w + 2\pi \cdot \operatorname{sgn}(\nu)\}.$$

We also denote by Z_0 the set of equilibria of (1). By (5), we can easily see that the semiflow $\{\Phi_t\}_{t\in(0,\infty]}$ can be extended to a one-parameter group (or a flow) $\{\Phi_t\}_{t\in\mathbb{R}}$ acting on Z_ν for each $\nu \in \mathbb{R}$ and that

$$Z_\nu = \{w \in C(\overline{\Omega}) \mid \Phi_\nu(w) = w + 2\pi\}$$

holds for $\nu \neq 0$.

Proposition 3.2.

(i) *If $\zeta^* = +\infty$, then $Z_{\nu^*} \neq \emptyset$ for some $\nu^* > 0$.*

(ii) *If $\zeta_* = -\infty$, then $Z_{\nu_*} \neq \emptyset$ for some $\nu_* < 0$.*

(iii) *If both ζ^* and ζ_* are finite, then $Z_0 \neq \emptyset$.*

Proof. (i) When $\zeta^* = +\infty$, there exists a sequence $0 < t_1 < t_2 < \cdots \to +\infty$ such that $\max_{x\in C(\overline{\Omega})} \zeta(x, t_j) \to +\infty$. By Lemma 3.1, we can take a positive integer m_j such that

$$0 \le \zeta(x, t_j) - 2m_j\pi \le C + 2\pi, \quad x \in \overline{\Omega}$$

for all $j \in \mathbb{N}$. We fix a positive constant δ and put

$$w_j = \Phi_\delta(\zeta(\cdot, t_j) - 2m_j\pi) = \zeta(\cdot, t_j + \delta) - 2m_j\pi.$$

Since the map Φ_δ is compact, replacing $\{t_j\}$ by its subsequence if necessary, we have $\lim_{j\to\infty} w_j = \varphi$ for some $\varphi \in C(\overline{\Omega})$. We define

$$l(t) = \inf\{\nu \ge 0 \mid \zeta(\cdot, t) + 2\pi \le \zeta(\cdot, t + \nu)\}.$$

Since $\zeta^* = +\infty$, the function $l(t)$ is well-defined for each $t \ge 0$. By the comparison theorem, $l(t)$ is positive and is monotone decreasing in t. Put

$\nu^* = \lim_{t\to+\infty} l(t)$. Since $\zeta(\cdot,t)+2\pi \le \zeta(\cdot,t+l(t))$ for $t\ge 0$, letting $t=t_j+\delta$ and $j\to\infty$, we obtain $\varphi+2\pi \le \Phi_{\nu^*}(\varphi)$. This implies $\nu^*>0$.

Suppose that $\varphi+2\pi < \Phi_{\nu^*}(\varphi)$. Then for any fixed $s>0$, we have $\Phi_s(\varphi+2\pi) = \Phi_s(\varphi)+2\pi \ll \Phi_{\nu^*+s}(\varphi)$. From this, for a sufficiently large $j_0\in\mathbb{N}$, it follows that

$$\Phi_s(w_{j_0})+2\pi \ll \Phi_{\nu^*+s}(w_{j_0}).$$

Therefore, there exists a small positive constant $\varepsilon\in(0,\nu^*)$ such that

$$\Phi_s(w_{j_0})+2\pi \le \Phi_{\nu^*-\varepsilon+s}(w_{j_0}),$$

and hence

$$\zeta(\cdot,t_{j_0}+\delta+s)+2\pi \le \zeta(\cdot,t_{j_0}+\delta+s+\nu^*-\varepsilon).$$

This implies that $l(t_{j_0}+\delta+s)\le \nu^*-\varepsilon$, which contradicts the definition of ν^*. Therefore $\varphi+2\pi=\Phi_{\nu^*}(\varphi)$ holds and thus $Z_{\nu^*}\ne\emptyset$. We can prove the statement (ii) similarly.

(iii) When both ζ^* and ζ_* are finite, the set $\{\zeta(\cdot,t)\mid t\ge 0\}$ is bounded in $C(\overline{\Omega})$. Since equation (1) has a Lyapunov functional, by virtue of the results of Matano [9], there exist a sequence $0<t_1<t_2<\cdots\to+\infty$ and $v\in C(\overline{\Omega})$ such that $\zeta(\cdot,t_j)\to v$ in $C(\overline{\Omega})$ and that v is an equilibrium solution of (1). The proposition is proved. □

Lemma 3.2.

