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Termination of 4-fold Canonical Flips

By

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Abstract

There does not exist an infinite sequence of 4-fold canonical flips.

§1. Introduction

One of the most important conjectures in the minimal model program is (*log*) *Flip Conjecture* II. It claims that any sequence of (log) flips:

$$\begin{array}{ccccccc} (X_0, B_0) & \dashrightarrow & (X_1, B_1) & \dashrightarrow & (X_2, B_2) & \dashrightarrow & \cdots \\ & \searrow & \swarrow \quad \searrow & & \swarrow & & \\ & Z_0 & & Z_1 & & & \end{array},$$

has to terminate after finitely many steps. In this paper, we prove it for 4-dimensional canonical pairs. For the details of the log minimal model program, see [KMM, Introduction] or [KM, §3.7]. The following is the main theorem of this paper:

Theorem 1.1 (Termination of 4-fold canonical flips)**.** *Let X be a normal projective* 4*-fold and B an effective $\mathbb{Q}$-divisor such that (X, B) is canonical. Consider a sequence of log flips* (*see* Definition 2.2) *starting from* $(X, B) = (X_0, B_0)$:

$$\begin{array}{ccccccc} (X_0, B_0) & \dashrightarrow & (X_1, B_1) & \dashrightarrow & (X_2, B_2) & \dashrightarrow & \cdots \\ & \searrow & \swarrow \quad \searrow & & \swarrow & & \\ & Z_0 & & Z_1 & & & \end{array},$$

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where $\phi_i : X_i \longrightarrow Z_i$ *is a contraction and* ${\phi_i}^+ : {X_i}^+ = X_{i+1} \longrightarrow Z_i$ *is the log flip. Then this sequence terminates after finitely many steps.*

It is a slight generalization of [KMM, Theorem 5-1-15] and contains [M, Main Theorem 2.1]. We note that a D-flop is a log flip with respect to $K_X + \varepsilon D$ for $0 < \varepsilon \ll 1$.

Corollary 1.1 ([M, Main Theorem 2.1])**.** *Let* X *be a projective* 4*-fold with only terminal singularities and* D *an effective* $\mathbb{Q}$*-Cartier* $\mathbb{Q}$*-divisor. Then any sequence of* D*-flops is finite.*

The author believes that the main theorem is a first step to attack log Flip Conjecture II in dimension 4. In [Fj], we treat log Flip Conjecture II for 4-dimensional semi-stable log flips.

We will review several basic results and define *a weighted version of difficulty* in Section 2. The proof of the main theorem: Theorem 1.1, will be given in Section 3.

§2. Preliminaries

We will work over $\mathbb{C}$, the complex number field, throughout this paper.

First, let us recall the definitions of *discrepancies* and *singularities of pairs*. For the details, see [KM, §2.3].

Definition 2.1 (Discrepancies and singularities of pairs)**.** Let X be a normal variety and $\Delta = \sum \delta_i \Delta_i$ a $\mathbb{Q}$-divisor on X, where Δ_i is a prime divisor for every i and $\Delta_i \neq \Delta_j$ for $i \neq j$. We assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $f : Y \longrightarrow X$ be a proper birational morphism from a normal variety Y. Then we can write

$$K_Y = f^*(K_X + \Delta) + \sum a(E, X, \Delta)E,$$

where the sum runs over all the distinct prime divisors $E \subset Y$, and $a(E, X, \Delta) \in \mathbb{Q}$. This $a(E, X, \Delta)$ is called the *discrepancy* of E with respect to (X, Δ). We define

$$\operatorname{discrep}(X, \Delta) := \inf_E \{a(E, X, \Delta) \mid E \text{ is exceptional over } X\}.$$

From now on, we assume that Δ is effective. We say that (X, Δ) is

$$\begin{cases} \textit{terminal} \\ \textit{canonical} \end{cases} \quad \text{if} \quad \operatorname{discrep}(X, \Delta) \quad \begin{cases} > 0 \\ \geq 0. \end{cases}$$

If $a(E, X, \Delta) > -1$ for every E, then we say that (X, Δ) is a *klt* pair, where klt is short for *Kawamata log terminal* (or *log terminal* in the terminology of [KMM]).

Next, let us recall the definition of *canonical flips*, which is slightly different from the usual one (cf. [S, (2.11) Adjoint Diagram]).

Definition 2.2 (Canonical flip). Let X be a normal projective variety and B an effective $\mathbb{Q}$-divisor such that the pair (X, B) is canonical. Let $\phi : (X, B) \longrightarrow Z$ be a small contraction corresponding to a $(K_X + B)$-negative extremal face. If there exists a normal projective variety X^+ and a projective morphism $\phi^+ : X^+ \longrightarrow Z$ such that

1. ϕ^+ is small;
2. $K_{X^+} + B^+$ is ϕ^+-ample, where B^+ is the strict transform of B,

then we call ϕ^+ the *canonical flip* or *log flip* of ϕ. We call the following diagram a *flipping diagram*:

$$\begin{array}{ccc} (X, B) & \dashrightarrow & (X^+, B^+) \\ \phi \searrow & & \swarrow \phi^+ \\ & Z & \end{array} .$$

We introduce a variant of *difficulty*. This is slightly different from [K$^+$, Chapter 4]. It was inspired by [K$^+$, 4.14 Remark]. Note that the notion of difficulty was first introduced by Shokurov in [S, (2.15) Definition].

