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On the Stokes Equation with the Leak and Slip Boundary Conditions of Friction Type: Regularity of Solutions

By

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Abstract

We consider the Stokes equations under some nonlinear boundary conditions, which are described in terms of subdifferentials of maximal monotone graphs and are called leak and slip boundary conditions of friction type. The main objective is to show the existence of strong solutions, say $u \in H^2$ and $p \in H^1$, to these problems. We start with weak solutions to variational inequalities, and then study the regularity of weak solutions. Our main theorems imply the maximality of Stokes operators with such nonlinear boundary conditions in a suitable Hilbert space and they are of use in analysis of time-dependent problems. Linear boundary conditions of Neumann type, such as slip and penetration conditions, are also discussed.

§1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N = 2, 3$. We suppose that the boundary $\partial\Omega$ of Ω is composed of two connected components Γ and Γ_D which are assumed to be Lipschitz continuous, unless otherwise stated. Γ is not empty, whereas Γ_D may be empty. In the present paper, we shall mainly discuss the existence of a *strong solution* $u \in H^2(\Omega)^N$ and $p \in H^1(\Omega)$ to a modified Stokes equation

$$\lambda u - \Delta u + \nabla p = f, \quad \operatorname{div} u = 0 \qquad \text{in } \Omega \tag{1.1}$$

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under the standard Dirichlet boundary condition

$$u = 0 \qquad \text{on } \Gamma_D \tag{1.2}$$

together with the one of the following nonlinear boundary conditions:

$$u_\tau = 0, \quad -\sigma_n \in g\partial|u_n| \qquad \text{on } \Gamma, \tag{1.3}$$

$$u_n = 0, \quad -\sigma_\tau \in g\partial|u_\tau| \qquad \text{on } \Gamma. \tag{1.4}$$

Here, λ denotes a non-negative constant; $u = (u_1, \ldots, u_N)$ the velocity vector and p the pressure; f and $g(\geq 0)$ given vector and scalar functions; $u_n \equiv u \cdot n$ and $u_\tau \equiv u - nu_n$ are the normal and tangential components of the velocity, respectively, where $n = (n_1, \ldots, n_N)$ stands for the outer unit normal to Γ; $\sigma_n = \sigma_n(u, p)$ and $\sigma_\tau = \sigma_\tau(u)$ denote normal and tangential components of the stress vector (the precise definitions will be recalled in §2); and finally $\partial|z|$ denotes a graph

$$\partial|z| = \begin{cases} \dfrac{z}{|z|} & (z \neq 0,\ z \in \mathbb{R}^m) \\ \{w \in \mathbb{R}^m |\ |w| \leq 1\} & (z = 0,\ z \in \mathbb{R}^m), \end{cases} \tag{1.5}$$

where $m = 1$ for (1.3) and $m = N$ for (1.4). It is easy to see that the second conditions of (1.3) and (1.4) are equivalent to

$$|\sigma_n| \leq g, \quad \sigma_n u_n + g|u_n| = 0 \qquad \text{on } \Gamma,$$
$$|\sigma_\tau| \leq g, \quad \sigma_\tau \cdot u_\tau + g|u_\tau| = 0 \qquad \text{on } \Gamma,$$

respectively.

The problem composed of (1.1), (1.2) and (1.3), which we will refer as the leak boundary problem of friction type, or simply (Pr. LF), was introduced by H. Fujita ([9]) in order to study steady motions of viscous incompressible fluid involving a leak of the fluid through the surface or penetration into adjacent media. Applications to oil flow over or beneath sand layers are presented in Kawarada, Fujita and Suito [19], and Kawarada and Suito [20].

In [9], the existence and uniqueness/non-uniqueness of a weak solution, say $u \in H^1(\Omega)^N$ and $p \in L^2(\Omega)$, has been established by means of a variational inequality and an extended Hanh-Banach theorem. We shall review his results in §3.

The first purpose of the present paper is to study the regularity of Fujita's weak solution, and prove the following theorem.

Theorem 1.1. *Let $\lambda \geq 0$. Suppose that the following assumptions hold:*

$$\overline{\Gamma} \cap \overline{\Gamma_D} = \emptyset; \tag{1.6}$$

$$\Gamma_D \text{ and } \Gamma \text{ are of class } C^2 \text{ and } C^4, \text{ respectively}; \tag{1.7}$$

$$\Gamma_D \neq \emptyset \quad (\text{if } \lambda = 0); \tag{1.8}$$

$$g \in H^{1/2}(\Gamma), \qquad g \geq 0 \ \text{a.e. in} \ \ \Gamma; \tag{1.9}$$

$$f \in L^2(\Omega)^N. \tag{1.10}$$

Then there exists a solution $\{u, p\} \in H^2(\Omega)^N \times H^1(\Omega)$ of

$$\text{(Pr. LF)} \qquad \begin{cases} \lambda u - \Delta u + \nabla p = f, \quad \operatorname{div} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ u_\tau = 0, \quad -\sigma_n \in g\partial|u_n| & \text{on } \Gamma. \end{cases}$$

The velocity u is unique, while the pressure p is unique except for an additive constant. The range of the additive constant to p is limited to $\{0\}$ or to a finite closed interval. Furthermore there is a positive constant depending only on Ω such that

$$\|u\|_{H^2(\Omega)^N} + \|p\|_{H^1(\Omega)} \\ \leq C(\|f\|_{L^2(\Omega)^N} + \|g\|_{H^{1/2}(\Gamma)} + \|u\|_{H^1(\Omega)^N} + \|p\|_{L^2(\Omega)})$$

for any solution $\{u, p\}$ of (Pr. LF).

Several remarks are in order.

Remark. The description about non-uniqueness of the pressure is rather troublesome. Let $\{u, p\}$ solve (Pr. LF). For the sake of simplicity, we assume that $\sigma_n(u, p)$ and g are continuous. Following [9], we set

$$k_1 = \sup_\Gamma(\sigma_n(u, p) - g), \quad k_2 = \inf_\Gamma(\sigma_n(u, p) + g). \tag{1.11}$$

Obviously $-2\|g\|_{L^\infty(\Gamma)} \leq k_1 \leq k_2 \leq 2\|g\|_{L^\infty(\Gamma)}$. Then a function $p + k$, $k \in [k_1, k_2]$, is also a corresponding pressure of u; $\{u, p + k\}$ also solve (Pr. LF). If another corresponding pressure p^* has been taken first, then the value of k_1, k_2 are changed. On the other hand, if non-trivial movement ($u_n \neq 0$) takes place on a portion $\Gamma_1 \subset \Gamma$, then p is uniquely determined. This means that the range of the additive constant to p is limited to $\{0\}$. See, for more detail, Remark 3.2 of [9].

We here give a simple example to illustrate this issue. We employ the polar coordinates $x = (r, \theta)$ in $\mathbb{R}^2$. We assume that

$$\Omega = \{(r,\theta) |\ 1 < r < 2\}, \quad \Gamma_D = \{r = 1\}, \quad \Gamma = \{r = 2\},$$

and set $e_r = (\cos\theta, \sin\theta)$, $e_\theta = (-\sin\theta, \cos\theta)$. Put $u(r,\theta) = w(r)e_\theta$ and $p(r) = \kappa r$, where $w(r) = 4r^{-1} + 2r - 6$ and $\kappa > 0$ is a constant. Observe that $\{u, p\}$ solves $-\Delta u + \nabla p = -6e_\theta r^{-2} + \kappa e_r$, div $u = 0$ in Ω, $u|_{\Gamma_D} = 0$, $u_\tau|_\Gamma = 0$, and $u_n|_\Gamma = 0$. We have $\sigma_n(u,p) = -p(r)$ and hence $|\sigma_n(u,p)| = 2\kappa$. We define $g = 2\kappa + 1$. Then $|\sigma_n(u,p)| < g$ and u_τ, u_n vanish on Γ. Therefore $\{u, p\}$ is a solution of (Pr. LF). In this case, we have $k_1 = -1 - 4\kappa$ and $k_2 = 1$. Now put $p_k = p + k$ with a constant k. As long as k is taken from $[k_1, k_2]$, $\{u, p_k\}$ solves (Pr. LF). However, if $k \notin [k_1, k_2]$, then $\{u, p_k\}$ does not solve (Pr. LF), since $|\sigma_n(u, p_k)| > g$.

Remark. Assumption that $g \in H^{1/2}(\Gamma)$ is really optimal to derive the H^2-H^1 regularity of $\{u, p\}$. We shall revisit this issue in the paragraph (D) of §7, after having prepared a few materials to discuss it.

Remark. Theorem 1.1 has an important application on *evolution problems.* Actually, Theorem 1.1 implies the maximality of the operator A on $X = L^2(\Omega)^N$ defined as

$$Au = \{-\Delta u + \nabla p |\ p \in M(u)\} \qquad (u \in D(A))$$

with

$$D(A) = \{u \in H^2(\Omega)^N |\ \text{div}\ u = 0,\ u|_{\Gamma_D} = 0,\ u_\tau|_\Gamma = 0,\ M(u) \neq \emptyset\},$$

where

$$M(u) = \{p \in H^1(\Omega) |\ -\sigma_n \in g\partial|u_n| \text{ on } \Gamma\}.$$

That is, we have $u \in D(A)$ satisfying $\lambda u + Au = f$ in X for all $f \in X$ and $\lambda > 0$. On the other hand, it is easy to verify that A is monotone in X; $(f_1 - f_2, u_1 - u_2) \geq 0$ for all $u_1, u_2 \in D(A)$ and $f_1 \in Au_1$, $f_2 \in Au_2$. These enable us to apply the nonlinear semigroup theory in X ([21]) and lead to the unique solvability of a non-stationary Stokes problem

$$\frac{\partial u}{\partial t} = \Delta u - \nabla p, \quad \text{div}\ u = 0$$

with boundary conditions (1.2), (1.3) and the initial condition $u|_{t=0} = u_0(x)$. For more detail, see Fujita [10], [11] and [31]. Moreover, by developing Fujita's argument with the aid of monotonicity method ([3], [5]), under a suitable assumption on $f = f(x, t)$, we can solve

$$\frac{\partial u}{\partial t} = \Delta u - \nabla p + f, \quad \text{div } u = 0$$

with (1.2), (1.3), $u|_{t=0} = u_0(x)$. We shall report the detail and an application to the non-stationary Navier-Stokes equations in a forthcoming paper.

The second purpose of this paper is to study the slip boundary value problem of friction type, (Pr. SF), which is composed of (1.1), (1.2) and (1.4). This problem appears in modelling of blood flow in a vein of an arterial sclerosis patient and in that of avalanche of water and rocks ([9]). Concerning (Pr. SF), we have the following theorem where we should keep in mind that σ_τ does not explicitly contain p.

Theorem 1.2. *Let* $\lambda \geq 0$, *and suppose* (1.6), (1.7), (1.8), (1.9), *and* (1.10). *Then there exists a unique solution* $\{u, p\} \in H^2(\Omega)^N \times H^1(\Omega)$ *of*

$$\text{(Pr. SF)} \qquad \begin{cases} \lambda u - \Delta u + \nabla p = f, \quad \text{div } u = 0 & \textit{in } \Omega, \\ u = 0 & \textit{on } \Gamma_D, \\ u_n = 0, \quad -\sigma_\tau \in g\partial|u_\tau| & \textit{on } \Gamma \end{cases}$$

satisfying

$$\int_\Omega p \, dx = 0.$$

Moreover we have

$$\|u\|_{H^2(\Omega)^N} + \|p\|_{H^1(\Omega)} \\ \leq C(\|f\|_{L^2(\Omega)^N} + \|g\|_{H^{1/2}(\Gamma)} + \|u\|_{H^1(\Omega)^N} + \|p\|_{L^2(\Omega)}).$$

In §2, we shall introduce the notation used in this paper and describe some identities and lemmas. In order to prove Theorem 1.1, we begin by considering a weak formulation (Pr. WLF) to (Pr. LF) by means of a variational inequality which was proposed by [9]. However, following Brezis [4], we do not treat (Pr. WLF) directly and study a regularized problem (Pr. WLF$_\varepsilon$) to (Pr.

WLF), where $\varepsilon > 0$ denotes a regularized parameter. Actually, (Pr. WLF$_\varepsilon$) is defined via the Yosida regularization of $\partial|\cdot|$. Taking such detour allows us to avoid a redundant hypothesis on g. On the other hand, from the viewpoint of numerical analysis, a regularized problem itself is worth considering. We shall describe (Pr. WLF) and (Pr. WLF$_\varepsilon$) in §3. The unique existence of a solution to (Pr. WLF$_\varepsilon$) is also mentioned there. In §4, we investigate the regularity of a solution to (Pr. WLF$_\varepsilon$) and then establish the proof of Theorem 1.1. The strategy is standard in regularization theory. That is, we employ a localization technique, partition of unity and difference quotients. The method of the proof of Theorem 1.2 is given in §5.

In §6, although digressing from the main subject, we consider a few kinds of linear boundary value problems of Neumann type for the Stokes equations. Specifically, we shall deal with (1.1) and (1.2) under one of the following conditions to be posed on Γ:

$$u_\tau = 0, \quad \sigma_n = \omega_n \qquad \text{(non-slip and penetration)}; \tag{1.12}$$

$$u_n = 0, \quad \sigma_\tau = \omega_\tau \qquad \text{(non-penetration and slip)}; \tag{1.13}$$

$$\sigma = \omega \qquad \text{(prescribed stress)}, \tag{1.14}$$

where ω_n, ω_τ and ω are given functions defined on Γ. Here, for example, the condition (1.12) means that the fluid under consideration does not slip at the boundary and the penetration of the fluid through the boundary is controlled by a prescribed "force of stream" which is expressed as the normal component of the stress. We are concerned with the regularity of weak solutions to these problems such as $u \in H^{k+2}$ and $p \in H^{k+1}$, $k \geq 0$ (Theorems 6.1, 6.2, and 6.3 described below). To accomplish these, in addition to the argument of §4, following Bello [2], we adopt a well-known theorem on abstract variational problems with constraints, which gives a better viewpoint in the proof.