(i) *If $w\in Z_\nu$ for some $\nu\in\mathbb{R}$, then $Z_\nu=\{g_\alpha w\mid \alpha\in\mathbb{R}\}$ and $g_\alpha w$ is strongly monotone increasing in α, that is, $g_\alpha w\ll g_\beta w$ holds for $\alpha<\beta$.*

(ii) *If $w\in Z_\nu$ for some $\nu\ne 0$, then $Z_\nu=\{\Phi_t(w)\mid t\in\mathbb{R}\}$. Furthermore, $\Phi_t(w)$ is strongly monotone increasing in t if $\nu>0$, and it is strongly monotone decreasing in t if $\nu<0$.*

Proof. (i) Set a metric space $X=C(\overline{\Omega})$ and a group $G=\{g_\alpha\mid\alpha\in\mathbb{R}\}$ acting on X. Then conditions (G1) and (G2) in Appendix hold. We also put $Y=Z_\nu$ and $\overline{\varphi}=w$. Clearly (H1) and (H2) are fulfilled. We show condition (H3) is also satisfied. Suppose that $v\in Z_\nu$ satisfies $v<g_\alpha w$ for some $\alpha\in\mathbb{R}$. In the case where $\nu\ne 0$, since $\Phi_{|\nu|}(v)\ll\Phi_{|\nu|}(g_\alpha w)=g_\alpha\Phi_{|\nu|}(w)$, we have

$$v+2\pi\cdot\operatorname{sgn}(\nu)\ll g_\alpha(w+2\pi\cdot\operatorname{sgn}(\nu))=g_\alpha w+2\pi\cdot\operatorname{sgn}(\nu),$$

hence $v \ll g_\alpha w$. In the case where $\nu = 0$, we also have $v \ll g_\alpha w$ since $\Phi_t(v) \ll \Phi_t(g_\alpha w)$ for $t > 0$. Thus condition (H3) holds. Therefore, applying Proposition B1 in [14], we see that $Z_\nu = Gw = \{g_\alpha w \mid \alpha \in \mathbb{R}\}$ and that Z_ν is a totally ordered set. These imply that $g_\alpha w < g_\beta w$ holds for $\alpha < \beta$, since $\max_{x\in\overline{\Omega}}(g_\alpha w)(x) < \max_{x\in\overline{\Omega}}(g_\beta w)(x)$. Therefore $g_\alpha w \ll g_\beta w$ holds as above.

(ii) Set metric spaces $X = C(\overline{\Omega})$ and $Y = Z_\nu$, a group $G = \{\Phi_t\}_{t\in\mathbb{R}}$ acting on Y and $\overline{\varphi} = w$. In the same way as in the proof of (i), all conditions (G1), (G2) and (H1)-(H3) are fulfilled, since the flow $\{\Phi_t\}_{t\in\mathbb{R}}$ is strongly order-preserving on Z_ν. It follows from Proposition B1 in [14] that $Z_\nu = Gw = \{\Phi_t(w) \mid t \in \mathbb{R}\}$ and that Z_ν is homeomorphic and order-isomorphic to $\mathbb{R}$. This implies that $\Phi_t(w)$ is strongly monotone in t. By the definition of Z_ν we obtain the desired conclusion. □

Lemma 3.3. *If $\varphi = \varphi(r,\theta) \in Z_\nu$ for some $\nu \neq 0$, then $\varphi\left(r, \theta - \frac{\omega}{\sigma}t\right) + \omega t$ is a solution of* (1) *with $\omega = 2\pi/\nu$.*

Proof. We put $\alpha_m = 2\pi/m$ for $m \in \mathbb{N}$. By Lemma 3.2, there exists some $s_m \in \mathbb{R}$ such that $\Phi_{s_m}(\varphi) = g_{\alpha_m}\varphi$. This implies that

$$\Phi_{ks_m}(\varphi) = g_{k\alpha_m}\varphi$$

for all $k \in \mathbb{Z}$, in particular, $\Phi_{ms_m}(\varphi) = g_{2\pi}\varphi = \varphi + 2\sigma\pi$. On the other hand, $\Phi_{\sigma\nu}(\varphi) = \varphi + 2\sigma\pi$ since $\varphi \in Z_\nu$. If $ms_m \neq \sigma\nu$, then $\{\Phi_t(\varphi) \mid t \in \mathbb{R}\}$ is a periodic orbit with period $|ms_m - \sigma\nu|$, which contradicts the fact that the orbit $\{\Phi_t(\varphi) \mid t \in \mathbb{R}\}$ is not bounded in $C(\overline{\Omega})$. Hence $ms_m = \sigma\nu$ for any $m \in \mathbb{N}$. Thus we have, for any $k \in \mathbb{Z}$,