Definition 2.3 (A weighted version of difficulty). Let (X, B) be a pair with only canonical singularities, where $B = \sum_{j=1}^{l} b_j B^j$ with $0 < b_1 < \cdots < b_l \leq 1$ and B^j is a reduced divisor for every j. We note that B^j is not necessarily irreducible. We put $b_0 = 0$, and $S := \sum_{j \geq 0} b_j \mathbb{Z}_{\geq 0} \subset \mathbb{Q}$. Note that $S = 0$ if $B = 0$.

We call a divisor E over X *essential* if E is exceptional over X and is not obtained from blowing up the generic point of a subvariety $W \subset B \subset X$ such that B and X are generically smooth along W (and thus only one of the irreducible components of $\sum_{j \geq 1} B^j$ contains W) and $\dim W = \dim X - 2$. We set

$$d_{S,b}(X, B) := \sum_{\xi \in S, \xi \geq b} \sharp\{E | E \text{ is essential and } a(E, X, B) < 1 - \xi\}.$$

Then $d_{S,b_j}(X, B)$ is finite by Lemma 2.1 below. We note that the pair (X, B) is canonical if and only if $\operatorname{discrep}(X, B) \geq 0$ (see Definition 2.1).

Lemma 2.1 ([K$^+$, (4.12.2.1)]). *Let (X, B) be a klt pair. Then*

$$\sharp\{E|E \text{ is essential and } a(E, X, B) < \min\{1, 1 + \mathrm{discrep}(X, B)\}\}$$

is finite.

Definition 2.4. Let $\varphi : (X, B) \dashrightarrow (X^+, B^+)$ be a canonical flip. We say that this flip is *of type* $(\dim A, \dim A^+)$, where A (resp. A^+) is the exceptional locus of $\phi : X \longrightarrow Z$ (resp. $\phi^+ : X^+ \longrightarrow Z$). We call A (resp. A^+) the *flipping* (resp. *flipped*) locus of φ. When $\dim X = 4$, the log flip is either of type $(1, 2)$, $(2, 2)$ or $(2, 1)$ by [KMM, Lemma 5-1-17].

Remark. In [KMM, §5-1], the variety is $\mathbb{Q}$-factorial and every flipping contraction corresponds to a negative extremal ray. However, the above properties were not used in the proof of [KMM, Lemma 5-1-17]. Note that [KMM, Lemma 5-1-17] holds in more general setting. For the details, see the original article [KMM].

Lemma 2.2 (cf. [KM, Lemma 6.21]). *Let $\varphi : (X, \Delta) \dashrightarrow (X^+, \Delta^+)$ be a canonical flip of n-folds. Let $\Delta := \sum \delta_i \Delta_i$ be the irreducible decomposition and Δ^+ (resp. Δ_i^+) the strict transform of Δ (resp. Δ_i). Let F be an $(n-2)$-dimensional irreducible component of A^+, and E_F the exceptional divisor obtained by blowing up F near the generic point of F. Then X^+ is generically smooth along F and*

$$0 \le a(E_F, X, \Delta) < a(E_F, X^+, \Delta^+) = 1 - \sum \delta_i \mathrm{mult}_F(\Delta_i^+),$$

where $\mathrm{mult}_F(\Delta_i^+)$ *is the multiplicity of Δ_i^+ along F.*

Proof. The pair (X^+, Δ^+) is terminal near the generic point of F by the negativity lemma. Therefore, X^+ is generically smooth along F, and the rest is an obvious computation. □

2.1. Let X be a projective variety and X^{an} the underlying analytic space of X. Let $H_k^{BM}(X^{an})$ be the *Borel-Moore* homology. For the details, see [Fl, 19.1 Cycle Map]. Then there exists a cycle map;

$$cl : A_k(X) \longrightarrow H_{2k}^{BM}(X^{an}),$$

where $A_k(X)$ is the group generated by rational equivalence classes of k-dimensional cycles on X. For the details about $A_k(X)$, see [Fl, Chapter 1].

We note that the cycle map *cl* commutes with push-forward for proper morphisms, and with restriction to open subschemes. From now on, we omit the superscript *an* for simplicity.

The following is in the proof of [KMM, Theorem 5-1-15].

