

The Aron-Berner Extension for Polynomials Defined in the Dual of a Banach Space

By

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Abstract

Let $E = F'$ where F is a complex Banach space and let $\pi_1 : E'' = E \oplus F^\perp \rightarrow E$ be the canonical projection. Let $P(^n E)$ be the space of the complex valued continuous n -homogeneous polynomials defined in E . We characterize the elements $P \in P(^n E)$ whose Aron-Berner extension coincides with $P \circ \pi_1$. The case of weakly continuous polynomials is considered. Finally we also study the same problem for holomorphic functions of bounded type.

Introduction

Let E be a complex Banach space and let $P(^n E)$ denote the space of all continuous n -homogeneous polynomials from E into \mathbb{C} . The problem of extending every element of $P(^n E)$ to a continuous n -homogeneous polynomial \bar{P} on the bidual E'' of E was first studied by Aron and Berner [2] in 1978, who showed that such extensions always exist. The Aron-Berner extension has received most attention since then. Beyond being of interest in its own sight, it is very useful in many branches of complex analysis. Meanwhile it is sometimes hard to work

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with. For instance, it was not until 1989 that Davie and Gamelin [8] showed that the norm of the Aron-Berner extension \bar{P} of $P \in P(^nE)$ to E'' coincides with the norm of P . In 1990 Zalduendo [14] gave a characterization of this extension in terms of continuity properties of the first order differentials.

If E is a dual Banach space, it is known that E is complemented in E'' and clearly the projection $\pi_1 : E'' \rightarrow E$ provides a norm-preserving extension of every element of $P(^nE)$ to E'' . Our main goal in this paper is to determine necessary and sufficient conditions for the equality $\bar{P} = P \circ \pi_1$ be true. As a consequence we solve some related problems. For instance, in a recent paper Aron, Boyd and Choi [3] investigate the unicity of the norm-preserving extensions of elements of $P(^nE)$ to E'' . We will get here that if E is a non-reflexive dual Banach space, then for every $n > 1$ there exists always an element $P \in P(^nE)$ such that $\bar{P} \neq P \circ \pi_1$ and so the norm-preserving extension is not always unique in such spaces.

Our notation is as follows. Throughout, E and G will be complex Banach spaces. For $n \in \mathbb{N}$, $L(^nE, G)$ (resp. $L_s(^nE, G)$) denotes the space of all continuous (resp. continuous symmetric) n -linear mappings $A : E^n \rightarrow G$. For each $(x_1, \dots, x_k) \in E^k$ and each $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}^k$ with $\alpha_1 + \dots + \alpha_k = n$ we will write $A(x_1)^{\alpha_1} \dots (x_k)^{\alpha_k}$ for $A(x_1, \dots, x_1, \dots, x_k, \dots, x_k)$. For $n \in \mathbb{N}$, let $P(^nE, G)$ be the space of all $P : E \rightarrow G$ such that $P(x) = A(x)^n \forall x \in E$ for some $A \in L(^nE, G)$. The elements of this space are called the continuous n -homogeneous polynomials. The space $P(^nE, G)$ is a Banach space under the norm $\|P\| = \sup \{\|P(x)\| : x \in E, \|x\| \leq 1\}$. By definition $L(^0E, G) = G = P(^0E, G)$. Given $P \in P(^nE, G)$, it is straightforward that we may take the associated n -linear mapping A to be symmetric; in fact $P(^nE, G)$ and $L_s(^nE, G)$ are isomorphic as Banach spaces. The unique associated symmetric n -linear mapping \check{P} is given by the Polarization Formula. It is well known that $\|P\| \leq \|\check{P}\| \leq \frac{n^n}{n!} \|P\|$.