$$\Phi_{k\sigma\nu/m}(\varphi) = g_{2\pi k/m}\varphi,$$

and further, for any $q \in \mathbb{Q}$,

$$\Phi_{\sigma\nu q}(\varphi) = g_{2\pi q}\varphi.$$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, if we set $\omega = 2\pi/\nu$ then

$$\Phi_t(\varphi) = g_{\omega t/\sigma}\varphi$$

holds for any $t \in \mathbb{R}$. This implies that $(g_{\omega t/\sigma}\varphi)(r,\theta) = \varphi\left(r, \theta - \frac{\omega}{\sigma}t\right) + \omega t$ is a solution of (1). □

Lemma 3.4. *If $Z_\nu \neq \emptyset$, then* $\operatorname{sgn}(\langle f \rangle) = \operatorname{sgn}(\nu)$.

Proof. When $\nu \neq 0$, equation (1) possesses a spiral traveling wave solution $U(x,t) = \varphi\left(r, \theta - \frac{\omega}{\sigma} t\right) + \omega t$ with $\omega = 2\pi/\nu$. Hence $\varphi = \varphi(r,\xi)$ satisfies

$$-\frac{\omega}{\sigma}\varphi_\xi + \omega = \Delta\varphi + f(\varphi - \sigma\xi). \tag{11}$$

Multiplying both sides of (11) by $\varphi_\xi - \sigma$ and integrating over Ω, we have

$$-\frac{\omega}{\sigma}\int_\Omega (\varphi_\xi - \sigma)^2 dx = -\sigma\pi(b^2 - a^2)\langle f \rangle,$$

and hence $\operatorname{sgn}(\langle f \rangle) = \operatorname{sgn}(\omega) = \operatorname{sgn}(\nu)$.

When $\nu = 0$, $\varphi \in Z_0$ is an equilibrium solution of (1), that is, φ satisfies (11) with $\omega = 0$. Therefore arguing as above, we obtain $\langle f \rangle = 0$. □

Proof of Theorem 2.1. First we note that by virtue of Lemma 3.1 one of the following holds:

$$\text{(a) } \zeta^* = \zeta_* = +\infty, \quad \text{(b) } \zeta^* = \zeta_* = -\infty, \quad \text{(c) } \zeta^*, \zeta_* \text{ are finite.}$$

It follows from Proposition 3.2 and Lemma 3.4 that (a), (b), (c) holds if $\langle f \rangle > 0$, $\langle f \rangle < 0$, $\langle f \rangle = 0$, respectively.

(i) Suppose that $\langle f \rangle > 0$. Then by Lemma 3.3, (1) possesses a spiral traveling wave solution of the form $U(x,t) = \varphi\left(r, \theta - \frac{\omega}{\sigma} t\right) + \omega t$ with positive growth speed ω. If $\widetilde{U}$ is also a spiral traveling wave solution, then by Proposition 3.1, we have

$$\widetilde{U}(x,t) = (g_{\alpha_0}\varphi)\left(r, \theta - \frac{\omega}{\sigma} t\right) + \omega t$$

for some $\alpha_0 \in \mathbb{R}$. This equality yields

$$\widetilde{U}(\cdot,t) = U(\cdot, t + t_0), \quad t \geq 0$$

with $t_0 = \sigma\alpha_0/\omega$. By Lemma 3.3, we have $\varphi \in Z_{2\pi/\omega}$, namely, $\Phi_{2\pi/\omega}(\varphi) = g_{2\pi/\sigma}\varphi = \varphi + 2\pi$. The latter equality means

$$\varphi\left(r, \theta - \frac{2\pi}{\sigma}\right) = \varphi(r,\theta)$$

for all $(r,\theta) \in \overline{\Omega}$. We can prove the statement for $\langle f \rangle < 0$ in the same way.

