Publ. RIMS, Kyoto Univ.
40 (2004), 125–162

Fermionic Formulas for $(k, 3)$-admissible Configurations

By

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Abstract

We obtain the fermionic formulas for the characters of (k, r)-admissible configurations in the case of $r = 2$ and $r = 3$. This combinatorial object appears as a label of a basis of certain subspace $W(\Lambda)$ of level-k integrable highest weight module of $\widehat{\mathfrak{sl}}_r$. The dual space of $W(\Lambda)$ is embedded into the space of symmetric polynomials. We introduce a filtration on this space and determine the components of the associated graded space explicitly by using vertex operators. This implies a fermionic formula for the character of $W(\Lambda)$.

§1. Introduction

Let $\widehat{\mathfrak{sl}}_r$ be the affine Lie algebra $\mathfrak{sl}_r \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ and $L(\Lambda)$ the integrable highest weight module for dominant integral weight Λ of level k. We denote by $\mathfrak{a}$ the commutative Lie subalgebra of $\widehat{\mathfrak{sl}}_r$ generated by the elements

$$e_{21}[n], e_{31}[n], \ldots, e_{r1}[n], \quad n \in \mathbb{Z}.$$

Communicated by M. Kashiwara. Revised December 27, 2002.
2000 Mathematics Subject Classification(s): 05A15, 17B67, 17B69
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Consider the $\mathfrak{a}$-submodule

$$W(\Lambda) := U(\mathfrak{a})v_\Lambda, \tag{1.1}$$

where $v_\Lambda \in L(\Lambda)$ is the highest weight vector satisfying $e_{ij}[n]v_\lambda = 0, (n > 0)$. Our problem is to find some formulas for the character of $W(\Lambda)$. In [P], Primc constructed a basis of $W(\Lambda)$. His basis consists of vectors parametrized by the combinatorial object called (k, r)-admissible configurations. We can introduce some degrees on (k, r)-admissible configurations and define the character, which is equal to that of $W(\Lambda)$. In [FJLMM1] certain formulas, called 'bosonic formulas', for the character of (k, r)-admissible configurations are obtained (see also [FJLMM2]). Connections to Jack and Macdonald polynomials are discussed in [FJMM1], [FJMM2], [FJMMT1]. In this paper we find another type of formulas in the cases of $r = 2$ and $r = 3$.

We start from an algebra E_Λ isomorphic to $W(\Lambda)$ as vector spaces. The algebra E_Λ is constructed by generators $e_{21}[n], \ldots, e_{r1}[n], (n \le 0)$ with some relations. We will obtain the Gordon-type (or 'fermionic') formulas for the character of E_Λ by using vertex operators.

Let $\mathcal{W}$ be a vector space with non-degenerate quadratic form $\langle \cdot, \cdot \rangle$, and let Γ be an integral lattice in $\mathcal{W}$, i.e., $\langle \gamma_1, \gamma_2 \rangle \in \mathbf{Z}$ for any $\gamma_1, \gamma_2 \in \Gamma$. With such data we can associate a lattice vertex operator algebra $\mathcal{V}_\Gamma$. The algebra $\mathcal{V}_\Gamma$ is generated by vertex operators $V(\gamma, z)$ $(\gamma \in \Gamma, z \in \mathbf{C})$. Let us take a set $\{p_1, \ldots, p_n\}$ of linearly independent vectors from $\mathcal{W}$, and consider the subalgebra $\mathcal{C}$ generated by the vertex operators $a_1(z), \ldots, a_n(z)$ where $a_i(z) = V(p_i, z)$. The operators $a_i(z)$ satisfy quadratic relations. It can be easily formulated in the case when $\langle p_i, p_j \rangle \ge 0$ for all i, j. In this case, we have

$$[a_\alpha(z), a_\alpha(w)]_\pm = 0, \tag{1.2}$$
$$\text{where } + \text{(resp., } -\text{) if } \langle p_\alpha, p_\alpha \rangle \text{ is odd (resp., even),}$$

$$[a_\alpha(z), a_\beta(w)] = 0, \text{ if } \alpha \ne \beta, \tag{1.3}$$

$$a_\alpha(z)\partial_z^l a_\beta(z) = 0 \text{ for } l < \langle p_\alpha, p_\beta \rangle. \tag{1.4}$$

If $\langle p_\alpha, p_\beta \rangle < 0$, then the relations are also quadratic, but $a_\alpha(z)$ and $a_\beta(w)$ are not commutative.

Let us continue the discussion under the condition $\langle p_i, p_j \rangle \ge 0$. It is important that the relations (1.2)–(1.4) are the set of defining relations. This fact actually is equivalent to the following statement about representations of $\mathcal{C}$. Let $a_\alpha[i]$ be components of $a_\alpha(z)$, i.e., $a_\alpha(z) = \sum_{i \in \mathbb{Z}} a_\alpha[i] z^i$. Choose the irreducible representation of $\mathcal{V}_\Gamma$ with the vacuum vector v satisfying $a_\alpha[i]v = 0$

for $i \le 0$. Consider the space $W = \mathcal{C}v$. Let $\theta : W \to \mathbf{C}$ be a linear functional. Define the function

$$\Psi_\theta^{\alpha_1,\dots,\alpha_m}(z_1,\dots,z_m) = \langle \theta, a_{\alpha_1}(z_1)\cdots a_{\alpha_m}(z_m)v\rangle.$$

It has a form

$$\Psi_\theta^{\alpha_1,\dots,\alpha_m}(z_1,\dots,z_m) = F(z_1,\dots,z_m)\prod_i z_i \prod_{i<j}(z_i - z_j)^{\langle p_{\alpha_i}, p_{\alpha_j}\rangle}, \tag{1.5}$$

where F is a polynomial which is symmetric with respect to the transposition of z_i with z_j if $\alpha_i = \alpha_j$.

Let S be the space of functions of the form (1.5). More precisely, we have a direct sum $S = \oplus S_{\alpha_1,\dots,\alpha_m}$, where the set of indices $(\alpha_1 \dots, \alpha_m)$ is defined up to permutations. The statement is that the map $W^* \to S$ is an isomorphism. This fact is equivalent to the relations (1.2)–(1.4), and also gives a possibility of writing down the character of the space W. The space W is naturally graded by $\deg a_\alpha[i] = i$ as well as the function space S by $\deg z_i = 1$, and we have the equality of the corresponding characters: $\operatorname{ch} W = \sum_{(\alpha_1,\dots,\alpha_m)} \operatorname{ch} S_{\alpha_1,\dots,\alpha_m}$. We have

$$\operatorname{ch} S_{\alpha_1,\dots,\alpha_m} = \sum_{m_1,\dots,m_n \ge 0} \frac{q^{\sum_j m_j + \sum_{i<j}\langle p_{\alpha_i}, p_{\alpha_j}\rangle m_i m_j}}{(q)_{m_1}\cdots(q)_{m_n}},$$

where m_j is the number of i such that $\alpha_i = j$. Using this formula, we get a Gordon-type formula for the character of the space W.

Our strategy is to compare the more complicated algebras with the algebras like $\mathcal{C}$. Let us consider the simplest example.

In the algebra $E = \mathbf{C}[e[0], e[-1], e[-2], \dots]$ there are a sequence of ideals $E \supset J_1 \supset J_2 \cdots$. Here J_s is the ideal generated by the components $e^{(s)}[i]$ of the current $e(z)^s = (\sum e[i]z^i)^s = \sum e^{(s)}[i]z^i$. We want to study the quotient $E/J_{k+1} = E_k$. Filter E_k by ideals

$$E_k \supset J_k \supset J_k^2 \supset \cdots,$$

and construct the corresponding associated graded algebra

$$E_k^{(1)} = E_k/J_k \oplus J_k/J_k^2 \oplus J_k^2/J_k^3 \oplus \cdots.$$

We denote by the same symbol J_k the image of $J_k \subset E$ in E_k. Note that $E_k/J_k \simeq E_{k-1}$ and the algebra $E_k^{(1)}$ is generated by the components of the currents $e(z) \in E_k/J_k$ and $e^{(k)}(z) \in J_k/J_k^2$. The current $e^{(k)}(z)$ corresponds to the current $e(z)^k$ from E. In the algebra $E_k/J_k \simeq E_{k-1}$ we have ideals $E_{k-1} \supset J_1 \supset J_2 \supset \cdots \supset J_{k-1}$. Let $J_s^{(1)}$ be an ideal in $E_k^{(1)}$ generated by $J_s \subset E_{k-1}$. In $E_k^{(1)}$ there are ideals $J_1^{(1)} \supset J_2^{(1)} \supset \cdots \supset J_{k-1}^{(1)}$. We can repeat such a construction and get the algebra

$$E_k^{(2)} = E_k^{(1)}/J_{k-1}^{(1)} \oplus J_{k-1}^{(1)}/(J_{k-1}^{(1)})^2 \oplus \cdots .$$

Obviously, the algebra $E_k^{(2)}$ is generated by the component of the currents $e(z)$, $e^{(k)}(z) \in E_k^{(1)}/J_{k-1}^{(1)}$ and $e^{(k-1)}(z) \in J_{k-1}^{(1)}/(J_{k-1}^{(1)})^2$. In $E_{k-2} = E^{(1)}/J_{k-1}^{(1)}$ we have its own sequence of ideals $E_{k-2} \supset J_1 \supset J_2 \supset \cdots \supset J_{k-2}$, and we can repeat what we did before. As a result we get an algebra $E_k^{(2)}$ which is generated by $e(z), e^{(k-1)}(z), e^{(k)}(z)$. Then, filter $E_k^{(2)}$ again, and so on. In the end we construct an algebra $E_k^{(k)}$, which we denote by B_k. The algebra B_k is generated by $e(z) = e^{(1)}(z), e^{(2)}(z), \ldots, e^{(k)}(z)$. It has many gradings. Surely, it inherits the q-grading, $\deg e_i = i$. It has also $\mathbf{Z}^k$-grading: each of the generators $e^{(\alpha)}(z)$ is homogeneous and has grading $(0, \ldots, \overset{\alpha-\mathrm{th}}{1}, \ldots, 0)$. By a simple calculation, one can check that the generators $e^{(\alpha)}(z)$ satisfy the quadratic relations,

$$e^{(\alpha)}(z)\partial_z^l e^{(\beta)}(z) = 0 \quad \text{for} \quad l < 2\min(\alpha, \beta). \tag{1.6}$$

Actually, these relations are defining relations for B_k. One way to prove this statement is to compare B_k with some algebra generated by vertex operators. Now, we explain how to do it.

Consider an integrable representation of $\widehat{\mathfrak{sl}}_2$ of level k. It is known that in such a representation the current $e_{21}(z)$ satisfies the relation $e_{21}(z)^{k+1} = 0$. Here e_{21} is the nilpotent generator of $\mathfrak{sl}_2$ and $e_{21}(z)$ is the corresponding current.

The explicit construction of such an $e_{21}(z)$ uses the so-called vertex operator realization. To do it consider the vector space $\mathcal{W}$ with a base $p_1, \ldots, p_k$ and a bilinear form $\langle p_i, p_j \rangle = 2\delta_{i,j}$. Let $a_i(z) = V(p_i, z)$ and $b_i(z) = V(-p_i, z)$. Let $e_{21}(z) = a_1(z) + \cdots + a_k(z)$ and $e_{12}(z) = b_1(z) + \cdots + b_k(z)$. It is well-known that such $e_{21}(z)$ and $e_{12}(z)$ generate $\widehat{\mathfrak{sl}}_2$ of level k. The whole construction is nothing but the tensor product of k copies of the standard vertex operator realization of $\widehat{\mathfrak{sl}}_2$ of level 1.

The representation of the corresponding vertex operator algebra after restriction to $\widehat{\mathfrak{sl}}_2$ is a sum of integrable representations of level k. Choose the vacuum vector v in the representation $\mathcal{F}$ of the vertex operator algebra which generates the vacuum module for $\widehat{\mathfrak{sl}}_2$. (Our convention is such that $e_{ij}[n]v = 0$ for $n > 0$.) There is a map $\varphi : E_k \to \mathcal{F}$ such that $P(e[0], e[-1], \ldots) \overset{\varphi}{\mapsto} P(e[0], e[-1], \ldots)v$. We will prove that φ is an embedding.

Consider the family of maps $\varphi_\varepsilon : E_k \to \mathcal{F}$ where $\varepsilon \in \mathbf{C}, \varepsilon \neq 0$ which send $e_{21}[i]$ to the i-th component of the current $e_\varepsilon(z) = a_1(z) + \varepsilon a_2(z) + \cdots + \varepsilon^{k-1} a_k(z)$. Let φ_0 be the limit of φ_ε when $\varepsilon \to 0$. More precisely, we want to study the limit W_0 of the image of φ_ε when $\varepsilon \to 0$. First consider the limit of operators

$$\begin{aligned}
&\lim_{\varepsilon\to 0} e_\varepsilon(z) = a_1(z),\\
&\lim_{\varepsilon\to 0} \varepsilon^{-1} e_\varepsilon(z)^2 = 2a_1(z)a_2(z),\\
&\qquad \dots,\\
&\lim_{\varepsilon\to 0} \varepsilon^{1-s} e_\varepsilon(z)^s = s!\, a_1(z)\dots a_s(z).
\end{aligned}$$

Note that $\rho_s(z) = a_1(z)\dots a_s(z)$ are vertex operators $V(q_s, z)$ where q_s's are vectors such that $\langle q_\alpha, q_\beta\rangle = 2\min(\alpha, \beta)$. It means that they satisfy the same quadratic relations as generators $e_{21}^{(\alpha)}(z)$ in the algebra B_k. Looking more carefully at the limit $\varepsilon \to 0$, it is possible to show that we have a surjection $B_k \to \lim_{\varepsilon\to 0}\varphi_\varepsilon(E_k)$. It means that there is a family of algebras U_ε such that $U_0 \simeq B_k$, $U_\varepsilon \simeq E_k$ for $\varepsilon \neq 0$, and $\varphi_\varepsilon : U_\varepsilon \to \mathcal{F}$. Therefore, we have a surjection $B_k \to W_0$, and in W_0 there is a subspace $\widetilde{W_0} = \mathcal{C}v$ where $\mathcal{C}$ is the algebra generated by $\rho_s(z)$.

Comparing the characters of $B_k, \widetilde{W_0}$ and W_0 we get that actually they are all isomorphic. As a corollary, we establish the Gordon-type formula for the character of E_k.