Given any $P \in P(^nE, G)$, let $T_P : E \rightarrow L_s(^{n-1}E, G)$ be the linear bounded operator given by $T_P(x)(x_1, \dots, x_{n-1}) := \check{P}(x, x_1, \dots, x_{n-1})$ and let $dP \in P(^{n-1}E, L(E, G))$ its derivative. By the symmetry of \check{P} it is obvious that $dP(x)(y) = n\check{P}(x)^{n-1}(y) = nT_P(y)(x)^{n-1}$. Its associated operator $T_{dP} : E \rightarrow L_s(^{n-2}E, L(E, G))$ is given by $T_{dP}(x)(x_1, \dots, x_{n-2}) = n\check{P}(x, x_1, \dots, x_{n-2}, \bullet)$.

Throughout, if the range space is not specifically identified, then it will be assumed that the range is \mathbb{C} ; thus, for example, $L(^nE)$ denotes $L(^nE, \mathbb{C})$, $P(^nE)$ denotes $P(^nE, \mathbb{C})$, etc.

If $E = F'$ for some Banach space F , we have $E'' = E \oplus F^\perp$. In this case the weak topology associated to the dual pair (E, F) will be denoted

by w^* . Let $\pi_1 : E'' \rightarrow E$ be the canonical projection. As usual we denote by $\widehat{\otimes}_{n,\pi} E$ (resp. $\widehat{\otimes}_{n,\pi,s} E$) the completion of the n -fold (resp. symmetric n -fold) tensor product of E , with the projective norm. We remark that the projective norm in $\widehat{\otimes}_{n,\pi,s} E$ is defined by $\pi_s(u) = \inf\{\sum_{k=1}^\infty \|x_k\|^m : u = \sum_{k=1}^\infty \varepsilon_k x_k \otimes \cdots \otimes x_k; \varepsilon_k = \pm 1\}$, where the infimum is taken over all such representations $u = \sum_{k=1}^\infty \varepsilon_k x_k \otimes \cdots \otimes x_k \in \widehat{\otimes}_{n,\pi,s} E$. This norm is different of the restriction to $\widehat{\otimes}_{n,\pi,s} E$ of the usual projective norm defined in $\widehat{\otimes}_{n,\pi} E$. We refer to [9] or [11] for details. It is known that $L^{(n-1)}E = (\widehat{\otimes}_{n-1,\pi} E)'$ and $L_s^{(n-1)}E = (\widehat{\otimes}_{n-1,\pi,s} E)'$. The weak topology on $L^{(n-1)}E$ associated to the dual pair $(L^{(n-1)}E, \widehat{\otimes}_{n-1,\pi} E)$ will be also denoted by w^* . Given any $P \in P^n(E)$, \overline{P} will always denote the Aron-Berner extension of P to E'' (cf. [2]).

For basic results and unclear notation we refer to [13].

§1. The Extension of Polynomials

Theorem 1.1. *Suppose that $E = F'$ for some Banach space F . Then for $P \in P^n(E)$ the following conditions are equivalent:*

- (1) T_P is $w^* - w^*$ -continuous.
- (2) $dP(x)$ is w^* -continuous $\forall x \in E$.
- (3) $dP(x) \in F \ \forall x \in E$.
- (4) $\overline{P} = P \circ \pi_1$.

Proof. (1) \Rightarrow (2). Let $(x_\alpha) \subset E$ such that $(x_\alpha) \xrightarrow{w^*} z \in E$. By (1) we have that $(x \otimes \cdots \otimes x)(T_P(x_\alpha)) \rightarrow (x \otimes \cdots \otimes x)(T_P(z)) \ \forall x \otimes \cdots \otimes x \in \widehat{\otimes}_{n-1,\pi} E$ and consequently $dP(x)(x_\alpha) = nT_P(x_\alpha)(x, \dots, x) = (x \otimes \cdots \otimes x)(nT_P(x_\alpha)) \rightarrow (x \otimes \cdots \otimes x)(nT_P(z)) = dP(x)(z)$.

(2) \Rightarrow (3) is clear.