(ii) When $\langle f \rangle = 0$, it follows from Proposition 3.2 and Lemma 3.2 that there exists an equilibrium solution $v(x)$ and that $Z_0 = \{g_\alpha v \mid \alpha \in \mathbb{R}\}$. Since $v(r, \theta + 2\pi/\sigma)$ is also an equilibrium solution, we have

$$v(r, \theta + 2\pi/\sigma) = v(r, \theta - \alpha_0) + \sigma\alpha_0, \quad (r, \theta) \in \overline{\Omega}$$

for some $\alpha_0 \in \mathbb{R}$. Comparing the maximum of both sides, we get $\alpha_0 = 0$, in other words, v is $2\pi/\sigma$-periodic in θ. □

Proof of Theorem 2.2. (i) In the case where $\langle f \rangle > 0$, Lemma 3.2(ii) yields that $U_t(x,t) \geq 0$ and $U_t(x,t) \not\equiv 0$ for all $x \in \overline{\Omega}$, $t > 0$. Therefore, it follows from the strong maximum principle that $U_t(x,t) > 0$ for $x \in \overline{\Omega}$, $t > 0$.

Next we show that U is stable in the sense of Lyapunov. For any $\varepsilon > 0$, take $\delta_0 > 0$ satisfying $\|U(\cdot, \delta_0) - U(\cdot, -\delta_0)\|_{C(\overline{\Omega})} < \varepsilon$ and set

$$\delta = \min\{U(x, \delta_0) - U(x, 0) \mid x \in \overline{\Omega}\} = \min\{U(x, 0) - U(x, -\delta_0) \mid x \in \overline{\Omega}\} > 0.$$

Then, for any solution u of (1) satisfying $\|u(\cdot, 0) - U(\cdot, 0)\|_{C(\overline{\Omega})} < \delta$, we have

$$U(\cdot, -\delta_0) < u(\cdot, 0) < U(\cdot, \delta_0).$$

Therefore, by the positivity of U_t and the strong maximum principle, we obtain

$$\begin{aligned} U(\cdot, t - \delta_0) &< U(\cdot, t) < U(\cdot, t + \delta_0), \\ U(\cdot, t - \delta_0) &< u(\cdot, t) < U(\cdot, t + \delta_0), \end{aligned}$$

hence

$$\begin{aligned} \|u(\cdot, t) - U(\cdot, t)\|_{C(\overline{\Omega})} &< \|U(\cdot, t + \delta_0) - U(\cdot, t - \delta_0)\|_{C(\overline{\Omega})} \\ &= \|U(\cdot, \delta_0) - U(\cdot, -\delta_0)\|_{C(\overline{\Omega})} < \varepsilon \end{aligned}$$

for all $t > 0$. We can treat the case $\langle f \rangle < 0$ in the same manner.

(ii) By Lemma 3.2(i), we have $\frac{\partial}{\partial \alpha}(g_\alpha v)(x) \geq 0$ for all $x \in \overline{\Omega}$. Hence the statement (ii) immediately follows from the strong maximum principle. □

§4. Asymptotic Stability

In this section we study the asymptotic stability of spiral traveling wave solutions. For the proof, monotonicity of spiral traveling wave solutions plays a crucial role.

The following lemma is a modified version of Property (B2) in [2], where Xinfu Chen has studied, among other things, the asymptotic stability of traveling waves in one space dimensional evolution equations.

Lemma 4.1. *There exists a constant $d \in (0, 1/|\Omega|)$ such that for any supersolution $w^+(x,t)$ and any subsolution $w^-(x,t)$ of (1) satisfying $w^+(x,0) \geq w^-(x,0)$ for $x \in \overline{\Omega}$, we have*

$$w^+(x,1) - w^-(x,1) \geq d \int_\Omega \{w^+(y,0) - w^-(y,0)\} dy \tag{12}$$

for all $x \in \overline{\Omega}$.

Proof. Let $\Gamma(x,y,t)$ be the fundamental solution ([5], [7]) of the problem

$$\begin{cases} u_t = \Delta u & \text{in } \Omega \times [0,+\infty), \\ u_r = 0 & \text{on } \partial\Omega \times [0,+\infty). \end{cases}$$

Note that the fundamental solution Γ is continuous and positive in $\overline{\Omega} \times \overline{\Omega} \times (0,+\infty)$. We also remark that

$$\int_\Omega \Gamma(x,y,1) dy = 1, \quad x \in \overline{\Omega},$$

since we impose the homogeneous Neumann boundary conditions.