There are many cases that can be studied in a similar manner. We can replace $\widehat{\mathfrak{sl}}_2$ by $\widehat{\mathfrak{g}}$ for any simply-laced semi-simple Lie algebra $\mathfrak{g}$. Let $\mathfrak{n}$ be a maximal nilpotent subalgebra in $\mathfrak{g}$, L_k the vacuum representation of $\widehat{\mathfrak{g}}$ of level k, v the vacuum vector of L_k and $W_0 = U(\widehat{\mathfrak{n}})v \subset L_k$.

Following [FK] we can realize L_1 as a representation of some lattice vertex operator algebra. In this construction, simple root generators $g_\alpha(z) \in \widehat{\mathfrak{n}}$ are just vertex operators (up to some twisting, which is not essential in our argument). Operators $g_\alpha(z)$ in representation of level k can be represented as a sum of vertex operators: $g_\alpha(z) = g_\alpha^{(1)}(z) + g_\alpha^{(2)}(z) + \cdots + g_\alpha^{(k)}(z)$. Now let us use the same ε-method. Namely, introduce operators

$$g_{\alpha,\varepsilon}(z) = g_\alpha^{(1)}(z) + \varepsilon g_\alpha^{(2)}(z) + \cdots + \varepsilon^{k-1} g_\alpha^{(k)}(z).$$

Again, we consider the limit $\varepsilon \to 0$, and repeating the process in the $\widehat{\mathfrak{sl}}_2$ case we get the following result, which was formulated in [FS].

$$\operatorname{ch} W_0 = \sum_{m_1,\dots,m_{kr}} \frac{q^{\frac{1}{2}\langle D\mathbf{m},\mathbf{m}\rangle}}{(q)_{m_1}\cdots(q)_{m_{kr}}}.$$

Here r is the rank of $\mathfrak{g}$ and D is the tensor product of two matrices $C \otimes G$, where C is the Cartan matrix of $\mathfrak{g}$ and G is the $k \times k$ Gordon matrix, i.e., $G = (G_{\alpha,\beta})$ where $G_{\alpha,\beta} = \min(\alpha, \beta)$.

In a slightly different manner, the same method is used for the problem which we will consider in this paper.

Let L_k be the vacuum representation of $\widehat{\mathfrak{sl}}_3$ of level k. By $e_{ij}(z)$ we denote the standard basis of $\widehat{\mathfrak{sl}}_3$ ($1 \le i, j \le 3$). Let $a(z) = e_{21}(z)$ and $b(z) = e_{31}(z)$. It is known that in L_k the currents $a(z)$ and $b(z)$ satisfy the relations

$$a(z)^\alpha b(z)^\beta = 0 \text{ if } \alpha + \beta = k + 1.$$

These are equivalent to the integrability of representation. For $k = 1$, $a(z)$ and $b(z)$ can be realized by vertex operators: $a(z) = V(q_1, z)$ and $b(z) = V(q_2, z)$ where $\langle q_1, q_1\rangle = \langle q_2, q_2\rangle = 2$ and $\langle q_1, q_2\rangle = 1$. Again, for a bigger k, we consider $a(z) = V(t_1, z) + \cdots + V(t_k, z)$ and $b(z) = V(s_1, z) + \cdots + V(s_k, z)$ where $t_1, \ldots, t_k$ and $s_1, \ldots, s_k$ are vectors with the scalar products $\langle t_i, t_i\rangle = \langle s_i, s_i\rangle = 2$, $\langle s_i, t_i\rangle = 1$ and $\langle t_i, t_j\rangle = \langle s_i, s_j\rangle = \langle s_i, t_j\rangle = 0$ for $i \ne j$.

Degeneration is given by formulas

$$\begin{aligned} a(\varepsilon, z) &= V(t_1, z) + \varepsilon V(t_2, z) + \cdots + \varepsilon^{k-1} V(t_k, z), \\ b(\varepsilon, z) &= \varepsilon^{k-1} V(t_1, z) + \varepsilon^{k-2} V(t_2, z) + \cdots + V(t_k, z). \end{aligned}$$

The Gordon-type formula for the character of $W_0 = \mathcal{C}v$ where $\mathcal{C}$ is generated by $a(z), b(z)$ and v is the vacuum vector in L_k, can be found in Theorem 8.1. Again we have vertex operators which represent the currents $a(z)^\alpha$ and $b(z)^\beta$, $\alpha, \beta \le k$. Now $a(z)^\alpha a(w)^\beta \sim (z - w)^{2\min(\alpha,\beta)}$. By this we mean that in the representation an arbitrary matrix element $\langle \theta^\vee, a(z)^\alpha a(w)^\beta \theta\rangle$ has the form $(z - w)^{2\min(\alpha,\beta)} f(z, w)$ where $f(z, w)$ is a Laurent polynomial. The currents $b(z)^\beta$ have the same properties and $a(z)^\alpha b(w)^\beta \sim (z-w)^{(\alpha+\beta-k)_+}$, with $(m)_+ = \max(m, 0)$.

In our paper we use the same ε-method to study $\widehat{\mathfrak{sl}}_3$ representations a little differently. Let us combine the currents $a(z)$ and $b(z)$ into a single one as $e(z) = a(z^2) + zb(z^2)$. The relations $a^\alpha(z)b^\beta(z) = 0$ ($\alpha + \beta = k + 1$) can be written in terms of $e(z)$ as $e^\alpha(z)e^\beta(-z) = 0$ ($\alpha+\beta = k+1$). The algebra $\widehat{\mathfrak{sl}}_3$ has a vertex operator realization where the current $e(z)$ is a sum of vertex operators, and all previous techniques can be used. For $k = 1$, $e(z)$ satisfies the relations $e(z)^2 = 0$ and $e(z)e(-z) = 0$. The matrix elements $\langle \theta^\vee, e(z_1)e(z_2)\theta\rangle$ have the form $(z_1 - z_2)^2(z_1 + z_2)f(z_1, z_2)$ where $f(z_1, z_2)$ is a Laurent polynomial. Such an $e(z)$ can be realized as a vertex operator. An explicit formula is given by $E_{l+1}(z)$ in (9.7). For $k = 2$, $e(z)$ satisfies $e(z)^3 = 0$ and $e(z)^2 e(-z) = 0$.

Such an operator can be constructed as a sum $e(z) = a_1(z) + a_2(z)$, with $[a_i(z), a_i(w)] = 0$, $a_i(z)^2 = 0$ and $a_1(z)a_2(-z) = 0$. Since the relations for $a_i(z)$ are quadratic, they can be realized as vertex operators.

In general, for even $k = 2s$, we set

$$\begin{aligned}
&e(z) = a_1(z) + \cdots + a_s(z) + b_1(z) + \cdots + b_s(z),\\
&[a_i(z), a_j(w)] = 0, \quad [b_i(z), b_j(w)] = 0, \quad [a_i(z), b_j(w)] = 0,\\
&a_i(z)^2 = 0, \quad b_i(z)^2 = 0, \quad a_i(z)b_i(-z) = 0.
\end{aligned}$$

For odd $k = 2s + 1$,

$$e(z) = a_1(z) + \cdots + a_s(z) + c(z) + b_1(z) + \cdots + b_s(z),$$

where $a_i(z), b_i(z)$ satisfy the same relations as above, and

$$[c(z), a_i(w)] = 0, \quad [c(z), b_i(w)] = 0, \quad c(z)^2 = 0, \quad c(z)c(-z) = 0.$$

Such $a_i(z), b_i(z), c(z)$ can be constructed as vertex operators, and these operators are a part of a vertex operator realization of the entire algebra $\widehat{\mathfrak{sl}}_3$. The ε-deformation for even k is given by

$$e_\varepsilon(z) = a_1(z) + \varepsilon a_2(z) + \cdots + \varepsilon^{s-1}a_s(z) + \varepsilon^s b_1(z) + \cdots + \varepsilon^{2s-1}b_s(z),$$

and similarly for odd k.

The plan of this paper is as follows. Throughout this paper we consider $(k, 2)$ or $(k, 3)$-admissible configurations with the initial condition $a_0 \le b_0$, see (2.6). This corresponds to the case of $\Lambda = (k - b_0)\Lambda_0 + b_0\Lambda_1$ for $r = 2$, and $\Lambda = b_0\Lambda_1 + (k - b_0)\Lambda_2$ for $r = 3$. Here Λ_i's are the fundamental weights of $\widehat{\mathfrak{sl}}_r$. As we mentioned above, the case $r = 2$ has been studied in [FS]. Nevertheless we give here the details in order to illustrate the method of vertex operators. The fermionic formulas for $r = 3$ are new. First we introduce the algebra $E_\Lambda^{(k,r)}$ in Section 2. From Section 3 to Section 7 we consider the case of $r = 2$. In Section 3 the dual space $(E_\Lambda^{(k,2)})^*$ is realized as the space of functions $F^{(k,2)}$. In order to calculate the character of $F^{(k,2)}$ we define certain filtration $\{\Gamma_\lambda\}$ on $F^{(k,2)}$ in Section 4. Each component of the associated graded space determined by this filtration is embedded into a space of functions $G_\lambda^{(2)}\mathcal{S}_\lambda$, see Proposition 6.2. We will prove that this embedding is surjective by using vertex operators. We summarize some properties of vertex operators constructed with k-dimensional bosons in Section 5. In Section 6 we give the current $e_{21}(z)$ using the vertex operator and prove that the dual space $\widetilde{W}_0^*$ is isomorphic to the space $G_\lambda^{(2)}\mathcal{S}_\lambda$. This implies surjectivity of the embedding and we get the fermionic formula for the character of $E_\Lambda^{(k,2)}$, which is given in Section 7. In Section 8 we apply the argument above to the case of $r = 3$. We use the two currents $e_{21}(z)$ and $e_{31}(z)$ and obtain the fermionic formula. As mentioned before we can construct a little different realization of the representation of $\widehat{\mathfrak{sl}}_3$ by using the mixed current $e(z) = e_{21}(z^2) + ze_{31}(z^2)$. This representation is of highest weight $\Lambda = \left[\frac{k+1}{2}\right]\Lambda_1 + \left[\frac{k}{2}\right]\Lambda_2$. In Section 9 we obtain another type of fermionic formula in this special case using the current $e(z)$. This fermionic formula is the one obtained from a combinatorial point of view in [FJMMT2]. We give additional results and discuss some remaining problems in Section 10.

§2. Preliminaries

§2.1. A polynomial algebra $\boldsymbol{E}_{\boldsymbol{\Lambda}}^{(k,r)}$

Let r be a positive integer. Consider the polynomial ring

$$E^{(r)} := \mathbb{C}[e_1[-n], e_2[-n], \ldots, e_{r-1}[-n]; n \geq 0].$$

We define formal power series $e_j(z)$ in z by

$$e_j(z) := \sum_{n=0}^{\infty} e_j[-n]z^n, \quad (j = 1, \ldots, r-1). \tag{2.1}$$

Denote by $\{\Lambda_i\}_{i=0}^{r-1}$ the set of the fundamental weights of $\widehat{\mathfrak{sl}}_r$. Let k be a positive integer and $\mathbf{b} = (b_0, \ldots, b_{r-2})$ a vector with non-negative integer entries such that

$$0 \leq b_0 \leq \cdots \leq b_{r-2} \leq k.$$

We set the dominant integral weight Λ of level k by

$$\Lambda = (k - b_{r-2})\Lambda_0 + b_0\Lambda_1 + (b_1 - b_0)\Lambda_2 + \cdots + (b_{r-2} - b_{r-3})\Lambda_{r-1}. \tag{2.2}$$

Denote by $J_\Lambda^{(k,r)}$ the ideal of $E^{(r)}$ generated by the elements

$$e_1[0]^{c_0} \cdots e_{i+1}[0]^{c_i}, \quad (i = 0, \ldots, r-2), \tag{2.3}$$

where c_i's are non-negative integers such that

$$c_0 + \cdots + c_i > b_i, \tag{2.4}$$

and all the coefficients of the power series in the following form:

$$e_1(z)^{p_1} \cdots e_{r-1}(z)^{p_{r-1}},$$

where $p_1, \dots, p_{r-1}$ are non-negative integers satisfying

$$p_1 + \cdots + p_{r-1} = k + 1.$$

Set

$$E_\Lambda^{(k,r)} := E^{(r)}/J_\Lambda^{(k,r)}.$$

Now we give a basis of the vector space $E_\Lambda^{(k,r)}$. For $e \in E^{(r)}$ we denote by $\overline{e} \in E_\Lambda^{(k,r)}$ the image of e by the projection $E^{(r)} \twoheadrightarrow E_\Lambda^{(k,r)}$. Let $\mathbf{a} = (a_i)_{i=0}^\infty$ be a sequence of non-negative integers with finitely many non-zero entries. We define $e(\mathbf{a}) \in E^{(r)}$ by

$$\begin{aligned} e(\mathbf{a}) &:= \prod_{n\geq 0} \prod_{i=1}^{r-1} e_i[-n]^{a_{(r-1)n+i-1}} \\ &= \cdots e_{r-1}[-1]^{a_{2r-3}} \cdots e_1[-1]^{a_{r-1}} e_{r-1}[0]^{a_{r-2}} \cdots e_1[0]^{a_0}. \end{aligned}$$

A sequence $\mathbf{a} = (a_i)_{i=0}^\infty$ of integers with finitely many non-zero entries is called (k,r)-*admissible* if

$$0 \leq a_i \leq k, \quad a_i + \cdots + a_{i+r-1} \leq k \tag{2.5}$$

for all $i \geq 0$. Denote by $C_{\mathbf{b}}^{(k,r)}$ the set of all (k,r)-admissible sequences such that

$$a_0 \leq b_0,\ a_0 + a_1 \leq b_1,\ \dots, a_0 + \cdots + a_{r-2} \leq b_{r-2}. \tag{2.6}$$

Proposition 2.1. *The set*

$$\{\overline{e(\mathbf{a})}; \mathbf{a} \in C_{\mathbf{b}}^{(k,r)}\} \tag{2.7}$$

is a basis of $E_\Lambda^{(k,r)}$.