(3) \Rightarrow (4). By [14], all we have to show is

- (i) $\forall x \in E, \ d(P \circ \pi_1)(x)$ is w^* -continuous.

and

(ii) $\forall z \in E''$ and $\forall (x_\alpha) \subset E$ such that $(x_\alpha) \xrightarrow{w^*} z$ we have $d(P \circ \pi_1)(z)(x_\alpha) \rightarrow d(P \circ \pi_1)(z)(z)$. Take $(z_\alpha) \subset E''$ such that $(z_\alpha) \xrightarrow{w^*} z \in E''$. As $E'' = E \oplus F^\perp$ we can write $z = \pi_1(z) + z_2$ where $z_2 \in F^\perp$ and $z_\alpha = \pi_1(z_\alpha) + z'_\alpha$ where $z'_\alpha \in F^\perp \ \forall \alpha$. For every $x \in E, \ d(P \circ \pi_1)(x)(z_\alpha) = dP(x)(\pi_1(z_\alpha)) =$

$\pi_1(z_\alpha)(dP(x)) = z_\alpha(dP(x))$ since $z'_\alpha \in F^\perp$ and $dP(x) \in F$. But $(z_\alpha) \xrightarrow{w^*} z$ and $dP(x) \in E'$ implies $z_\alpha(dP(x)) \rightarrow z(dP(x)) = (\pi_1(z) + z_2)(dP(x)) = dP(x)(\pi_1(z)) = d(P \circ \pi_1)(z)$ and this gives (i).

Now fix $z \in E''$ arbitrary and take $(x_\alpha) \subset E$ such that $(x_\alpha) \xrightarrow{w^*} z$. Let $z = \pi_1(z) + z_2$ where $z_2 \in F^\perp$. Then we have $d(P \circ \pi_1)(z)(x_\alpha) = dP(\pi_1(z))(\pi_1(x_\alpha)) = dP(\pi_1(z))(x_\alpha) \rightarrow z(dP(\pi_1(z))) = (\pi_1(z) + z_2)(dP(\pi_1(z))) = dP(\pi_1(z))(\pi_1(z)) = d(P \circ \pi_1)(z)(z)$ and this completes the proof of (ii).

(4) \Rightarrow (3). Take $x \in E$ and $z_2 \in F^\perp$ arbitrary and consider $z = x + z_2 \in E''$. By [14] we have that $d(P \circ \pi_1)(z)(x_\alpha) \rightarrow d(P \circ \pi_1)(z)(z)$ for every $(x_\alpha) \subset E$ such that $(x_\alpha) \xrightarrow{w^*} z$. Since $dP(x) \in E'$ we have $d(P \circ \pi_1)(z)(x_\alpha) = dP(x)(x_\alpha) \rightarrow z(dP(x)) = (\pi_1(z) + z_2)(dP(x)) = dP(x) \circ \pi_1(z) + z_2(dP(x))$ and the unicity of the limit gives $z_2(dP(x)) = 0$. As $z_2 \in F^\perp$ was arbitrary, by using the Hahn-Banach theorem we get $dP(x) \in F$.

(2) \Rightarrow (1) Let $(x_\alpha) \subset E$ such that $(x_\alpha) \xrightarrow{w^*} z \in E$. By (2) we have $(x \otimes \cdots \otimes x)(T_P(x_\alpha)) = T_P(x_\alpha)(x, \dots, x) = \dot{P}(x, \dots, x, x_\alpha) = \frac{1}{n} dP(x)(x_\alpha) \rightarrow \frac{1}{n} dP(x)(z) = T_P(z)(x, \dots, x) = (x \otimes \cdots \otimes x)(T_P(z)) \quad \forall x \in E$. Now, by using the polarization formula we have that $(x_1 \otimes \cdots \otimes x_{n-1})(T_P(x_\alpha)) \rightarrow (x_1 \otimes \cdots \otimes x_{n-1})(T_P(z)) \quad \forall x_1 \otimes \cdots \otimes x_{n-1} \in \widehat{\otimes}_{n-1, \pi} E$ and this implies (1). \square