As is well-known, Neumann boundary value problems described above play a fundamental role in analysis of some actual and practical problems including free boundary problems ([24], [27], [28]) and a technique of numerical methods ([18]). However, concerning regularity results of weak solutions, we could find no explicit reference to them in the literature. In fact, (1.1), (1.2) with the condition (1.12) in the case of $\omega_n \equiv 0$ was described in Solonnikov and Ščadilov [28]. But it seems that the complete proof for the case of $\omega_n \not\equiv 0$ is not explicitly stated there. This is why we decide to state an explicit and a somewhat elementally proof in this paper, although they seem not to be new for specialists.

Finally, in §7, we mention several additional remarks including a commentary to the full stress problem ([26]) and a relation with a general theory of Agmon, Douglis and Nirenberg [1].

§2. Preliminaries

(A) Notation. The deformation tensor and stress tensor associated with a velocity field $u = (u_1, \ldots, u_N)$ and pressure p are denoted by

$$[e_{ij}(u)] \quad e_{ij}(u) = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \quad \text{and} \quad [S_{ij}(u,p)] \quad S_{ij}(u) = -p\delta_{ij} + e_{ij}(u),$$

respectively, where δ_{ij} denotes Kroneker's delta. The stress vector $\sigma(u,p)$ is defined by $\sigma(u,p) = [S_{ij}(u,p)]n$ of which the ith component is

$$\sum_{j=1}^{N} S_{ij}(u,p) n_j.$$

The normal and tangential components of a vector field u are defined as $u_n = u \cdot n$ and $u_\tau = u - n u_n$, respectively. In particular,

$$\begin{aligned} &\sigma_n(u,p) = \sigma(u,p) \cdot n && \text{(the normal component of the stress vector);} \\ &\sigma_\tau(u) = \sigma(u,p) - n\sigma_n(u,p) && \text{(the tangential component of the stress vector).} \end{aligned}$$

If there is no possibility of confusion, we simply write σ, σ_n and σ_τ to express $\sigma(u,p)$, $\sigma_n(u,p)$ and $\sigma_\tau(u)$, respectively.

We will use the $L^2(\Omega)$ space and the usual Sobolev spaces $H^m(\Omega)$ for a non-negative integer m. $H^0(\Omega)$ is understood as $L^2(\Omega)$. We put

$$L_0^2(\Omega) = \left\{ v \in L^2(\Omega) \,\middle|\, \int_\Omega v \; dx = 0 \right\}.$$

We write $\|\cdot\|_m = \|\cdot\|_{m,\Omega}$ instead of $\|\cdot\|_{H^m(\Omega)}$, and set $\|\cdot\| = \|\cdot\|_0$.

We also use the Sobolev space $H^s(\Gamma)$ defined on the boundary Γ, where $s \in \mathbb{R}$. We write $\|\cdot\|_{s,\Gamma} = \|\cdot\|_{H^s(\Gamma)}$. $H^0(\Gamma)$ is understood as $L^2(\Gamma)$. The surface element of Γ is denoted by ds, that is

$$\|\eta\|_{0,\Gamma}^2 = \int_\Gamma |\eta|^2 \; ds.$$

We write $(\cdot,\cdot) = (\cdot,\cdot)_{L^2(\Omega)}$ and $(\cdot,\cdot)_\Gamma = (\cdot,\cdot)_{L^2(\Gamma)}$.

Let Tr be the trace operator from $H^1(\Omega)$ into $H^{1/2}(\Gamma)$. Then the trace Tr v on Γ of $v \in H^1(\Omega)$ is denoted by $v|_\Gamma$. If it is clear from the context, we will not distinguish v from $v|_\Gamma$. The meaning of $v|_{\Gamma_D}$ is similar.

In general, for a Hilbert space X, the adjoint space is denoted by X^*, and X^N denotes the set of vector $v = (v_1, \ldots, v_N)$, $v_j \in X$. For vector functions, we use same symbol to indicate their inner product and norm; $(\cdot,\cdot)_X = (\cdot,\cdot)_{X^N}$ and $\|\cdot\|_X = \|\cdot\|_{X^N}$.

We use closed subspaces of $H^1(\Omega)^N$:

$$
\begin{aligned}
&K = \begin{cases} H^1(\Omega)^N & \text{if } \Gamma_D = \emptyset, \\ \{v \in H^1(\Omega)^N \mid v|_{\Gamma_D} = 0\} & \text{if } \Gamma_D \neq \emptyset; \end{cases} \\
&K_\tau = \{v \in K \mid v_\tau|_\Gamma = 0\}; \quad \hat{K}_\tau = \{v \in K_\tau \mid \operatorname{div} v = 0 \text{ in } \Omega\}; \\
&K_n = \{v \in K \mid v_n|_\Gamma = 0\}; \quad \hat{K}_n = \{v \in K_n \mid \operatorname{div} v = 0 \text{ in } \Omega\}.
\end{aligned}
$$

If $\Gamma_D \neq \emptyset$, $\|\cdot\|_1$ is equivalent to Dirichlet's norm $\|\nabla \cdot\|$ in K by Poincaré's inequality. We shall not emphasise this in what follows.

Let ψ be a proper ($\psi \not\equiv \infty$) lower semi-continuous convex function defined on $\mathbb{R}^m$, $m = 1$ or N. Then, for any $z \in \mathbb{R}^m$, $\partial\psi(z)$ denotes the set

$$\partial\psi(z) = \{h \in \mathbb{R}^m \mid \psi(t) - \psi(z) \geq h \cdot (t - z) \ \forall t \in \mathbb{R}^m\}$$

which is called the subdifferential of ψ at z. It is easy to see that the right-hand side of (1.5) coincides with $\partial\psi(z)$, when $\psi(z) = |z|$ for $z \in \mathbb{R}^m$.

The symbol C denotes various generic positive constant *depending on* Ω. When we need to specify the dependence of other parameters $q_1, \ldots, q_M$, which may not be numbers, we shall write as $C = C(q_1, \ldots, q_M)$.

(B) Bilinear forms. We introduce a bilinear form on $H^1(\Omega)^N \times H^1(\Omega)^N$ defined as

$$a_\lambda(v, w) = \lambda \int_\Omega v \cdot w \, dx + \frac{1}{2} \int_\Omega e_{ij}(v) e_{ij}(w) \, dx \quad (v, w \in H^1(\Omega)^N) \tag{2.1}$$

for $\lambda \geq 0$. Here and hereafter the summation convection is employed. We put

$$a(v, w) = a_0(v, w).$$

Clearly a_λ is continuous on $H^1(\Omega)^N \times H^1(\Omega)^N$:

$$|a_\lambda(v, w)| \leq C(\lambda) \|v\|_1 \|w\|_1 \qquad (\forall v, w \in H^1(\Omega)^N). \tag{2.2}$$

If $\lambda > 0$, a_λ is coercive on $H^1(\Omega)^N \times H^1(\Omega)^N$, that is,

$$a_\lambda(v, v) \geq C(\lambda)\|v\|_1^2 \qquad (\forall v \in H^1(\Omega)^N). \tag{2.3}$$

In fact, (2.3) is a consequence of Korn's inequality (for example [8], [30])

$$\int_\Omega e_{ij}(v)e_{ij}(v)\ dx + \int_\Omega |v|^2\ dx \geq C\|v\|_1^2 \qquad (\forall v \in H^1(\Omega)^N). \tag{2.4}$$

In the case of $\lambda = 0$, we assume $\Gamma_D \neq \emptyset$. Then, it holds that

$$\int_\Omega e_{ij}(v)e_{ij}(v)\ dx \geq C\|v\|_1^2 \qquad (\forall v \in K). \tag{2.5}$$

(See, for example, Theorem 6.3-4 of Ciarlet [30]. Although the proof only in the case of $N = 3$ is explicitly stated there, it is valid in the case of $N = 2$, too.) This, together with Poincaré's inequality, implies

$$a(v, v) \geq C\|v\|_1^2 \qquad (\forall v \in K). \tag{2.6}$$

We shall also use a continuous bilinear form on $H^1(\Omega)^N \times L^2(\Omega)$

$$b(v, \chi) = -\int_\Omega \chi \operatorname{div} v\ dx \qquad (v \in H^1(\Omega)^N,\ \chi \in L^2(\Omega)). \tag{2.7}$$

(C) Green's formula. If a smooth vector field u and a smooth scalar field p solve (1.1), then by integration by parts,

$$a_\lambda(u, \varphi) + b(\varphi, p) = \int_{\partial\Omega} \sigma \cdot \varphi\ ds + (f, \varphi) \qquad (\forall \varphi \in H^1(\Omega)^N).$$

In particular,

$$a_\lambda(u, \varphi) + b(\varphi, p) = \int_\Gamma \sigma_n \varphi_n\ ds + (f, \varphi) \qquad (\forall \varphi \in K_\tau), \tag{2.8}$$

$$a_\lambda(u, \varphi) + b(\varphi, p) = \int_\Gamma \sigma_\tau \cdot \varphi_\tau\ ds + (f, \varphi) \qquad (\forall \varphi \in K_n). \tag{2.9}$$

Variational inequalities (Pr. WLF) and (Pr. WSF) which will appeared in the subsequent sections are based on these identities and the definition of subdifferentials.

(D) Lemmas. With the bilinear form b, we associate the bounded linear operator $B_\tau : L^2(\Omega) \to K_\tau$ defined as

$$(B_\tau \chi, v)_1 = b(v, \chi) \qquad (\forall \chi \in L^2(\Omega),\ \forall v \in K_\tau). \tag{2.10}$$

Clearly we have $\|B_\tau \chi\|_1 \leq \|\chi\|$ for all $\chi \in L^2(\Omega)$.

Lemma 2.1. *The range $R(B_\tau)$ of B_τ is a closed set in K_τ. Moreover we have the orthogonal decomposition $K_\tau = R(B_\tau) \bigoplus \hat{K}_\tau$.*

The proof of this lemma depends on

Lemma 2.2. *Any $\chi \in L^2(\Omega)$ admits $w \in K_\tau$ satisfying* $\operatorname{div} w = \chi$ *in Ω and $\|w\|_1 \leq C\|\chi\|$.*

Proof. We only state the case of $\Gamma_D \neq \emptyset$. Firstly we assume that $\overline{\Gamma} \cap \overline{\Gamma_D} = \emptyset$. Let B an open ball which includes $\overline{\Omega}$ and take $v \in H^2(\Omega)$ satisfying

$$\Delta v = \begin{cases} \chi & \text{in } \Omega \\ 0 & \text{in } B\backslash\Omega, \end{cases} \qquad v|_{\partial B} = 0, \quad \|v\|_{2,B} \leq C\|\chi\|.$$

Put

$$b = \begin{cases} (\nabla v)|_\Gamma + cn & \text{on } \Gamma \\ (\nabla v)|_{\Gamma_D} & \text{on } \Gamma_D \end{cases} \qquad \text{with} \qquad c = -\frac{1}{|\Gamma|}\int_{\partial\Omega} \frac{\partial v}{\partial n}\, ds,$$

where $|\Gamma|$ denotes the Lebesgue measure of Γ. Then $b \in H^{1/2}(\partial\Omega)^N$ and it satisfies

$$\int_{\partial\Omega} b_n\, ds = 0 \quad \text{and} \quad \|b\|_{1/2,\partial\Omega} \leq C\|\nabla v\|_{1,B} \leq C\|v\|_{2,B} \leq C\|\chi\|.$$

Hence, as is well-known (for example Lemma I-3.2 of [17]), there is a $\psi \in H^1(\Omega)^N$ such that $\operatorname{div} \psi = 0$ in Ω, $\psi|_{\partial\Omega} = b$ and $\|\psi\|_1 \leq C\|b\|_{1/2,\partial\Omega}$. We set $w = \nabla v - \psi$, which is the desired function. In fact, $\operatorname{div} w = 0$, $w|_{\Gamma_D} = 0$, $w_\tau|_\Gamma = cn - nc = 0$, and $\|w\|_1 \leq \|\nabla v\|_1 + \|\psi\|_1 \leq C\|v\|_{2,B} \leq C\|\chi\|$.

Next we consider the case of $\overline{\Gamma} \cap \overline{\Gamma_D} \neq \emptyset$. Fix $\tilde{\varphi} \in C^\infty(\overline{\Omega})$ satisfying $\tilde{\varphi}|_{\Gamma_D} = 0$ and $\tilde{\varphi}|_\Gamma > 0$. Then put $\varphi = \tilde{\varphi}|_{\partial\Omega}$. In this case, we define

$$b = \begin{cases} (\nabla v)|_\Gamma + cn\varphi & \text{on } \Gamma \\ (\nabla v)|_{\Gamma_D} & \text{on } \Gamma_D \end{cases} \qquad \text{with} \qquad c = -\left(\int_\Gamma \varphi\, ds\right)^{-1} \int_{\partial\Omega} \frac{\partial v}{\partial n}\, ds.$$

The rest is the same as before. □

Proof of Lemma 2.1. Let $\chi \in L^2(\Omega)$. By Lemma 2.2, we can take $w \in K_\tau$ such that $\operatorname{div} w = q$ and $\|w\|_1 \leq C\|\chi\|$. Substituting $v = w$ into (2.10), we obtain

$$\|\chi\|^2 \leq \|B_\tau \chi\| \cdot \|w\| \leq C\|B_\tau \chi\| \cdot \|\chi\|.$$

Hence it follows from the closed range theorem that $R(B_\tau)$ is a closed set. To prove the second assertion, it suffices to see $R(B_\tau)^\perp = \hat{K}_\tau$, where $R(B_\tau)^\perp$ stands for the orthogonal complement of $R(B_\tau)$ in K_τ. Let $\varphi \in \hat{K}_\tau$. Then $(B_\tau\chi, \varphi)_1 = (\chi, \operatorname{div} \varphi) = 0$ for all $\chi \in L^2(\Omega)$ so that $\varphi \in R(B_\tau)^\perp$. Conversely, let $\varphi \in R(B_\tau)^\perp$. We have $0 = (B_\tau\chi, \varphi) = (\chi, \operatorname{div} \varphi)_1$ for all $\chi \in L^2(\Omega)$. Taking $\chi = \operatorname{div} \varphi$, we obtain $\operatorname{div} \varphi = 0$ a.e. in Ω. $\square$

Remark. Lemma 2.2 guarantees the *inf-sup condition*

$$\exists\, \beta > 0 : \quad \inf_{\chi \in L^2(\Omega)} \sup_{v \in K_\tau} \frac{b(v, \chi)}{\|v\|_1 \|\chi\|} \geq \beta, \tag{2.11}$$

which will be used in §6. In fact, for $\chi \in X$, we take $w \in K_\tau$ such as described in Lemma 2.2. Then

$$\sup_{v \in K_\tau} \frac{b(v, \chi)}{\|v\|_1} \geq \frac{b(w, \chi)}{\|w\|_1} \geq \frac{\|\chi\|^2}{\|w\|_1} \geq \frac{1}{C}\|\chi\|.$$

Next, let $B_n : L^2_0(\Omega) \to K_n$ be the bounded linear operator associated with b defined by

$$(B_n\chi, v)_1 = b(v, \chi) \qquad (\forall \chi \in L^2_0(\Omega),\ \forall v \in K_n). \tag{2.12}$$

Then we have

Lemma 2.3. *$R(B_n)$ is closed, and $K_n = R(B_n) \bigoplus \hat{K}_n$.*

The proof, which we omit, depends on the following well-known result; For example, Corollary I-2.4 of [17].