By the comparison theorem we get $w^+(x,t) \geq w^-(x,t)$ for $x \in \overline{\Omega}$ and $t \geq 0$. Hence $W(x,t) = w^+(x,t) - w^-(x,t)$ satisfies

$$\begin{cases} W_t \geq \Delta W - \rho W & \text{in } \Omega \times [0,+\infty), \\ W_r \geq 0 & \text{on } \partial\Omega \times [0,+\infty), \end{cases}$$

where $\rho = \sup_{u \in \mathbb{R}} |f'(u)| > 0$. Again, by the comparison theorem, we obtain

$$W(x,t) \geq e^{-\rho t} \int_\Omega \Gamma(x,y,t) W(y,0) dy, \quad x \in \overline{\Omega}, \ t \geq 0. \tag{13}$$

Therefore, we see that

$$W(x,1) \geq d \int_\Omega W(y,0) dy, \quad x \in \overline{\Omega}, \tag{14}$$

where d is a positive constant defined by

$$d = e^{-\rho} \inf_{(x,y) \in \overline{\Omega} \times \overline{\Omega}} \Gamma(x,y,1) < \frac{1}{|\Omega|}.$$

□

Let $U(x,t)$ be the spiral traveling wave solution obtained in Theorem 2.1. We define positive constants M, m and δ_* by

$$\begin{aligned} M &= \max\{U_t(x,t) \mid x \in \overline{\Omega}, t \in \mathbb{R}\},\\ m &= \min\{U_t(x,t) \mid x \in \overline{\Omega}, t \in \mathbb{R}\},\\ \delta_* &= \frac{dm|\Omega|}{2M}. \end{aligned}$$

By Lemma 4.1, the constant δ_* satisfies $0 < \delta_* < 1/2$.

Lemma 4.2. *Let $u(x,t)$ be a solution of* (1) *such that*

$$U(x,t_0+T_0) \le u(x,t_0) \le U(x,t_0+T_0+h_0), \quad x \in \overline{\Omega}$$

for some $t_0 \ge 0$, $T_0 \in \mathbb{R}$ and $h_0 > 0$. Then, for any $t \ge t_0+1$ it holds that

$$U(x,t+T_1) \le u(x,t) \le U(x,t+T_1+h_1), \quad x \in \overline{\Omega}, \tag{15}$$

where $T_1 \in \{T_0, T_0+\delta_* h_0\}$ and $h_1 = (1-\delta_*)h_0$.

Proof. We may assume $t_0 = 0$ without loss of generality. By the comparison theorem,

$$U(x,t+T_0) \le u(x,t) \le U(x,t+T_0+h_0), \quad x \in \overline{\Omega}, \ \ t \ge 0. \tag{16}$$

Since

$$\int_\Omega \{U(y,T_0+h_0) - U(y,T_0)\}dy \ge m|\Omega|h_0,$$

either of the following holds:

$$\begin{aligned} &\text{(i)} \int_\Omega \{u(y,0) - U(y,T_0)\}dy \ge m|\Omega|h_0/2,\\ &\text{(ii)} \int_\Omega \{U(y,T_0+h_0) - u(y,0)\}dy \ge m|\Omega|h_0/2. \end{aligned}$$

Here we consider only the case (i), since the other is treated similarly. By Lemma 4.1,

$$u(x,1) - U(x,1+T_0) \ge d\int_\Omega \{u(y,0) - U(y,T_0)\} \ge dm|\Omega|h_0/2$$

for $x \in \overline{\Omega}$. Since $U(x,1+T_0+\delta_* h_0) - U(x,1+T_0) \le M\delta_* h_0 = dm|\Omega|h_0/2$, we have $u(x,1) \ge U(x,1+T_0+\delta_* h_0)$ for $x \in \overline{\Omega}$, hence

$$u(x,t) \ge U(x,t+T_0+\delta_* h_0), \quad x \in \overline{\Omega}, \ \ t \ge 1. \tag{17}$$

Combining (16) and (17), we obtain the inequality (15) with $T_1 = T_0 + \delta_* h_0$ and $h_1 = T_0 + h_0 - T_1 = (1 - \delta_*)h_0$. □

