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Q-reflexive Locally Convex Spaces

By

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Abstract

For a locally convex space E we use the Aron-Berner extension to define canonical mappings from $\widehat{\bigotimes_{s,n,\pi}} E''_e$ into different duals of $\mathcal{P}(^nE)$. We investigate necessary and sufficient conditions for the continuity of these mappings, paying particular attention to three special cases — Fréchet spaces, DF spaces and reflexive A-nuclear spaces. We define Q-reflexive spaces as spaces where a certain canonical mapping can be extended to an isomorphism between $\widehat{\bigotimes_{s,n,\pi}} E''_e$ and $\overline{(\mathcal{P}(^nE), \tau_b)'_i}$. We find examples of such spaces.

§1. Introduction

In [3] R. Aron and S. Dineen considered the problem of obtaining a polynomial functional representation of the bidual of the space of continuous n-homogeneous polynomials on a Banach space E. More precisely, they asked when the space $\mathcal{P}(^nE)''$ is isomorphic to $\mathcal{P}(^nE'')$ in a canonical way. Spaces with this property are called *Q-reflexive*. A reflexive Banach space E with the approximation property is Q-reflexive if and only if $\mathcal{P}(^nE)$ is reflexive.

In this article we consider the analogous problem when E is a locally convex space. When E is a Banach space, $\mathcal{P}(^nE)$ endowed with the topology of uniform convergence over the unit ball of E is a Banach space. The situation becomes complicated in the more general setting due to the increased choice of topologies on $\mathcal{P}(^nE)$ and the dual of E. To arrive at a suitable definition of Q-reflexive locally convex space we examine three classes of spaces which

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have shown themselves to be interesting from polynomial and holomorphic viewpoints — Fréchet spaces, DF spaces and fully nuclear spaces. We refer to [11] and [15] for background information on polynomials over locally convex spaces and the theory of locally convex spaces respectively.

§2. Biduals of Spaces of Homogeneous Polynomials with the Compact Open Topology

In this section we discuss spaces of polynomials endowed with the compact open topology τ_0. Biduality, when the domain space is either DF or Fréchet, is relatively straightforward in this case. We first, however, introduce some notation that will be used throughout the article. Let E be a locally convex space over the complex numbers $\mathbb{C}$. We will denote by $\overline{E}$ the completion of E, and by E' the space of all continuous linear functionals on E. If E' is endowed with the strong topology (i.e. the topology of uniform convergence over the bounded subsets of E) we denote it by E'_β. We say that E is *infrabarrelled* (or *quasibarrelled*) if the canonical inclusion of E into $E''_{\beta\beta} := (E'_\beta)'_\beta$ is continuous. Let $\mathcal{V}$ be a fundamental 0-neighbourhood basis of E, the collection $(V^{\circ\circ})_{V\in\mathcal{V}}$ is a fundamental 0-neighbourhood basis for the *natural topology* on E''. The bidual of E endowed with the natural topology is denoted by E''_e. It is well known that E is infrabarrelled if and only if $E''_e = E''_{\beta\beta}$, or, equivalently, if and only if the bounded subsets of E'_β are equicontinuous. A locally convex space E is barrelled if and only if the $\sigma(E', E)$-bounded subsets in E' are equicontinuous (thus every barrelled space is infrabarrelled). A locally convex space is distinguished if its strong dual is barrelled.

For E a locally convex space we let $\mathcal{P}_a({}^nE)$ denote the vector space of all n-homogeneous polynomials on E, and $\mathcal{P}({}^nE)$ denote the space of all *continuous* n-homogeneous polynomials on E. The topology on $\mathcal{P}({}^nE)$ of uniform convergence over the compact (respectively bounded) subsets of E is denoted by τ_0 (respectively τ_b). A third topology on $\mathcal{P}({}^nE)$ can be defined in the following way. A semi-norm p on $\mathcal{P}({}^nE)$ is τ_ω-continuous if for every zero neighbourhood V in E there exists a positive constant $C(V)$ such that

$$p(P) \leq C(V)\|P\|_V$$

for all $P \in \mathcal{P}({}^nE)$. The topology generated by all such semi-norms is denoted by τ_ω. When $n = 1$, $E'_i := (\mathcal{P}({}^1E), \tau_\omega)$ is the inductive dual of E, $E'_\beta := (\mathcal{P}({}^1E), \tau_b)$ is the strong dual of E and, if E is quasi-complete, $E'_c := (\mathcal{P}({}^1E), \tau_0)$.

If $\widehat{\bigotimes_{s,n,\pi}} E$ denotes the completed symmetric n-fold tensor product of E endowed with the projective tensor topology, then $(\widehat{\bigotimes_{s,n,\pi}} E)'_i$ and $(\mathcal{P}(^nE), \tau_\omega)$ are isomorphic. The space E has the $(BB)_n$ property if the closed convex hull of $\otimes_{n,s} B$ forms a fundamental system of bounded subsets of $\widehat{\bigotimes_{s,n,\pi}} E$ as B ranges over the bounded subsets of E. Clearly E has $(BB)_n$ if and only if $(\widehat{\bigotimes_{s,n,\pi}} E)'_\beta$ and $(\mathcal{P}(^nE), \tau_b)$ are isomorphic. A locally convex space in which all closed bounded sets are compact is called semi-Montel. A semi-Montel Fréchet space is called Fréchet-Montel and a semi-Montel DF space is called a DFM space.

Proposition 2.1. *Let E be a Fréchet space and n a positive integer. Then*

(a) $((\mathcal{P}(^nE), \tau_0)'_\beta)'_\beta = (\mathcal{P}(^nE), \tau_w)$ *if and only if* $\widehat{\bigotimes_{s,n,\pi}} E$ *is a distinguished Fréchet space.*

(b) $((\mathcal{P}(^nE), \tau_0)'_\beta)'_\beta = (\mathcal{P}(^nE), \tau_0)$ *if and only if E is a Fréchet-Montel space with the $(BB)_n$ property.*

Proof. By ([11], Proposition 2.20),

$$((\mathcal{P}(^nE), \tau_0)'_\beta)'_\beta = (\widehat{\bigotimes_{s,n,\pi}} E)'_\beta. \tag{2.1}$$

Hence (a) holds if and only if $(\widehat{\bigotimes_{s,n,\pi}} E)'_i = (\widehat{\bigotimes_{s,n,\pi}} E)'_\beta$. Since E is Fréchet $\widehat{\bigotimes_{s,n,\pi}} E$ is also Fréchet. As the strong and inductive duals of a Fréchet space have the same bounded sets, a result of Grothendieck ([14], Theorem 3.16.1) implies that $(\widehat{\bigotimes_{s,n,\pi}} E)'_i = (\widehat{\bigotimes_{s,n,\pi}} E)'_\beta$ if and only if $\widehat{\bigotimes_{s,n,\pi}} E$ is distinguished. This proves (a).