This proposition is a special case of the result by Primc [P] which we will explain below. Now set $e_i[n] = e_{i+1,1}[n] \in \widehat{\mathfrak{sl}}_r$. Then the elements (2.3) satisfy

$$e_1[0]^{c_0} \cdots e_{i+1}[0]^{c_i} v_\Lambda = 0 \tag{2.8}$$

for non-negative integers $\{c_i\}$ satisfying (2.4) and the formal power series (2.1) satisfy

$$e_1(z)^{p_1} \cdots e_{r-1}(z)^{p_{r-1}} = 0 \tag{2.9}$$

on $L(\Lambda)$ for non-negative integers $p_1, \ldots, p_{r-1}$ such that $p_1 + \cdots + p_{r-1} = k+1$. Hence the map

$$E_\Lambda^{(k,r)} \ni \overline{e} \ \mapsto \ \overline{e} v_\Lambda \in W(\Lambda) \tag{2.10}$$

is well-defined. Here $W(\Lambda)$ is the subspace defined by (1.1). This map is also surjective.

In [P], Primc constructed a basis of $W(\Lambda)$. For $\mathbf{a} = (a_i)_{i=0}^\infty \in C^{(k,r)}$, define the vector $M(\mathbf{a})$ of $W(\Lambda)$ by

$$M(\mathbf{a}) := \overline{e(\mathbf{a})} v_\Lambda.$$

Theorem 2.1 ([P]). *Let Λ be the dominant integral weight given by (2.2). Then the set*

$$\mathcal{M}(\Lambda) := \{M(\mathbf{a}); \mathbf{a} \in C_{\mathbf{b}}^{(k,r)}\}$$

constitutes a basis of $W(\Lambda)$.

From this theorem, the map (2.10) is injective and this implies Proposition 2.1.

§2.2. Characters of (k, r)-admissible configurations

Now we introduce two kinds of degrees on $E^{(r)}$. First we define the *q-degree* by

$$\deg_q e_i[-n] := (r-1)n + i - 1.$$

Next define the *z-degree* by

$$\deg_z e_i[-n] := 1$$

for all $i = 1, \ldots, r-1$ and $n \geq 0$.

Note that the ideal $J_\Lambda^{(k,r)}$ is generated by homogeneous elements with respect to both of the degrees. Hence $E_\Lambda^{(k,r)}$ is a graded vector space with $\deg_q$ and $\deg_z$.

Denote by $E_{\Lambda;i,j}^{(k,r)}$ the subspace spanned by homogeneous elements of q-degree i and z-degree j. Consider the character

$$\chi_{E_\Lambda^{(k,r)}}(q, z) := \sum_{i,j \geq 0} (\dim E_{\Lambda;i,j}^{(k,r)}) q^i z^j.$$

From Proposition 2.1, we have

$$\chi_{E_\Lambda^{(k,r)}}(q,z) = \sum_{\mathbf{a}\in C_{\mathbf{b}}^{(k,r)}} q^{\sum_{j\geq 0} j a_j} z^{\sum_{j\geq 0} a_j}.$$

This is nothing but the *character of* (k,r)*-configurations* $\chi_{k,r;\mathbf{b}}(q,z)$ [FJLMM1].

In the following we give fermionic formulas for the characters $\chi_{k,r;\mathbf{b}}$ in the two cases:

$$\begin{aligned}&(\mathrm{I}) r = 2, (\Lambda = (k-b_0)\Lambda_0 + b_0\Lambda_1),\\ &(\mathrm{II}) r = 3, b_1 = k, \ (\Lambda = b_0\Lambda_1 + (k-b_0)\Lambda_2).\end{aligned}$$

In other words we consider $(k,2)$ or $(k,3)$-admissible configurations with the initial condition $a_0 \leq b_0$.

§3. Functional Realization

From this section to Section 7, we consider $(k,2)$-admissible configurations. In the following we fix $\Lambda = (k-b_0)\Lambda_0 + b_0\Lambda_1$ and abbreviate $E_\Lambda^{(k,2)}$ and $J_\Lambda^{(k,2)}$ to $E^{(k,2)}$ and $J^{(k,2)}$, respectively.

Denote by F_n the space of symmetric polynomials with n variables:

$$F_n := \mathbb{C}[x_1,\ldots,x_n]^{\mathfrak{S}_n}.$$

Let $E_n^{(2)}$ be the graded component of $E^{(2)}$ with z-degree n.

We introduce a pairing

$$\langle\cdot,\cdot\rangle : E_n^{(2)} \otimes F_n \longrightarrow \mathbb{C}$$

as follows. Set $e(z) := e_1(z)$. Then we define the pairing by

$$\langle e(z_1)\cdots e(z_n), f(x_1,\ldots,x_n)\rangle := f(z_1,\ldots,z_n). \tag{3.1}$$

It is easy to see that the pairing $\langle\cdot,\cdot\rangle$ is a bilinear non-degenerate pairing. Moreover, it respects the grading on $E_n^{(2)}$ defined by the q-degree and the one on F_n defined by the usual degree: $\deg x_i = 1$.

Denote by $J_n^{(k,2)}$ the graded component of $J^{(k,2)}$ with z-degree n.

Proposition 3.1. *The orthogonal complement* $F_n^{(k,2)} := (J_n^{(k,2)})^\perp \subset F_n$ *is given as follows*:

$$F_n^{(k,2)} := \left\{ f \in F_n; f(x_1,\ldots,x_n) = 0 \text{ if } \begin{array}{l} x_1 = \cdots = x_{k+1} \text{ or} \\ x_1 = \cdots = x_{b_0+1} = 0 \end{array} \right\}.$$

Proof. From the conditions (2.9) and (2.8) we have $e(z)^{k+1} = 0$ and $e(0)^{b_0+1} = 0$. Note that

$$\left\langle e(z)^{k+1} \prod_{j=k+2}^{n} e(z_j), f(x_1, \ldots, x_n) \right\rangle = f(z, \ldots, z, z_{k+2}, \ldots, z_n) \tag{3.2}$$

and

$$\left\langle e(0)^{b_0+1} \prod_{j=b_0+2}^{n} e(z_j), f(x_1, \ldots, x_n) \right\rangle = f(0, \ldots, 0, z_{b_0+2}, \ldots, z_n). \tag{3.3}$$

Both of (3.2) and (3.3) equal zero if and only if $f \in F_n^{(k,2)}$. □

Note that the graded components $E_n^{(2)}$ and F_n are finite-dimensional and the pairing respects the grading. Therefore $(F_n^{(k,2)})^{\perp} = (J_n^{(k,2)})^{\perp\perp} = J_n^{(k,2)}$ and we obtained the following.

Proposition 3.2. *The pairing $\langle \cdot, \cdot \rangle$ defined by* (3.1) *induces a well-defined non-degenerate bilinear pairing of graded spaces*

$$\langle \cdot, \cdot \rangle : E_n^{(k,2)} \otimes F_n^{(k,2)} \longrightarrow \mathbb{C},$$

where $E_n^{(k,2)}$ is the graded component of $E^{(k,2)}$ with z-degree n.

Hence the character $\chi_{k,2;b_0}(q, z)$ is represented in terms of the character of $F_n^{(k,2)}$ as follows. The character $\mathrm{ch}\, F_n^{(k,2)}(q)$ is defined by

$$\mathrm{ch}\, F_n^{(k,2)}(q) := \sum_{m=0}^{\infty} q^m \dim(F_n^{(k,2)})_m,$$

where $(F_n^{(k,2)})_m$ is the graded component of degree m. Then we get

Corollary 3.1.

$$\chi_{k,2;b_0}(q, z) = \sum_{n=0}^{\infty} z^n \, \mathrm{ch}\, F_n^{(k,2)}(q).$$

§4. Gordon Filtration

Let $k \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{\geq k}$. Let λ be a level-k restricted partition of n, that is

$$\lambda = (1^{m_1}, 2^{m_2}, \ldots, k^{m_k}), \quad \sum_{a=1}^{k} a m_a = n.$$

Denote by $m_a(\lambda)$ the number of rows of length a in the partition (or Young diagram) λ. Set $\mathbf{m}(\lambda) := (m_1(\lambda), \dots, m_k(\lambda))$.

For a sequence of non-negative integers $\mathbf{m} = (m_1, \dots, m_r)$, we define the space of functions $\mathcal{S}_\mathbf{m}$ by

$$\mathcal{S}_\mathbf{m} := \mathbb{C}[x_1^{(1)}, \dots, x_{m_1}^{(1)}]^{\mathfrak{S}_{m_1}} \otimes \cdots \otimes \mathbb{C}[x_1^{(r)}, \dots, x_{m_r}^{(r)}]^{\mathfrak{S}_{m_r}}. \tag{4.1}$$

In particular, for a level-k restricted partition λ of n, we abbreviate $\mathcal{S}_{\mathbf{m}(\lambda)}$ to $\mathcal{S}_\lambda$. Now we define a map

$$\varphi_\lambda : \mathbb{C}[x_1, \dots, x_n]^{\mathfrak{S}_n} \longrightarrow \mathcal{S}_\lambda \tag{4.2}$$

as follows. Fix a numbering from 1 to n of the set of indices

$$\{(a, i, j); 1 \le a \le k, 1 \le i \le m_a(\lambda), 1 \le j \le a\}.$$

We set $\varphi(x_m) := x_i^{(a)}$ where (a, i, j) is the m-th index in this numbering. Then the map φ_λ is defined by

$$\varphi_\lambda(f(x_1, \dots, x_n)) := f(\varphi(x_1), \dots, \varphi(x_n)).$$

Since f is symmetric, this map does not depend on the numbering.

Introduce the lexicographical order on partitions of n by

$$\lambda \succ \mu \Longleftrightarrow \lambda_j = \mu_j \ (j < p) \text{ and } \lambda_p > \mu_p \text{ , for some } p.$$

We define the subspaces of $F_n^{(k,2)}$ by

$$\begin{aligned} \mathcal{F}_\lambda &:= \operatorname{Ker} \varphi_\lambda \cap F_n^{(k,2)}, \\ \Gamma_\lambda &:= \cap_{\nu \succ \lambda} \mathcal{F}_\nu, \\ \Gamma'_\lambda &:= \Gamma_\lambda \cap \operatorname{Ker} \varphi_\lambda. \end{aligned}$$

The subspaces Γ_λ give a filtration of $F_n^{(k,2)}$ and we have

$$\operatorname{ch} F_n^{(k,2)} = \sum_\lambda \operatorname{ch}(\Gamma_\lambda / \Gamma'_\lambda), \tag{4.3}$$

where the right hand side is the summation over all level-k restricted partitions of n.

For an integer s we set $(s)_+ := \max(s, 0)$.

Proposition 4.1. *Let λ be a level-k restricted partition of n. The image of the map $\varphi_\lambda|_{\Gamma_\lambda}$ is contained in the principal ideal $G_\lambda^{(2)} \mathcal{S}_\lambda$, where the function*

$G^{(2)}_\lambda$ is defined by

$$G^{(2)}_\lambda := \prod_{a=1}^{k}\prod_{j}(x^{(a)}_j)^{(a-b_0)_+}\prod_{1\le a<b\le k}\prod_{i,j}(x^{(a)}_i - x^{(b)}_j)^{2a} \times \prod_{a=1}^{k}\prod_{i<j}(x^{(a)}_i - x^{(a)}_j)^{2a}. \tag{4.4}$$

Hence the map $\varphi_\lambda|_{\Gamma_\lambda}$ induces the embedding of the subquotient $\Gamma_\lambda/\Gamma'_\lambda$ into the principal ideal $G^{(2)}_\lambda\mathcal{S}_\lambda$.

Proof. Similar to the proof of Lemma 3.5.1 and Lemma 3.5.3 in [FKLMM]. □

Our goal is to prove that the image of $\varphi_\lambda|_{\Gamma_\lambda}$ is equal to $G^{(2)}_\lambda\mathcal{S}_\lambda$.

§5. Vertex Operators

§5.1. Definitions

Let N be a positive integer. We fix a non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ on the N-dimensional $\mathbb{C}$-vector space $\mathbb{C}^N$.

We denote by $\widehat{\mathcal{H}}_N$ the Heisenberg algebra with unit 1 generated by the elements $a_m(\alpha)$ and $e^{Q(\alpha)}$ ($m\in\mathbb{Z},\alpha\in\mathbb{C}^N$) satisfying the relations

$$[a_m(\alpha),a_n(\beta)] = m\langle\alpha,\beta\rangle\delta_{m+n,0},$$
$$[a_m(\alpha),e^{Q(\beta)}] = \delta_{m,0}\langle\alpha,\beta\rangle e^{Q(\beta)},\quad e^{Q(\alpha)}e^{Q(\beta)} = e^{Q(\alpha+\beta)}.$$

Here the generators $a_m(\alpha)$ are linear on α.

We define the Fock space $\mathcal{F}$ by

$$\mathcal{F} := \mathbb{C}[a_{-m}(\alpha); m>0, \alpha\in\mathbb{C}^N]\otimes\mathbb{C}[e^{Q(\beta)};\beta\in\mathbb{C}^N].$$

The algebra $\widehat{\mathcal{H}}_N$ acts on $\mathcal{F}$ as follows:

$$a_m(\alpha)(f\otimes e^{Q(\beta)}) = \begin{cases}(a_m(\alpha)f)\otimes e^{Q(\beta)}, & (m<0),\\ [a_m(\alpha),f]\otimes e^{Q(\beta)}, & (m>0),\\ \langle\alpha,\beta\rangle f\otimes e^{Q(\beta)}, & (m=0),\end{cases}$$
$$e^{Q(\alpha)}(f\otimes e^{Q(\beta)}) = f\otimes e^{Q(\alpha+\beta)},$$

where $f\in\mathbb{C}[a_{-m}(\alpha); m>0,\alpha\in\mathbb{C}^N]$.