Lemma 2.4. *Any $\chi \in L^2_0(\Omega)$ admits $w \in H^1_0(\Omega)^N$ satisfying* $\operatorname{div} w = \chi$ *in Ω and $\|w\|_1 \leq C\|\chi\|$.*

§3. Leak Problem of Friction Type. Variational Inequality

We introduce

$$j(\eta) = \int_\Gamma g|\eta|\, ds \qquad (\eta \in H^{1/2}(\Gamma)), \tag{3.1}$$

which we call a friction functional. Then, as a weak form of (Pr. LF), we consider the following variational inequality:

(Pr. WLF). Find $u \in K_\tau$ and $p \in L^2(\Omega)$ satisfying

$$a_\lambda(u, v-u) + b(v-u, p) + j(v_n) - j(u_n) \geq (f, v-u) \qquad (\forall v \in K_\tau), \tag{3.2}$$

$$b(u, \chi) = 0 \qquad (\forall \chi \in L^2(\Omega)). \tag{3.3}$$

Fujita [9] proved the following

Proposition 3.1. *Let $\lambda = 0$ and $\overline{\Gamma} \cap \overline{\Gamma}_D = \emptyset$. Suppose that $\Gamma_D \neq \emptyset$,*

$$f \in L^2(\Omega)^N, \quad g \in L^2(\Gamma), \tag{3.4}$$

and $g > 0$ a. e. in Γ. Then there is a solution $\{u, p\}$ of (Pr. WLF). *The velocity u is unique, but the uniqueness of p depends on cases.*

Remark. We can extend Proposition 3.1 to the case where $\lambda \geq 0$ and $\overline{\Gamma} \cap \overline{\Gamma}_D \neq \emptyset$ in the almost same way as in [9]. However, in the argument of [9], the positively of g is essential to treat (Pr. WLF).

A solution $\{u, p\}$ of (Pr. WLF) is actually a strong solution of (Pr. LF) under some reasonable assumptions. In order to prove it, we approximate a solution $\{u, p\}$ of the *inequality* (3.2) by solutions $\{u_\varepsilon, p_\varepsilon\}$ of *equations* which are obtained by replacing j by a regularized functional j_ε. Then the regularity of $\{u_\varepsilon, p_\varepsilon\}$ is studied.

Let $\varepsilon > 0$. We define a regularization of j as

$$j_\varepsilon(\eta) = \int_\Gamma g\rho_\varepsilon(\eta)\, ds \qquad (\eta \in H^{1/2}(\Gamma)),$$

where ρ_ε denotes the Yosida regularization of $\rho(z) = |z|$ $(z \in \mathbb{R})$;

$$\rho_\varepsilon(z) = \left(\frac{1 - (1+\varepsilon\rho)^{-1}}{\varepsilon}\right)(z) = \begin{cases} z - \varepsilon/2 & (z > \varepsilon) \\ z^2/(2\varepsilon) & (-\varepsilon \leq z \leq \varepsilon) \\ -z + \varepsilon/2 & (z < -\varepsilon). \end{cases}$$

Actually, j_ε is an approximation of j in the sense that

$$|j_\varepsilon(\eta) - j(\eta)| \leq \frac{\varepsilon}{2}\|g\|_{L^1(\Gamma)} \qquad (\eta \in H^{1/2}(\Gamma)). \tag{3.5}$$

Moreover, j_ε is Gâteaux differentiable in $H^{1/2}(\Gamma)$, namely

$$\lim_{h\to 0} \frac{1}{h}\Big[j_\varepsilon(\eta + h\xi) - j_\varepsilon(\eta)\Big] = \int_\Gamma g\alpha_\varepsilon(\eta)\xi\, ds \quad (\eta, \xi \in H^{1/2}(\Gamma)), \tag{3.6}$$

where

$$\alpha_\varepsilon(z) = \partial\rho_\varepsilon(z) = \begin{cases} 1 & (z > \varepsilon) \\ z/\varepsilon & (-\varepsilon \le z \le \varepsilon) \\ -1 & (z < -\varepsilon). \end{cases}$$

Remark. Since ρ is proper, convex and lower semi-continuous, $\alpha \equiv \partial\rho$ is a maximal monotone graph on $\mathbb{R}$. The function α_ε is nothing but the Yosida regularization of α; $\alpha_\varepsilon = (1 - (1 + \varepsilon\alpha)^{-1})/\varepsilon$.

We state a regularized problem to (Pr. WLF):

(Pr. WLF$_\varepsilon$). Find $u_\varepsilon \in K_\tau$ and $p_\varepsilon \in L^2(\Omega)$ satisfying

$$\begin{aligned} &(3.7) \quad a_\lambda(u_\varepsilon, v - u_\varepsilon) + b(v - u_\varepsilon, p_\varepsilon) \\ &\qquad\qquad + j_\varepsilon(v_n) - j_\varepsilon(u_{\varepsilon,n}) \ge (f, v - u_\varepsilon) \quad (\forall v \in K_\tau), \\ &(3.8) \quad b(u_\varepsilon, \chi) = 0 \qquad\qquad (\forall \chi \in L^2(\Omega)). \end{aligned}$$

Lemma 3.1. *Let $\lambda \ge 0$ and $\varepsilon > 0$. Suppose that $\Gamma_D \ne \emptyset$ if $\lambda = 0$. Assume that* (3.4) *and $g \ge 0$ a.e. in Γ. Then* (Pr. WLF$_\varepsilon$) *admits a unique solution $\{u_\varepsilon, p_\varepsilon\} \in K_\tau \times L^2(\Omega)$ satisfying*

$$\|u_\varepsilon\|_1 + \|p_\varepsilon\| \le C(\lambda)(\|f\| + \|g\|_\Gamma). \tag{3.9}$$

The proof is divided into the following two lemmas.

Lemma 3.2. *Under the same assumption of Lemma* 3.1, *there is a unique solution $\{u_\varepsilon, p_\varepsilon\} \in K_\tau \times L^2(\Omega)$ of*

$$a_\lambda(u_\varepsilon, \varphi) + b(\varphi, p_\varepsilon) + \int_\Gamma g\alpha_\varepsilon(u_{\varepsilon,n})\varphi_n \, ds = (f, \varphi) \quad (\forall \varphi \in K_\tau), \tag{3.10}$$

$$b(u_\varepsilon, \chi) = 0 \qquad (\forall \chi \in L^2(\Omega)), \tag{3.11}$$

and $\{u_\varepsilon, p_\varepsilon\}$ satisfies (3.9).

Lemma 3.3. *Suppose that the same assumption of Lemma* 3.1 *holds. Then $\{u_\varepsilon, p_\varepsilon\} \in K_\tau \times L^2(\Omega)$ is a solution of* (Pr. WLF$_\varepsilon$) *if and only if $\{u_\varepsilon, p_\varepsilon\}$ is a one of* (3.10) *and* (3.11).

Proof of Lemma 3.2. We shall drop the subscript ε for the sake of simplicity and put $V = \hat{K}_\tau$. We start with the minimization problem:

$$\text{Find } u \in V: \quad J(u) \le J(v) \quad (\forall v \in V), \tag{3.12}$$

where

$$J(v) = \frac{1}{2}a_\lambda(v,v) - (f,v) + j(v_n).$$

Since j is a proper, convex, and lower semi-continuous functional, from standard theory of convex analysis (for example, Lemma I-4.1 of [14]), (3.12) admits a unique solution u which is characterized by

$$a_\lambda(u, v-u) + j(v_n) - j(u_n) \geq (f, v-u), \quad (\forall v \in V). \tag{3.13}$$

Substituting into (3.13) $v = u \pm t\psi$ with arbitrary $\psi \in V$, $t > 0$ and letting $t \downarrow 0$, we obtain by (3.6)

$$a_\lambda(u,\psi) + \int_\Gamma g\alpha(u_n)\psi_n \, ds = (f,\psi), \qquad (\forall \psi \in V). \tag{3.14}$$

In order to prove the existence of p associated with u, we introduce a linear functional on K_τ by setting

$$F(\varphi) = a_\lambda(u,\varphi) + \int_\Gamma g\alpha(u_n)\varphi_n \, ds - (f,\varphi), \qquad (\varphi \in K_\tau).$$

It is bounded. In particular,

$$|F(\varphi)| \leq C(\lambda)(\|u\|_1 + \|g\|_{0,\Gamma} + \|f\|)\|\varphi\|_1, \qquad (\forall \varphi \in K_\tau)$$

because of $|\alpha(u_n)| \leq 1$.

Let B_τ be the bounded linear operator associated with b defined as (2.10). We apply Riesz's representation theorem to F on a closed subspace $R(B_\tau)$ of K_τ. Thus we have $w \in R(B_\tau)$ such that $F(v) = (w,v)_1$ for any $v \in R(B_\tau)$. Furthermore, by the definition of B_τ, there is a unique $p \in L^2(\Omega)$ satisfying $F(v) = (p, \operatorname{div} v)_1$ for any $v \in R(B_\tau)$.

Now let $\varphi \in K_\tau$ and, according to Lemma 2.1, we decompose φ as $\varphi = v + \psi$, where $v \in R(B_\tau)$ and $\psi \in V$. Then, by virtue of (3.14),

$$F(\varphi) = F(v) + F(\psi) = F(v) = (p, \operatorname{div} v) = (p, \operatorname{div} \varphi),$$

which means (3.10).

Next we derive (3.9). Substituting $\phi = u$ into (3.10), we get

$$C(\lambda)\|u\|_1^2 - C\|g\|_{0,\Gamma}\|u\|_1 \leq C\|f\| \cdot \|u\|_1.$$

Hence we have $\|u\|_1 \le C(\lambda)(\|g\|_{0,\Gamma} + \|f\|)$. We again take $w \in K_\tau$ such that div $w = p$ and $\|w\|_1 \le C\|p\|$. Then by choosing $\varphi = w$ in (3.10), we obtain

$$\begin{aligned}\|p\|^2 &\le a_\lambda(u, w) + \int_\Gamma g\alpha(u_n)w_n - (f, w)\\ &\le C(\lambda)(\|u\|_1\|w\|_1 + \|g\|_{0,\Gamma}\|w\|_1 + \|f\| \cdot \|w\|_1)\\ &\le C(\lambda)(\|u\|_1 + \|g\|_{0,\Gamma} + \|f\|)\|p\|.\end{aligned}$$

This completes the proof of Lemma 3.2. □

Proof of Lemma 3.3. We have by (3.6) and the convexity of j_ε

$$\int_\Gamma g\alpha_\varepsilon(v_n)(\varphi_n - v_n)\, ds \le j_\varepsilon(\varphi_n) - j_\varepsilon(v_n) \quad (\forall v, \varphi \in H^1(\Omega)^N). \tag{3.15}$$

From this, we see that a solution $\{u_\varepsilon, p_\varepsilon\}$ of (3.10) with (3.11) also solves (Pr. WLF$_\varepsilon$). To check the converse assertion, in (3.7), we take $v = u_\varepsilon \pm t\psi$ with $\psi \in K_\tau$, $t > 0$. Then let $t \downarrow 0$ to obtain the result. □

§4. Leak Problem of Friction Type. Regularity

In this section, we firstly study the regularity of a solution to (Pr. WLF$_\varepsilon$) and then give the proof of Theorem 1.1.

Lemma 4.1. *Let* $\lambda \ge 0$ *and suppose* (1.6), (1.7), (1.9), *and* (1.10). *Let* $\varepsilon > 0$ *and let* $\{u_\varepsilon, p_\varepsilon\} \in K_\tau \times L^2(\Omega)$ *solve*

$$a_\lambda(u_\varepsilon, \varphi) + b(\varphi, p_\varepsilon) + \int_\Gamma g\alpha_\varepsilon(u_{\varepsilon,n})\varphi_n\, ds = (f, \varphi) \qquad (\forall \varphi \in K_\tau), \tag{4.1}$$

$$b(u_\varepsilon, \chi) = 0 \qquad (\forall \chi \in L^2(\Omega)). \tag{4.2}$$

Then $u_\varepsilon \in H^2(\Omega)^N$, $p_\varepsilon \in H^1(\Omega)$ *and they satisfy*

$$\lambda u_\varepsilon - \Delta u_\varepsilon + \nabla p_\varepsilon = f, \quad \text{div } u_\varepsilon = 0 \qquad \textit{in } \Omega, \tag{4.3}$$

$$u_\varepsilon = 0, \qquad \textit{on } \Gamma_D, \tag{4.4}$$

$$u_{\varepsilon,\tau} = 0, \quad -\sigma_n(u_\varepsilon, p_\varepsilon) = g\alpha_\varepsilon(u_{\varepsilon,n}) \qquad \textit{on } \Gamma. \tag{4.5}$$

Moreover there is a constant $C > 0$ *independent of* ε *and* λ *such that*

$$\|u_\varepsilon\|_2 + \|p_\varepsilon\|_1 \le C\left(\|f\| + \|g\|_{1/2,\Gamma} + \|u_\varepsilon\|_1 + \|p_\varepsilon\|\right). \tag{4.6}$$

Remark. Inequality (4.6), together with (3.9), implies

$$\|u_\varepsilon\|_2 + \|p_\varepsilon\|_1 \le C(\lambda)\left(\|f\| + \|g\|_{1/2,\Gamma}\right). \tag{4.7}$$

That is, $\|u_\varepsilon\|_2$ and $\|p_\varepsilon\|_1$ are bounded sequences as $\varepsilon \downarrow 0$.