By (2.1), (b) holds if and only if $(\widehat{\bigotimes_{s,n,\pi}} E)'_\beta$ and $(\mathcal{P}(^nE), \tau_0) = (\widehat{\bigotimes_{s,n,\pi}} E)'_c$ are isomorphic, i.e. if and only if $\widehat{\bigotimes_{s,n,\pi}} E$ is Fréchet-Montel. By ([11], Proposition

1.35) and [1], $\widehat{\bigotimes_{s,n,\pi}} E$ is Fréchet-Montel space if and only if E is a Fréchet-Montel space with $(BB)_n$. This completes the proof of (b). $\square$

Proposition 2.2. *Let E be a complete infrabarrelled DF space. Then*

(a) $((\mathcal{P}(^nE),\tau_0)'_\beta)'_\beta = (\mathcal{P}(^nE),\tau_\omega)$ *for every n.*

(b) $((\mathcal{P}(^nE),\tau_0)'_\beta)'_\beta = (\mathcal{P}(^nE),\tau_0)$ *for every n if and only if E is a DFM space.*

Proof. (a) By ([16], p. 264), $(\mathcal{P}(^nE),\tau_0)' = \widehat{\bigotimes_{s,n,\pi}} E$ algebraically. The topology on $\widehat{\bigotimes_{s,n,\pi}} E$ is the topology of uniform convergence on the equicontinuous subsets of the dual $\big(\widehat{\bigotimes_{s,n,\pi}} E\big)' = \mathcal{P}(^nE)$, while the topology on $(\mathcal{P}(^nE),\tau_0)'_\beta$ is the topology of uniform convergence on the τ_0-bounded, or, by ([11], Lemma 1.23), the τ_b-bounded subsets of $\mathcal{P}(^nE)$. Let E be infrabarrelled, by ([15]), Proposition 15.6.8) $\widehat{\bigotimes_{s,n,\pi}} E$ is infrabarrelled and hence the strongly bounded and the equicontinuous subsets of its dual $\mathcal{P}(^nE)$ coincide. Since every DF space has $(BB)_n$, this means that the τ_b-bounded subsets and the equicontinuous subsets of $\mathcal{P}(^nE)$ coincide and $(\mathcal{P}(^nE),\tau_0)'_\beta = \widehat{\bigotimes_{s,n,\pi}} E$. By ([4], Corollary 3.4) $\big(\widehat{\bigotimes_{s,n,\pi}} E\big)'_\beta = \big(\widehat{\bigotimes_{s,n,\pi}} E\big)'_i$, hence

$$(\mathcal{P}(^nE),\tau_0)''_{\beta\beta} = \big(\widehat{\bigotimes_{s,n,\pi}} E\big)'_i = (\mathcal{P}(^nE),\tau_\omega).$$

This completes the proof of (a).

(b) Since E is a complete infrabarrelled DF space, by (a) $((\mathcal{P}(^nE),\tau_0)'_\beta)'_\beta = (\mathcal{P}(^nE),\tau_\omega)$ for every n. Suppose E is DFM, by ([11], Example 1.32) $\tau_0 = \tau_\omega$ on $\mathcal{P}(^nE)$, hence $((\mathcal{P}(^nE),\tau_0)'_\beta)'_\beta = (\mathcal{P}(^nE),\tau_0)$.

Conversely, suppose $((\mathcal{P}(^nE),\tau_0)'_\beta)'_\beta = (\mathcal{P}(^nE),\tau_0)$. By (a), $((\mathcal{P}(^nE),\tau_0)'_\beta)'_\beta = (\mathcal{P}(^nE),\tau_\omega)$, so $\tau_\omega = \tau_0$ on $\mathcal{P}(^nE)$. Since $\tau_\omega \geq \tau_b \geq \tau_0$, the Hahn-Banach Theorem implies that E is a DFM space. This completes the proof. $\square$

Remark 1. The space $E = \varprojlim(c_0(\Gamma'), \|\cdot\|_{\Gamma'})$, where the projective limit is over all countable $\Gamma' \subset \Gamma$ for an uncountable Γ, is a DF space which is not infrabarrelled. Nevertheless, it can be shown that $((\mathcal{P}(^nE),\tau_0)'_\beta)'_\beta = (\mathcal{P}(^nE),\tau_\omega)$ for every n.

§3. The Canonical Map J_n

In this section we consider $\mathcal{P}(^nE)$ endowed with the τ_ω and τ_b topologies. If $P \in \mathcal{P}(^nE)$ let $AB_n(P)$ denote the Aron-Berner extension of P to $E'' := (E'_\beta)'$ (see [2]). If $x'' \in E''$ then there exists a bounded subset B of E such that

$$|AB_n(P)(x'')| \leq \|P\|_B$$

for all $P \in \mathcal{P}(^nE)$. Thus the mapping

$$J_n : \bigotimes_{s,n,\pi} E'' \longrightarrow (\mathcal{P}(^nE), \tau_b)', \tag{3.1}$$

given by $[J_n(\otimes_n x'')](P) = [AB_n(P)](x'')$ for all $P \in \mathcal{P}(^nE)$ and all $x'' \in E''$, and extended by linearity, is well defined. Since the topology τ_ω is finer than τ_b, the mapping J_n is also well defined with range space $(\mathcal{P}(^nE), \tau_\omega)'$.

We are interested in turning J_n into a continuous mapping. To proceed we need to label the different topologies that we consider. The following diagram fixes our notation:

$$\begin{array}{ccccc} (\mathcal{P}(^nE), \tau_b)'_i & & \xrightarrow{\ i_n\ } & & (\mathcal{P}(^nE), \tau_\omega)'_i \\ & \nwarrow^{J_n^{bw}} & & \nearrow^{J_n^{ww}} & \\ \big\downarrow k_n & & \bigotimes_{s,n,\pi} E''_e & & \big\downarrow K_n \\ & \swarrow_{J_n^{bb}} & & \searrow_{J_n^{wb}} & \\ (\mathcal{P}(^nE), \tau_b)'_\beta & & \xrightarrow{\ I_n\ } & & (\mathcal{P}(^nE), \tau_\omega)'_\beta \end{array} \tag{3.2}$$

The diagonal mappings are just the mapping J_n with superscripts used to denote the structure of the range space. The mappings along the horizontal and vertical arrows are always well defined and continuous.

Remark 2. The continuity of $J_1^{bb} : E''_e \longrightarrow (\mathcal{P}(^1E), \tau_b)'_\beta = E''_{\beta\beta}$ implies that E is infrabarrelled.

We first consider the lower diagonal mappings in Diagram (3.2).

Proposition 3.1. *Let E be a locally convex space such that the τ-bounded sets of $\mathcal{P}(^nE)$, $\tau = \tau_b$ or τ_ω, are locally bounded for some n. Then the* *mapping*

$$J_n : \bigotimes_{s,n,\pi} E''_e \longrightarrow (\mathcal{P}(^nE), \tau)'_\beta$$

is continuous. If $\tau = \tau_\omega$ *then* J_n^{wb} *can be extended to the completion* $\widehat{\bigotimes_{s,n,\pi}} E''_e$.

Proof. By our hypothesis the topology on $(\mathcal{P}(^nE), \tau)'_\beta$ is generated by the semi-norms

$$\alpha_V(\phi) = \sup\big\{|\phi(P)| : \|P\|_V \le 1\big\},$$

where V ranges over the convex balanced neighbourhoods of zero in E. Let $P \in \mathcal{P}(^nE)$ and $\overset{\vee}{AB}_n(P)$ be the symmetric n-linear form associated with $AB_n(P)$. The mapping

$$j_n : (x''_1, \dots, x''_n) \longrightarrow [P \to [\overset{\vee}{AB}_n(P)](x''_1, \dots, x''_n)],$$

where $x''_i \in E''_e$ for $1 \le i \le n$, is symmetric, n-linear, and has linearization J_n. If V is a convex balanced neighbourhood of zero in E, then by ([11], Proposition 1.53) and the Polarization Formula,

$$\big|[\overset{\vee}{AB}_n(P)](x''_1, \dots, x''_n)\big| \le \frac{n^n}{n!}\|x''_1\|_{V^{\circ\circ}} \cdots \|x''_1\|_{V^{\circ\circ}}\|P\|_V$$

where $x''_i \in E''_e$ for $1 \le i \le n$ and $P \in \mathcal{P}(^nE)$. Hence

$$\begin{aligned}\alpha_V\big(j_n(x''_1, \dots, x''_n)\big) &= \sup\{\big([\overset{\vee}{AB}_n(P)](x''_1, \dots, x''_n)\big) : \|P\|_V \le 1\}\\ &\le \frac{n^n}{n!}\|x''_1\|_{V^{\circ\circ}} \cdots \|x''_1\|_{V^{\circ\circ}},\end{aligned}$$

and j_n is continuous. By the definition of the projective tensor product this implies that J_n is also continuous.