Let $\boldsymbol{\alpha} = (\{\alpha_m\}_{m\in\mathbb{Z}}, \alpha^0)$ be a sequence of vectors in $\mathbb{C}^N$. The vertex operator $X_{\boldsymbol{\alpha}}(z)$ is defined by

$$X_{\boldsymbol{\alpha}}(z) := \exp\left(\sum_{m>0} \frac{a_{-m}(\alpha_{-m})}{m} z^m\right) \exp\left(-\sum_{m>0} \frac{a_m(\alpha_m)}{m} z^{-m}\right) e^{Q(\alpha^0)} z^{a_0(\alpha_0)}.$$

Introduce the normal ordering $:\cdot:$ on $\widehat{\mathcal{H}}_N$:

$$: a_m(\alpha)a_n(\beta) := \begin{cases} a_m(\alpha)a_n(\beta), (m<0), \\ a_n(\beta)a_m(\alpha), (m>0), \end{cases}$$

$$: a_0(\alpha)e^{Q(\beta)} := : e^{Q(\beta)}a_0(\alpha) := e^{Q(\beta)}a_0(\alpha).$$

Then we have

$$X_{\boldsymbol{\alpha}}(z)X_{\boldsymbol{\beta}}(w) = g(z,w;\boldsymbol{\alpha},\boldsymbol{\beta}) : X_{\boldsymbol{\alpha}}(z)X_{\boldsymbol{\beta}}(w) :,$$

where

$$g(z,w;\boldsymbol{\alpha},\boldsymbol{\beta}) := z^{\langle \alpha_0,\beta^0\rangle} \exp\left(-\sum_{m>0} \frac{\langle \alpha_m, \beta_{-m}\rangle}{m} \left(\frac{w}{z}\right)^m\right)$$

for $\boldsymbol{\alpha} = (\{\alpha_m\}, \alpha^0)$ and $\boldsymbol{\beta} = (\{\beta_m\}, \beta^0)$.

§5.2. Matrix elements

Set

$$|\beta\rangle := 1 \otimes e^{Q(\beta)} \in \mathcal{F}.$$

Note that

$$a_m(\alpha)|\beta\rangle = 0, \quad \text{if} \quad m > 0.$$

Let $\langle\beta| \in \mathcal{F}^*$ be the dual vector defined by

$$\langle\beta|(f\otimes e^{Q(\gamma)}) = \begin{cases} c, \text{ if } f = c \in \mathbb{C} \text{ and } \beta = \gamma, \\ 0, \text{otherwise.} \end{cases}$$

Denote by $\widehat{\mathcal{H}}_N^+$ the commutative subalgebra of $\widehat{\mathcal{H}}_N$ generated by the generators $a_m(\alpha), (m > 0, \alpha \in \mathbb{C}^N)$ and 1. Consider the matrix element

$$\langle\beta'|hX_{\boldsymbol{\alpha}_1}(x_1)\cdots X_{\boldsymbol{\alpha}_n}(x_n)|\beta\rangle, \qquad h \in \widehat{\mathcal{H}}_N^+,$$

for $\boldsymbol{\alpha}_a = (\{\alpha_{a,m}\}, \alpha_a^0), (a = 1,\ldots,n)$. From the definition of $X_{\boldsymbol{\alpha}}(z)$, it is easy to see that

$$\langle\beta'|hX_{\boldsymbol{\alpha}_1}(x_1)\cdots X_{\boldsymbol{\alpha}_n}(x_n)|\beta\rangle = 0 \quad \text{unless} \quad \beta' - \beta = \sum_{a=1}^n \alpha_a^0.$$

Theorem 5.1. *Let $\boldsymbol{\alpha}_a = (\{\alpha_{a,m}\}, \alpha_a^0)$, $(a = 1,\dots,N)$ be sequences of vectors in $\mathbb{C}^N$ and $\mathbf{m} = (m_1,\dots,m_N)$ a sequence of non-negative integers. Denote by $S_{\mathbf{m}}(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_N;\beta)$ the set of functions given by*

(5.1)
$$\{\langle\beta+\alpha^*|hX_{\boldsymbol{\alpha}_1}(x_1^{(1)})\cdots X_{\boldsymbol{\alpha}_1}(x_{m_1}^{(1)})\cdots X_{\boldsymbol{\alpha}_N}(x_1^{(N)})\cdots X_{\boldsymbol{\alpha}_N}(x_{m_N}^{(N)})|\beta\rangle; h\in\widehat{\mathcal{H}}_N^+\},$$

where $\alpha^* := \sum_a m_a\alpha_a^0$ and $\beta\in\mathbb{C}^N$.

Suppose that the vectors $\alpha_{a,-m}$ $(a = 1,\dots,N)$ are linearly independent for each $m > 0$. Then we have

$$S_{\mathbf{m}}(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_N;\beta) = \prod_{a=1}^N\prod_j (x_j^{(a)})^{\langle\alpha_a^0,\beta\rangle}\prod_{1\le a<b\le N}\prod_{i,j} g(x_i^{(a)},x_j^{(b)};\boldsymbol{\alpha}_a,\boldsymbol{\alpha}_b)$$
$$\times\prod_{a=1}^N\prod_{i<j} g(x_i^{(a)},x_j^{(a)};\boldsymbol{\alpha}_a,\boldsymbol{\alpha}_a)\cdot\mathcal{S}_{\mathbf{m}}$$

for any $\mathbf{m}$.

Proof. Fix $\mathbf{m} = (m_1,\dots,m_N)$. For $h\in\widehat{\mathcal{H}}_N^+$, we set

$$F(h) := \langle\beta+\alpha^*|hX_{\boldsymbol{\alpha}_1}(x_1^{(1)})\cdots X_{\boldsymbol{\alpha}_1}(x_{m_1}^{(1)})\cdots X_{\boldsymbol{\alpha}_N}(x_1^{(N)})\cdots X_{\boldsymbol{\alpha}_N}(x_{m_N}^{(N)})|\beta\rangle.$$

Then it is easy to see that

$$F(1) = \prod_{a=1}^N\prod_j (x_j^{(a)})^{\langle\alpha_a^0,\beta\rangle}\prod_{1\le a<b\le N}\prod_{i,j} g(x_i^{(a)},x_j^{(b)};\boldsymbol{\alpha}_a,\boldsymbol{\alpha}_b)$$
$$\times\prod_{a=1}^N\prod_{i<j} g(x_i^{(a)},x_j^{(a)};\boldsymbol{\alpha}_a,\boldsymbol{\alpha}_a).$$

For $r > 0$ the vertex operator $X_{\boldsymbol{\alpha}}(z)$ satisfies

$$[a_r(\gamma), X_{\boldsymbol{\alpha}}(z)] = \langle\gamma,\boldsymbol{\alpha}_{-r}\rangle z^r X_{\boldsymbol{\alpha}}(z).$$

Hence we have

$$F(a_{r_1}(\gamma_1)\cdots a_{r_l}(\gamma_l)) = \prod_{i=1}^l\left(\sum_{a=1}^N\langle\gamma_i,\alpha_{a,-r_i}\rangle p_{r_i}^{(a)}\right)F(1),\quad(\forall r_i>0),$$

where $p_r^{(a)}$ is the r-th power sum of $x_j^{(a)}$'s, that is $p_r^{(a)} := \sum_j (x_j^{(a)})^r$. Therefore, if the vectors $\alpha_{a,-r}, (a = 1,\dots,N)$ are linearly independent for each $r > 0$, we can obtain any polynomial in $\mathcal{S}_{\mathbf{m}}$ as $F(h)/F(1)$ by taking a suitable $h\in\widehat{\mathcal{H}}_N^+$. □

§6. Construction of Vertex Operators

Fix a basis $\{\epsilon_a\}_{a=1}^k$ of $\mathbb{C}^k$ satisfying

$$\langle \epsilon_a, \epsilon_b \rangle = 2\delta_{a,b}.$$

For $1 \le a \le k$, define a sequence of vectors $\boldsymbol{\alpha}_a = (\{\alpha_{a,m}\}, \alpha_a^0)$ by

$$\alpha_{a,m} = \alpha_a^0 = \epsilon_a, \quad (\forall m \in \mathbb{Z}).$$

Now we set

$$E_a(z) := X_{\boldsymbol{\alpha}_a}(z).$$

Then we have

$$E_a(z)E_b(w) = \begin{cases} :E_a(z)E_b(w):, & a \ne b, \\ (z-w)^2 :E_a(z)E_a(w):, & a = b. \end{cases}$$

In particular the operators $E_a(z)$ are commutative and satisfy $E_a(z)^2 = 0$.

Set

$$E_\varepsilon(z) := \varepsilon_1 E_1(z) + \cdots + \varepsilon_k E_k(z).$$

Let λ be a level-k restricted partition of n and $\lambda' = (\lambda'_1, \dots, \lambda'_k)$ its conjugate (or transpose). Define the operator E_λ by

$$E_\lambda(x_1, \dots, x_n) := \prod_{a=1}^k \frac{1}{\lambda'_a!} \left(\frac{\partial}{\partial \varepsilon_1}\right)^{\lambda'_1} \cdots \left(\frac{\partial}{\partial \varepsilon_k}\right)^{\lambda'_k} E_\varepsilon(x_1) \cdots E_\varepsilon(x_n)\Big|_{\forall \varepsilon_a = 0}.$$

In other words the operator $E_\lambda(x_1, \dots, x_n)$ is the symmetrization of

$$\prod_{a=1}^k (E_a(x_{n'_{a-1}+1}) \cdots E_a(x_{n'_a})), \tag{6.1}$$

where $n'_0 = 0$ and $n'_a := \sum_{j=1}^a \lambda'_j$.

Set $\epsilon^*_\lambda := \sum_a \lambda'_a \epsilon_a$. Note that

$$\langle \beta' | h E_\lambda(x_1, \dots, x_n) | \beta \rangle = 0 \ (\forall h \in \widehat{\mathcal{H}}_k^+), \quad \text{unless } \beta' - \beta = \epsilon^*_\lambda.$$

Consider the space of symmetric polynomials

$$U_\lambda := \{ \langle \beta_0 + \epsilon^*_\lambda | h E_\lambda(x_1, \dots, x_n) | \beta_0 \rangle; h \in \widehat{\mathcal{H}}_k^+ \},$$

where β_0 is the vector in $\mathbb{C}^k$ uniquely determined by

$$\langle \epsilon_a, \beta_0 \rangle = \begin{cases} 0, & \text{if } 1 \le a \le b_0, \\ 1, & \text{if } a > b_0. \end{cases}$$

Proposition 6.1.

$$U_\lambda \subset \Gamma_\lambda.$$

Proof. Set

$$F_\lambda(h; x_1, \dots, x_n) := \langle \beta_0 + \epsilon^*_\lambda | h E_\lambda(x_1, \dots, x_n) | \beta_0 \rangle.$$

It suffices to prove

$$\varphi_\nu(F_\lambda(h; x_1, \dots, x_n)) = 0 \tag{6.2}$$

for any $\nu \succ \lambda$ and

$$F_\lambda(h; 0, \dots, 0, x_{b_0+2}, \dots, x_n) = 0. \tag{6.3}$$

First we prove (6.2). Note that E_λ is also the symmetrization of

$$\prod_{j=1}^{\lambda'_1} (E_1(x_{n_{j-1}+1}) E_2(x_{n_{j-1}+2}) \cdots E_{\lambda_j}(x_{n_j})), \tag{6.4}$$

where $n_0 = 0$ and $n_j := \sum_{i=1}^{j} \lambda_i$. From this expression and the relation $E_a(z)^2 = 0$, it is easy to see $\varphi_\nu(F_\lambda) = 0$ for $\nu \succ \lambda$.

Next we prove (6.3). Note that E_λ is the symmetrization of (6.1). Consider the function

$$\Big\langle \beta_0 + \epsilon^*_\lambda | h \prod_{a=1}^{k} (E_a(x_{n'_{a-1}+1}) \cdots E_a(x_{n'_a})) | \beta_0 \Big\rangle. \tag{6.5}$$

From Theorem 5.1 this function (6.5) is a polynomial with the factor

$$\prod_{j=n'_{b_0}+1}^{n} x_j \prod_{\substack{1 \le a \le k \\ n'_{a-1} < i < j \le n'_a}} (x_i - x_j)^{2a}.$$

Hence (6.5) becomes zero if $b_0 + 1$ variables of $x_1, \dots, x_n$ are equal to zero. This implies (6.3) because the function F_λ is the symmetrization of (6.5). □

Proposition 6.2.

$$\varphi_\lambda(U_\lambda) = S_{\mathbf{m}(\lambda)}(\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_k; \beta_0),$$

where $S_{\mathbf{m}(\lambda)}$ *is the space defined in Theorem* 5.1 *and the sequences of vectors* $\boldsymbol{\gamma}_a = (\{\gamma_{a,m}\}, \gamma^0_a), (a = 1, \dots, k)$ *are defined by*

$$\gamma_{a,m} = \gamma^0_a = \sum_{j=1}^{a} \epsilon_j, \quad (\forall m \in \mathbb{Z}). \tag{6.6}$$

Proof. From the relation $E_a(z)^2 = 0$, we have

$$\varphi_\lambda(E_\lambda(x_1,\dots,x_n)) = z_\lambda \prod_{a=1}^{k} \prod_{j=1}^{m_a(\lambda)} (E_1(x_j^{(a)}) \cdots E_a(x_j^{(a)})),$$

where z_λ is a constant defined by $z_\lambda := \prod_{a=1}^{k} (a!)^{m_a(\lambda)}$. Moreover, we see that

$$E_1(x) \cdots E_a(x) = :E_1(x) \cdots E_a(x): = X_{\boldsymbol{\gamma}_a}(x).$$

This completes the proof. □

Note that the vectors $\gamma_{a,-m}, (a = 1,\dots,k)$ defined in (6.6) are linearly independent for each $m > 0$. It is easy to check that

$$g(z,w;\boldsymbol{\gamma}_a,\boldsymbol{\gamma}_b) = (z-w)^{2\min(a,b)}, \quad \langle \gamma_a^0, \beta_0 \rangle = (a-b_0)_+.$$

Therefore, from Proposition 4.1, Theorem 5.1 and Proposition 6.2, we see

Corollary 6.1.

$$\varphi_\lambda(\Gamma_\lambda) = G_\lambda^{(2)} \mathcal{S}_\lambda.$$

Example 1. Consider the case of $k = 2, b_0 = 1$ and $n = 3$. Then the Gordon filtration is

$$F_3^{(2,2)} = \Gamma_{(2,1)} \supset \Gamma_{(1,1,1)} \supset \{0\},$$

where

$$F_3^{(2,2)} = \left\{ f(x_1,x_2,x_3) \in \mathbb{C}[x_1,x_2,x_3]^{\mathfrak{S}_3}; f = 0 \quad \text{if} \quad \begin{array}{l} x_1 = x_2 = x_3 \text{ or} \\ x_1 = x_2 = 0 \end{array} \right\}.$$

$$\begin{aligned} \Gamma_{(1,1,1)} &= \{ f(x_1,x_2,x_3) \in F_3^{(2,2)}; f = 0 \text{ if } x_1 = x_2 \} \\ &= \{ f(x_1,x_2,x_3) \in \mathbb{C}[x_1,x_2,x_3]^{\mathfrak{S}_3}; f = 0 \text{ if } x_1 = x_2 \}. \end{aligned}$$