Proof of Lemma 4.1. The regularity in interior and near Γ_D is well-known; for example [7] and [22]. Specifically, let $k \ge 0$ and $\omega \subset \Omega$ be an open set subject to

(I) $\operatorname{dist}(\overline{\omega}, \partial\Omega) \ge \delta > 0$ (δ: constant);

or (II) $\omega \cap \Gamma_D \ne \emptyset$, $\omega \cap \Gamma = \emptyset$, Γ_D is of class C^{k+2}

then we have

$$u \in H^{k+2}(\overline{\omega})^N,\ p \in H^{k+1}(\overline{\omega}), \quad \|u\|_{k+2,\overline{\omega}} + \|p\|_{k+1,\overline{\omega}} \le C(\lambda, \overline{\omega})\|f\|_{k,\overline{\omega}}$$

if $f \in H^k(\Omega)^N$. Therefore it suffices to study the regularity near Γ. Let $x_0 \in \Gamma$ and $U_0 \subset \mathbb{R}^N$ be a neighbourhood of x_0. Taking a cut-off function $\theta \in C^\infty(\mathbb{R}^N)$ subject to

$$0 \le \theta \le 1, \quad \operatorname{supp}\theta \subset U_0, \quad \operatorname{supp}\theta \cap (\mathbb{R}^N \backslash \overline{\Omega}) \ne \emptyset. \tag{4.8}$$

Substituting $\theta^2\varphi$ and $\theta^2\chi$, where $\varphi \in K_\tau$ and $\chi \in L^2(\Omega)$, into (4.1) and (4.2) as test functions, we obtain

$$\begin{aligned}
&(4.9) \quad a_\lambda(\theta^2 u, \varphi) + b(\varphi, \theta^2 p) + \int_\Gamma (\theta^2 g)\alpha(u_n)\varphi_n \, ds \\
&\qquad\qquad\qquad\qquad = (\theta^2 f, \varphi) + F^*(\varphi), \quad (\forall \varphi \in K_\tau), \\
&(4.10) \quad b(\theta^2 u, \chi) = G^*(\chi), \qquad\qquad\qquad (\forall \chi \in L^2(\Omega)),
\end{aligned}$$

where

$$\begin{aligned}
F^*(\varphi) &= \frac{1}{2}\int_\Omega \left[\Big(u_i \frac{\partial\theta^2}{\partial x_j} + \frac{\partial\theta^2}{\partial x_i}u_j\Big)e_{ij}(\varphi)\right. \\
&\qquad\qquad \left. - e_{ij}(u)\Big(\varphi_i \frac{\partial\theta^2}{\partial x_j} + \frac{\partial\theta^2}{\partial x_i}\varphi_j\Big)\right] dx + \int_\Omega p\varphi_i \frac{\partial\theta^2}{\partial x_i}\, dx, \\
G^*(\chi) &= -\int_\Omega \chi u_i \frac{\partial\theta^2}{\partial x_i}\, dx.
\end{aligned}$$

Here and hereafter, we drop ε for simplicity. We take $R > 0$, $U \subset U_0$ and a one-to-one mapping $\Phi = (\Phi_1, \ldots, \Phi_N)$ from U onto $\tilde{U} \subset \mathbb{R}^N_y$ enjoying the following properties (see, for example, §I-2 of Wolka [29]):

1. Φ is a C^3-diffeomorphism;
2. $\Phi(x_0) = 0$;
3. $\Phi(U \cap \Omega) = Q_R \equiv \{y = (y', y_N) \in \mathbb{R}^{N-1} \times \mathbb{R} |\ |y'| < R,\ 0 < y_N < R\}$;
4. $\Phi(U \cap \Gamma) = S_R \equiv \{y = (y', y_N) \in \mathbb{R}^{N-1} \times \mathbb{R} |\ |y'| < R,\ y_N = 0\}$;
5. $\dfrac{\partial \Phi_N}{\partial x_j} = \dfrac{\partial \Phi_j}{\partial x_N} = 0$ and $\dfrac{\partial \Phi_N}{\partial x_N} = -1$ on $U \cap \Gamma$ $(j = 1, \ldots, N-1)$;
6. $\Phi:\ n(x) \mapsto \tilde{n}(y) \equiv (0, \ldots, 0, -1)$ for $x \in U \cap \Gamma$.

We introduce

$$K(Q_R) = \{\varphi \in H^1(Q_R)^N |\ \varphi(y) = 0 \text{ for } |y| = R,\ y_N = R\}; \tag{4.11}$$
$$K_\tau(Q_R) = \{\varphi \in K(Q_R) |\ \varphi_j = 0 \text{ on } S_R\ (j = 1, \ldots, N-1)\} \tag{4.12}$$

and set $y = \Phi(x) = (\Phi_1(x), \ldots, \Phi_N(x))$,

$$\tilde{u}(y) = (\theta^2 u)(x), \qquad \tilde{p}(y) = (\theta^2 p)(x)$$

and

$$\tilde{f}(y) = (\theta^2 f)(x), \quad \tilde{g}(y) = (\theta^2 g)(x).$$

It should be kept in mind that for any $\tilde{\varphi} \in K_\tau(Q_R)$ the function on Ω defined as

$$\varphi(x) = \begin{cases} \tilde{\varphi}(y) & \text{for } y \in Q_R \\ 0 & \text{otherwise} \end{cases}$$

is in K_τ. Hence we obtain by (4.9) and (4.10)

$$\begin{aligned} \tilde{a}_\lambda(\tilde{u}, \tilde{\varphi}) + \tilde{b}(\tilde{\varphi}, \tilde{p}) - \int_{S_R} \tilde{g}\alpha(-\tilde{u}_N)\tilde{\varphi}_N\ dy' \\ = \int_{Q_R} \tilde{f} \cdot \tilde{\varphi} |\mathrm{Jac}\ \Phi|\ dy + \tilde{F}(\tilde{\varphi}), \quad (\forall \tilde{\varphi} \in K_\tau(Q_R)), \end{aligned} \tag{4.13}$$

$$\tilde{b}(\tilde{u}, \tilde{\chi}) = \tilde{G}(\tilde{\chi}), \qquad (\forall \tilde{\chi} \in L^2(Q_R)). \tag{4.14}$$

Here we have put:

$$\tilde{a}_\lambda(\tilde{u},\tilde{\varphi}) = \lambda\int_{Q_R}\tilde{u}\cdot\tilde{\varphi}\;dy + \frac{1}{2}\int_{Q_R}\tilde{e}_{ij}(\tilde{u})\tilde{e}_{ij}(\tilde{\varphi})dy;$$

$$\tilde{e}_{ij}(\tilde{u})\tilde{e}_{ij}(\tilde{\varphi}) = \left(\frac{\partial\Phi_k}{\partial x_j}\frac{\partial\tilde{u}_i}{\partial y_k} + \frac{\partial\Phi_k}{\partial x_i}\frac{\partial\tilde{u}_j}{\partial y_k}\right)\left(\frac{\partial\Phi_l}{\partial x_j}\frac{\partial\tilde{\varphi}_i}{\partial y_l} + \frac{\partial\Phi_l}{\partial x_i}\frac{\partial\tilde{\varphi}_j}{\partial y_l}\right);$$

$$\tilde{b}(\tilde{\varphi},\tilde{\chi}) = -\int_{Q_R}\tilde{\chi}\frac{\partial\Phi_k}{\partial x_i}\frac{\partial\tilde{\varphi}_i}{\partial y_k}|\mathrm{Jac}\;\Phi|\;dy;$$

$$\tilde{F}(\tilde{\varphi}) = \int_{Q_R}\left[\sum_{i,j,k}\left(d^{(1)}_{ijk}\tilde{u}_i\frac{\partial\tilde{\varphi}_i}{\partial y_k} + d^{(2)}_{ijk}\frac{\partial\tilde{u}_i}{\partial y_j}\tilde{\varphi}_k\right)\right.$$

$$\left. + \sum_{i,j}d^{(3)}_{ij}\tilde{u}_i\tilde{\varphi}_j + \sum_i d^{(4)}_i\tilde{p}\tilde{\varphi}_i\right]|\mathrm{Jac}\;\Phi|\;dy;$$

$$\tilde{G}(\tilde{\chi}) = \int_{Q_R}d^{(5)}_i\tilde{\chi}\tilde{u}_i|\mathrm{Jac}\;\Phi|\;dy,$$

where $d^{(1)}_{ijk}$, $d^{(2)}_{ijk}$, $d^{(3)}_{ij}$, $d^{(4)}_i$, and $d^{(5)}_i$ are $C^\infty(\overline{Q_R})$ functions composed of θ and $\partial\theta/\partial x_l$.

Now take $R_1\in(0,R)$ and put $R_2 = R-\varepsilon$, where $\varepsilon = (R-R_1)/2$. We simply write $Q_i = Q_{R_i}$, $S_i = S_{R_i}$ $(i=1,2)$, and $Q = Q_R$, $S = S_R$. We will not distinguish $\varphi\in K(Q_i)$ with its zero extension $\hat{\varphi}\in K(Q)$ ($\hat{\varphi} = \varphi$ in Q_i, $\hat{\varphi} = 0$ in $Q\backslash Q_i$) for the sake of simplicity.

By re-choosing the cut-off function θ if necessary, we may assume that

$$\tilde{u}\in K_\tau(Q_1), \qquad \tilde{g}\in H^{1/2}_{00}(S_1),$$

where

$$H^{1/2}_{00}(S_1) = \left\{\xi\in H^{1/2}(\Gamma)\;\middle|\;(R_1-|y'|)^{-1/2}\xi\in L^2(S_1)\right\}$$

which is a Hilbert space equipped with the norm

$$\|\xi\|_{00,S_1} = \|\xi\|_{H^{1/2}_{00}(S_1)} = \left[\|\xi\|^2_{Q_1} + \int_{S_1}\frac{\xi^2}{R_1-|y'|}dy'\right]^{1/2}.$$

We shall review a more general definition of $H^{1/2}_{00}$ and its properties in Appendix. At this stage we only mention that

$$\tilde{g}\in H^{1/2}_{00}(S), \qquad \|\tilde{g}\|_{00,S}\le\|g\|_{1/2,\Gamma}.$$

(The definition of $H^{1/2}_{00}(S)$ is similar.)

Now we recall the notation of the finite difference quotients. Let $h \in \mathbb{R}$ and $\{e_i\}_{i=1,\dots,N}$ be the canonical basis of $\mathbb{R}^N_y$. Shift and forward difference operators are defined by

$$s^i_h v(y) = v(y + he_i) \quad \text{and} \quad D^i_h v(y) = \frac{s^i_h v(y) - v(y)}{|h|}. \tag{4.15}$$

It is well-known (for example [6]) that

$$\begin{aligned} D^i_h(uv) &= u(D^i_h v) + (D^i_h u)(s^i_h v) \qquad (u, v \in K(Q_j)), \\ \int_Q u \cdot D^i_{-h} v \; dy &= \int_Q (D^i_h) u \cdot v \; dy \qquad (u, v \in K(Q_j)), \\ \|D^i_h u\|_Q &\le \|\nabla u\|_Q \qquad (u \in K(Q_j)) \end{aligned}$$

for $j = 1, 2$, $i = 1, \dots, N-1$ and a suitably small h.

In the following, for $k \ge 0$, C_k denotes a positive constant depending on $\partial^k \Phi / \partial x^k_j$. Let $0 < h < \varepsilon$. We fix $i \in \{1, 2, \dots, N-1\}$ and write $D_{\pm h} = D^i_{\pm h}$. Then $D_h \tilde{u} \in K_\tau(Q_2)$. By choosing

$$\tilde{\varphi} = v \equiv D_{-h}\left(D_h \tilde{u}\right) \in K_\tau(Q_1),$$

in (4.13), we obtain

$$\begin{aligned} &\|\nabla_y \left(D_h \tilde{u}\right)\|^2_Q \\ &\quad \le C_3(\|\tilde{u}\|_{1,Q} + \|\tilde{p}\|_Q + \|\tilde{g}\|_{00,S} + \|\tilde{f}\|_Q)\left(\|\tilde{u}\|_{1,Q} + \|\nabla_y \left(D_h \tilde{u}\right)\|_Q\right) \end{aligned} \tag{4.16}$$

by making use of

$$\tilde{a}_\lambda(\tilde{u}, v) \ge C_1 \|\nabla_y \left(D_h \tilde{u}\right)\|^2_Q - C_2 \|\tilde{u}\|_{1,Q} \|\nabla_y \left(D_h \tilde{u}\right)\|_Q ; \tag{4.17}$$

$$|\tilde{b}(v, \tilde{p})| \le C_3 \|\tilde{p}\|_{Q_R} \left(\|\tilde{u}\|_{1,Q} + \|\nabla_y (D_h \tilde{u})\|_Q\right); \tag{4.18}$$

$$\int_S \tilde{g}\alpha(-\tilde{u}_N) v_N(y', 0) \; dy' \le C_2 \|\tilde{g}\|_{00,S} \|\nabla_y \left(D_h \tilde{u}\right)\|_Q , \tag{4.19}$$

and

$$\begin{aligned} &\left|\int_Q (\tilde{f} \cdot v)|\mathrm{Jac}\ \Phi| \; dy + \tilde{F}(v)\right| \\ &\qquad \le C_2(\|\tilde{f}\|_Q + \|\tilde{u}\|_{1,Q} + \|\tilde{p}\|_Q)\left(\|\tilde{u}\|_{1,Q} + \|\nabla_y (D_h \tilde{u})\|_Q\right), \end{aligned} \tag{4.20}$$

where $\nabla_y \left(D_h \tilde{u}\right)$ denotes the tensor product. We shall verify inequalities (4.17)–(4.20) later and continue the proof of Lemma 4.1. Equality (4.16) implies

$$\|\nabla_y (D_h \tilde{u})\|_Q \le C_3(\|\tilde{u}\|_{1,Q} + \|\tilde{p}\|_Q + \|\tilde{g}\|_{00,S} + \|\tilde{f}\|_Q),$$

and then letting $h \downarrow 0$,

$$\sum_{i=1}^{N-1} \sum_{j=1}^{N} \left\| \frac{\partial}{\partial y_i} \frac{\partial \tilde{u}}{\partial y_j} \right\|_Q \leq C_3(\|\tilde{u}\|_{1,Q} + \|\tilde{p}\|_Q + \|\tilde{g}\|_{00,S} + \|\tilde{f}\|_Q).$$

This and the trace theorem imply that

$$\|\xi_j\|_{1/2,S} \leq C_3(\|\tilde{u}\|_{1,Q} + \|\tilde{p}\|_Q + \|\tilde{g}\|_{00,S} + \|\tilde{f}\|_Q)$$

for $j = 1, \ldots, N-1$, where ξ_j's are defined as $\xi_j = (\partial \tilde{u}_N / \partial y_j)|_S$. This means in particular that all tangential derivatives of $\tilde{u}_N|_S$ belongs to $H^{1/2}(S)$. Therefore, $\tilde{u}_N|_S \in H^{3/2}(S)$ and

$$\|\tilde{u}_N|_S\|_{3/2,S} \leq C_3(\|\tilde{u}\|_{1,Q} + \|\tilde{p}\|_Q + \|\tilde{g}\|_{00,S} + \|\tilde{f}\|_Q).$$

Summing up the above estimates, by the argument of the partition of unity, we finally have

$$\|\beta\|_{3/2,\Gamma} \leq C_3(\|u\|_{1,\Omega} + \|p\|_\Omega + \|g\|_{1/2,\Gamma} + \|f\|_\Omega),$$

where $\beta = u_n|_\Gamma$. Therefore, in accordance with a well-known regularity result on the Dirichlet boundary value problem for the Stokes equations by Cattabriga [7], we deduce (4.6).