Example 1.1. Suppose that $E = F'$ for some nonreflexive Banach space F . Consider $P : E \rightarrow \mathbb{K}$ defined by $P(x) := \varphi^n(x) \quad \forall x \in E$ where $\varphi \in E'$. It is clear that $\overline{P} = P \circ \pi_1$ if $\varphi \in F$ and $\overline{P} \neq P \circ \pi_1$ if $\varphi \notin F$. \square

Example 1.2. For $E = l_1 = c'_0$ every diagonal n -homogeneous polynomial $P : l_1 \rightarrow \mathbb{K}$ satisfies $\overline{P} = P \circ \pi_1$. Indeed: We recall that a diagonal n -homogeneous polynomial P is defined by $P(x) = \sum_{j=1}^{\infty} a_j x_j^n$ where $(a_j) \in l_\infty$. Now, $T_P : l_1 \rightarrow (\widehat{\otimes}_{n-1, \pi} l_1)'$ is defined by $T_P(x)(z^1 \otimes \cdots \otimes z^{n-1}) = \sum_{j=1}^{\infty} a_j x_j z_j^1 \cdots z_j^{n-1}$ where $x = (x_j)_{j=1}^{\infty}$ and $z^i = (z_j^i)_{j=1}^{\infty} \forall 1 \leq i \leq n-1$ and it is clear that $T_P = B^*$ where $B : \widehat{\otimes}_{n-1, \pi} l_1 \rightarrow c_0$ is defined by $B(z^1 \otimes \cdots \otimes z^{n-1}) = (a_i z_i^1 \cdots z_i^{n-1})_{i=1}^{\infty}$. Consequently T_P is $w^* - w^*$ continuous. \square

Example 1.3. Suppose that $k \in \mathbb{N}$ and consider the sequence $(A_j)_{j=1}^{\infty}$, $A_j \in L_s(k-1, l_1)$, satisfying:

(1.3.1) A_j is symmetric in the sense that $A_{i_1}(e_{i_2}, \dots, e_{i_k})$ is invariant under permutation of the indices i_1, \dots, i_k .

(1.3.2) $(A_j)_{j=1}^\infty$ is equivalent to the unit vector basis of l_1 .

(1.3.3) For all indices i_1, \dots, i_k , $A_{i_1}(e_{i_2}, \dots, e_{i_k}) \in \{-1, 1\}$.

We refer to [12] for the proof of the existence of such a sequence.

If $P \in P(kl_1)$ is given by

$$P(\alpha) := \sum_{i_1, \dots, i_k=1}^\infty \alpha_{i_1} \dots \alpha_{i_k} A_{i_1}(e_{i_2}, \dots, e_{i_k}); \text{ for } \alpha = (\alpha_j)_{j=1}^\infty \in l_1$$

it follows that $\overline{P} \neq P \circ \pi_1$. Indeed, it is enough to observe that for all $i \in \mathbb{N}$, $A_j(\overbrace{e_i, \dots, e_i}^{k-1}) \in \{-1, 1\}$ and therefore $T_P : l_1 \rightarrow (\widehat{\otimes}_{k-1, \pi} l_1)'$ is given by $T(e_j) = A_j$ and clearly is not $w^* - w^*$ continuous. \square

Aron and Prolla [6] showed that the finite type polynomials on a Banach space are exactly the weakly continuous polynomials. The proof of next lemma will be omitted as it is very similar to the proof given by Aron and Prolla for their result. This lemma will be used in the proof of Theorem 1.3.

Lemma 1.2. *Suppose that $E = F'$ for some Banach space F and let $P \in P(nE)$. Then P is w^* -continuous on E if and only if there exists $\phi_1, \dots, \phi_M \in F$ such that $P = \sum_{i=0}^M \phi_i^n$.*

Theorem 1.3. *Suppose that $E = F'$ for some Banach space F and P is a weakly continuous n -homogeneous polynomial from E into \mathbb{C} . Then P is w^* -continuous if and only if $dP(x) \in F \ \forall x \in E$.*

Proof. We are going to use the induction.