The map φ_λ is defined by

$$\begin{aligned} \varphi_{(2,1)} &: f(x_1,x_2,x_3) \mapsto f(x_1^{(2)},x_1^{(2)},x_1^{(1)}) \in \mathbb{C}[x_1^{(1)}] \otimes \mathbb{C}[x_1^{(2)}], \\ \varphi_{(1,1,1)} &: f(x_1,x_2,x_3) \mapsto f(x_1^{(1)},x_2^{(1)},x_3^{(1)}) \in \mathbb{C}[x_1^{(1)},x_2^{(1)},x_3^{(1)}]^{\mathfrak{S}_3}. \end{aligned}$$

Corollary 6.1 shows that

$$\begin{aligned} \varphi_{(2,1)}(\Gamma_{(2,1)}) &= x_1^{(2)}(x_1^{(1)} - x_1^{(2)})^2 \mathbb{C}[x_1^{(1)}] \otimes \mathbb{C}[x_1^{(2)}], \\ \varphi_{(1,1,1)}(\Gamma_{(1,1,1)}) &= \prod_{1 \le i < j \le 3} (x_i^{(1)} - x_j^{(1)}) \, \mathbb{C}[x_1^{(1)},x_2^{(1)},x_3^{(1)}]^{\mathfrak{S}_3}. \end{aligned}$$

§7. Fermionic Formula

Recall Corollary 3.1: we have

$$\chi_{k,2;b_0}(q,z)=\sum_{n=0}^{\infty} z^n \,\mathrm{ch}\, F_n^{(k,2)}(q).$$

Now let us write down the character of $F_n^{(k,2)}$. From Proposition 4.1 and Corollary 6.1, we find

$$\mathrm{ch}(\Gamma_\lambda/\Gamma'_\lambda)=\mathrm{ch}(G_\lambda^{(2)}\mathcal{S}_\lambda).$$

It is easy to obtain the formula for $\mathrm{ch}(G_\lambda^{(2)}\mathcal{S}_\lambda)$. Introduce the $k\times k$ matrix $A^{(2)}$ defined by

$$A^{(2)}=(A^{(2)}_{ab})_{1\le a,b\le k},\quad A^{(2)}_{ab}:=2\mathrm{min}(a,b). \tag{7.1}$$

Denote by $\mathbf{c}^{(2)}_{b_0}$ the vector defined by

$$\mathbf{c}^{(2)}_{b_0}:=(0,\ldots,0,1,2,\ldots,k-b_0). \tag{7.2}$$

Then we have

$$\mathrm{ch}(\Gamma_\lambda/\Gamma'_\lambda)=\mathrm{ch}(G_\lambda^{(2)}\mathcal{S}_\lambda)=\frac{q^{\frac{1}{2}({}^t\mathbf{m}A^{(2)}\mathbf{m}-(\mathrm{diag}A^{(2)})\cdot\mathbf{m})+\mathbf{c}^{(2)}_{b_0}\cdot\mathbf{m}}}{(q)_{m_1(\lambda)}\cdots(q)_{m_k(\lambda)}}, \tag{7.3}$$

where $\mathbf{m}={}^t\mathbf{m}(\lambda)={}^t(m_1(\lambda),\ldots,m_k(\lambda))$ and $(q)_n:=\prod_{j=1}^n(1-q^j)$. Here the numerator in (7.3) represents the degree of $G_\lambda^{(2)}$ and the part

$$\frac{1}{(q)_{m_1(\lambda)}\cdots(q)_{m_k(\lambda)}}$$

is the character of $\mathcal{S}_\lambda$.

By substituting (7.3) into (4.3), we get the fermionic formula for $(k,2)$-admissible configurations:

Theorem 7.1.

$$\chi_{k,2;b_0}(q,z)=\sum_{n=0}^{\infty}\sum_{\substack{m_1+2m_2+\cdots+km_k=n\\ m_1,\ldots,m_k\ge 0}}\frac{q^{\frac{1}{2}({}^t\mathbf{m}A^{(2)}\mathbf{m}-(\mathrm{diag}A^{(2)})\cdot\mathbf{m})+\mathbf{c}^{(2)}_{b_0}\cdot\mathbf{m}}}{(q)_{m_1}\cdots(q)_{m_k}}z^n,$$

where $A^{(2)}$ is the $k\times k$ matrix defined by (7.1), *$\mathbf{c}^{(2)}_{b_0}$ is the vector defined by* (7.2) *and $\mathbf{m}={}^t(m_1,\ldots,m_k)$.*

§8. Fermionic Formula for $\chi_{k,3}$

In this section we consider the case where $r = 3$ and $\mathbf{b} = (b_0, k)$. We fix $\Lambda = b_0\Lambda_1 + (k - b_0)\Lambda_2$ and abbreviate $E_\Lambda^{(k,3)}$ and $J_\Lambda^{(k,3)}$ to $E^{(k,3)}$ and $J^{(k,3)}$, respectively.

§8.1. Functional realization of $W^{(k,3)}$

Consider the space of polynomials

$$F_{l_1,l_2} := \mathbb{C}[x_1^2, \dots, x_{l_1}^2]^{\mathfrak{S}_{l_1}} \otimes \mathbb{C}[y_1^2, \dots, y_{l_2}^2]^{\mathfrak{S}_{l_2}} \cdot \prod_{j=1}^{l_2} y_j.$$

Let us introduce a pairing

$$\langle \cdot, \cdot \rangle : E_n^{(3)} \otimes \left(\bigoplus_{\substack{l_1+l_2=n \\ l_1,l_2\geq 0}} F_{l_1,l_2} \right) \longrightarrow \mathbb{C}$$

as follows. Set

$$a(z) := e_1(z^2), \quad b(z) := ze_2(z^2).$$

Then we define the pairing by

$$\begin{aligned} (8.1) \qquad & \langle a(z_1)\cdots a(z_{l_1}) b(w_1)\cdots b(w_{l_2}), f(x_1, \dots, x_{m_1}; y_1, \dots, y_{m_2}) \rangle \\ & := \delta_{l_1 m_1} \delta_{l_2 m_2} f(z_1, \dots, z_{l_1}; w_1, \dots, w_{l_2}) \end{aligned}$$

for $f \in F_{m_1,m_2}$. This pairing is non-degenerate and respects the gradings on $E_n^{(3)}$ and $\oplus_{l_1+l_2=n} F_{l_1,l_2}$. Here the grading on $\oplus_{l_1+l_2=n} F_{l_1,l_2}$ is the usual one defined by $\deg x_i = 1 = \deg y_i$.

Let us determine the orthogonal complement $F_{l_1,l_2}^{(k,3)} := (J_n^{(k,3)})^\perp \cap F_{l_1,l_2}$ with respect to the pairing defined above. Denote by $I_{l_1,l_2}^{(k,3)}$ the space of functions

$$g(x_1, \dots, x_{l_1}; y_1, \dots, y_{l_2}) \in \mathbb{C}[x_1, \dots, x_{l_1}]^{\mathfrak{S}_{l_1}} \otimes \mathbb{C}[y_1, \dots, y_{l_2}]^{\mathfrak{S}_{l_2}}$$

such that

$$\begin{aligned} (8.2) \qquad g = 0 \text{ if } & x_1 = \cdots = x_a = y_1 = \cdots = y_b, \ (a \geq 0, b \geq 0, a + b = k + 1), \\ \text{or } & x_1 = \cdots = x_{b_0+1} = 0. \end{aligned}$$

Proposition 8.1.

$$F^{(k,3)}_{l_1,l_2} = \left\{ g(x_1^2,\ldots,x_{l_1}^2; y_1^2,\ldots,y_{l_2}^2) \prod_{j=1}^{l_2} y_j ; g \in I^{(k,3)}_{l_1,l_2} \right\}. \tag{8.3}$$

The proof is quite similar to that of Proposition 3.1.

From this proposition, we have

Proposition 8.2. *The pairing defined by* (8.1) *induces a well-defined non-degenerate bilinear pairing of the graded spaces*

$$\langle \cdot, \cdot \rangle : E^{(k,3)}_n \otimes \left(\bigoplus_{\substack{l_1+l_2=n \\ l_1,l_2 \geq 0}} F^{(k,3)}_{l_1,l_2} \right) \longrightarrow \mathbb{C}.$$

Introduce the usual grading on $I^{(k,3)}_{l_1,l_2}$ and denote by $\mathrm{ch}\, I^{(k,3)}_{l_1,l_2}(q)$ the character of the graded space $I^{(k,3)}_{l_1,l_2}$ with the formal variable q. From Proposition 8.1 we have

$$\mathrm{ch}\, F^{(k,3)}_{l_1,l_2}(q) = q^{l_2}\, \mathrm{ch}\, I^{(k,3)}_{l_1,l_2}(q^2).$$

Hence the character of $(k,3)$-admissible configurations is given as follows.

Corollary 8.1.

$$\chi_{k,3;(b_0,k)}(q,z) = \sum_{n=0}^{\infty} \sum_{\substack{l_1+l_2=n \\ l_1,l_2 \geq 0}} z^n q^{l_2}\, \mathrm{ch}\, I^{(k,3)}_{l_1,l_2}(q^2).$$

§8.2. Gordon filtration

Let us introduce a filtration on $I^{(k,3)}_{l_1,l_2}$.

For a partition λ of n, let us write clearly the variables in $\mathcal{S}_\lambda$ by $\mathcal{S}_\lambda = \mathcal{S}_\lambda(x)$. Let λ and μ be level-k restricted partitions of l_1 and l_2, respectively. We denote by $\varphi_{\lambda,\mu}$ the tensor product of φ_λ and φ_μ:

$$\varphi_{\lambda,\mu} := \varphi_\lambda \otimes \varphi_\mu : \mathbb{C}[x_1,\ldots,x_{l_1}]^{\mathfrak{S}_{l_1}} \otimes \mathbb{C}[y_1,\ldots,y_{l_2}]^{\mathfrak{S}_{l_2}} \longrightarrow \mathcal{S}_\lambda(x) \otimes \mathcal{S}_\mu(y).$$

We define the lexicographical order on pairs of partitions by

$$(\lambda^{(1)}, \mu^{(1)}) \succ (\lambda^{(2)}, \mu^{(2)}) \iff \lambda^{(1)} \succ \lambda^{(2)}, \text{ or } \lambda^{(1)} = \lambda^{(2)} \text{ and } \mu^{(1)} \succ \mu^{(2)}.$$

Now let us define the subspaces of $I^{(k,3)}_{l_1,l_2}$ by

$$\mathcal{F}_{\lambda,\mu} := \operatorname{Ker}\varphi_{\lambda,\mu} \cap I^{(k,3)}_{l_1,l_2}, \tag{8.4}$$

$$\Gamma_{\lambda,\mu} := \cap_{(\nu,\kappa)\succ(\lambda,\mu)}\mathcal{F}_{\nu,\kappa}, \tag{8.5}$$

$$\Gamma'_{\lambda,\mu} := \Gamma_{\lambda,\mu} \cap \operatorname{Ker}\varphi_{\lambda,\mu}. \tag{8.6}$$

The subspaces $\Gamma_{\lambda,\mu}$ give a filtration of $I^{(k,3)}_{l_1,l_2}$ and we have

$$\operatorname{ch} I^{(k,3)}_{l_1,l_2} = \sum_{(\lambda,\mu)} \operatorname{ch}(\Gamma_{\lambda,\mu}/\Gamma'_{\lambda,\mu}).$$

In the same way as the proof of Proposition 4.1, we can show the following:

Proposition 8.3. *Let λ and μ be level-k restricted partitions of l_1 and l_2, respectively. The image of the map $\varphi_{\lambda,\mu}|_{\Gamma_{\lambda,\mu}}$ is contained in the subspace $G_{\lambda,\mu}\cdot(\mathcal{S}_\lambda(x)\otimes\mathcal{S}_\mu(y))$, where the function $G_{\lambda,\mu}$ is defined by*

$$\begin{aligned} G_{\lambda,\mu} := & \prod_{1\le a<a'\le k}\prod_{i,j}(x_i^{(a)}-x_j^{(a')})^{2a}\prod_{a=1}^{k}\prod_{i<j}(x_i^{(a)}-x_j^{(a)})^{2a} \\ & \times \prod_{1\le b<b'\le k}\prod_{i,j}(y_i^{(b)}-y_j^{(b')})^{2b}\prod_{b=1}^{k}\prod_{i<j}(y_i^{(b)}-y_j^{(b)})^{2b} \\ & \times \prod_{\substack{1\le a,b\le k\\ a+b>k}}\prod_{i,j}(x_i^{(a)}-y_j^{(b)})^{a+b-k}\prod_{a=1}^{k}\prod_{j}(x_j^{(a)})^{(a-b_0)_+}. \end{aligned}$$

In the following, we prove that the image of $\varphi_{\lambda,\mu}|_{\Gamma_{\lambda,\mu}}$ is equal to $G_{\lambda,\mu}\cdot(\mathcal{S}_\lambda(x)\otimes\mathcal{S}_\mu(y))$ by using vertex operators in the same way as before.

§8.3. Construction of vertex operators

Decompose $\mathbb{C}^{2k}$ into k orthogonal components

$$\mathbb{C}^{2k} = V_1\oplus\cdots\oplus V_k, \quad V_j\simeq\mathbb{C}^2, (j=1,\dots,k).$$

We define a basis of $\mathbb{C}^{2k}$ as follows. Take a basis $\{\epsilon_j^+,\epsilon_j^-\}$ of $V_j\simeq\mathbb{C}^2$ such that

$$\langle\epsilon_j^\pm,\epsilon_j^\pm\rangle = 2, \quad \langle\epsilon_j^\pm,\epsilon_j^\mp\rangle = 1. \tag{8.7}$$

Then the set of vectors $\{\epsilon_1^+,\epsilon_1^-,\dots,\epsilon_k^+,\epsilon_k^-\}$ is a basis of $\mathbb{C}^{2k}$.