To verify that a solution $\{u_\varepsilon, p_\varepsilon\} \in H^2(\Omega)^N \times H^1(\Omega)$ of (4.1) and (4.2) also satisfies (4.3), (4.4) and (4.5) is standard.

It remains to prove inequalities (4.17), (4.18), (4.19) and (4.20).

Proof of (4.17). We have

$$\begin{aligned} \tilde{a}_\lambda(\tilde{u}, v) &= \tilde{a}_\lambda(D_h\tilde{u}, D_h\tilde{u}) \\ &\quad + \int_Q (s_h \tilde{e}_{ij}(\tilde{u})) \left[\left(D_h \frac{\partial \Phi_l}{\partial x_i} \right) \frac{\partial (D_h \tilde{u}_j)}{\partial y_l} + \left(D_h \frac{\partial \Phi_l}{\partial x_j} \right) \frac{\partial (D_h \tilde{u}_i)}{\partial y_l} \right] \, dy. \end{aligned}$$

The absolute value of the second term of the right-hand side is estimated by

$$C_2 \|\tilde{u}\|_{1,Q} \, \|\nabla_y (D_h \tilde{u})\|_Q \, .$$

On the other hand, we have

$$\tilde{a}_\lambda(D_h\tilde{u}, D_h\tilde{u}) \geq C_1 \, \|\nabla_y (D_h \tilde{u})\|_Q^2 \, ,$$

since

$$\tilde{a}_\lambda(w, w) \geq C_1 \int_{Q_R} |\nabla w|^2 \, dy \qquad (\forall w \in K(Q)), \tag{4.21}$$

which is proved in the following way. Recall the change of variables $\Phi\colon x \to y$ and write

$$\hat{w}(x) = \begin{cases} w(y) & y \in Q_R \\ 0 & \text{otherwise.} \end{cases}$$

Then $\hat{w} \in K$ and $\hat{w} = 0$ on $\Gamma \backslash U$. (2.5) gives us

$$\frac{1}{2} \int_\Omega e_{ij}(\hat{w}) e_{ij}(\hat{w})\ dx \geq C \int_\Omega |\nabla \hat{w}|^2\ dx.$$

Noting that there is $0 < \underline{\mu} < \overline{\mu}$ such that $\underline{\mu} \leq \text{Jac}\ \Phi \leq \overline{\mu}$, where Jac Φ denotes the Jacobian of $(\partial \Phi_i / \partial x_j)$, we can estimate as

$$\begin{aligned} &\frac{1}{2} \int_\Omega e_{ij}(\hat{w}) e_{ij}(\hat{w})\ dx \\ &\qquad \leq \frac{\overline{\mu}}{2} \int_{Q_R} \sum_{k,l} \left(a^0_{k,l} \frac{\partial w}{\partial y_k} \frac{\partial w}{\partial y_l} + \sum_{\nu,\nu'} \frac{\partial \Phi_l}{\partial x_\nu} \frac{\partial \Phi_k}{\partial x_{\nu'}} \frac{\partial w_\nu}{\partial y_k} \frac{\partial w_{\nu'}}{\partial y_l} \right) dy \end{aligned}$$

and

$$\int_\Omega |\nabla \hat{w}|^2\ dx \geq \underline{\mu} \int_{Q_R} |\nabla w|^2\ dy.$$

Hence we obtain (4.21).

Proof of (4.18). Putting $\eta = |\text{Jac}\ \Phi| \partial \Phi_k / \partial x_i$, we have

$$\begin{aligned} \tilde{b}(v, \tilde{p}) &= \int_{Q_R} (D_h \tilde{p}) \left(\eta \frac{\partial}{\partial y_k} D_h \tilde{u} \right)\ dy + \int_{Q_R} (s_h \tilde{p})(D_h \eta) \frac{\partial}{\partial y_k} D_h \tilde{u}\ dy \\ &\equiv I_1 + I_2. \end{aligned} \tag{4.22}$$

On the other hand, substituting $\tilde{\chi} = D_{-h}(D_h \tilde{p})$ into (4.14), we obtain

$$I_3 + I_1 = I_4, \tag{4.23}$$

where

$$\begin{aligned} I_3 &= - \int_{Q_R} \tilde{p} D_{-h} \left[(D_h \eta) \left(s_h \frac{\partial \tilde{u}_i}{\partial y_k} \right) \right]\ dy, \\ I_4 &= \int_{Q_R} \tilde{p} D_{-h} \left[D_h \left(d_i^{(5)} \tilde{u}_i |\text{Jac}\ \Phi| \right) \right]\ dy. \end{aligned}$$

It is easy to see that

$$|I_2| \leq C_2 \|\tilde{p}\|_Q \left(\|\tilde{u}\|_{1,Q} + \|\nabla_y (D_h \tilde{u})\|_Q \right), \tag{4.24}$$

$$|I_3|,\ |I_4| \leq C_3 \|\tilde{p}\|_Q \left(\|\tilde{u}\|_{1,Q} + \|\nabla_y (D_h \tilde{u})\|_Q \right). \tag{4.25}$$

(4.18) then follows from (4.22), (4.23), (4.24) and (4.25).

Proof of (4.20). We have

$$\begin{aligned}\int_S \tilde{g}\alpha(-\tilde{u}_N)D_{-h}(D_h\tilde{u}_N)\ dy' \\ &= \int_S (D_h\tilde{g})[s_h\alpha(-\tilde{u}_N)]D_h\tilde{u}_N\ dy' + \int_S \tilde{g}\,[D_h\alpha(-\tilde{u}_N)]\,D_h\tilde{u}_N\ dy' \\ &\equiv J_1 + J_2.\end{aligned}$$

Since α is nondecreasing, if $\tilde{u}_N(y+he) \neq \tilde{u}_N(y)$,

$$[D_h\alpha(-\tilde{u}_N)]\,D_h\tilde{u}_N = -\frac{\alpha(-\tilde{u}_N(y+he)) - \alpha(-\tilde{u}_N(y))}{-\tilde{u}_N(y+he) - (-\tilde{u}_N(y))}\,(D_h\tilde{u}_N)^2 \leq 0 \tag{4.26}$$

at $y = (y', 0) \in S$. Hence $J_2 \leq 0$. In order to estimate J_1, we need

Lemma 4.2. *Let* $0 < h < \varepsilon$ *and* $i = 1, \ldots, N-1$. *Suppose* $g \in H_{00}^{1/2}(S_{R_1})$. *Then*

$$\left|\int_{S_R} (D_h^i g)\varphi\ dy'\right| \leq \int_{S_R} \left|(D_h^i g)\varphi\right|\ dy' \leq C(R)\|g\|_{00,S_1}\|\nabla\varphi\|_{Q_R} \tag{4.27}$$

for all $\varphi \in K(Q_{R_2})$.

The proof of this lemma will be given in Appendix. In view of this, by $|\alpha(-\tilde{u}_N)| \leq 1$, we have

$$J_1 \leq C_2\|\tilde{g}\|_{00,S_R}\,\|\nabla_y D_h\tilde{u}\|_Q\,,$$

therefore (4.20) is proved.

Proof of (4.19). It is not new; For example, see [15] or [2].

This completes the proof of Lemma 4.1. ☐

Now we can state

Proof of Theorem 1.1. Let $\varepsilon > 0$, and let $\{u_\varepsilon, p_\varepsilon\} \in K_\tau \times L^2(\Omega)$ be a unique solution of (Pr. WLF$_\varepsilon$). By (4.7), sequences $\|u_\varepsilon\|_2$ and $\|p_\varepsilon\|_1$ are bounded as $\varepsilon \downarrow 0$. Hence, they admit a sequence ε_j ($0 < \varepsilon_j \downarrow 0$ as $j \uparrow \infty$), $u \in H^2(\Omega)^N \cap K_\tau$, and $p \in H^1(\Omega)$ such that

$$\begin{aligned} u_{\varepsilon_j} &\to u && \text{weakly in } H^2(\Omega)^N, \\ p_{\varepsilon_j} &\to p && \text{weakly in } H^1(\Omega). \end{aligned}$$

Moreover, from the trace theorem,

$$u_{\varepsilon_j}|_{\Gamma_D} \to u|_{\Gamma_D} \qquad \text{weakly in } H^{3/2}(\Gamma_D)^N,$$
$$u_{\varepsilon_j,n}|_{\Gamma} \to u_n|_{\Gamma} \qquad \text{weakly in } H^{3/2}(\Gamma), \tag{4.28}$$
$$\sigma_n(u_{\varepsilon_j}, p_{\varepsilon_j})|_{\Gamma} \to \sigma_n(u,p)|_{\Gamma} \qquad \text{weakly in } H^{1/2}(\Gamma). \tag{4.29}$$

By virtue of (4.3) and (4.5), $\{u,p\}$ satisfies (1.1) and (1.2). Similarly we have $u_\tau|_\Gamma = 0$. In order to see that $\{u,p\}$ satisfies the nonlinear boundary condition, we recall $\alpha_\varepsilon(z) \in \alpha\left((1+\varepsilon\alpha)^{-1}z\right)$ for all $z \in \mathbb{R}$, where $\alpha = \partial|\cdot|$. (This is a property of the Yosida regularization.) Hence, by the maximality of α, (4.28) and (4.29), we obtain

$$-\sigma_n(u,p) \in g\alpha(u_n).$$

On the other hand, (4.6), together with (3.9), leads to

$$\|u\|_2 + \|p\|_1 \le C(\|f\| + \|g\|_{1/2,\Gamma} + \|u\|_1 + \|p\|).$$

Now we take $k \in [k_1, k_2]$, where k_1 and k_2 are defined as (1.11). Then $\{u, p+k\}$ also solves (Pr. LF). By the definition,

$$|k_1|,\ |k_2| \le |\sigma_n(u,p)| + |g| \le 2g \text{ on } \Gamma.$$

This implies $|k| \le 2|\Gamma|^{-1/2}\|g\|_{0,\Gamma}$. As a result, we have

$$\begin{aligned}\|u\|_2 + \|p+k\|_1 &\le \|u\|_2 + \|p\|_1 + |k|\sqrt{|\Omega|} \\ &\le C(\|f\| + \|g\|_{1/2,\Gamma} + \|u\|_1 + \|p\|),\end{aligned}$$

which completes the proof. □

§5. Slip Problem of Friction Type

In this section, we consider (Pr. SF), which is composed of (1.1), (1.2) with (1.4), and give the proof of Theorem 1.2. The strategy is the same as the previous sections.

We begin with a weak formulation using a variational inequality.

(Pr. WSF). Find $u \in K_n$ and $p \in L^2(\Omega)$ satisfying

$$\begin{aligned}&a_\lambda(u, v-u) + b(v-u, p) \\ &\qquad + j(v_\tau) - j(u_\tau) \ge (f, v-u) \qquad (\forall v \in K_n), \\ &b(u,\chi) = 0 \qquad (\forall \chi \in L^2(\Omega)),\end{aligned}$$

where j denotes

$$j(\eta) = \int_\Gamma g|\eta|\ ds \qquad (\eta \in H^{1/2}(\Gamma)^N).$$

Fujita [9] proved the following

Proposition 5.1. *Assume* (3.4), *and* $g > 0$ *a. e. in* Γ. *Then there is a solution* $\{u, p\}$ *of* (Pr. WSF). *Moreover*, u *is unique*; *and* p *is also unique except for an additive constant. In particular*, p *such that* $(p, 1) = 0$ *is unique.*

Let $\varepsilon > 0$. We introduce

$$j_\varepsilon(\eta) = \int_\Gamma g\rho_\varepsilon(\eta)\ ds \qquad (\eta \in H^{1/2}(\Gamma)^N),$$

where ρ_ε denotes the Yosida regularization of $\rho = |z| = \sqrt{z_1^2 + \cdots + z_N^2}$ for $z \in \mathbb{R}^N$:

$$\rho_\varepsilon(z) = \begin{cases} |z| - \varepsilon/2 & (|z| > \varepsilon) \\ |z|^2/(2\varepsilon) & (|z| \le \varepsilon). \end{cases}$$

We have

$$|j_\varepsilon(\eta) - j(\eta)| \le \frac{\varepsilon}{2}\|g\|_{L^1(\Omega)} \qquad (\eta \in H^{1/2}(\Gamma)^N), \tag{5.1}$$

$$\lim_{h\to 0} \frac{1}{h}\Big[j_\varepsilon(\eta + h\xi) - j_\varepsilon(\eta)\Big] = \int_\Gamma g\alpha_\varepsilon(\eta)\cdot\xi\ ds \qquad (\eta, \xi \in H^{1/2}(\Gamma)^N), \tag{5.2}$$

where

$$\alpha_\varepsilon(z) = \begin{cases} z/|z| & (|z| > \varepsilon) \\ z/\varepsilon & (|z| \le \varepsilon). \end{cases}$$

A regularized problem to (Pr. WLF) is

(Pr. WSF$_\varepsilon$). Find $u_\varepsilon \in K_n$ and $p_\varepsilon \in L^2(\Omega)$ satisfying

$$\begin{aligned} &a(u_\varepsilon, v - u_\varepsilon) + b(v - u_\varepsilon, p_\varepsilon) \\ &\qquad\qquad + j_\varepsilon(v_{\varepsilon,t}) - j_\varepsilon(u_{\varepsilon,t}) \ge (f, v - u_\varepsilon) \qquad (\forall v \in K_n), \\ &b(u_\varepsilon, \chi) = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad (\forall \chi \in L^2(\Omega)). \end{aligned}$$

As Theorem 1.1 is so, Theorem 1.2 is a consequence of the following lemmas; Since the proof of Theorem 1.2 is almost same as that of Theorem 1.1, we skip it.