The statement is clearly true for $n = 1$. Assume that for every $k = 1, 2, \dots, n - 1$ it is true. Since P is weakly continuous, there are functionals $\phi_1, \dots, \phi_N \in E'$ such that $P = \sum_{i=1}^N \phi_i^n$. For all $x \in E$, $dP(x) = \sum_{i=1}^N n\phi_i^{n-1}(x)\phi_i$. Therefore the condition $dP(x) \in F \ \forall x \in E$ means that there exists a linearly independent subset $\{\psi_1, \dots, \psi_M\}$ of F such that $dP(E) \subset \text{span}\{\psi_1, \dots, \psi_M\}$. Let $x_1, \dots, x_M \in E$ such that $\psi_i(x_j) = \delta_{ij}, (1 \leq i, j \leq M)$. For each $x \in E$ we have $dP(x) = \sum_{j=1}^M \lambda_j(x)\psi_j$ where for all $j = 1, \dots, M$ we have $\lambda_j(x) = dP(x)(x_j) = nT_P(x_j)(x)^{n-1}$ for all $x \in E$. It is clear that each $\lambda_j \in P(n-1E)$ and λ_j is w -continuous. Now, as $d\lambda_j(x) = d^2P(x)(x_j) \in F \ \forall x \in E$, by the hypothesis of induction it follows that λ_j is w^* -continuous and this is true for all $j = 1, \dots, M$. Finally, $P(x) = T_P(x)(x)^{n-1} =$

$\frac{1}{n}dP(x)(x) = \frac{1}{n} \sum_{j=0}^M \lambda_j(x)\psi_j(x)$ is w^* -continuous since $\psi_j \in F$ and λ_j is w^* -continuous $\forall j = 1, \dots, M$. So the proof is complete. The proof of the converse follows trivially as a consequence of Lemma 1.2. \square

Corollary 1.4. *Suppose that $E = F'$ for some Banach space F . Then for every weakly continuous n -homogeneous polynomial P from E into \mathbb{C} we have that $\overline{P} = P \circ \pi_1$ if and only if P is w^* -continuous.*

Remark 1.1. In connection with Theorem 1.3, it is reasonable to ask if every sequentially weakly continuous n -homogeneous polynomials P from E into \mathbb{C} such that $dP(x) \in F \ \forall x \in E$ is sequentially w^* -continuous. The answer is negative. Indeed: Recall that every continuous mapping defined in l_1 is sequentially weakly continuous. Now take $P : l_1 \rightarrow \mathbb{C}$ defined by $P(x) = \sum_{n=1}^{\infty} x_n^2 \ \forall x = (x_n)_{n=1}^{\infty} \in l_1$. By Example 1.2, $dP(x) \in F \ \forall x \in l_1$, but P is not w^* -sequentially continuous as $P(e_n) = 1 \ \forall n \in \mathbb{N}$.

We denote by $P_{wu}(^nE, G)$ the subspace of those elements of $P(^nE, G)$ whose restrictions to the bounded subsets of E are (uniformly) weakly continuous. It is well known that every such polynomial is compact and, in case $n = 1, T \in P_{wu}(^1E, G)$ if and only if $T : E \rightarrow G$ is a compact linear operator (see [6] Lemma 2.2 and Proposition 2.5). For every $n \in \mathbb{N}$ it is true that $P \in P_{wu}(^nE)$ if and only if the associated operator T_P is compact (see [4] Theorem 2.9). We will use the fact that $P \in P_{wu}(^nE)$ if and only if the associated operator T_P is compact (see [4], Theorem 2.9).