When $\tau = \tau_\omega$ the space $(\mathcal{P}(^nE), \tau_\omega)'_\beta$ is complete as the strong dual of a bornological space, and consequently J_n can be extended to $\widehat{\bigotimes_{s,n,\pi}} E''_e$ by continuity. □

Next we consider the mapping J_n^{bw}, concentrating on some special cases.

If E is a Fréchet space $(\mathcal{P}(^nE), \tau_\omega)$ is a barrelled DF space, hence its strong and inductive duals coincide by [4]. Thus K_n is an isomorphism and $J_n^{wb} = J_n^{ww}$ for every n. Moreover, the τ_ω-bounded and the τ_b-bounded subsets of $\mathcal{P}(^nE)$ are locally bounded and hence, by Proposition 3.1, the mappings $J_n^{wb} = J_n^{ww}$ and J_n^{bb} are continuous.

Proposition 3.2. *Let E be a Fréchet space with $(BB)_n$ for some n. Then J_n^{bw} is continuous, k_n is an isomorphism and $J_n^{bw} = J_n^{bb}$.*

Proof. Since E has $(BB)_n$ we have

$$(\mathcal{P}(^nE), \tau_b)'_i = \big((\widehat{\bigotimes_{s,n,\pi} E})'_\beta\big)'_\beta = (\mathcal{P}(^nE), \tau_b)'_\beta,$$

hence $J_n^{bw} = J_n^{bb}$ and, in particular, J_n^{bw} is continuous. Moreover, as

$$(\mathcal{P}(^nE), \tau_b)'_i = (\mathcal{P}(^nE), \tau_b)'_\beta,$$

k_n is an isomorphism and $J_n^{bw} = J_n^{bb}$. □

Next suppose E is a DF space. By ([11], Example 1.32) $\tau_b = \tau_\omega$ on $\mathcal{P}(^nE)$ for every n, hence i_n and I_n are isomorphisms. Thus $J_n^{wb} = J_n^{bb}$ and $J_n^{bw} = J_n^{ww}$ for every n.

Proposition 3.3. *Let E be a DF space.*

(a) *The mapping*

$$J_n^{bb} : \bigotimes_{s,n,\pi} E''_e \longrightarrow (\mathcal{P}(^nE), \tau_b)'_\beta$$

is continuous for every n if and only if E is infrabarrelled. In this case J_n^{bb} can be extended to the completion $\widehat{\bigotimes_{s,n,\pi}} E''_e = \widehat{\bigotimes_{s,n,\pi}} E''_{\beta\beta}$.

(b) *The mapping*

$$J_n^{bw} : \bigotimes_{s,n,\pi} E''_e \longrightarrow (\mathcal{P}(^nE), \tau_b)'_i$$

is continuous for every n if and only if E and $E''_{\beta\beta}$ are infrabarrelled. In this case J_n^{bw} can be extended to the completion $\widehat{\bigotimes_{s,n,\pi}} E''_e = \widehat{\bigotimes_{s,n,\pi}} E''_{\beta\beta}$.

Proof. (a) If J_n^{bb} is continuous E is infrabarrelled by Remark 2. Conversely, if E is an infrabarrelled DF space then by ([15], Proposition 15.6.8) $\bigotimes_{s,n,\pi} E$ is an infrabarrelled DF space and consequently the bounded subsets of $\big(\bigotimes_{s,n,\pi} E\big)'_\beta$ are equicontinuous. As a DF space E has $(BB)_n$, this implies

$\big(\bigotimes_{s,n,\pi} E\big)'_\beta = (\mathcal{P}(^nE), \tau_b)$. By Proposition 3.1, J_n^{bb} is continuous. Since $(\mathcal{P}(^nE), \tau_b)$ is Fréchet, it is bornological. Hence $(\mathcal{P}(^nE), \tau_b)'_\beta$ is complete and the continuous mapping J_n^{bb} can be extended to $\widehat{\bigotimes_{s,n,\pi}} E''_{\beta\beta}$.

(b) Let E and $E''_{\beta\beta}$ be infrabarrelled DF spaces. As $E''_{\beta\beta}$ is the strong dual of a metrizable space, it is barrelled and bornological ([15], Corollary 13.4.4). By ([15], 15.6.8) $\bigotimes_{s,n,\pi} E''_{\beta\beta}$ is a bornological DF space. By (a) J_n^{bb} is continuous and hence maps the bounded sets of $\bigotimes_{s,n,\pi} E''_{\beta\beta}$ onto bounded sets in $(\mathcal{P}(^nE), \tau_b)'_\beta$. Since $\big(\widehat{\bigotimes_{s,n,\pi}} E\big)'_\beta$ is Fréchet, $(\mathcal{P}(^nE), \tau_b)'_\beta = \big(\big(\widehat{\bigotimes_{s,n,\pi}} E\big)'_\beta\big)'_\beta$ and $(\mathcal{P}(^nE), \tau_b)'_i = \big(\big(\widehat{\bigotimes_{s,n,\pi}} E\big)'_\beta\big)'_i$ have the same bounded sets ([11], Example 1.24). Hence J_n^{bw} maps bounded sets onto bounded sets and by ([14], Proposition 3.7.1) is continuous.

Conversely, let J_n^{bw} be continuous. Then J_n^{bb} is continuous and E is infrabarrelled by Remark 2. When $n = 1$ we obtain that $E''_{\beta\beta} = E''_{\beta i}$, and since inductive duals are barrelled, $E''_{\beta\beta}$ is barrelled and hence infrabarrelled. Moreover, since $(\mathcal{P}(^nE), \tau_b) = \big(\widehat{\bigotimes_{s,n,\pi}} E\big)'_\beta$ is Fréchet, its inductive dual is complete ([15], Corollary 13.4.3). Hence we can extend J_n^{bw} from $\bigotimes_{s,n,\pi} E''_{\beta\beta}$ to $\widehat{\bigotimes_{s,n,\pi}} E''_{\beta\beta}$ by continuity. □

Now we consider polynomials on reflexive A-nuclear spaces (a number of these results also hold for fully nuclear spaces and for fully nuclear spaces with a basis). A locally convex space E is *A-nuclear* if it has an absolute basis $(e_n)_n$ and there exists a sequence of positive real numbers $(\delta_n)_n$, $\sum_{i=1}^\infty \frac{1}{\delta_i} < \infty$, such that for each $p \in cs(E)$ the semi-norm

$$q\left(\sum_{i=1}^\infty x_i e_i\right) = \sum_{i=1}^\infty \delta_i p(x_i e_i)$$

is continuous. By the Grothendieck-Pietsch criterion every A-nuclear space is nuclear. Since the closed bounded subsets of a complete A-nuclear space E are compact, $\tau_0 = \tau_b$ on $\mathcal{P}(^nE)$ for every n.