Lemma 5.1. *Let $\lambda \geq 0$ and $\varepsilon > 0$. Suppose that $\Gamma_D \neq \emptyset$ if $\lambda = 0$. Assume that* (3.4) *and $g \geq 0$ a. e. in Γ. Then* (Pr. WSF$_\varepsilon$) *admits a unique solution $u_\varepsilon \in K_n$ and $p_\varepsilon \in L^2_0(\Omega)$ characterized by*

$$a(u_\varepsilon, \varphi) + b(\varphi, p_\varepsilon) + \int_\Gamma g\alpha_\varepsilon(u_{\varepsilon,\tau}) \cdot \varphi_\tau \, ds = (f, \varphi) \quad (\forall \varphi \in K_n).$$

Moreover we have

$$\|u_\varepsilon\|_1 + \|p_\varepsilon\| \leq C(\lambda)(\|f\| + \|g\|_\Gamma).$$

Lemma 5.2. *Let $\lambda \geq 0$. Suppose* (1.6), (1.7), (1.9), *and* (1.10). *Let $\varepsilon > 0$ and let $\{u_\varepsilon, p_\varepsilon\} \in K_n \times L^2_0(\Omega)$ solve*

$$\begin{aligned} &a_\lambda(u_\varepsilon, \varphi) + b(\varphi, p_\varepsilon) + \int_\Gamma g\alpha_\varepsilon(u_{\varepsilon,\tau}) \cdot \varphi_\tau \, ds = (f, \varphi) && (\forall \varphi \in K_n), \\ &b(u_\varepsilon, \chi) = 0 && (\forall \chi \in L^2(\Omega)). \end{aligned}$$

Then $u_\varepsilon \in H^2(\Omega)^N$, $p_\varepsilon \in H^1(\Omega)$ and they satisfy

$$\begin{aligned} &\lambda u_\varepsilon - \Delta u_\varepsilon + \nabla p_\varepsilon = f, \quad \operatorname{div} u_\varepsilon = 0 && \text{in } \Omega, \\ &u_\varepsilon = 0, && \text{on } \Gamma_D, \\ &u_{\varepsilon,n} = 0, \quad -\sigma_\tau(u_\varepsilon, p_\varepsilon) = g\alpha_\varepsilon(u_{\varepsilon,\tau}) && \text{on } \Gamma. \end{aligned}$$

Moreover there is a constant $C > 0$ independent of ε and λ such that

$$\|u_\varepsilon\|_2 + \|p_\varepsilon\|_1 \leq C(\|f\| + \|g\|_{1/2,\Gamma} + \|u_\varepsilon\|_1 + \|p_\varepsilon\|).$$

We can show Lemma 5.1 in the similar line as that of Lemma 3.1 by making use of Lemma 2.3. We only state

Sketch of the proof of Lemma 5.2. We follow the notation of the proof of Lemma 4.1. We investigate the regularity near Γ and only state the case of $N = 3$. Suppose that we have reached

$$\begin{aligned} &\tilde{a}_\lambda(\tilde{u}, \tilde{\varphi}) + \tilde{b}(\tilde{\varphi}, \tilde{p}) + \int_S \tilde{g}\alpha(\tilde{u}_\tau) \cdot \tilde{\varphi}_\tau \, dy' \\ &\qquad\qquad = \int_Q \tilde{f} \cdot \tilde{\varphi} |\text{Jac } \Phi| \, dy + \tilde{F}(\tilde{\varphi}), \qquad (\forall \tilde{\varphi} \in K_\tau(Q)), \\ &\tilde{b}(\tilde{u}, \tilde{\chi}) = \tilde{G}(\tilde{\chi}), \qquad\qquad\qquad\qquad\qquad\qquad (\forall \tilde{\chi} \in L^2(Q)), \end{aligned}$$

where $\tilde{u}_\tau = (\tilde{u}_1, \tilde{u}_2, 0)$, $\tilde{\varphi}_\tau = (\tilde{\varphi}_1, \tilde{\varphi}_2, 0)$, and

$$K_\tau(Q) = \{\varphi \in K(Q) |\ \varphi_3 = 0 \text{ on } S\}.$$

Put $\hat{u} = (\tilde{u}_1, 0, 0)$. Taking

$$\tilde{\varphi} = v \equiv D_{-h}\,(D_h \hat{u}) \in K_n(Q_1),$$

we obtain by (4.17)–(4.20)

$$\|\nabla_y\,(D_h \hat{u})\|_Q \le C_3(\|\tilde{u}\|_{1,Q} + \|\tilde{p}\|_Q + \|\tilde{g}\|_{00,S} + \|\tilde{f}\|_Q),$$

in other words

$$\|\nabla_y\,(D_h \tilde{u}_1)\|_Q \le C_3(\|\tilde{u}\|_{1,Q} + \|\tilde{p}\|_Q + \|\tilde{g}\|_{00,S} + \|\tilde{f}\|_Q).$$

(Here $\nabla_y\,(D_h \hat{u})$ denotes the tensor product, whereas $\nabla_y\,(D_h \tilde{u}_1)$ denotes the usual gradient.) There we note that α which appeared in (4.19) should be replaced by α_1, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. In the same way, we have

$$\|\nabla_y\,(D_h \tilde{u}_2)\|_Q \le C_3(\|\tilde{u}\|_{1,Q} + \|\tilde{p}\|_Q + \|\tilde{g}\|_{00,S} + \|\tilde{f}\|_Q)$$

by choosing $\hat{u} = (0, \tilde{u}_2, 0)$. These lead to

$$\|\nabla_y\,(D_h \tilde{u})\|_Q \le C_3(\|\tilde{u}\|_{1,Q} + \|\tilde{p}\|_Q + \|\tilde{g}\|_{00,S} + \|\tilde{f}\|_Q)$$

and the rest is the same as the proof of Lemma 4.1. □

§6. Remarks on Regularity Results for Stokes Problems

This section is devoted to the boundary value problems of Neumann type composed of

$$\begin{cases} \lambda u - \Delta u + \nabla p = f, \quad \operatorname{div} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D \end{cases} \tag{6.1}$$

with one of the following conditions:

$$u_\tau = 0, \quad \sigma_n = \omega_n \qquad \text{on } \Gamma; \tag{6.2}$$

$$u_n = 0, \quad \sigma_\tau = \omega_\tau \qquad \text{on } \Gamma; \tag{6.3}$$

$$\sigma = \omega \qquad \text{on } \Gamma, \tag{6.4}$$

where ω_n, ω_τ and ω are given functions defined on Γ. If a smooth vector field u and a smooth scalar filed p satisfy (6.1) and (6.2), then by (2.8)

$$a_\lambda(u,\varphi) + b(\varphi,p) = \int_\Gamma \omega_n\varphi_n \, ds + (f,\varphi) \qquad (\forall\varphi \in K_\tau).$$

From this observation, $\{u,p\} \in \hat{K}_\tau \times L^2(\Omega)$ satisfying this equation may be regarded as a weak solution of (6.1) with (6.2). Furthermore we have

Theorem 6.1. *Let $\lambda \geq 0$ and $0 \leq k \in \mathbb{Z}$. Suppose that*

$$\overline{\Gamma_D} \cap \overline{\Gamma} = \emptyset, \qquad \Gamma_D,\ \Gamma \text{ are of class } C^{k+2},\ C^{k+3}; \tag{6.5}$$

$$f \in H^k(\Omega)^N. \tag{6.6}$$

Let $\omega_n \in H^{k+1/2}(\Gamma)$. Moreover, suppose that $\{u,p\} \in K_\tau \times L^2(\Omega)$ is a solution of

$$a_\lambda(u,\varphi) + b(\varphi,p) = \int_\Gamma \omega_n\varphi_n \, ds + (f,\varphi), \qquad (\forall\varphi \in K_\tau), \tag{6.7}$$

$$b(u,\chi) = 0 \qquad (\forall\chi \in L^2(\Omega)). \tag{6.8}$$

Then $u \in H^{k+2}(\Omega)^N$, $p \in H^{k+1}(\Omega)$, and they satisfy

$$\lambda u - \Delta u + \nabla p = f, \quad \operatorname{div} u = 0 \qquad \text{in } \Omega, \tag{6.9}$$

$$u = 0, \qquad \text{on } \Gamma_D, \tag{6.10}$$

$$u_\tau = 0, \quad \sigma_n(u,p) = \omega_n \qquad \text{on } \Gamma. \tag{6.11}$$

Moreover

$$\|u\|_{k+2} + \|p\|_{k+1} \leq C(\lambda,k)(\|f\|_k + \|\omega_n\|_{k+1/2,\Gamma}).$$

Remark. In the above theorem, we have assumed the existence of $\{u,p\}$ satisfying (6.7) and (6.8). However, if $\Gamma_D \neq \emptyset$, such $\{u,p\}$ always exists under the condition that $f \in L^2(\Omega)$ and $\omega_n \in L^2(\Gamma)$. Actually, a linear functional on K_τ defined as the right-hand side of (6.7) is bounded and its bound is given by $C(\|f\| + \|\omega_n\|_{0,\Gamma})$. The bilinear form a is coercive in $K_\tau \times K_\tau$ as was stated before, and moreover the bilinear form b satisfies the inf-sup condition (2.11). Therefore a well-known result on abstract variational problem with constraints (Theorem 5.6 of [13], Corollary I-4.1 of [17]) is applicable. Hence we immediately obtain a unique existence of $\{u, p\} \in K_\tau \times L^2(\Omega)$ of a solution of (6.7), (6.8) with

$$\|u\|_1 + \|p\| \le C(\|f\| + \|\omega_n\|_{0,\Gamma}).$$

Of course, it can be verified in the same manner as the proof of Lemma 3.2 with the aid of Lemma 2.1. In the case of $\Gamma_D = \emptyset$, an additional assumption $\lambda > 0$ is required.

Theorem 6.2. *Let $\lambda \ge 0$ and $0 \le k \in \mathbb{Z}$. Suppose that* (6.5) *and* (6.6) *hold. Let $\omega_\tau \in H^{k+1/2}(\Gamma)^N$ and let $\{u, p\} \in K_n \times L_0^2(\Omega)$ be a solution of*

$$\begin{aligned} &a_\lambda(u, \varphi) + b(\varphi, p) = \int_\Gamma \omega_\tau \cdot \varphi_\tau \, ds + (f, \varphi), && (\forall \varphi \in K_n), \\ &b(u, \chi) = 0 && (\forall \chi \in L^2(\Omega)). \end{aligned}$$

Then $u \in H^{k+2}(\Omega)^N$, $p \in H^{k+1}(\Omega)$, and they satisfy

$$\begin{aligned} &\lambda u - \Delta u + \nabla p = f, \quad \operatorname{div} u = 0 && \text{in } \Omega, \\ &u = 0, && \text{on } \Gamma_D, \\ &u_n = 0, \quad \sigma_\tau(u, p) = \omega_\tau && \text{on } \Gamma. \end{aligned}$$

Moreover

$$\|u\|_{k+2} + \|p\|_{k+1} \le C(\lambda, k)(\|f\|_k + \|\omega_\tau\|_{k+1/2,\Gamma}).$$

Theorem 6.3. *Let $\lambda \ge 0$ and $0 \le k \in \mathbb{Z}$. Suppose that* (6.5) *and* (6.6) *hold. Let $\omega \in H^{k+1/2}(\Gamma)^N$ and let $\{u, p\} \in K \times L^2(\Omega)$ be a solution of*

$$\begin{aligned} &a_\lambda(u, \varphi) + b(\varphi, p) = \int_\Gamma \omega \cdot \varphi \, ds + (f, \varphi), && (\forall \varphi \in K), \\ &b(u, \chi) = 0 && (\forall \chi \in L^2(\Omega)). \end{aligned}$$

Then $u \in H^{k+2}(\Omega)^N$, $p \in H^{k+1}(\Omega)$, and they satisfy

$$\begin{aligned} &\lambda u - \Delta u + \nabla p = f, \quad \operatorname{div} u = 0 && \text{in } \Omega, \\ &u = 0, && \text{on } \Gamma_D, \\ &\sigma(u, p) = \omega && \text{on } \Gamma. \end{aligned}$$

Moreover

$$\|u\|_{k+2} + \|p\|_{k+1} \le C(\lambda, k)(\|f\|_k + \|\omega\|_{k+1/2,\Gamma}).$$

Since proofs of these three theorems are done in the essentially same way, we prove the first one.

Proof of Theorem 6.1. We first prove the case of $k = 0$. As is stated before, it suffices to investigate the regularity near Γ. Let $x_0 \in \Gamma$ and $U_0 \subset \mathbb{R}^N$ be a neighbourhood of x_0. Again we take $R > 0$, $U \subset U_0$ and a one-to-one mapping $\Phi = (\Phi_1, \dots, \Phi_N)$ from U onto $\tilde{U} \subset \mathbb{R}^N_y$ described in the proof of Lemma 4.1. (We follow the notation of the proof of Lemma 4.1.) However in this case we assume that Φ is only of class C^2.