Lemma 2.3 (Log flips of type $(2,1)$). *When the* 4*-dimensional log flip*

$$\begin{array}{ccc} (X,B) & \dashrightarrow & (X^+,B^+) \\ \phi\searrow & & \swarrow\phi^+ \\ & Z & \end{array}$$

is of type $(2,1)$*, we study the rank of the* $\mathbb{Z}$*-module* $cl(A_2(X))$*, where* $cl(A_2(X))$ *is the image of the cycle map* $cl : A_2(X) \longrightarrow H_4^{BM}(X)$*. By the following commutative diagrams*;

$$\begin{array}{ccccccc} A_2(A) & \longrightarrow & A_2(X) & \longrightarrow & A_2(X\setminus A) & \longrightarrow & 0 \\ \downarrow cl & & \downarrow cl & & \downarrow cl & & \\ H_4^{BM}(A) & \longrightarrow & H_4^{BM}(X) & \longrightarrow & H_4^{BM}(X\setminus A), & & \end{array}$$

and

$$\begin{array}{ccc} A_2(Z) & \xrightarrow{\sim} & A_2(Z\setminus\phi(A)) \\ \downarrow cl & & \downarrow cl \\ H_4^{BM}(Z) & \xrightarrow{\sim} & H_4^{BM}(Z\setminus\phi(A)), \end{array}$$

we have the surjective homomorphism;

$$cl(A_2(X)) \longrightarrow cl(A_2(Z)).$$

We note that $X\setminus A \simeq Z\setminus\phi(A)$ *and* [Fl, p. 371 (6) and Lemma 19.1.1]*. For any closed algebraic subvariety* V *on* X *of complex dimension* 2*,* V *is not numerically trivial since* X *is projective. Therefore,* $cl(V)\neq 0$ *in* $cl(A_2(X))$ *(see* [Fl, Definition 19.1]*). Thus the kernel of the surjection above is not zero. By the similar arguments, we obtain that*

$$cl(A_2(X^+)) \simeq cl(A_2(Z)).$$

We note that A^+ *is one-dimensional and* $X^+\setminus A^+ \simeq Z\setminus\phi^+(A^+)$*. Therefore, since the rank* $\mathrm{rk}_{\mathbb{Z}}cl(A_2(X))$ *is finite, we finally have the result*

$$\mathrm{rk}_{\mathbb{Z}}cl(A_2(X)) > \mathrm{rk}_{\mathbb{Z}}cl(A_2(X^+)).$$

§3. Proof of the Main Theorem

Let us start the proof of Theorem 1.1.

Proof of Theorem 1.1. By the definition of the weighted version of difficulty and the negativity lemma, $d_{S,b_l}(X_i, B_i)$ does not increase. We note that if E is exceptional over X_i and $a(E, X_i, B_i) < 1 - b_l$, then E is essential. If B^l_i, which is the strict transform of B^l on X_i, contains 2-dimensional flipped locus, then $d_{S,b_l}(X_i, B_i)$ decreases by easy computations (see Lemma 2.2). So, after finitely many flips, B^l_i does not contain 2-dimensional flipped locus. Thus, we can assume that B^l_i does not contain 2-dimensional flipped locus for every $i > 0$ by shifting the index i. Let $\overline{B^l_i}$ be the normalization of B^l_i. We consider a sequence of birational maps

$$\overline{B^l_0} \dashrightarrow \overline{B^l_1} \dashrightarrow \cdots .$$

By [M, Lemma 2.11], after finitely many flips, B^l_i does not contain 2-dimensional flipping locus. Thus, we can assume that B^l_i does not contain 2-dimensional flipping locus for every $i \geq 0$. In particular, $B^l_i \dashrightarrow B^l_{i+1}$ is an isomorphism in codimension one for every $i \geq 0$.

Next, we look at $d_{S,b_{l-1}}(X_i, B_i)$. We note that if E is essential over X_{i+1} and $a(E, X_{i+1}, B_{i+1}) < 1 - b_{l-1}$, then E is essential over X_i and $a(E, X_i, B_i) \leq a(E, X_{i+1}, B_{i+1})$. It is because $B^l_i \dashrightarrow B^l_{i+1}$ is an isomorphism in codimension one. So, $d_{S,b_{l-1}}(X_i, B_i)$ does not increase by log flips. Thus, by Lemma 2.2, B^{l-1}_i does not contain 2-dimensional flipped locus after finitely many flips. By using [M, Lemma 2.11] again, we see that B^{l-1}_i does not contain 2-dimensional flipping locus after finitely many steps. Therefore, we can assume that $B^{l-1}_i \dashrightarrow B^{l-1}_{i+1}$ is an isomorphism in codimension one for every $i \geq 0$ by shifting the index i.

By repeating this argument, we can assume that $B^j_i \dashrightarrow B^j_{i+1}$ is an isomorphism in codimension one for every i, j.

If the log flip is of type $(1, 2)$ or $(2, 2)$, then $d_{S,b_0}(X_i, B_i) = d_{S,0}(X_i, B_i)$ decreases by Lemma 2.2. Therefore, we can assume that all the flips are of type $(2, 1)$ after finitely many steps. This sequence terminates by Lemma 2.3.

So we complete the proof of the main theorem. □

Remark (3-fold case). By using the weighted version of difficulty, we can easily prove the termination of 3-fold canonical flips without using $\mathbb{Q}$-factoriality. This is [K$^+$, 4.14 Remark].

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