A polynomial $P \in \mathcal{P}(^nE)$ has *finite rank* if there exists a finite subset $\{\varphi_i\}_{i=1}^{l}$ in E' such that

$$P(x) = \sum_{i=1}^{l} \varphi_i^n(x)$$

for all $x \in E$. We let $\mathcal{P}_f(^nE)$ denote the space of all n-homogeneous polynomials of finite rank on E. By ([7], p. 186),

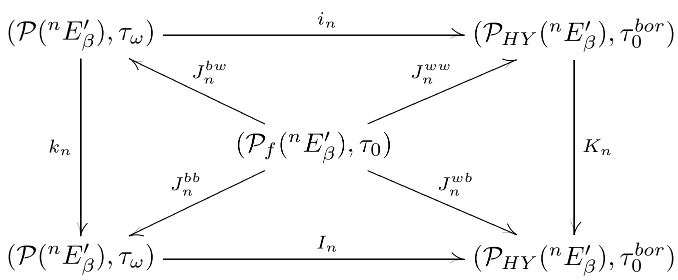
$$\mathcal{P}_f(^nE) = \bigotimes_{s,n} E'_\beta. \tag{3.3}$$

Polynomials in $\mathcal{P}_A(^nE)$, the closure of $\mathcal{P}_f(^nE)$ in $(\mathcal{P}(^nE), \tau_b)$, are called *continuous approximable polynomials.*

An element in $\mathcal{P}_a(^nE, F)$ is *hypocontinuous* if its restriction to each compact set is continuous. We let $\mathcal{P}_{HY}(^nE, F)$ denote the vector space of all hypocontinuous n-homogeneous polynomials from E into F.

Proposition 3.4. *If E is a reflexive A-nuclear space then Diagram (3.2) takes the following form*:

(3.4)



where k_n and K_n are isomorphisms.

Proof. If E is a reflexive A-nuclear space then in a way similar to ([11], Proposition 3.46) it can be shown that $(\mathcal{P}(^nE), \tau_0)$ and $(\mathcal{P}(^nE), \tau_\omega)$ are A-nuclear. Thus $\big(\overline{\mathcal{P}(^nE), \tau_0}\big)$ and $\big(\overline{\mathcal{P}(^nE), \tau_\omega}\big)$ are complete A-nuclear and, by [4], are reinforced regular, i.e. $\big(\overline{\mathcal{P}(^nE), \tau_0}\big)'_\beta = \big(\overline{\mathcal{P}(^nE), \tau_0}\big)'_i$ and $\big(\overline{\mathcal{P}(^nE), \tau_\omega}\big)'_\beta = \big(\overline{\mathcal{P}(^nE), \tau_\omega}\big)'_i$. By ([10], Corollary 5.7) $\big(\overline{\mathcal{P}(^nE),\ \tau_0}\big)'_\beta = (\mathcal{P}(^nE), \tau_0)'_\beta$ and $\big(\overline{\mathcal{P}(^nE), \tau_\omega}\big)'_\beta = \big(\mathcal{P}(^nE), \tau_\omega\big)'_\beta$. Since similar equalities hold for the inductive duals ([15], p. 200), k_n and K_n are isomorphisms.

By ([10], Proposition 1.56) $(\mathcal{P}(^nE), \tau_0)'_\beta = (\mathcal{P}(^nE'_\beta), \tau_\omega)$ and by ([10], Proposition 1.48) the Borel transform is an algebraic isomorphism from

$(\mathcal{P}(^nE),\tau_\omega)'$ onto $\mathcal{P}_{HY}(^nE'_\beta)$ under which the equicontinuous subsets of $(\mathcal{P}(^nE),\tau_\omega)'$ can be identified with the τ_0-bounded subsets of $\mathcal{P}_{HY}(^nE'_\beta)$. Let $(\mathcal{P}(^nE),\tau_\omega)'_\beta = (\mathcal{P}_{HY}(^nE'_\beta),\tau)$ for some topology τ. Since $(\mathcal{P}(^nE),\tau_\omega)$ is barrelled, the equicontinuous subsets of its dual coincide with the τ-bounded subsets of $\mathcal{P}_{HY}(^nE'_\beta)$. Hence τ and τ_0 define the same bounded sets on $\mathcal{P}_{HY}(^nE'_\beta)$. Since K_n is an isomorphism $(\mathcal{P}(^nE),\tau_\omega)'_\beta$ is bornological and hence τ is the bornological topology associated with τ_0, τ_0^{bor}.

Finally, since E is infrabarrelled $E''_e = E''_{\beta\beta}$, and, by (3.3), $\bigotimes_{s,n} E''_{\beta\beta} = \mathcal{P}_f(^nE'_\beta)$. As E is a reflexive nuclear space, $\bigotimes_{s,n,\pi} E''_{\beta\beta} = (\mathcal{P}_f(^nE'_\beta),\tau_0)$ ([11], Proposition 2.13). □

Let E be a reflexive A-nuclear space and n a positive integer. By Diagram (3.4) and the proof of Proposition 3.4 we have established the following identifications:

$$(\mathcal{P}(^nE),\tau_\omega)'_\beta = (\mathcal{P}(^nE),\tau_w)'_i = (\mathcal{P}_{HY}(^nE'_\beta),\tau_0^{bor}), \tag{3.5}$$

$$(\mathcal{P}(^nE),\tau_0)'_\beta = (\mathcal{P}(^nE),\tau_0)'_i = (\mathcal{P}(^nE'_\beta),\tau_\omega). \tag{3.6}$$

By (3.5), $(\mathcal{P}_{HY}(^nE'_\beta),\tau_0^{bor})$ is complete as a strong dual of bornological space.

Corollary 3.1. *Let E be a reflexive A-nuclear space and n a positive integer. Then*

(a) *J_n^{wb} is continuous if and only if $\tau_0 = \tau_0^{bor}$ on $\mathcal{P}_{HY}(^nE'_\beta)$.*

(b) *J_n^{bw} is continuous if and only if $\tau_0 = \tau_\omega$ on $\mathcal{P}(^nE'_\beta)$.*

(c) *I_n is an isomorphism if and only if $\tau_0 = \tau_\omega$ on $\mathcal{P}(^nE)$.*

Proof. (a) If $\tau_0 = \tau_0^{bor}$ on $\mathcal{P}_{HY}(^nE'_\beta)$ then J_n^{wb} is continuous by Diagram (3.4). Conversely, if J_n^{wb} is continuous then it extends to a continuous mapping $\tilde{J}_n^{wb}$ from $(\mathcal{P}_{HY}(^nE'_\beta),\tau_0)$ into $(\mathcal{P}_{HY}(^nE'_\beta),\tau_0^{bor})$. Since $J_n^{wb}(P) = P$ for all P on a dense subspace of $\mathcal{P}_{HY}(^nE'_\beta)$, we have $\tilde{J}_n^{wb}(P) = P$ for all $P \in \mathcal{P}_{HY}(^nE'_\beta)$. Hence $\tau_0 = \tau_0^{bor}$ on $\mathcal{P}_{HY}(^nE'_\beta)$.