Theorem 1.5. *Suppose that $E = F'$ for some Banach space F . If $P \in P(^nE)$ is weakly continuous on bounded sets and $dP(x) \in F \ \forall x \in E$ then P is w^* -continuous on the bounded subsets of E .*

Proof. If $P \in P(^nE)$, its derivative $dP \in P(^{n-1}E, E')$ and $dP(x)(y) = nT_P(y)(x)^{n-1} = nT_P(x)(x)^{n-2}(y)$. The condition $dP(x) \in F \ \forall x \in E$, means that the continuous linear operator $\widetilde{dP} : \widehat{\bigotimes}_{n-1, \pi, s} E \rightarrow F$, $\widetilde{dP}(x \otimes \dots \otimes x) = dP(x)$ is well defined. We consider now the adjoint operator

$$\widetilde{dP}^* : F' \rightarrow \left(\widehat{\bigotimes}_{n-1, \pi, s} E \right)' = L_s(^{n-1}E) = P(^{n-1}E).$$

It is clear that $\widetilde{dP}^*(x)(y \otimes \dots \otimes y) = dP(y)(x)$, $x, y \in E$. Also dP is compact if and only if \widetilde{dP}^* is compact (see [7]).

If $(x_\alpha) \subset E$ is a bounded net such that $(x_\alpha) \rightarrow x$ weakly, and $y \in E$ we have $\frac{1}{n}|dP(x_\alpha)(y) - dP(x)(y)| = |T_P(x_\alpha)(x_\alpha)^{n-2}(y) - T_P(x)(x)^{n-2}(y)| = |T_P(x_\alpha - x)(x_\alpha)^{n-2}(y) + T_P(x_\alpha - x)(x_\alpha)^{n-3}(x)(y) + \dots + T_P(x_\alpha - x)(x_\alpha)(x)^{n-3}(y) + T_P(x_\alpha - x)(x)^{n-2}(y)|.$

The compactness of T_P means that T_P is weakly- $\|\cdot\|$ continuous on bounded sets, and therefore

$$\|dP(x_\alpha) - dP(x)\| = \sup_{y \in E, \|y\| \leq 1} \|dP(x_\alpha)(y) - dP(x)(y)\| \xrightarrow{\alpha} 0$$

and dP is compact. The compactness of dP means that \widetilde{dP} is compact since every Banach space has $(BB)_\infty$ (see [10] p. 32) and therefore \widetilde{dP}^* is w^* - $\|\cdot\|$ continuous on w^* -compact sets in $F' = E$. We remark that for $x \in E$,

$$\begin{aligned} \|\widetilde{dP}^*(x)\| &= \sup_{y \in E, \|y\| \leq 1} |dP(y)(x)| \\ &= n \sup_{y \in E, \|y\| \leq 1} |T_P(x)(y)^{n-1}| = n\|T_P(x)\|. \end{aligned}$$

If $(x_\alpha) \subset E$ is a bounded net such that $(x_\alpha) \xrightarrow{w^*} x$, we have that $\widetilde{dP}^*(x_\alpha) \rightarrow \widetilde{dP}^*(x)$ in norm and therefore

$$\begin{aligned} |P(x_\alpha) - P(x)| &= |T_P(x_\alpha)(x_\alpha)^{n-1} - T_P(x)(x)^{n-1}| \\ &= |T_P(x_\alpha - x)(x_\alpha)^{n-1} + T_P(x_\alpha - x)(x)(x_\alpha)^{n-2} \\ &\quad + \dots + T_P(x_\alpha - x)(x)^{n-2}(x_\alpha) + T_P(x_\alpha - x)(x)^{n-1}| \xrightarrow{\alpha} 0, \end{aligned}$$

because

$$\|T_P(x_\alpha - x)\| = \|T_P(x_\alpha) - T_P(x)\| \leq \frac{1}{n} \|\widetilde{dP}^*(x_\alpha) - \widetilde{dP}^*(x)\| \xrightarrow{\alpha} 0,$$

and the proof is complete. □

Corollary 1.6. *Suppose that $E = F'$ for some Banach space F . For $P \in P_{wb}(^n E)$ we have that P is w^* -continuous on the bounded subsets of E whenever $\overline{P} = P \circ \pi_1$.*

Let $P_{w^*b}(^n E)$ be the space of the elements of $P(^n E)$ that are w^* -continuous on the bounded subsets of E .