We set $y = \Phi(x) = (\Phi_1(x), \dots, \Phi_N(x))$,

$$\tilde{u}(y) = (\theta^2 u)(x), \qquad \tilde{p}(y) = (\theta^2 p)(x)$$

and

$$\tilde{f}(y) = (\theta^2 f)(x), \quad \tilde{\omega}(y) = (\theta^2 \omega_n)(x),$$

where $\theta \in C^\infty(\mathbb{R}^N)$ is a cut-off function subject to (3.3). Then we deduce

$$\begin{aligned} &(6.12) \quad \tilde{a}_\lambda(\tilde{u}, \tilde{\varphi}) + \tilde{b}(\tilde{\varphi}, \tilde{p}) = -\int_{S_R} \tilde{\omega}\tilde{\varphi_N}\, dy' \\ &\qquad\qquad + \int_{Q_R} \tilde{f} \cdot \tilde{\varphi} |\mathrm{Jac}\ \Phi|\, dy + \tilde{F}(\tilde{\varphi}), \quad (\forall \tilde{\varphi} \in K_\tau(Q_R)), \\ &(6.13) \quad \tilde{b}(\tilde{u}, \tilde{\chi}) = \tilde{G}(\tilde{\chi}), \qquad (\forall \tilde{\chi} \in L^2(Q_R)). \end{aligned}$$

If we proceed in the same way as in the proof of Lemma 4.1, we should assume that Φ is a C^3 mapping. However we can avoid such issue by employing a trick of Bello [2].

Let $R_1 \in (0, R)$. We set $\varepsilon = R - R_1$, $R_2 = R - \varepsilon/2$, $R_3 = R - \varepsilon/4$ and simply write $Q_i = Q_{R_i}$, $S_i = S_{R_i}$ for $i = 0, 1, 2, 3$. (R_0 is understood as R.) We shall not distinguish $v \in K(Q)$ with their restrictions into Q_i ($i = 1, 2, 3$); $v \in K(Q_i)$.

We may assume that

$$\tilde{u} \in K_\tau(Q_1), \qquad \tilde{\omega} \in H^{1/2}_{00}(S_1).$$

Let $0 < h < \varepsilon/4$. We fix $i \in \{1, 2, \dots, N-1\}$ and write $D_{\pm h} = D^i_{\pm h}$ for simplicity. Suppose $\tilde{\varphi} \in K_\tau(Q_2)$, $\tilde{\chi} \in L^2(Q_2)$ and substitute $D_{-h}\tilde{\varphi}$, $D_{-h}\tilde{\chi}$ into (6.12), (6.13) as test functions. Then

$$(6.14) \qquad \tilde{a}_\lambda(D_h\tilde{u}, \tilde{\varphi}) + \tilde{b}(\tilde{\varphi}, D_h\tilde{p}) = \tilde{F}_0(\tilde{\varphi}), \qquad (\forall \tilde{\varphi} \in K_\tau(Q_2)),$$

$$(6.15) \qquad \tilde{b}(D_h\tilde{u}, \tilde{\chi}) = \tilde{G}_0(\tilde{\chi}), \qquad (\forall \tilde{\chi} \in L^2(Q_2)),$$

where

$$\tilde{F}_0(\tilde{\varphi}) = -\int_{S_R} \tilde{\omega} D_{-h}\tilde{\varphi}_N \, dy' + \int_{Q_R} \tilde{f}\cdot(D_{-h}\tilde{\varphi})|\text{Jac }\Phi| \, dy + \tilde{F}(D_{-h}\tilde{\varphi})$$
$$- \int_{Q_R} (s_h \tilde{e}_{ij}(\tilde{u})) \left[\left(D_h \frac{\partial \Phi_l}{\partial x_j}\right)\frac{\partial \tilde{\varphi}_i}{\partial y_l} + \left(D_h \frac{\partial \Phi_l}{\partial x_i}\right)\frac{\partial \tilde{\varphi}_j}{\partial y_l}\right] dy;$$
$$\tilde{G}_0(\tilde{\chi}) = \tilde{G}(D_{-h}\chi) + \int_{Q_R} (s_h\tilde{\chi})\, D_h\left(\frac{\partial \Phi_k}{\partial y_i}|\text{Jac }\Phi|\right)\frac{\partial \varphi_i}{\partial y_k}\, dy.$$

We claim:

$$|\tilde{a}_\lambda(v,w)| \le C(\lambda)\|v\|_{1,Q_2}\|w\|_{1,Q_2} \qquad (\forall v,w \in K_\tau(Q_2)); \tag{6.16}$$
$$\tilde{a}_\lambda(v,v)| \ge C\|v\|^2_{1,Q_2} \qquad (\forall v \in K_\tau(Q_2)); \tag{6.17}$$
$$C\|\chi\|_{Q_2} \le \sup_{v\in K_\tau(Q_2)} \frac{\tilde{b}(v,\chi)}{\|v\|_{1,Q_2}} \qquad (\forall \chi \in L^2(\Omega)); \tag{6.18}$$
$$|\tilde{F}_0(\varphi)| \le C(\|\tilde{u}\|_{1,Q_2} + \|\tilde{p}\|_{Q_2} + \|\tilde{f}\|_{Q_3} + \|\tilde{\omega}\|_{00,S_1})\|\varphi\|_{1,Q_2} \qquad (\forall \varphi \in K_\tau(Q_2)); \tag{6.19}$$
$$|\tilde{G}_0(\chi)| \le C\|\tilde{u}\|_{1,Q_2}\|\chi\|_{Q_2} \qquad (\forall \chi \in L^2(\Omega)). \tag{6.20}$$

(6.16) is obvious. (6.17) is a consequence of (4.21). To see (6.18), let $v \in K_\tau(Q_2)$, $\chi \in L^2(Q_2)$, and set

$$\hat{v}(x) = \begin{cases} v(y) & y\in Q_2 \\ 0 & \text{otherwise}, \end{cases} \qquad \hat{\chi}(x) = \begin{cases} \chi(y) & y\in Q_2 \\ 0 & \text{otherwise}. \end{cases}$$

Since $\hat{v}\in K_\tau$, $\hat{\chi}\in L^2(\Omega)$, by (2.11), we have

$$\beta\|\chi\|_{Q_2} = \beta\|\hat{\chi}\| \le \frac{|b(\hat{v},\hat{\chi})|}{\|\hat{v}\|_1} \le C\frac{|\tilde{b}(v,\chi)|}{\|v\|_{1,Q_2}}.$$

Finally (6.19) and (6.20) are easily obtained in the standard manner.

Now, (6.16)–(6.20) enable us to apply to (6.14)(6.15) the theorem on abstract variational problem with constrains; Theorem 5.6 of [13] or Corollary I-4.1 of [17]. As a result, $D_h\tilde{u}\in K_\tau(Q_2)$, $D_h\tilde{p}\in L^2(Q_2)$ and

$$\|D_h\tilde{u}\|_{1,Q_2} + \|D_h\tilde{p}\|_{Q_2} \le C(\|\tilde{u}\|_{1,Q_2} + \|\tilde{p}\|_{Q_2} + \|\tilde{f}\|_{Q_3} + \|\tilde{\omega}\|_{00,S_1}). \tag{6.21}$$

Letting $h\downarrow 0$, we obtain

$$\left\|\frac{\partial \tilde{u}}{\partial y_j}\right\|_{1,Q_2} + \left\|\frac{\partial \tilde{p}}{\partial y_j}\right\|_{Q_2} \le C(\|\tilde{u}\|_{1,Q_2} + \|\tilde{p}\|_{Q_2} + \|\tilde{f}\|_{Q_3} + \|\tilde{\omega}\|_{00,S_1})$$

for $j = 1, \dots, N-1$, and hence

$$\sum_{j=1}^{N-1} \|\zeta_j\|_{1,Q_2} \le C(\|\tilde{u}\|_{1,Q_2} + \|\tilde{p}\|_{Q_2} + \|\tilde{f}\|_{Q_3} + \|\tilde{\omega}\|_{00,S_1}),$$

where $\zeta_j = \partial\tilde{u}/\partial y_j$. This means that $\eta = \tilde{u}_N|_{S_2} \in H^{3/2}(S_2)$ and

$$\|\eta\|_{3/2,S_2} \le C(\|\tilde{u}\|_{1,Q_2} + \|\tilde{p}\|_{Q_2} + \|\tilde{f}\|_{Q_3} + \|\tilde{\omega}\|_{00,S_1}).$$

Summing up the above estimates, by the argument of the partition of unity, we have

$$\|\beta\|_{3/2,\Gamma} \le C(\|u\|_{1,\Omega} + \|p\|_{\Omega} + \|\omega_n\|_{1/2,\Gamma} + \|f\|_{\Omega}),$$

where $\beta = u_n|_{\Gamma}$. Therefore, by Cattabriga's result, we get the desired result for $k = 0$.

We proceed to the general case $k \ge 0$. Let $\alpha = (\alpha_1, \dots, \alpha_{N-1})$ be a multi-index with $|\alpha| = k$ and set

$$\partial^\alpha = \left(\frac{\partial}{\partial y_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial y_{N-1}}\right)^{\alpha_{N-1}}.$$

Suppose that we have obtained the desired result for $k \ge 0$. We may assume

$$\partial^\alpha \tilde{u} \in K_\tau(Q_1), \qquad \partial^\alpha \tilde{\omega} \in H^{1/2}_{00}(S_1),$$

then we have

$$\tilde{a}_\lambda\left(D_h(\partial^\alpha \tilde{u}), \tilde{\varphi}\right) + \tilde{b}\left(\tilde{\varphi}, D_h(\partial^\alpha \tilde{p})\right) = \tilde{F}_k(\tilde{\varphi}), \qquad (\forall \tilde{\varphi} \in K_\tau(Q_2)), \tag{6.22}$$

$$\tilde{b}\left(D_h(\partial^\alpha \tilde{u}), \tilde{\chi}\right) = \tilde{G}_k(\hat{\chi}), \qquad (\forall \tilde{\chi} \in L^2(Q_2)), \tag{6.23}$$

where $\tilde{F}_k$ and $\tilde{G}_k$ are functionals such as

$$\begin{aligned} |\tilde{F}_k(\varphi)| &\le C(\|\tilde{u}\|_{k+1,Q_2} + \|\tilde{p}\|_{k,Q_2} \\ &\qquad + \|\tilde{f}\|_{k,Q_3} + \|\partial^\alpha \tilde{\omega}\|_{00,S_1})\|\varphi\|_{1,Q_2} && (\forall \varphi \in K_\tau(Q_2)), \\ |\tilde{G}_k(\chi)| &\le C\|\tilde{u}\|_{k+1,Q_2}\|\chi\|_{Q_2} && (\forall \chi \in L^2(\Omega)). \end{aligned}$$

In fact, we firstly assume that $\tilde{\varphi} \in K_\tau(Q_2)$, $\tilde{\chi} \in L^2(Q_2)$ are sufficiently smooth and then substitute $(-1)^{|\alpha|}\partial^\alpha D_{-h}\hat{\varphi}$, $(-1)^{|\alpha|}\partial^\alpha D_{-h}\hat{\chi}$ into (6.12), (6.13) as new $\tilde{\varphi}$, $\tilde{\chi}$. Then, by virtue of integration by parts and the density argument, we have (6.22) and (6.23). Again repeating the argument for $k = 0$, we obtain $D_h(\partial^\alpha \tilde{u}) \in K_\tau(Q_2)$, $D_h(\partial^\alpha \tilde{p}) \in L^2(Q_2)$ and

$$\|D_h(\partial^\alpha \tilde{u})\|_{1,Q_2} + \|D_h(\partial^\alpha \tilde{p})\|_{Q_2} \leq C(\|\tilde{u}\|_{k+1,Q_2} + \|\tilde{p}\|_{k,Q_2} + \|\tilde{f}\|_{k,Q_3} + \|\partial^\alpha \tilde{\omega}\|_{00,S_1}).$$

Therefore, as before, we deduce the desired result for $k + 1$. The proof is completed by the induction. □

§7. Concluding Remarks

(A) Convergence rate. The following theorem provides how fast $\{u_\varepsilon, p_\varepsilon\}$ converges to $\{u, p\}$, where $\{u_\varepsilon, p_\varepsilon\}$ and $\{u, p\}$ are solutions of (Pr. WLF$_\varepsilon$) and (Pr. WLF), respectively. This result gives useful information in the numerical approximation of these problems.

Theorem 7.1. *Let $\lambda \geq 0$ and $\varepsilon > 0$. Suppose $\Gamma_D \neq \emptyset$ if $\lambda = 0$. Assume that* (3.4) *and $g \geq 0$ a. e. in Γ. Let $\{u, p\}$ and $\{u_\varepsilon, p_\varepsilon\}$ be solutions of* (Pr. WLF) *and* (Pr. WLF$_\varepsilon$), *respectively. Then*

$$\|u_\varepsilon - u\|_1 + \|\tilde{p}_\varepsilon - \tilde{p}\| \leq C\sqrt{\varepsilon}\|g\|_{L^1(\Gamma)}, \tag{7.1}$$

where $\tilde{p} = p - |\Omega|^{-1}(p, 1)$, $\tilde{p}_\varepsilon = p_\varepsilon - |\Omega|^{-1}(p_\varepsilon, 1)$ with $|\Omega|$ being the Lebesgue measure of Ω in $\mathbb{R}^N$.