(b) The method used for (a) can be adapted to prove (b). We give, however, an alternative proof. Clearly, by Diagram (3.4), if $\tau_0 = \tau_\omega$ on $\mathcal{P}(^nE'_\beta)$ then J_n^{bw} is continuous. Conversely, let J_n^{bw} be continuous. If p is a τ_ω-continuous semi-norm on $\mathcal{P}(^nE'_\beta)$ then there exist a compact polydisc $K \subset E'_\beta$ such that $p(P) \leq \|P\|_K$ for all $P \in \mathcal{P}_f(^nE'_\beta)$. If $\delta = (\delta_n)_n$ is the sequence defining

A-nuclearity, as in ([11], p. 205) it can be shown that there exists $C(\delta) > 0$ such that

$$\sum_{m\in\mathbb{N}^{(\mathbb{N})},\,|m|=n} |a_m|\|z^m\|_K \le C(\delta)\Big\|\sum_{m\in\mathbb{N}^{(\mathbb{N})},\,|m|=n} a_m z^m\Big\|_{\delta K}$$

for all $\sum_{m\in\mathbb{N}^{(\mathbb{N})},\,|m|=n} a_m z^m \in \mathcal{P}(^nE'_\beta)$. The set δK is a compact polydisc in E'_β. By the proof of ([11], Proposition 3.45) the semi-norm

$$\tilde{p}\Big(\sum_{m\in\mathbb{N}^{(\mathbb{N})},\,|m|=n} a_m z^m\Big) := \sum_{m\in\mathbb{N}^{(\mathbb{N})},\,|m|=n} |a_m| p(z^m)$$

is τ_ω-continuous and $p \le \tilde{p}$. Hence for all $P = \sum_{m\in\mathbb{N}^{(\mathbb{N})},\,|m|=n} a_m z^m \in \mathcal{P}(^nE'_\beta)$ we have

$$\begin{aligned} p(P) \le \tilde{p}(P) &= \sum_{m\in\mathbb{N}^{(\mathbb{N})},\,|m|=n} |a_m| p(z^m) \\ &\le \sum_{m\in\mathbb{N}^{(\mathbb{N})},\,|m|=n} |a_m|\|z^m\|_K \le C(\delta)\|P\|_{\delta K}. \end{aligned}$$

Hence $\tau_\omega = \tau_0$ on $\mathcal{P}(^nE'_\beta)$.

(c) If I_n is an isomorphism then, by Diagram (3.4), $\mathcal{P}(^nE'_\beta) = \mathcal{P}_{HY}(^nE'_\beta)$. By ([10], Propositions 1.47 and 1.48) this implies $(\mathcal{P}(^nE), \tau_0)' = (\mathcal{P}(^nE), \tau_\omega)'$. Since the monomials form an absolute basis for both $(\mathcal{P}(^nE), \tau_0)$ and $(\mathcal{P}(^nE), \tau_\omega)$, by ([11], Lemma 4.41) $\tau_0 = \tau_\omega$ on $\mathcal{P}(^nE)$.

Conversely, if $\tau_0 = \tau_\omega$ on $\mathcal{P}(^nE)$ then, by Diagram (3.2), I_n is an isomorphism. □

Proposition 3.5. *If E is a reflexive A-nuclear space then the following are equivalent*:

(a) J_n^{wb} *is continuous.*

(b) *The τ_ω-bounded sets of $\mathcal{P}(^nE)$ are locally bounded.*

(c) $(\mathcal{P}(^nE), \tau_\omega)'_\beta = (\mathcal{P}_{HY}(^nE'_\beta), \tau_0)$.

(d) $(\mathcal{P}(^nE), \tau_\omega)$ *is quasi-complete.*

(e) $(\mathcal{P}(^nE), \tau_\omega)$ *is semi-reflexive.*

Proof. If the τ_ω-bounded sets of $\mathcal{P}(^nE)$ are locally bounded then J_n^{wb} is continuous by Proposition 3.1, hence (b)⇒(a).

Conversely, suppose J_n^{wb} is continuous. By Corollary 3.1, $\tau_0 = \tau_0^{bor}$ on $\mathcal{P}_{HY}({}^nE'_\beta)$ and (a)⇒(c) by (3.5).

(b)⇔(c) follows from ([10], Proposition 1.57). By the proof of ([10], Proposition 5.37) conditions (b), (d) and (e) are equivalent. □

Further equivalent conditions can be found in [12]. By Corollary 3.1(b) and the proof of ([11], Corollary 4.46) we obtain the following result.

Proposition 3.6. *If E is a reflexive A-nuclear space then the following are equivalent*:

(a) J_n^{bw} *is continuous.*

(b) *The τ_0-bounded sets of $\mathcal{P}({}^nE)$ are locally bounded.*

(c) $\mathcal{P}({}^nE) = \mathcal{P}_{HY}({}^nE)$.

(d) $(\mathcal{P}({}^nE), \tau_0)$ *is complete.*

Remark 3.

(a) Proposition 3.5 shows that the hypothesis in Proposition 3.1 is both necessary and sufficient when E is a reflexive A-nuclear space.

(b) If E is a reflexive A-nuclear space and J_n^{wb} (respectively J_n^{bw}) is continuous, then it extends to define an isomorphism onto $\mathcal{P}_{HY}({}^nE'_\beta)$ (respectively $\mathcal{P}({}^nE'_\beta)$).

(c) If E is Fréchet nuclear (or DFN) with basis, then E is a reflexive A-nuclear space and $\tau_0 = \tau_\omega$ on $\mathcal{P}({}^nE'_\beta)$ for every n ([11], Example 2.18). Hence both J_n^{wb} and J_n^{bw} extend to isomorphisms from the respective completions of their domains.

(d) Countable direct sums and products of reflexive A-nuclear spaces are again reflexive A-nuclear spaces.

Example 1.

1. Let $E = \prod_{k=1}^{\infty} E_k$ where each E_k is a DFN space. Then J_n^{wb} is always continuous and J_n^{bw} is continuous if and only if each $(E_k)'_\beta$ admits a continuous norm.

2. Let $E = \bigoplus_{k=1}^{\infty} E_j$ where each E_j is Fréchet nuclear space with a basis. Then

(a) J_n^{wb} is continuous if and only if E is isomorphic to one of the spaces $\mathbb{C}^{(\mathbb{N})}$, $\mathbb{C}^{(\mathbb{N})} \times \mathbb{C}^{\mathbb{N}}$ or $(\mathbb{C}^{\mathbb{N}})^{(\mathbb{N})}$.

(b) J_n^{bw} is continuous if and only if E is isomorphic to $\mathbb{C}^{(\mathbb{N})}$.

Proof. (1) The τ_ω-bounded subsets of $\mathcal{P}(^nE)$ are locally bounded ([11], Example 3.24(c)), hence by Proposition 3.5 J_n^{wb} is continuous. By ([18], Proposition 2) $\mathcal{P}(^nE) = \mathcal{P}_{HY}(^nE)$ if and only if there exists a continuous norm on $(E_k)'_\beta$ for every k. It suffices to apply Proposition 3.6 to obtain the required result for J_n^{bw}.

(2) Part (a) follows from Proposition 3.5 and ([8], Theorem 1); part (b) follows from Proposition 3.6, ([8], Theorem 1) and ([11], Example 3.24(b)). □

Example 2. Let $\mathcal{D} = \bigoplus_{k=1}^\infty s_j$ where each s_j is the Fréchet nuclear space of rapidly decreasing sequences. By Example 1 neither of J_n^{bw} or J_n^{wb} are continuous. By ([6], Proposition 9) $\tau_0 = \tau_\omega$ on $\mathcal{P}(^n\mathcal{D})$, hence, by Corollary, 3.1(c) I_n is an isomorphism.