Definition 1.1. *A Banach space E is said to have the Mazur's property if w^* -sequentially continuous functionals on E' are actually w^* -continuous, i.e. belong to E .*

Every separable Banach space enjoy Mazur’s property while l_∞ doesn’t have it.

Proposition 1.7. *Suppose that $E = F'$ for some Banach space F . If F has the Mazur’s property, for every $P \in P_{w^*b}(^nE)$ we have that $dP(x)$ is w^* -continuous $\forall x \in E$ and, consequently, $\overline{P} = P \circ \pi_1$.*

Proof. Since F has the Mazur’s property, it is enough to show that $dP(x)$ is w^* -sequentially continuous in E . Let $(y_k) \xrightarrow{w^*} y$ in E and let $\epsilon > 0$. If there exists $k_0 \in \mathbb{N}$ such that

$$(*) \quad \sup_{|\lambda|=1} |P(x + \lambda y_k) - P(x + \lambda y)| < \epsilon \quad \forall k \geq k_0,$$

the thesis follows by noting that

$$|dP(x)(y_k) - dP(x)(y)| = \frac{1}{2\pi} \left| \int_{|\lambda|=1} \frac{P(x + \lambda y_k) - P(x + \lambda y)}{\lambda^2} d\lambda \right|.$$

Suppose that $(*)$ is not true for every $k_0 \in \mathbb{N}$. In this case, for infinitely many $k \in \mathbb{N}$ there exist $|\lambda_k| = 1$ such that $|P(x + \lambda_k y_k) - P(x + \lambda_k y)| \geq \epsilon$. But (λ_k) has a subsequence (λ_{k_i}) converging to some λ satisfying $|\lambda| = 1$ and then, since $(x + \lambda_{k_i} y_{k_i}) \xrightarrow{w^*} x + \lambda y$, and P is w^* -continuous on bounded subsets of E we have $|P(x + \lambda_{k_i} y_{k_i}) - P(x + \lambda_{k_i} y)| \leq |P(x + \lambda_{k_i} y_{k_i}) - P(x + \lambda y)| + |P(x + \lambda y) - P(x + \lambda_{k_i} y)| \rightarrow 0$, a contradiction. \square

Proposition 1.8. *Suppose that $E = F'$ for some Banach space F . Then for $P \in P(^nE)$ the following conditions are equivalent:*

- (1) $\overline{P} = P \circ \pi_1$.
- (2) \overline{P} factors throughout F^\perp .

Proof. (1) \Rightarrow (2). Take any $z \in E''$. If $\overline{P} = P \circ \pi_1$, $\forall w \in F^\perp$ we have $d\overline{P}(z)(w) = dP(\pi_1(z))(w) = dP(\pi_1(z))(0) = 0$. So $d\overline{P}(z) \in (F^\perp)^\perp \forall z \in E''$ and \overline{P} factors throughout F^\perp by [5], Theorem 2.2.

(2) \Rightarrow (1). We first remark that \overline{P} factors throughout F^\perp means that \overline{P} factors throughout $j(F^\perp)$ where $j : E \rightarrow E''$ is the canonical immersion.