Let $\boldsymbol{\alpha}_j = (\{\alpha_{j,m}\},\alpha_j^0)$ and $\boldsymbol{\beta}_j = (\{\beta_{j,m}\},\beta_j^0), (j=1,\dots,k)$ be sequences of vectors in $V_j\subset\mathbb{C}^{2k}$ defined by

$$\alpha_{j,m} = \alpha_j^0 = \epsilon_j^+, \quad \beta_{j,m} = \beta_j^0 = \epsilon_j^-, \quad (\forall m\in\mathbb{Z}).$$

We define the vertex operators $A_a(z)$ and $B_b(z), (a, b = 1, \ldots, k)$ by

$$A_a(z) := X_{\boldsymbol{\alpha}_a}(z), \quad B_b(z) := X_{\boldsymbol{\beta}_b}(z).$$

These operators satisfy

$$\begin{aligned} A_a(z)A_b(w) &= (z-w)^{2\delta_{a,b}} :A_a(z)A_b(w):, \\ A_a(z)B_b(w) &= (z-w)^{\delta_{a,b}} :A_a(z)B_b(w):, \\ B_a(z)A_b(w) &= (z-w)^{\delta_{a,b}} :B_a(z)A_b(w):, \\ B_a(z)B_b(w) &= (z-w)^{2\delta_{a,b}} :B_a(z)B_b(w): . \end{aligned}$$

In particular, we have

$$A_a(z)A_b(w) = A_b(w)A_a(z), \quad B_a(z)B_b(w) = B_b(w)B_a(z) \tag{8.8}$$

for $a, b = 1, \ldots, k$, and

$$A_a(z)^2 = 0, \quad B_b(z)^2 = 0, \quad A_a(z)B_a(z) = 0 = B_a(z)A_a(z) \tag{8.9}$$

for $a = 1, \ldots, k$.

Now we set

$$A_\varepsilon(z) := \varepsilon_1 A_1(z) + \cdots + \varepsilon_k A_k(z), \tag{8.10}$$

$$B_\varepsilon(z) := \varepsilon_1 B_k(z) + \cdots + \varepsilon_k B_1(z). \tag{8.11}$$

Note that the ordering of operators is reversed in (8.10) and (8.11).

Let λ and μ be level-k restricted partitions of l_1 and l_2, respectively. Define the vertex operators $A_\lambda(x_1, \ldots, x_n)$ and $B_\mu(y_1, \ldots, y_l)$ by

$$A_\lambda(x_1, \ldots, x_n) := \prod_{a=1}^{k} \frac{1}{\lambda'_a!} \left(\frac{\partial}{\partial \varepsilon_1}\right)^{\lambda'_1} \cdots \left(\frac{\partial}{\partial \varepsilon_k}\right)^{\lambda'_k} A_\varepsilon(x_1) \cdots A_\varepsilon(x_n)\Big|_{\forall \varepsilon_a = 0},$$

$$B_\mu(y_1, \ldots, y_l) := \prod_{b=1}^{k} \frac{1}{\mu'_b!} \left(\frac{\partial}{\partial \varepsilon_1}\right)^{\mu'_1} \cdots \left(\frac{\partial}{\partial \varepsilon_k}\right)^{\mu'_k} B_\varepsilon(y_1) \cdots B_\varepsilon(y_l)\Big|_{\forall \varepsilon_a = 0},$$

where $\lambda' = (\lambda'_1, \ldots, \lambda'_k)$ and $\mu' = (\mu'_1, \ldots, \mu'_k)$ are the conjugates of λ and μ, respectively.

Set

$$\epsilon^*_{\lambda,\mu} := \sum_{a=1}^{k} \lambda'_a \epsilon^+_a + \sum_{b=1}^{k} \mu'_b \epsilon^-_{k+1-b} \in \mathbb{C}^{2k}.$$

Let γ_0 be a vector in $\mathbb{C}^{2k}$ uniquely determined by

$$\langle \epsilon_a^+, \gamma_0\rangle = \begin{cases} 0, (a \le b_0), \\ 1, (a > b_0), \end{cases} \quad \langle \epsilon_b^-, \gamma_0\rangle = 0, \ (1 \le b \le k).$$

Consider the space of functions

$$U_{\lambda,\mu} := \{\langle \gamma_0 + \epsilon^*_{\lambda,\mu} | h A_\lambda(x_1, \dots, x_n) B_\mu(y_1, \dots, y_l) | \gamma_0\rangle; h \in \widehat{\mathcal{H}}^+_{2k}\}.$$

From (8.8) it is easy to see that

$$U_{\lambda,\mu} \subset \mathbb{C}[x_1, \dots, x_n]^{\mathfrak{S}_n} \otimes \mathbb{C}[y_1, \dots, y_l]^{\mathfrak{S}_l}.$$

Moreover, in the same way as Proposition 6.1 we have

Proposition 8.4.

$$U_{\lambda,\mu} \subset \Gamma_{\lambda,\mu}.$$

The image $\varphi_{\lambda,\mu}(U_{\lambda,\mu})$ is given as follows.

Proposition 8.5.

$$\varphi_{\lambda,\mu}(U_{\lambda,\mu}) = S_{\mathbf{m}(\lambda),\mathbf{m}(\mu)}(\boldsymbol{\gamma}_1^+, \dots, \boldsymbol{\gamma}_k^+, \boldsymbol{\gamma}_1^-, \dots, \boldsymbol{\gamma}_k^-; \gamma_0).$$

Here the right hand side is defined by (5.1) *with the substitution* $x_j^{(k+b)} := y_j^{(b)}$, $(b = 1, \dots, k)$. *The sequences* $\boldsymbol{\gamma}_a^\pm = (\{\gamma_{a,m}^\pm\}, \gamma_a^{\pm,0})$ *are given by*

$$\gamma_{a,m}^+ = \gamma_a^{+,0} = \sum_{j=1}^a \epsilon_j^+, \quad \gamma_{a,m}^- = \gamma_a^{-,0} = \sum_{j=1}^a \epsilon_{k+1-a}^-, \quad (\forall m \in \mathbb{Z}).$$

Note that the vectors $\gamma_{a,-m}^\pm, (a = 1, \dots, k)$ are linearly independent for each $m > 0$. Hence we can apply Theorem 5.1. The functions $g(z, w; \boldsymbol{\gamma}_a^\pm, \boldsymbol{\gamma}_b^\pm)$ are given by

$$g(z, w; \boldsymbol{\gamma}_a^\pm, \boldsymbol{\gamma}_b^\pm) = (z - w)^{2\min(a,b)},$$

$$g(z, w; \boldsymbol{\gamma}_a^\pm, \boldsymbol{\gamma}_b^\mp) = \begin{cases} (z - w)^{a+b-k}, & \text{if } a + b > k, \\ 1, & \text{if } a + b \le k, \end{cases}$$

$$\langle \gamma_a^{+,0}, \gamma_0\rangle = (a - b_0)_+, \quad \langle \gamma_a^{-,0}, \gamma_0\rangle = 0.$$

Therefore we see that

Corollary 8.2.

$$\varphi_{\lambda,\mu}(\Gamma_{\lambda,\mu}) = G_{\lambda,\mu} \cdot (\mathcal{S}_\lambda(x) \otimes \mathcal{S}_\mu(y)).$$

§8.4. Fermionic formula

From Proposition 8.3 and Corollary 8.2, we have

$$\mathrm{ch}(\Gamma_{\lambda,\mu}/\Gamma'_{\lambda,\mu}) = \mathrm{ch}(G_{\lambda,\mu}\cdot(\mathcal{S}_\lambda(x)\otimes\mathcal{S}_\mu(y))).$$

The character of $G_{\lambda,\mu}\cdot(\mathcal{S}_\lambda(x)\otimes\mathcal{S}_\mu(y))$ is given as follows. Introduce the $2k\times 2k$ matrix A defined by

$$A := \left(\begin{array}{c|c} A^{(2)} & B^{(3)} \\ \hline B^{(3)} & A^{(2)} \end{array}\right), \tag{8.12}$$

where $A^{(2)}$ is the matrix defined by (7.1) and $B^{(3)}$ is defined by

$$B^{(3)} = (B^{(3)}_{ab})_{1\le a,b\le k}, \quad B^{(3)}_{ab} := \max(0, a+b-k). \tag{8.13}$$

For example, the matrix A for $k=1,2$ and 3 is given by

$$\begin{pmatrix} 2&1\\1&2 \end{pmatrix}, \begin{pmatrix} 2&2&0&1\\2&4&1&2\\0&1&2&2\\1&2&2&4 \end{pmatrix} \text{ and } \begin{pmatrix} 2&2&2&0&0&1\\2&4&4&0&1&2\\2&4&6&1&2&3\\0&0&1&2&2&2\\0&1&2&2&4&4\\1&2&3&2&4&6 \end{pmatrix},$$

respectively. We denote by $\mathbf{c}^{(3)}_{b_0}$ the vector defined by

$$\mathbf{c}^{(3)}_{b_0} := (\underbrace{0,\ldots,0,1,2,\ldots,k-b_0}_{k},\underbrace{0,\ldots,0}_{k}). \tag{8.14}$$

Then we have

$$\begin{aligned} &\mathrm{ch}(G_{\lambda,\mu}\cdot(\mathcal{S}_\lambda(x)\otimes\mathcal{S}_\mu(y))) \\ &= \frac{q^{\frac{1}{2}({}^t\mathbf{m}A\mathbf{m}-(\mathrm{diag}A)\cdot\mathbf{m})+\mathbf{c}^{(3)}_{b_0}\cdot\mathbf{m}}}{(q)_{m_1(\lambda)}\cdots(q)_{m_k(\lambda)}(q)_{m_1(\mu)}\cdots(q)_{m_k(\mu)}}, \end{aligned} \tag{8.15}$$

where $\mathbf{m} := {}^t(m_1(\lambda),\ldots,m_k(\lambda),m_1(\mu),\ldots,m_k(\mu))$.

From (8.15) and Corollary 8.1, we obtain the fermionic formula for the character of $(k,3)$-admissible configurations:

Theorem 8.1.

$$\begin{aligned} &\chi_{k,3;(b_0,k)}(q,z) \\ &= \sum_{n=0}^{\infty} \sum_{\substack{l_1+l_2=n\\ l_1,l_2\ge 0}} \sum_{\substack{\sum_j jm_{i,j}=l_i,\\ (i=1,2)}} \frac{q^{{}^t\mathbf{m}A\mathbf{m}-(\mathrm{diag}A)\cdot\mathbf{m}+2\mathbf{c}^{(3)}_{b_0}\cdot\mathbf{m}}}{(q^2)_{m_{1,1}}\cdots(q^2)_{m_{1,k}}(q^2)_{m_{2,1}}\cdots(q^2)_{m_{2,k}}} q^{l_2} z^n, \end{aligned}$$

where A is the matrix defined by (8.12), $\mathbf{c}_{b_0}^{(3)}$ *is the vector defined by* (8.14), $\mathbf{m} = {}^t(m_{1,1}, \dots, m_{1,k}, m_{2,1}, \dots, m_{2,k})$ *and* $(q^2)_m := \prod_{j=1}^{m}(1-q^{2j})$.

§9. Another Fermionic Formula for $\chi_{k,3}$ in a Special Case

In this section we consider $(k,3)$-admissible configurations with the initial condition $\mathbf{b} = (\left[\frac{k+1}{2}\right], k)$, that is, $a_0 \le \left[\frac{k+1}{2}\right]$. In this special case we can find another fermionic formula. As a consequence we get non-trivial equality between the different fermionic formulas for the character with $\mathbf{b} = (\left[\frac{k+1}{2}\right], k)$.

§9.1. Functional realization

First we give another functional realization of $E_\Lambda^{(k,3)}$ for $\Lambda = b_0\Lambda_1 + (k - b_0)\Lambda_2$. We fix Λ and abbreviate $E_\Lambda^{(k,3)}$ and $J_\Lambda^{(k,3)}$ to $E^{(k,3)}$ and $J^{(k,3)}$, respectively.

Let $F_n = \mathbb{C}[x_1, \dots, x_n]^{\mathfrak{S}_n}$. Define a pairing

$$\langle \cdot, \cdot \rangle : E_n^{(3)} \otimes F_n \longrightarrow \mathbb{C}$$

by

$$\langle e(z_1) \cdots e(z_n), f(x_1, \dots, x_n) \rangle := f(z_1, \dots, z_n), \tag{9.1}$$

where

$$e(z) := e_1(z^2) + z e_2(z^2).$$

This pairing is non-degenerate and respects the grading on $E_n^{(3)}$ and F_n.

Proposition 9.1. *The orthogonal complement $F_n^{(k,3)} := (J_n^{(k,3)})^\perp$ is the space of functions $f(x_1, \dots, x_n) \in F_n$ such that*

$$f = 0 \quad \text{if} \quad \begin{array}{l} x_1 = \cdots = x_a = -x_{a+1} = \cdots = -x_{k+1} \ (0 \le \forall a \le k+1) \text{ or} \\ x_1 = \cdots = x_{b_0+1} = 0. \end{array}$$

Proof. Recall that $J_n^{(k,3)}$ is the ideal generated by the coefficients of $e_1(z)^\alpha e_2(z)^\beta, (\alpha + \beta = k+1)$ and the element $e_1(0)^{b_0+1}$. It is easy to see that the condition

$$e_1(z)^\alpha e_2(z)^\beta = 0, \text{ for } \alpha + \beta = k+1$$

is equivalent to

$$e(z)^a e(-z)^{k+1-a} = 0, \text{ for } 0 \le a \le k+1.$$

From this observation the proposition follows in the same way as Proposition 3.1. □

Hence we have the following.

Proposition 9.2. *The pairing defined by* (9.1) *induces a well-defined non-degenerate bilinear pairing of the graded spaces*

$$\langle \cdot, \cdot \rangle : E_n^{(k,3)} \otimes F_n^{(k,3)} \longrightarrow \mathbb{C},$$

where $E_n^{(k,3)}$ *is the graded component* $E_n^{(k,3)} := E_n^{(3)} / J_n^{(k,3)}$.

Therefore the character of $(k,3)$-admissible configurations is given as follows.