Proof of Theorem 2.3. Let $u_0 \in C(\overline{\Omega})$ and $u(x,t)$ be the solution of (1) with initial data u_0. We take $T_0 \in \mathbb{R}$ and $h_0 > 0$ satisfying

$$U(x, T_0) \leq u_0(x) \leq U(x, T_0 + h_0), \quad x \in \overline{\Omega}.$$

It follows from Lemma 4.2 and a mathematical induction that for any $k \in \mathbb{N}$, $t \in [k, k+1)$ and $x \in \overline{\Omega}$,

$$U(x, t + T_k) \leq u(x,t) \leq U(x, t + T_k + h_k)$$

with $T_k \in \{T_{k-1}, T_{k-1} + \delta_* h_{k-1}\}$, $h_k = (1 - \delta_*)h_{k-1}$. Therefore we obtain

$$U(x, t + T(t)) \leq u(x,t) \leq U(x, t + T(t) + h(t)), \quad x \in \overline{\Omega}, \ \ t \geq 0,$$

where $T(t) = T_{[t]}$, $h(t) = h_{[t]}$ and $[t]$ is the largest integer less than or equal to t. By the definition of $T(t)$ and $h(t)$,

$$\begin{aligned} &h(t) = (1 - \delta_*)^{[t]} h_0, \\ &0 \leq T(t_1) - T(t_2) \leq \{(1 - \delta_*)^{[t_2]} - (1 - \delta_*)^{[t_1]}\} h_0, \end{aligned}$$

for any $t \geq 0$ and $t_1 > t_2 \geq 0$. Thus the limit $\lim_{t \to +\infty} T(t) = \tau_0$ exists and satisfies $0 \leq \tau_0 - T(t) \leq (1 - \delta_*)^{[t]} h_0$. Hence, letting $\mu = -\log(1 - \delta_*) > 0$, we have

$$\|u(\cdot\,, t) - U(\cdot\,, t + \tau_0)\|_{C(\overline{\Omega})} \leq M_0 e^{-\mu t}, \quad t \geq 0$$

with $M_0 = M h_0 / (1 - \delta_*)$.

Note that $U(\cdot, t) = \Phi_t(\varphi) = g_{\frac{\omega}{\sigma} t} \varphi$. Therefore, setting

$$M = \max_{x \in \overline{\Omega}, \alpha \in \mathbb{R}} \frac{\partial}{\partial \alpha} (g_\alpha v)(x), \quad m = \min_{x \in \overline{\Omega}, \alpha \in \mathbb{R}} \frac{\partial}{\partial \alpha} (g_\alpha v)(x),$$

and replacing $U(\cdot, \cdot + s)$ by $g_s v$ in the above argument, we obtain the statement for the case where $\langle f \rangle = 0$. □

Appendix

In this appendix we present a proposition in [14] concerning with the structure of a subset of an ordered metric space under a group action.

Let X be an ordered metric space. In other words, X is a metric space on which a closed partial order relation is defined. We will denote by $\leq$ the order relation in X. Here, we say that a partial order relation in X is *closed* if $\varphi_n \leq \psi_n$ $(n = 1, 2, 3, \cdots)$ implies $\lim_{n\to\infty} \varphi_n \leq \lim_{n\to\infty} \psi_n$ provided that both limits exist. We write $\varphi < \psi$ if $\varphi \leq \psi$ and $\varphi \neq \psi$.

Let G be a metrizable topological group acting on some subset X_1 of X. We say G acts on X_1 if there exists a continuous mapping $\gamma\colon G \times X_1 \to X_1$ such that $g \mapsto \gamma(g, \cdot)$ is a group homomorphism of G into $Hom(X_1)$, the group of homeomorphisms of X_1 onto itself. For brevity, we write $\gamma(g, \varphi) = g\varphi$ and identify the element $g \in G$ with its action $\gamma(g, \cdot)$. We assume that

(G1) γ is order-preserving (that is, $\varphi \leq \psi$ implies $g\varphi \leq g\psi$ for any $g \in G$);

(G2) G is connected.

Let Y be a subset of X and $\overline{\varphi}$ be an element of $Y \cap X_1$ such that

(H1) $g\overline{\varphi} \in Y$ for any $g \in G$;

(H2) for any $\psi \in Y$, there exist some $g_1, g_2 \in G$ satisfying $g_1\overline{\varphi} < \psi < g_2\overline{\varphi}$;

(H3) for any $\psi \in Y$ with $\psi < h\overline{\varphi}$ for some $h \in G$, there exists some neighborhood B of the unit element of G such that $\psi < gh\overline{\varphi}$ for any $g \in B$.

Proposition 4.1 [14, Proposition B1]**.** *Let G satisfy* (G1), (G2) *and Y, $\overline{\varphi}$ satisfy* (H1), (H2), (H3). *Then Y is a totally-ordered connected set and $Y = G\overline{\varphi}$. Furthermore, if Y is locally precompact, then Y is homeomorphic and order-isomorphic to $\mathbb{R}$.*

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