§4. Continuity of J_n

In Section 3 we concentrated on continuity of the mappings $J_n^{\cdot\cdot}$. In this section we discuss injectivity. Let $x'' \in E''_e$ and $\varphi \in E'$. Then $\otimes_n x'' \in \bigotimes_{s,n,\pi} E''_e$, $\varphi^n \in \mathcal{P}_f(^nE)$, and we have the duality

$$\langle \otimes_n x'', \varphi^n \rangle = x''(\varphi)^n. \tag{4.1}$$

Suppose

$$[J_n(\otimes_n x'')](\varphi^n) = 0$$

for every $\varphi \in E'$. Then, by (4.1), $x''(\varphi)^n = 0$ for all $x'' \in E''_e$ and hence $\otimes_n x'' = 0$. This motivates us to restrict our attention to $\mathcal{P}_f(^nE)$, and our results in Section 5 show that this is indeed a good choice. Let $R(T) := T|_{\mathcal{P}_f(^nE)}$ for $T \in (\mathcal{P}(^nE), \tau_b)'$. We let $J_n^f := R \circ J_n^{bw}$. By ([15], Proposition 10.3.4)

$$R : (\mathcal{P}(^nE), \tau_b)'_i \longrightarrow (\mathcal{P}_f(^nE), \tau_b)'_i$$

is continuous and open, hence if J_n^{bw} is continuous then J_n^f is continuous.

In order to investigate the continuity of J_n^f we require some further definitions. An n-homogeneous polynomial P on E is called *nuclear* if there exist an equicontinuous sequence $(\psi_i)_i$ in E' and $(\lambda_i)_i$ in l_1 such that

$$P(x) = \sum_{i=1}^\infty \lambda_i {\psi_i}^n(x)$$

for all x in E. Let $\mathcal{P}_N({}^nE)$ denote the space of all nuclear polynomials on E. If A is a subset of E let

$$\pi_{N,A}(P) = \|P\|_{N,A} := \inf\left[\sum_{i=1}^{\infty}|\lambda_i|\|\psi_i\|_A^n : P = \sum_{i=1}^{\infty}\lambda_i\psi_i^n\right]$$

As A ranges over the bounded sets of E we obtain the π_b topology. We also let

$$(\mathcal{P}_N({}^nE), \pi_\omega) = \varinjlim_{\alpha\in cs(E)} (\mathcal{P}_N({}^nE_\alpha), \pi_b).$$

The space of all n-homogeneous (algebraic) polynomials on E' which are bounded on the equicontinuous subsets of E' is denoted by $\mathcal{P}_\xi({}^nE')$. An n-homogeneous polynomial P on a locally convex space E is *integral* if there is an absolutely convex closed neighbourhood of 0, U, and a finite regular Borel measure μ on U° endowed with the w^*-topology, such that

$$P(x) = \int_{U^\circ}\psi^n(x)d\mu(\psi)$$

for all $x \in E$. The space of all n-homogeneous integral polynomials on E is denoted by $\mathcal{P}_I({}^nE)$, and the topology τ_I is defined as the locally convex inductive limit

$$(\mathcal{P}_I({}^nE), \tau_I) = \varinjlim_{U\in\mathcal{U}} (\mathcal{P}({}^nE_U), \|\cdot\|_{U,I}),$$

where

$$\|P\|_{U,I} = \inf\left\{\|\mu\|_{U^\circ} : P(x) = \int_{U^\circ}\psi^n(x)d\mu(\psi)\right\}.$$

Clearly every polynomial of finite rank is nuclear, hence $\mathcal{P}_f({}^nE)$ is a subset of both $\mathcal{P}_N({}^nE)$ and $\mathcal{P}_I({}^nE)$. Moreover, by ([7], p. 186) the algebraic representation (3.3) can be extended to give

$$(\mathcal{P}_f({}^nE), \pi_b) = \bigotimes_{s,n,\pi} E'_\beta. \tag{4.2}$$

The space $\mathcal{P}_f({}^nE)$ is dense both in $(\mathcal{P}_N({}^nE), \pi_b)$ and $(\mathcal{P}_N({}^nE), \pi_w)$. This often allows us to use finite polynomials in place of nuclear polynomials and to avoid the approximation property. Clearly $\pi_\omega \geq \pi_b$ and, since in the Banach space case $\|\cdot\|_I \leq \|\cdot\|_N$, the topology π_ω is finer than τ_I.

Lemma 4.1. *Let E be an infrabarrelled locally convex space and n be a positive integer. The mapping J_n^f is continuous if and only if π_b is finer then τ_I on $\mathcal{P}_f({}^nE'_\beta)$.*

Proof. By ([7], Proposition 2)

$$(\mathcal{P}_f({}^nE), \tau_b)'_i = (\mathcal{P}_A({}^nE), \tau_b)'_i = (\mathcal{P}_I({}^nE'_\beta), \tau_I).$$

By (4.2), $\bigotimes_{s,n,\pi} E''_{\beta\beta} = (\mathcal{P}_f({}^nE'_\beta), \pi_b)$, and hence J_n^f is the identity mapping

$$(\mathcal{P}_f({}^nE'_\beta), \pi_b) \longrightarrow (\mathcal{P}_f({}^nE'_\beta), \tau_I).$$

This completes the proof. □

A locally convex space E is *locally Asplund* if for every probability space (Ω, Σ, μ) all operators $T : L^1(\mu) \to E'$ which map some neighbourhood of 0 into an equicontinuous set are locally representable. By [9] locally Asplund spaces include Schwartz spaces, reflexive quasinormable spaces and DF spaces with separable duals. By [7] if E is locally Asplund then $(\mathcal{P}_I({}^nE), \tau_I) = (\mathcal{P}_N({}^nE), \pi_w)$. By Lemma 4.1 this implies the following result.

Corollary 4.1. *If E is an infrabarrelled locally convex space and E'_β is locally Asplund, then J_n^f is continuous if and only if $\pi_w = \pi_b$ on $\mathcal{P}_f({}^nE'_\beta)$.*

Proposition 4.1. *If E is an infrabarrelled locally convex space. Then $\pi_w = \pi_b$ on $\mathcal{P}_f({}^nE'_\beta)$ if and only if $\mathcal{P}_\xi({}^nE''_{\beta\beta}) = \mathcal{P}({}^nE''_{\beta\beta})$ and the subsets of $\mathcal{P}({}^nE''_{\beta\beta})$ which are bounded on the equicontinuous subsets of $E''_{\beta\beta}$ are locally bounded.*

If these conditions are satisfied, J_n^f is continuous for every n.

Proof. See [10], Propositions 1.47 and 1.48. The final remark follows from Lemma 4.1. □

Proposition 4.2. *Let E be a Fréchet space, then J_n^f is continuous for every positive integer n and extends to $\widehat{\bigotimes}_{s,n,\pi} E''_{\beta\beta}$.*

Proof. If E is Fréchet then $E''_{\beta\beta}$ is Fréchet. Thus every convergent sequence in $E''_{\beta\beta}$ is equicontinuous (see [14], p. 293), hence $\mathcal{P}_\xi({}^nE''_{\beta\beta}) = \mathcal{P}({}^nE''_{\beta\beta})$. Moreover, subsets of $\mathcal{P}({}^nE''_{\beta\beta})$ which are bounded on convergent sequences in $E''_{\beta\beta}$ are locally bounded by ([11], Example 1.24). By Proposition 4.1 this implies the continuity of J_n^f.