Corollary 9.1.

$$\chi_{k,3;(b_0,k)} = \sum_{n=0}^{\infty} z^n \operatorname{ch} F_n^{(k,3)}(q).$$

§9.2. Gordon filtration

For a level-k restricted partition λ we defined the map

$$\varphi_\lambda : \mathbb{C}[x_1, \ldots, x_n]^{\mathfrak{S}_n} \longrightarrow \mathcal{S}_\lambda$$

in (4.2). Using this map we define the subspaces $\mathcal{F}_\lambda, \Gamma_\lambda$ and Γ'_λ as in the case of $r = 2$, that is,

$$\begin{aligned} \mathcal{F}_\lambda &:= \operatorname{Ker} \varphi_\lambda \cap F_n^{(k,3)}, \\ \Gamma_\lambda &:= \cap_{\nu \succ \lambda} \mathcal{F}_\nu, \\ \Gamma'_\lambda &:= \Gamma_\lambda \cap \operatorname{Ker} \varphi_\lambda. \end{aligned}$$

Then we have

$$\operatorname{ch} F_n^{(k,3)} = \sum_\lambda \operatorname{ch}(\Gamma_\lambda / \Gamma'_\lambda). \tag{9.2}$$

Proposition 9.3. *Let λ be a level-k restricted partition of n. Then the image of the map $\varphi_\lambda|_{\Gamma_\lambda}$ is contained in the principal ideal $G^{(3)}_\lambda \mathcal{S}_\lambda$, where $G^{(3)}_\lambda$ is defined by*

$$G^{(3)}_\lambda := G^{(2)}_\lambda \overline{G}^{(3)}_\lambda,$$
$$\overline{G}^{(3)}_\lambda := \prod_{\substack{1\le a<b\le k\\ a+b>k}} \prod_{i,j} (x^{(a)}_i + x^{(b)}_j)^{a+b-k} \prod_{a>\frac{k}{2}} \prod_{i<j} (x^{(a)}_i + x^{(a)}_j)^{2a-k}.$$

Here $G^{(2)}_\lambda$ is the function defined by (4.4).

Proof. It suffices to prove that the function in the image of $\varphi_\lambda|_{\Gamma_\lambda}$ is divisible by $\overline{G}^{(3)}_\lambda$.

Denote the variables x_p such that $\varphi_\lambda(x_p) = x^{(a)}_i$ by $x^{(a)}_{i,l}, (l = 1, \dots, a)$. We can carry out the evaluation of φ_λ in two steps: $\varphi_\lambda(F) = \varphi_2(\varphi_1(F))$, where φ_1 is the evaluation of all variables except $\{x^{(a)}_{i,l}\}_{l=1}^a$ and φ_2 is the evaluation of the variables $\{x^{(a)}_{i,l}\}_{l=1}^a$. Let $F_1 := \varphi_1(F)$ for $F \in \Gamma_\lambda$. As a polynomial of $x^{(a)}_{i,l}, (l = 1, \dots, a)$, F_1 is symmetric. Moreover, F_1 equals zero if $(k - b + 1)$ variables of $\{x^{(a)}_{i,l}\}$ are equal to $-x^{(b)}_j$ for $b = 1, \dots, k$ such that $a + b > k$. Therefore, the following lemma implies that $\varphi_\lambda(F) = \varphi_2(F_1)$ is divisible by $\overline{G}^{(3)}_\lambda$. □

Lemma 9.1. *Let $f(x_1, \dots, x_m)$ be a symmetric polynomial satisfying*

$$f(x_1, \dots, x_m) = 0, \quad \text{if } x_1 = \dots = x_s = a \tag{9.3}$$

for some constant a. Then $f(x, \dots, x)$ is divisible by $(x - a)^{m-s+1}$.

This lemma is easy to prove.

If the induced map $\varphi_\lambda : \Gamma_\lambda/\Gamma'_\lambda \to G^{(3)}_\lambda \mathcal{S}_\lambda$ is surjective, we get the fermionic formula for $\mathrm{ch}\, F^{(k,3)}_n$ by (9.2). In fact the induced map is surjective. We can see this fact from the formula for the character of $(k, 3)$-admissible configurations obtained in [FJMMT2]. Here we do not assume this result. We have proved the surjectivity by using vertex operators only in the case of $b_0 = \left[\frac{k+1}{2}\right]$. In the following we consider this special case.

§9.3. Construction of vertex operators

First we consider the case that k is odd. Set $k = 2l + 1$. Note that $b_0 = \left[\frac{k+1}{2}\right] = l + 1$.

Decompose $\mathbb{C}^k$ into $l+1$ orthogonal components

$$\mathbb{C}^k = V \oplus V_1 \oplus \cdots \oplus V_l, \quad V \simeq \mathbb{C}, \quad V_j \simeq \mathbb{C}^2, (j = 1, \dots, l).$$

Take a vector $\epsilon_0 \in V \simeq \mathbb{C}$ such that $\langle \epsilon_0, \epsilon_0 \rangle = 1$. Next we take a basis $\{\epsilon_j^+, \epsilon_j^-\}$ of $V_j \simeq \mathbb{C}^2$ satisfying (8.7). Then the set of vectors $\{\epsilon, \epsilon_1^+, \epsilon_1^-, \dots, \epsilon_l^+, \epsilon_l^-\}$ is a basis of $\mathbb{C}^k$.

Let $\boldsymbol{\alpha} = (\{\alpha_m\}, \alpha^0)$ be a sequence of vectors in $V \subset \mathbb{C}^k$ defined by

$$\alpha^0 = \alpha_{2m} := \sqrt{3}\epsilon_0, \quad \alpha_{2m+1} := \epsilon_0, \quad (m \in \mathbb{Z}), \tag{9.4}$$

and $\boldsymbol{\alpha}_j^\pm = (\{\alpha_{j,m}^\pm\}, \alpha_j^{\pm,0}), (j = 1, \dots, l)$ sequences of vectors in $V_j \subset \mathbb{C}^k$ defined by

$$\alpha_{j,m}^\pm := (\pm 1)^m \epsilon_j^\pm, \quad \alpha_j^{\pm,0} := \epsilon_j^\pm. \tag{9.5}$$

We rename the sequences $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}_j^\pm, (j = 1, \dots, l)$ to $\boldsymbol{\beta}_a, (a = 1, \dots, k)$ by

$$\boldsymbol{\beta}_a := \begin{cases} \boldsymbol{\alpha}_a^+, & 1 \le a \le l, \\ \boldsymbol{\alpha}, & a = l+1, \\ \boldsymbol{\alpha}_{k-a+1}^-, & l+2 \le a \le k. \end{cases} \tag{9.6}$$

Now we define the vertex operators $E_a(z), (a = 1, \dots, k)$ by

$$E_a(z) := X_{\boldsymbol{\beta}_a}(z). \tag{9.7}$$

These operators satisfy the following:

$$E_a(z)E_b(w) = \begin{cases} (z-w)^2 : E_a(z)E_b(w) :, & a = b \ne l+1, \\ (z-w)^2(z+w) : E_a(z)E_b(w) :, & a = b = l+1, \\ (z+w) : E_a(z)E_b(w) :, & a+b = k+1, a \ne l+1, \\ : E_a(z)E_b(w) :, & \text{otherwise}. \end{cases}$$

In particular,

$$E_a(z)E_b(w) = E_b(w)E_a(z) \tag{9.8}$$

for $a, b = 1, \dots, k$, and

$$E_a(z)^2 = 0, \quad E_a(z)E_{k+1-a}(-z) = 0 \tag{9.9}$$

for $a = 1, \dots, k$.

As in the case of $r=2$, we set

$$E_\epsilon(z) := \epsilon_1 E_1(z) + \cdots + \epsilon_k E_k(z).$$

For a level-k restricted partition λ of n, we set

$$\begin{aligned}(9.10)\qquad & E_\lambda(x_1,\dots,x_n) \\ & := \prod_{a=1}^{k} \frac{1}{\lambda'_a!}\left(\frac{\partial}{\partial\epsilon_1}\right)^{\lambda'_1}\cdots\left(\frac{\partial}{\partial\epsilon_k}\right)^{\lambda'_k} E_t(x_1)\cdots E_t(x_n)\Big|_{\forall\epsilon_a=0},\end{aligned}$$

where $\lambda' = (\lambda'_1,\dots,\lambda'_k)$ is the conjugate of λ.

Set

$$\epsilon^*_\lambda := \sum_{a=1}^{l}\lambda'_a\epsilon^+_a + \sqrt{3}\lambda'_{l+1}\epsilon_0 + \sum_{a=l+2}^{k}\lambda'_a\epsilon^-_{k+1-a} \in \mathbb{C}^k.$$

Then we see that

$$\langle\beta|hE_\lambda(x_1,\dots,x_n)|0\rangle = 0,\ (\forall h\in\widehat{\mathcal{H}}^+_k),\quad \text{unless } \beta=\epsilon^*_\lambda.$$

Consider the space of symmetric polynomials

$$(9.11)\qquad U_\lambda := \{\langle\epsilon^*_\lambda|hE_\lambda(x_1,\dots,x_n)|0\rangle;\, h\in\widehat{\mathcal{H}}^+_k\}.$$

Proposition 9.4.

$$U_\lambda \subset \Gamma_\lambda.$$

Proof. In a similar way to the proof of Proposition 6.1, it can be shown that $\varphi_\mu(U_\lambda) = 0$ for any μ such that $\mu > \lambda$. Hence it suffices to prove that $U_\lambda \subset F_n^{(k,3)}$, and this is equivalent to

$$E_\lambda(\underbrace{x,\dots,x}_{p},\underbrace{-x,\dots,-x}_{k+1-p},x_{k+2},\dots,x_n) = 0$$

for $p = 0,\dots,k+1$, and

$$E_\lambda(\underbrace{0,\dots,0}_{l+2},x_{[(k+1)/2]+2},\dots,x_n) = 0.$$

This follows from the relation (9.9). □

From the relation $E_a(z)^2 = 0$, the following proposition holds as in the case of $r=2$:

Proposition 9.5.

$$\varphi_\lambda(U_\lambda) = S_{\mathbf{m}(\lambda)}(\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_k).$$

Here the sequences of vectors $\boldsymbol{\gamma}_a = (\{\gamma_{a,m}\}, \gamma_a^0)$, $(a = 1, \dots, k)$ *are defined by*

$$\gamma_{a,m} := \sum_{j=1}^{a} \beta_{j,m}, \ (\forall m \in \mathbb{Z}), \quad \gamma_a^0 := \sum_{j=1}^{a} \beta_j^0,$$

where $\boldsymbol{\beta}_j = (\{\beta_{j,m}\}, \beta_j^0)$, $(j = 1, \dots, k)$ *are defined in* (9.6).

Note that the vectors $\gamma_{a,-m}, (a = 1, \dots, k)$ are linearly independent for each $m > 0$. Hence we can apply Theorem 5.1. Then the function $g(z, w; \boldsymbol{\gamma}_a, \boldsymbol{\gamma}_b)$ is given by

$$g(z, w; \boldsymbol{\gamma}_a, \boldsymbol{\gamma}_b) = \begin{cases} (z-w)^{2\min(a,b)}, & \text{if } a+b \le k, \\ (z-w)^{2\min(a,b)}(z+w)^{a+b-k}, & \text{if } a+b > k. \end{cases} \tag{9.12}$$

Therefore we find

Corollary 9.2.

$$\varphi_\lambda(\Gamma_\lambda) = G_\lambda^{(3)} \mathcal{S}_\lambda.$$

Now we consider the case that k is even. Set $k = 2l$. The proof of Corollary 9.2 for this case is quite similar to the case that k is odd.

First introduce the vertex operators $E_a(z)$, $(a = 1, \dots, k)$ as follows.

We decompose $\mathbb{C}^k$ into l orthogonal components

$$\mathbb{C}^k = V_1 \oplus \cdots \oplus V_l, \quad V_j \simeq \mathbb{C}^2, (j = 1, \dots, l).$$

Take a basis $\{\epsilon_j^+, \epsilon_j^-\}$ of $V_j \simeq \mathbb{C}^2$ satisfying (8.7). Let $\boldsymbol{\alpha}_j^\pm = (\{\alpha_{j,m}^\pm\}, \alpha_j^{\pm,0}), (j = 1, \dots, l)$ be sequences of vectors in $V_j \subset \mathbb{C}^k$ defined by (9.5). We rename the sequences $\boldsymbol{\alpha}_j^\pm, (j = 1, \dots, l)$ to $\boldsymbol{\beta}_a, (a = 1, \dots, k)$ by

$$\boldsymbol{\beta}_a := \begin{cases} \boldsymbol{\alpha}_a^+, & 1 \le a \le l, \\ \boldsymbol{\alpha}_{k-a+1}^-, & l+1 \le a \le k. \end{cases}$$

Then we define the vertex operators $E_a(z)$, $(a = 1, \dots, k)$ by

$$E_a(z) := X_{\boldsymbol{\beta}_a}(z).$$

These operators satisfy the following:

$$E_a(z)E_b(w) = \begin{cases} (z-w)^2 : E_a(z)E_b(w) :, & a = b, \\ (z+w) : E_a(z)E_b(w) :, & a+b = k+1, \\ : E_a(z)E_b(w) :, & \text{otherwise.} \end{cases}$$

The commutation relations (9.8) and (9.9) hold also in this case.

Next we define the operator $E_\lambda(x_1,\ldots,x_n)$ by (9.10) for a level-k restricted partition λ, and consider the space of matrix elements U_λ defined by (9.11), where ϵ_λ^* is given by

$$\epsilon_\lambda^* = \sum_{a=1}^{l} \lambda'_a \epsilon_a^+ + \sum_{a=l+1}^{k} \lambda'_a \epsilon_{k+1-a}^-.$$

Then it is easy to see that Proposition 9.4 and Proposition 9.5 hold. The vectors $\gamma_{a,-m}, (a=1,\ldots,k)$ in Proposition 9.5 are linearly independent for each $m>0$ also in this case. The function $g(z,w;\boldsymbol{\gamma}_a,\boldsymbol{\gamma}_b)$ is given by (9.12). Therefore Corollary 9.2 holds also in the case that k is even.

§9.4. Fermionic formula

At last we write down the fermionic formula for $(k,3)$-admissible configurations with the initial condition $a_0 \le \left[\frac{k+1}{2}\right]$.

From Proposition 9.3 and Corollary 9.2, we have

$$\mathrm{ch}(\Gamma_\lambda/\Gamma'_\lambda) = \mathrm{ch}(G_\lambda^{(3)}\mathcal{S}_\lambda).$$

In order to write down the character of $G_\lambda^{(3)}\mathcal{S}_\lambda$ we introduce the $k\times k$ matrix B defined by $B := A^{(2)} + B^{(3)}$, that is,

$$B = (B_{ab})_{1\le a,b\le k}, \quad B_{ab} := 2\min(a,b) + (a+b-k)_+. \tag{9.13}$$

Then we have

$$\mathrm{ch}(G_\lambda^{(3)}\mathcal{S}_\lambda) = \frac{q^{\frac{1}{2}({}^t\mathbf{m}B\mathbf{m}-(\mathrm{diag}B)\cdot\mathbf{m})+\mathbf{c}^{(2)}_{\left[\frac{k+1}{2}\right]}\cdot\mathbf{m}}}{(q)_{m_1(\lambda)}\cdots(q)_{m_k(\lambda)}},$$

where $\mathbf{c}^{(2)}_{\left[\frac{k+1}{2}\right]}$ is defined by (7.2) with $b_0 = \left[\frac{k+1}{2}\right]$.