We are going to show that $dP(x) \in F \forall x \in E$ and then infer (1) by Theorem 1.1. It is enough to show that $\forall x \in E$ we have $dP(x)(y) = 0 \quad \forall y \in F^\perp$. By [5], Theorem 2.2 we have that $d\overline{P}(z) \in j(F^\perp)^\perp \quad \forall z \in E''$. For each $x \in E$ and $y \in F^\perp \subset E$ we have $dP(x)(y) = \check{P}(x, \dots, x, y) = \overline{P}(j(x))(j(y)) = 0$ as $j(y) \in j(F^\perp)$. So, $dP(x) \in F \quad \forall x \in E$. \square

§2. The Holomorphic Case

Let $H_b(E)$ be the space of the holomorphic mappings that are bounded in the bounded subset of E . We denote by $H_{wu}(E)$ (respectively $H_{w^*u}(E)$) the space of the holomorphic functions from E into \mathbb{C} that are uniformly w -continuous (respectively uniformly w^* -continuous) on the bounded subsets of E . Given $f = \sum_{n=0}^{\infty} P_n \in H_b(E)$, the Aron-Berner extension \bar{f} from f to E'' is, by definition, the function $\bar{f} := \sum_{n=0}^{\infty} \bar{P}_n \in H_b(E'')$.

Theorem 2.1. *Suppose that $E = F'$ for some Banach space F . Then for $f \in H_b(E)$ the following conditions are equivalent:*

- (1) $\bar{f} = f \circ \pi_1$
- (2) $df(x) \in F \ \forall x \in E$.
- (3) $df(x)$ is w^* -continuous $\forall x \in E$.

Proof. (1) \Rightarrow (2). $f = \sum_{n=0}^{\infty} P_n \in H_b(E)$. Let $B \subset E''$ be a bounded set. It is clear that $\pi_1(B) \subset E$ is bounded and since $f \in H_b(E, F)$ we have $\|f - \sum_{n=0}^k P_n\|_{\pi_1(B)} \rightarrow 0$. So, $\|f \circ \pi_1 - \sum_{n=0}^k P_n \circ \pi_1\|_{B \rightarrow 0} \ \forall B \subset B''$ bounded and consequently $f \circ \pi_1 = \sum_{n=0}^{\infty} P_n \circ \pi_1 \in H_b(E'')$.

But $\bar{f} = \sum_{n=0}^{\infty} \bar{P}_n$ and by the unicity of the Taylor representation, $\bar{f} = f \circ \pi_1$ if and only if $\bar{P}_n = P_n \circ \pi_1$ for every $n \in \mathbb{N}$. By Theorem 1.1 we have $dP_n(x) \in F \ \forall x \in E \ \forall n \in \mathbb{N}$ and consequently $df(x) \in F \ \forall x \in E$.

(2) \Rightarrow (1). The proof is similar to the proof of (3) \Rightarrow (4) in Theorem 1.1.

(2) \Leftrightarrow (3) is clear. □

Proposition 2.2. *Suppose that $E = F'$ for some Banach space F . If $f = \sum_{n=0}^{\infty} P_n \in H_{wu}(E)$ and $df(x) \in F \ \forall x \in E$ then $f \in H_{w^*u}(E)$.*

Proof. Let $f = \sum_{n=0}^{\infty} P_n \in H_{wu}(E) \subset H_b(E)$. Combining Proposition 1.5 (b) of [1] and Theorem 2.1 we obtain that $P_n \in P_{w^*u}(E) \ \forall n \in \mathbb{N}$ and the proposition follows immediately. □

Proposition 2.3. *Suppose that $E = F'$ for some Banach space F . If F has the Mazur's property, for every $f \in H_{w^*u}(E)$ we have that $df(x)$ is w^* -continuous $\forall x \in E$ and, consequently, $\bar{f} = f \circ \pi_1$.*

Proof. The same of Proposition 1.7. □

Proposition 2.4. *Suppose that $E = F'$ for some Banach space F . Then for $f = \sum_{n=0}^{\infty} P_n \in H_b(E)$ the following conditions are equivalent:*

- (1) $\bar{f} = f \circ \pi_1$
- (2) \bar{f} factors throughout F^\perp .

Proof. The proof is similar to the proof of Proposition 1.7. Use [5], Theorem 2.3 instead of [5], Theorem 2.2. \square

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