Let $\theta \in \widehat{\bigotimes}_{s,n,\pi} E''_{\beta\beta}$, the completion of $\bigotimes_{s,n,\pi} E''_{\beta\beta}$. Then θ has a representation $\sum_{i=1}^\infty \lambda_i \otimes_n x_i$, where $(x_i)_i$ is a null sequence in $E''_{\beta\beta}$ and $(\lambda_i)_i \in l_1$

([15], Corollary 15.6.4). Since $(x_i)_i$ is a countable bounded subset of $E''_{\beta\beta}$, it is equicontinuous and hence there exists a bounded subset B in E such that $(x_i)_i \subset B^{\circ\circ}$. By ([13], Theorem 1.5)

$$|\big(J_n^f(\otimes_n x_i)\big)(P)| \leq \|AB_n(P)\|_{B^{\circ\circ}} \leq \|P\|_B$$

and consequently for each $i \in \mathbb{N}$, $J_n^f(\otimes_n x_i)$ lies in $\big(P \in \mathcal{P}_f(^nE) : \|P\|_B \leq 1\big)^\circ$. Therefore $J_n^f(\sum_{i=1}^\infty \lambda_i \otimes_n x_i)$ belongs to $(\mathcal{P}_f(^nE), \tau_b)'_i$. This completes the proof. □

§5. Definition and Basic Properties of Q-reflexive Locally Convex Spaces

In this section we define Q-reflexive locally convex spaces and discuss their basic properties.

Definition 5.1. The locally convex space E is *Q-reflexive* if for every positive integer n:

1. The mapping J_n^{bw} is continuous.
2. The extension J_n of J_n^{bw} to the completion is an isomorphism between $\widehat{\bigotimes_{s,n,\pi}} E''_e$ and $\overline{(\mathcal{P}(^nE), \tau_b)'_i}$.

By Remark 2 every locally convex Q-reflexive space E is infrabarrelled.

A locally convex space E has the *strict approximation property* if it admits a fundamental system $\mathcal{A}$ of semi-norms such that $E_\alpha = (E, \alpha)/\alpha^{-1}(0)$ has the approximation property for each $\alpha \in \mathcal{A}$.

Proposition 5.1. *If E is an infrabarrelled locally convex space whose strong bidual has the strict approximation property, then the following conditions are equivalent*:

1. *E is Q-reflexive.*
2. $\overline{(\mathcal{P}_N(^nE'_\beta), \pi_b)} = (\mathcal{P}_I(^nE'_\beta), \tau_I)$ *and* $\mathcal{P}(^nE) = \mathcal{P}_A(^nE)$ *for every positive integer n.*

Proof. **(1)⇒(2)** Since J_n^{bw} is continuous, J_n^f is continuous and can be extended to a mapping

$$\overline{J}_n^f : \widehat{\bigotimes_{s,n,\pi}} E''_e \longrightarrow \overline{(\mathcal{P}_f(^nE), \tau_b)'_i}.$$

Suppose $\mathcal{P}_A({}^nE) \neq \mathcal{P}({}^nE)$. By the Hahn-Banach Theorem there exists a non-zero functional $\varphi \in (\mathcal{P}({}^nE), \tau_b)'$ such that $\varphi|_{\mathcal{P}_f({}^nE)} = 0$. Since E is Q-reflexive there exists $z \in \widehat{\bigotimes_{s,n,\pi}} E''_e$ such that $\overline{J}^f_n(z) = \varphi|_{\mathcal{P}_f({}^nE)} = 0$. Since $E''_{\beta\beta} = E''_e$ has the strict approximation property, E has a neighbourhood basis at the origin, $\mathcal{U}$, consisting of convex open balanced sets such that $E''_{U^{\circ\circ}}$ has the approximation property for all $U \in \mathcal{U}$. The space E can be written as $\lim\limits_{\overleftarrow{U\in\mathcal{U}}} E_U$. Then $E''_{\beta\beta} = \lim\limits_{\overleftarrow{U\in\mathcal{U}}} E''_{U^{\circ\circ}}$, so for every $U \in \mathcal{U}$ there exists a sequence $(x_i)_i$ in $E''_{U^{\circ\circ}}$ such that $z = \sum_{i=1}^{\infty} \otimes_n x_i$ and $\sum_{i=1}^{\infty}(\|x_i\|_{U^{\circ\circ}})^n < \infty$. For all $\xi \in (E_U)'$ we have

$$\left[J^f_n(z)\right](\xi^n) = \sum_{i=1}^{\infty} x_i{}^n(\xi) = 0.$$

By Goldstine's Theorem for all $\psi \in (E''_{U^{\circ\circ}})'$

$$\sum_{i=1}^{\infty} \big(\psi(x_i)\big)^n = 0$$

Hence $\|z\|_{U^{\circ\circ}} = 0$ for every $U \in \mathcal{U}$. As each $E''_{U^{\circ\circ}}$ has the approximation property, this implies $z = 0$ in $\widehat{\bigotimes_{s,n,\pi}} E''_{\beta\beta}$, hence $J_n(z) = \varphi = 0$. This contradicts our choice of φ and implies $\mathcal{P}_A({}^nE) = \mathcal{P}({}^nE)$.

Using Q-reflexivity and ([7], Proposition 2),

$$(\mathcal{P}_I({}^nE'_\beta), \tau_I) = \big(\overline{\mathcal{P}_A({}^nE), \tau_b}\big)'_i = \big(\overline{\mathcal{P}({}^nE), \tau_b}\big)'_i = \widehat{\bigotimes_{s,n,\pi}} E''_{\beta\beta} = \overline{(\mathcal{P}_N({}^nE'_\beta), \pi_b)}.$$

(2)⇒(1) By hypothesis

$$(\mathcal{P}({}^nE), \tau_b)'_i = \big(\overline{\mathcal{P}_A({}^nE), \tau_b}\big)'_i = (\mathcal{P}_I({}^nE'_\beta), \tau_I) = \overline{(\mathcal{P}_N({}^nE'_\beta), \pi_b)} = \widehat{\bigotimes_{s,n,\pi}} E''_{\beta\beta}.$$

□

Corollary 5.1. *If E is a Q-reflexive locally convex space whose strong bidual has the strict approximation property, then $J^f_n = J^{bw}_n$ for every positive integer n.*

Next we list some properties of Q-reflexive spaces. The proofs can be found in [19].

Proposition 5.2. *Let E be a Q-reflexive locally convex space whose strong bidual has the strict approximation property. Then*

(a) l_1 *is not a subspace of* E'_β *or* E.

(b) *If* E *is complete then* E'_β *does not contain a copy of* c_0.

(c) *If* E'_β *is barrelled then* E *does not contain a copy of* c_0.

(d) *If* E *is a complete DF space or a Fréchet space with* $(BB)_n$ *for every* n, *then* l_∞ *is not a subspace of* $(\mathcal{P}(^nE), \tau_b)$ *for any* n.

§6. Examples of Q-reflexive Spaces

In this section we give some examples of Q-reflexive locally convex spaces. Further examples are given in [19].

Every Q-reflexive Banach space satisfies Definition 5.1. On the other end of the spectrum, Fréchet nuclear and DFN spaces with a basis are Q-reflexive by Remark 3(c). This also is a special case of the following proposition.