Finally we get

Theorem 9.1.

$$\chi_{3,r;(\left[\frac{k+1}{2}\right],k)}(q,z) = \sum_{n=0}^{\infty} \sum_{\substack{m_1+2m_2+\cdots+km_k=n\\ m_1,\ldots,m_k\ge 0}} \frac{q^{\frac{1}{2}({}^t\mathbf{m}B\mathbf{m}-(\mathrm{diag}B)\cdot\mathbf{m})+\mathbf{c}^{(2)}_{\left[\frac{k+1}{2}\right]}\cdot\mathbf{m}}}{(q)_{m_1}\cdots(q)_{m_k}} z^n,$$

where B is the $k\times k$ matrix defined by (9.13), $\mathbf{c}^{(2)}_{\left[\frac{k+1}{2}\right]}$ *is the vector defined by* (7.2) *and* $\mathbf{m} = {}^t(m_1,\ldots,m_k)$.

§10. Discussion

§10.1.

The vertex operators constructed in Section 9.3 are a part of a vertex operator realization of $\widehat{\mathfrak{sl}}_3$ of level k. Here we describe the entire algebra $\widehat{\mathfrak{sl}}_3$ using the vertex operators in the cases of $k = 1$ and $k = 2$. For $k \geq 3$, the algebra is constructed as the tensor product of these algebras as mentioned in Introduction.

The $k = 1$ case. Set

$$\phi_-(z) := E_1(z) = X_{\boldsymbol{\alpha}}(z), \quad \phi_+(z) := X_{-\boldsymbol{\alpha}}(z),$$
$$\phi_0(z) := \sum_n a_n z^{-n-1}, \quad \overline{\phi}(z) := :\phi_-(-z)\phi_+(z):,$$

where $\boldsymbol{\alpha} = (\{\alpha_m\}, \alpha^0)$ is defined by (9.4) and $-\boldsymbol{\alpha} := (\{-\alpha_m\}, -\alpha^0)$. We abbreviated $a_n(\alpha_n)$ to a_n. The operator product expansion is given as follows:

$$\phi_-(z)\phi_+(w) \sim \begin{cases} \dfrac{1}{(z-w)^2}\dfrac{1}{2w} + \dfrac{1}{z-w}\left(\phi_0(w) - \dfrac{1}{2w}\right), & (z = w), \\ \dfrac{1}{z+w}\dfrac{\overline{\phi}(w)}{4w^2}, & (z = -w), \end{cases}$$

$$\phi_0(z)\phi_\pm(w) \sim \begin{cases} \mp\dfrac{2\phi_\pm(w)}{z-w}, & (z = w), \\ \mp\dfrac{\phi_\pm(w)}{z+w}, & (z = -w), \end{cases}$$

$$\phi_0(z)\overline{\phi}(w) \sim \begin{cases} -\dfrac{\overline{\phi}(w)}{z-w}, & (z = w), \\ \dfrac{\overline{\phi}(w)}{z+w}, & (z = -w), \end{cases}$$

$$\phi_\pm(z)\overline{\phi}(w) \sim \begin{cases} \mp\dfrac{2w\phi_\pm(\pm w)}{z-w}, & (z = w), \\ 0, & (z = -w) \end{cases}$$

$$\overline{\phi}(z)\overline{\phi}(w) \sim \begin{cases} 0, & (z = w), \\ \dfrac{4w^2}{(z+w)^2} - \dfrac{4w}{z+w}(1 + 2w(\phi_0(w) - \phi_0(-w))), & (z = -w), \end{cases}$$

$$\phi_0(z)\phi_0(w) \sim \begin{cases} \dfrac{2}{(z-w)^2}, & (z = w), \\ -\dfrac{1}{(z+w)^2}, & (z = -w). \end{cases}$$

The operator $\phi_\pm(z)\phi_\pm(w)$ is regular at $z = \pm w$.

The generators of $\widehat{\mathfrak{sl}}_3$ of level one are given by

$$\begin{aligned}
\phi_-(z) &= \sum_n e_{21}[n]z^{-2n} + \sum_n e_{31}[n]z^{-2n+1}, \\
\phi_+(z) &= \sum_n e_{12}[n]z^{-2n-3} + \sum_n e_{13}[n]z^{-2n-4}, \\
-\frac{5}{4} + \frac{1}{2}z\phi_0(z) + \frac{1}{4}\overline{\phi}(z) &= \sum_n e_{32}[n]z^{-2n+1} - \sum_n h_{13}[n]z^{-2n}, \\
-\frac{3}{4} + \frac{1}{2}z\phi_0(z) - \frac{1}{4}\overline{\phi}(z) &= \sum_n e_{23}[n]z^{-2n-1} - \sum_n h_{12}[n]z^{-2n}.
\end{aligned}$$

Here we set $h_{ij} := e_{ii} - e_{jj}$.

The $k = 2$ case. Set

$$\begin{aligned}
\phi_-(z) &:= E_1(z) + E_2(z) = X_{\boldsymbol{\alpha}_1^+}(z) + X_{\boldsymbol{\alpha}_1^-}(z), \\
\phi_+(z) &:= X_{-\boldsymbol{\alpha}_1^+}(z) + X_{-\boldsymbol{\alpha}_1^-}(z), \\
\phi_0(z) &:= \sum_n a_n z^{-n-1}, \\
\overline{\phi}(z) &:= \,:X_{\boldsymbol{\alpha}_1^+}(-z)X_{-\boldsymbol{\alpha}_1^-}(z): + :X_{\boldsymbol{\alpha}_1^-}(-z)X_{-\boldsymbol{\alpha}_1^+}(z):,
\end{aligned}$$

where $\boldsymbol{\alpha}_1^{\pm}$ is defined by (9.5) and $a_n := a_n(\alpha_{1,n}^+) + a_n(\alpha_{1,n}^-)$. Then the operator product expansion is given as follows:

$$\begin{aligned}
\phi_-(z)\phi_+(w) &\sim \begin{cases} \dfrac{2}{(z-w)^2} + \dfrac{\phi_0(w)}{z-w}, & (z = w), \\ \dfrac{\overline{\phi}(w)}{z+w}, & (z = -w), \end{cases} \\
\phi_0(z)\phi_\pm(w) &\sim \begin{cases} \mp\dfrac{2\phi_\pm(w)}{z-w}, & (z = w), \\ \mp\dfrac{\phi_\pm(w)}{z+w}, & (z = -w), \end{cases} \\
\phi_0(z)\overline{\phi}(w) &\sim \begin{cases} -\dfrac{\overline{\phi}(w)}{z-w}, & (z = w), \\ \dfrac{\overline{\phi}(w)}{z+w}, & (z = -w), \end{cases} \\
\phi_\pm(z)\overline{\phi}(w) &\sim \begin{cases} \dfrac{\phi_\pm(\pm w)}{z-w}, & (z = w), \\ 0, & (z = -w) \end{cases}
\end{aligned}$$

$$\overline{\phi}(z)\overline{\phi}(w) \sim \begin{cases} 0, & (z=w), \\ -\dfrac{2}{(z+w)^2} + \dfrac{1}{z+w}(\phi_0(w) - \phi_0(-w)), & (z=-w), \end{cases}$$

$$\phi_0(z)\phi_0(w) \sim \begin{cases} \dfrac{4}{(z-w)^2}, & (z=w), \\ -\dfrac{2}{(z+w)^2}, & (z=-w). \end{cases}$$

The operator $\phi_\pm(z)\phi_\pm(w)$ is regular at $z = \pm w$.

The generators of $\widehat{\mathfrak{sl}}_3$ of level two are given by

$$\phi_-(z) = \sum_n e_{21}[n]z^{-2n} + \sum_n e_{31}[n]z^{-2n+1},$$
$$\phi_+(z) = \sum_n e_{12}[n]z^{-2n-2} + \sum_n e_{13}[n]z^{-2n-3},$$
$$-2 + \frac{1}{2}\phi_0(z) + \frac{1}{2}\overline{\phi}(z) = \sum_n e_{32}[n]z^{-2n} - \sum_n h_{13}[n]z^{-2n-1},$$
$$-1 + \frac{1}{2}\phi_0(z) - \frac{1}{2}\overline{\phi}(z) = \sum_n e_{23}[n]z^{-2n-2} - \sum_n h_{12}[n]z^{-2n-1}.$$

§10.2.

Our problem is to obtain the fermionic formula for the character of (k,r)-admissible configurations with the initial condition (2.6). In previous sections we obtained the fermionic formulas for $(k,2)$ and $(k,3)$-admissible configurations with the condition $a_0 \le b_0$. For the case of $r=2$ our result is sufficient because the condition $a_0 \le b_0$ is the only initial condition. However, in the case of $r=3$, we should consider not only the condition $a_0 \le b_0$ but $a_0 + a_1 \le b_1$. The fermionic formula we obtained in Section 8 is for the case of $b_1 = k$. Here we consider the case of $b_1 < k$.

The definition (8.2) of the space $I^{(k,3)}_{l_1,l_2}$ is replaced by

$$\begin{aligned} g = 0 \quad \text{if } & x_1 = \cdots = x_a = y_1 = \cdots = y_b, \ (a \ge 0, b \ge 0, a+b = k+1), \\ \text{or } & x_1 = \cdots = x_{b_0+1} = 0, \\ \text{or } & x_1 = \cdots = x_s = y_1 = \cdots = y_t = 0, \ (s \ge 0, t \ge 0, s+t = b_1+1). \end{aligned}$$

The functional realization $F^{(k,3)}_{l_1,l_2}$ is given by (8.3) with this redefined space $I^{(k,3)}_{l_1,l_2}$.

Now introduce the filtration $\{\Gamma_{\lambda,\mu}\}$ on $I^{(k,3)}_{l_1,l_2}$ by (8.5) and consider the image of $\varphi_{\lambda,\mu}|_{\Gamma_{\lambda,\mu}}$ as in Proposition 8.3. Then the image is contained a space of functions described as follows. For a partition $\rho = (\rho_1, \rho_2, \dots)$ denote by $m_\rho(x_1, \dots, x_n)$ the monomial symmetric function:

$$m_\rho(x_1, \dots, x_n) := \mathrm{Sym}(x_1^{\rho_1} \cdots x_n^{\rho_n}).$$

Let $I_{\lambda,\mu}$ be the ideal of $\mathcal{S}_\lambda(x) \otimes \mathcal{S}_\mu(y)$ generated by the elements

$$m_{\rho^{(1)}}\left(x_1^{(a)}, \dots, x_{m_a(\lambda)}^{(a)}\right) m_{\rho^{(2)}}\left(y_1^{(b)}, \dots, y_{m_b(\mu)}^{(b)}\right)$$

such that

$$b_1 < a + b \le k, \quad m_a(\lambda) \ne 0, \quad m_b(\mu) \ne 0$$

and

$$\rho^{(1)}_{m_a(\lambda)} + \rho^{(2)}_{m_b(\lambda)} \ge \min(a, b - (b_1 - b_0)).$$

Then it can be shown that

$$\varphi_{\lambda,\mu}(\Gamma_{\lambda,\mu}) \subset G_{\lambda,\mu} \prod_{b=1}^{k} \prod_j (y_j^{(b)})^{(b-b_1)_+} \cdot I_{\lambda,\mu}. \tag{10.1}$$

In Section 8 we proved that the two spaces in (10.1) are equal in the case of $b_1 = k$ using the vertex operators. For the case of $b_1 < k$ we do not have proof or disproof of this equality.

Acknowledgements

BF is partially supported by the grants RFBR 02-01-01015 and INTAS-00-00055. JM is partially supported by the Grant-in-Aid for Scientific Research (B2) no. 12440039, and TM is partially supported by (A1) no. 13304010, Japan Society for the Promotion of Science. The work of EM is partially supported by NSF grant DMS-0140460. YT is supported by the Japan Society for the Promotion of Science.

References

[FJLMM1] Feigin, B., Jimbo, M., Loktev, S., Miwa, T. and Mukhin, E., Bosonic formulas for (k, l)-admissible partitions, math.QA/0107054.

[FJLMM2] ———, Addendum to 'Bosonic formulas for (k, l)-admissible partitions', math.QA/0112104.

[FJMM1] Feigin, B., Jimbo, M., Miwa, T. and Mukhin, E., A differential ideal of Symmetric polynomials spanned by Jack polynomials at $\beta = -(r-1)/(k+1)$, *Int. Math. Res. Notices*, **23** (2002), 1223-1237.

[FJMM2] Feigin, B., Jimbo, M., Miwa, T. and Mukhin, E., Symmetric polynomials vanishing on the shifted diagonals and Macdonald polynomials, *Int. Math. Res. Notices*, **18** (2003), 1015-1034.

[FJMMT1] Feigin, B., Jimbo, M., Miwa, T., Mukhin, E. and Takeyama, Y., Symmetric polynomials vanishing on the diagonals shifted by roots of unity, *Int. Math. Res. Notices*, **18** (2003), 999-1014.

[FJMMT2] ———, Particle content of the $(k, 3)$-configurations, to appear in *Publ. RIMS*.

[FK] Frenkel, I. B. and Kac, V. G., Basic representations of affine Lie algebras and dual resonance models, *Invent. Math.*, **62** (1980), 23-66.

[FKLMM] Feigin, B., Kedem, R., Loktev, S., Miwa, T. and Mukhin, E., Combinatorics of the $\widehat{sl}_2$ coinvariants: dual functional realization and recursion, *Compositio Math.*, **134** (2002), 193-241.

[FS] Feigin, B. and Stoyanovsky, A., Quasi-particles models for the representations of Lie algebras and geometry of flag manifold, hep-th/9308079, RIMS 942; Functional models for the representations of current algebras and the semi-infinite Schubert cells, *Funct. Anal. Appl.*, **28** (1994), 55-72.

[P] Primc, M., Vertex operator construction of standard modules for $A_n^{(1)}$, *Pacific J. Math.*, **162** (1994), 143-187.