Proposition 6.1. *Let E be a Fréchet-Montel space with $(BB)_n$ for every n. Then E is Q-reflexive.*

Proof. Since E is Fréchet-Montel it is reflexive, hence $\widehat{\bigotimes_{s,n,\pi}} E''_{\beta\beta} = \widehat{\bigotimes_{s,n,\pi}} E$ for every n. By ([11], Proposition 1.35) $(\mathcal{P}(^nE), \tau_b) = \big(\widehat{\bigotimes_{s,n,\pi}} E\big)'_\beta$ is a DFM space and in particular is reflexive and reinforced regular. Hence

$$(\mathcal{P}(^nE), \tau_b)'_i = \big(\widehat{\bigotimes_{s,n,\pi}} E\big)''_{\beta i} = \big(\widehat{\bigotimes_{s,n,\pi}} E\big)''_{\beta\beta} = \widehat{\bigotimes_{s,n,\pi}} E.$$

□

Proposition 6.2. *The space $\mathbb{C}^{(I)}$ is Q-reflexive if and only if I is countable.*

Proof. Every bounded subset of $\mathbb{C}^{(I)}$ is finite dimensional and consequently every polynomial on $\mathbb{C}^{(I)}$ is continuous on bounded sets. The nuclear space $\mathbb{C}^I$ is locally Asplund and consequently, by ([7], Theorem 3), $(\mathcal{P}_N(^n\mathbb{C}^I), \pi_\omega) = (\mathcal{P}_I(^n\mathbb{C}^I), \tau_I)$ for every n. Hence, by Proposition 5.1, $\mathbb{C}^{(I)}$ is Q-reflexive if and only if $\pi_\omega = \pi_b$ on $\mathcal{P}_N(^n\mathbb{C}^I)$. If $(\mathcal{P}_N(^n\mathbb{C}^I), \pi_\omega) = (\mathcal{P}_N(^n\mathbb{C}^I), \pi_b)$ then their duals will coincide, i.e. $\mathcal{P}_\xi(^n\mathbb{C}^{(I)}) = \mathcal{P}(^n\mathbb{C}^{(I)})$. Since $\mathbb{C}^I$ is barrelled, the equicontinuous and the bounded sets of $\mathbb{C}^{(I)}$ coincide, i.e. all equicontinuous sets are finite dimensional. Thus $\mathcal{P}_\xi(^n\mathbb{C}^{(I)}) = \mathcal{P}_a(^n\mathbb{C}^{(I)})$ and therefore $\mathcal{P}(^n\mathbb{C}^{(I)}) = \mathcal{P}_a(^n\mathbb{C}^{(I)})$, the space of all n-homogeneous (algebraic) polynomials on $\mathbb{C}^{(I)}$. Since $\mathcal{P}(^nE) = \mathcal{P}_a(^nE)$ if and only if $E = \mathbb{C}^{(\mathbb{N})}$, $\mathbb{C}^{(I)}$ is Q-reflexive if and only if $\mathbb{C}^{(I)} = \mathbb{C}^{(\mathbb{N})}$. □

The example $\mathbb{C}^{(\mathbb{N})} \times \mathbb{C}^{\mathbb{N}}$ shows that Q-reflexivity is not in general preserved by taking inductive or projective limits, direct sums or products. Indeed, $\mathbb{C}^{(\mathbb{N})} \times \mathbb{C}^{\mathbb{N}}$ is both a countable direct sum of Q-reflexive Fréchet spaces and a countable product of Q-reflexive DF spaces, but is not Q-reflexive. The following example shows that Q-reflexivity is preserved in the case of the Tsirelson-James space ${T_J}^*$.

Example 3. The direct sum $E := \bigoplus_{k=1}^\infty {T_J}^*$ and the product $F := \prod_{k=1}^\infty {T_J}^*$ are Q-reflexive spaces.

Proof. We note first that $E''_{\beta\beta}$ and $F''_{\beta\beta}$ have the strict approximation property (see [3]). Let $({T_J}^*)^k := \underbrace{{T_J}^* \times \cdots \times {T_J}^*}_{k}$. By ([20], Proposition 2.5.2), $({T_J}^*)^k$ is a Q-reflexive Banach space. The space $\big(({T_J}^*)^k\big)' = \underbrace{({T_J}^*)' \times \cdots \times ({T_J}^*)'}_{k}$ is Asplund and consequently locally Asplund. Since

$$E'_\beta = \prod_{k=1}^\infty ({T_J}^*)' = \varprojlim_k \big(({T_J}^*)^k\big)',$$

and projective limits of locally Asplund spaces are locally Asplund (see [9]), by ([7], Theorem 3) $(\mathcal{P}_N(^nE'_\beta), \pi_w) = (\mathcal{P}_I(^nE'_\beta), \tau_I)$ for every n. As a countable inductive limit of Banach spaces $E''_{\beta\beta}$ is a barrelled DF space, hence E'_β is a distinguished Fréchet space and by ([10], Corollary 1.53) $\pi_w = \pi_b$ on $\mathcal{P}_N(^nE'_\beta)$. Hence $(\mathcal{P}_N(^nE'_\beta), \pi_b) = (\mathcal{P}_I(^nE'_\beta), \tau_I)$.

Let $P \in \mathcal{P}(^nE)$ and B be a bounded subset of E. The countable strict inductive limit $\varinjlim_k \big(\widehat{\bigotimes_{n,\pi}} ({T_J}^*)^k\big)$ is regular, hence there exists positive integer k such that $B \subset ({T_J}^*)^k$. Since $({T_J}^*)^k$ is Q-reflexive $\mathcal{P}(^n({T_J}^*)^k) = \mathcal{P}_A(^n({T_J}^*)^k)$, hence for every $\epsilon > 0$ we can find $R \in \mathcal{P}_f(^n({T_J}^*)^k)$ such that

$$\big\| R - P|_{({T_J}^*)^k} \big\|_B < \epsilon.$$

Let $\tilde{R}(x+y) := R(x)$, where $x \in ({T_J}^*)^k$ and y belongs to the complement of $({T_J}^*)^k$ in E. Then $\tilde{R} \in \mathcal{P}_f({}^nE)$ and

$$\big\|\tilde{R} - P\big\|_B = \big\|R - P|_{({T_J}^*)^k}\big\|_B < \epsilon.$$

Hence $\mathcal{P}({}^nE) = \mathcal{P}_A({}^nE)$ and, by Proposition 5.1, E is Q-reflexive.

Since the countable inductive limit of locally Asplund spaces is locally Asplund ([9]), F'_β is locally Asplund, therefore by ([7], Theorem 3) $(\mathcal{P}_N({}^nF'_\beta), \pi_\omega) = (\mathcal{P}_I({}^nF'_\beta), \tau_I)$. As a countable inductive limit of Banach spaces F'_β is a barrelled DF space, hence by ([10], Corollary 1.53) $\pi_\omega = \pi_b$ on $\mathcal{P}_N({}^nF'_\beta)$. Thus $(\mathcal{P}_N({}^nF'_\beta), \pi_b) = (\mathcal{P}_I({}^nF'_\beta), \tau_I)$.

The implication $\mathcal{P}({}^nF) = P_A({}^nF)$ can be proved in a way similar to that used for E, where in place of the fact that the bounded subsets of E are contained in a finite product we can use the fact that every continuous polynomial factors through $({T_J}^*)^k$ for some integer k. Here we give an alternative proof.

By ([17], 44.5.6) $E\widehat{\bigotimes}_\epsilon\big(\bigoplus_{j=1}^\infty E_j\big)$ and $\bigoplus_{j=1}^\infty\big(E\widehat{\bigotimes}_\epsilon E_j\big)$ are isomorphic. Using this result n times and applying ([15], Theorem 8.8.5) we obtain

$$\widehat{\bigotimes_{n,\epsilon}} F'_\beta = \bigoplus_{j=1}^\infty \big(\widehat{\bigotimes_{n,\epsilon}}({T_J}^*)'\big) = \bigoplus_{j=1}^\infty \big(\widehat{\bigotimes_{n,\pi}} {T_J}^*\big)' = \big(\widehat{\bigotimes_{n,\pi}}({T_J}^*)\big)'_\beta = \big(\widehat{\bigotimes_{s,n,\pi}} F\big)'_\beta.$$

Applying the symmetrization operator we obtain $\mathcal{P}({}^nF) = \widehat{\bigotimes_{s,n,\epsilon}} F'_\beta = P_A({}^nF)$.

By Proposition 5.1 the proof is complete. □

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