Proof. Substituting $v = u_\varepsilon$ and $\varphi = u_\varepsilon - u$ into (3.2) and (3.10), respectively, we have

$$a_\lambda(u, u_\varepsilon - u) + j(u_{\varepsilon,n}) - j(u_n) \geq (f, u_\varepsilon - u),$$
$$a_\lambda(u_\varepsilon, u_\varepsilon - u) + \int_\Gamma g\alpha_\varepsilon(u_{\varepsilon,n})(u_{\varepsilon,n} - u_n)\, ds = (f, u_\varepsilon - u).$$

Hence

$$a_\lambda(u_\varepsilon - u, u_\varepsilon - u) \leq j(u_{\varepsilon,n}) - j(u_n) + \int_\Gamma g\alpha_\varepsilon(u_{\varepsilon,n})(u_n - u_{\varepsilon,n})\, ds,$$

and by (3.5), (3.15), (2.3) (or (2.6))

$$C\|u_\varepsilon - u\|_1^2 \leq |j(u_{\varepsilon,n}) - j(u_n)| + |j_\varepsilon(u_n) - j_\varepsilon(u_{\varepsilon,n})| \leq \varepsilon\|g\|_{L^1(\Gamma)}. \tag{7.2}$$

We proceed to the pressure part. Putting $q_\varepsilon = \tilde{p}_\varepsilon - \tilde{p}$, we have

$$a_\lambda(u - u_\varepsilon, \varphi) = b(\varphi, q_\varepsilon) \quad (\forall\varphi \in H_0^1(\Omega)^N). \tag{7.3}$$

In fact, for an arbitrary $\varphi \in H_0^1(\Omega)^N$, substituting $v = u \pm \varphi$ into (3.2), we get

$$a_\lambda(u, \varphi) + b(\varphi, p) = (f, \varphi).$$

Combining this with (3.10), we deduce (7.3). Since $(q_\varepsilon, 1) = 0$, by Lemma 2.4, we can take $\psi_\varepsilon \in H_0^1(\Omega)^N$ subject to $\operatorname{div} \psi_\varepsilon = q_\varepsilon$ in Ω and $\|\psi_\varepsilon\|_1 \leq C\|q_\varepsilon\|$. Now substituting into (7.3) $\varphi = \psi_\varepsilon$, we obtain

$$\|q_\varepsilon\|^2 = a_\lambda(u - u_\varepsilon, \psi_\varepsilon) \leq C\|u - u_\varepsilon\|_1 \|\psi_\varepsilon\|_1 \leq C\|u - u_\varepsilon\|_1 \|q_\varepsilon\|.$$

This, together with (7.2), implies (7.1). $\square$

Concerning (Pr. WSF$_\varepsilon$) and (Pr. WSF), we also have a corresponding result. The proof is omitted.

Theorem 7.2. *Let $\lambda \geq 0$ and $\varepsilon > 0$. Suppose that $\Gamma_D \neq \emptyset$ if $\lambda = 0$. Assume that* (3.4) *and $g \geq 0$ a. e. in Γ. Let $\{u, p\}$ and $\{u_\varepsilon, p_\varepsilon\}$ be solutions of* (Pr. WSF) *and* (Pr. WSF$_\varepsilon$), *respectively. Then*

$$\|u_\varepsilon - u\|_1 + \|\tilde{p}_\varepsilon - \tilde{p}\| \leq C\sqrt{\varepsilon}\|g\|_{L^1(\Gamma)}.$$

(B) Full stress problem of friction type. We consider the full stress problem of friction type, (Pr. F), which is composed of (1.1), (1.2) and

$$-\sigma(u, p) \in g\partial|u| \qquad \text{on } \Gamma, \tag{7.4}$$

where $\partial|\cdot|$ denotes a graph on $\mathbb{R}^N$ defined by (1.5) with $m = N$. It is equivalent to

$$|\sigma| \leq g, \qquad \sigma \cdot u + g|u| = 0 \quad \text{ on } \Gamma.$$

The existence and uniqueness/non-uniqueness of a weak solution to (Pr. F) are discussed by means of a variational inequality in Fujita and Kawarada [12]. In the previous paper [26], we assumed $g \in H^1(\Gamma) \cap L^\infty(\Gamma)$ to derive the H^2-H^1 regularity of the weak solution. However, in the same way as that of Theorem 1.2, we can prove

Theorem 7.3. *Let $\lambda \geq 0$. Assume that $\Gamma_D \neq \emptyset$ if $\lambda = 0$. Suppose that $\overline{\Gamma} \cap \overline{\Gamma_D} = \emptyset$ and that Γ_D, Γ are of class C^2, C^4, respectively. Suppose also that $g \in H^{1/2}(\Gamma)$ and $g \geq 0$ a.e., and finally let $f \in L^2(\Omega)^N$. Then there exists a solution $\{u, p\} \in H^2(\Omega)^N \times H^1(\Omega)$ of* (Pr. F). *u is unique, while p is* *unique except for an additive constant. The range of the additive constant to* p *is limited to* $\{0\}$ *or to a finite closed interval. Furthermore*

$$\|u\|_2 + \|p\|_1 \le C(\|f\| + \|g\|_{1/2,\Gamma} + \|u\|_1 + \|p\|)$$

for any solution $\{u, p\}$ *of* (Pr. F)*.*

Remark. As for the non-uniqueness of p, see Remark below Theorem 1.1 or an example in [26].

(C) Regularity for (Pr. WLF$_\varepsilon$). Concerning the regularity of a solution $\{u_\varepsilon, p_\varepsilon\}$ of (Pr. WLF$_\varepsilon$), we can easily derive $u_\varepsilon \in H^2$, $p_\varepsilon \in H^1$ by making use of

Lemma 7.1. *Let* α *be a mapping of* $\mathbb{R}^m \to \mathbb{R}^m$ *with* $\alpha(0) = 0$*. Moreover assume that there are* $L, M > 0$ *such that*

$$\begin{aligned} &|\alpha(z) - \alpha(z')| \le L|z - z'| && (\forall z,\ z' \in \mathbb{R}^m), \\ &|\alpha(z)| \le M && (\forall z \in \mathbb{R}^m). \end{aligned}$$

Let $\eta \in H^{1/2}(\Gamma)^m$*. Then, if* $g \in H^{1/2}(\Gamma) \cap L^\infty(\Gamma)$*, we have* $g\alpha(\eta) \in H^{1/2}(\Gamma)^m$ *and*

$$\|g\alpha(\eta)\|_{1/2,\Gamma} \le C(M)\|g\|_{1/2,\Gamma} + C(L, M, \|g\|_{L^\infty(\Gamma)})\|\eta\|_{1/2,\Gamma}.$$

Proof. Let $\eta \in H^{1/2}(\Gamma)^m$. The fact that

$$\alpha(\eta) \in H^{1/2}(\Gamma)^m \quad \text{and} \quad \|\alpha(\eta)\|_{1/2,\Gamma} \le C(L, M)\|\eta\|_{1/2,\Gamma} \tag{7.5}$$

is due to H. Brezis. (Lemme I.1 of [4]; The scalar-valued case is explicitly described there, but it is valid for the vector-valued case.)

Next let us denote by $\hat{g} \in H^1(\Omega)$ the weak harmonic extension of g. It follows from the maximum principle that $\|\hat{g}\|_\infty \le \|g\|_{L^\infty(\Gamma)}$. Let $\eta \in H^{1/2}(\Gamma)^m$. We take the weak harmonic extension $\hat{\alpha} \in H^1(\Omega)^m$ of $\alpha(\eta)$. That is, we extend an each component of $\alpha(\eta)$ by a harmonic function on Ω. Again by the maximum principle, we have $\|\hat{\alpha}\|_\infty \le \|\alpha(\eta)\|_{L^\infty(\Gamma)} \le M$. Since $\hat{g}\hat{\alpha} \in H^1(\Omega)^m$, by the trace theorem,

$$\|g\alpha(\eta)\|_{1/2,\Gamma} \le C\|\hat{g}\hat{\alpha}\|_1 \le CM\|\hat{g}\|_1 + \|g\|_{L^\infty(\Gamma)}\|\hat{\alpha}\|_1.$$

This, together with (7.5), implies the conclusion. □

Now, let $\lambda \geq 0$ and $\varepsilon > 0$. Suppose (1.6), (1.7), (1.9), and (1.10). (We follow the notation of §3.) Let $\{u_\varepsilon, p_\varepsilon\} \in K_\tau \times L^2(\Omega)$ solve (Pr. WLF$_\varepsilon$). Furthermore we assume $g \in L^\infty(\Gamma)$. We notice that

$$|\alpha_\varepsilon(z) - \alpha_\varepsilon(z')| \leq \frac{1}{\varepsilon}|z - z'|, \quad |\alpha_\varepsilon(z)| \leq 1 \quad (z, z' \in \mathbb{R}).$$

Then Lemmas 3.1 and 7.1 (in the case of $m = 1$) imply that $\sigma_n(u_\varepsilon, p_\varepsilon) = -g\alpha_\varepsilon(u_{\varepsilon,n}) \in H^{1/2}(\Gamma)$. Hence we have $u_\varepsilon \in H^2(\Omega)^N$, $p_\varepsilon \in L^2(\Omega)$, and

$$\begin{aligned}\|u_\varepsilon\|_2 + \|p_\varepsilon\| &\leq C(\|f\| + \|g\alpha_\varepsilon(u_{\varepsilon,n})\|_{1/2,\Gamma}) \\ &\leq C(\|f\| + C\|g\|_{1/2,\Gamma} + C(\varepsilon, \|g\|_{L^\infty(\Gamma)})\|u_{\varepsilon,n}\|_{1/2,\Gamma}) \\ &\leq C(\|f\| + C\|g\|_{1/2,\Gamma} + C(\varepsilon, \|g\|_{L^\infty(\Gamma)})\|u_\varepsilon\|_1)\end{aligned}$$

by Theorem 6.1. However in order to derive an estimate which is *independent of* ε like as (4.7), we need another device described in §4.

Remark. We meet the same issue when applying a general regularity theory by Agmon, Douglis and Nirenberg [1].

(D) Optimality of Assumption $g \in H^{1/2}(\Gamma)$. Theorem 6.1 claims that

$$g\alpha_\varepsilon(u_{\varepsilon,n}) \in H^{1/2}(\Gamma) \tag{7.6}$$

is essential to derive the $H^2(\Omega)^N \times H^1(\Omega)$-regularity of a solution $\{u_\varepsilon, p_\varepsilon\} \in K_\tau \times L^2(\Omega)$ to (Pr. WLF$_\varepsilon$). However, if $u_{\varepsilon,n} \geq 1/\varepsilon$ on Γ, then (7.6) means $g \in H^{1/2}(\Gamma)$.

Appendix A. The Space $H^{1/2}_{00}$

In this appendix, we establish the proof of Lemma 4.2 after having prepared some basic properties on the space $H^{1/2}_{00}$.

First we recall its definition. Let $\gamma \subset \mathbb{R}^{N-1}_y$ be a bounded domain with the Lipschitz boundary $\partial\gamma$. The distance from $y' \in \gamma$ to $\partial\gamma$ is denoted by $\delta = \delta(y')$. Then

$$H^{1/2}_{00}(\gamma) = \{\eta \in H^{1/2}(\gamma) |\ \delta^{-1/2}\eta \in L^2(\gamma)\}$$

is a Hilbert space equipped with the norm

$$\|\eta\|_{00,\gamma} = \|\eta\|_{H^{1/2}_{00}(\gamma)} = \left[\|\eta\|^2_{H^{1/2}(\gamma)} + \int_\gamma \frac{|\eta(y')|^2}{\delta(y')}dy'\right]^{1/2}.$$

We consider a finite cylinder $Q = \gamma \times (0, R) \subset \mathbb{R}_y^{N-1} \times \mathbb{R}_y$ with $R > 0$, and set

$$K(Q) = \{v \in H^1(Q) |\ v = 0 \text{ on } \partial Q \backslash \overline{\gamma}\}.$$

If $v \in K(Q)$, then $\eta = v|_\gamma \in H_{00}^{1/2}(\gamma)$ and $\|\eta\|_{00,\gamma} \leq C\|v\|_{H^1(Q)}$. Conversely, every $\eta \in H_{00}^{1/2}(\gamma)$ admits an extension $v \in K(Q)$ such that $v|_\gamma = \eta$ and $\|v\|_{H^1(Q)} \leq C\|\eta\|_{00,\gamma}$. Moreover the following equivalence holds true

$$\|\eta\|_{00,\gamma} \sim \inf\{\|v\|_{H^1(Q)} |\ v \in K(Q),\ v|_\gamma = \eta\} \sim \|\nabla w\|_{L^2(Q)},$$

where $w \in K(Q)$ denotes the weak harmonic extension η into Q; $\Delta w = 0$ in Q and $w|_\gamma = \eta$. See, for more detail, [23] or [25].

Lemma A.1. *Let γ, $\gamma' \subset \mathbb{R}_y^{N-1}$ be bounded domains with Lipschitz boundaries, and assume that $\gamma' \subset \gamma$ and $dist(\gamma', \gamma) > 0$. Let $g \in H_{00}^{1/2}(\gamma')$. Then the zero extension $\hat{g}$ of g into γ is in $H_{00}^{1/2}(\gamma)$. Furthermore we have*

$$\text{(A.1)} \qquad \frac{\partial \hat{g}}{\partial y_i} \in H^{-1/2}(\gamma) \equiv H_{00}^{1/2}(\gamma)^*, \qquad \left\|\frac{\partial \hat{g}}{\partial y_i}\right\|_{H^{-1/2}(\gamma)} \leq C(\gamma, \gamma')\|g\|_{00,\gamma'}$$

for $i = 1, 2, \ldots, N-1$.

Proof. The first assertion is obvious. We give the proof of the second one only when $N = 3$. Set $Q' = \gamma' \times (0, R/2)$, and take a smooth function w defined on $\overline{Q'}$ subject to $w = 0$ on $\partial Q' \backslash \overline{\gamma}$. Let $\hat{w}$ be the extension of w by 0 into $\overline{Q}$. Then $\partial \hat{w}/\partial y_j$, $j = 1, 2, 3$, are also smooth on $\overline{Q}$ and vanish on $\partial Q \backslash \overline{\gamma}$. We have by the integration by parts

$$\int_Q \text{curl } v \cdot \nabla \hat{w}\ dy = \int_\gamma \Big(v_2 \frac{\partial \hat{w}}{\partial y_1} - v_1 \frac{\partial \hat{w}}{\partial y_2}\Big)\ dy' \qquad (\forall v \in K(Q)^3).$$

Taking $\varphi \in K(Q)$ and substituting $v = (0, \varphi, 0)$ into the above, we obtain

$$\text{(A.2)} \qquad \int_\gamma \frac{\partial \hat{w}}{\partial y_1} \varphi(y', 0)\ dy' = \int_Q \Big(\frac{\partial \varphi}{\partial y_1}\frac{\partial \hat{w}}{\partial y_3} - \frac{\partial \varphi}{\partial y_3}\frac{\partial \hat{w}}{\partial y_1}\Big)\ dy.$$

By the density argument, the identity (A.2) is valid for any $w \in K(Q') = \{v \in H^1(Q') |\ v = 0 \text{ on } \partial Q' \backslash \overline{\gamma'}\}$. At this stage, we suppose that w is the weak harmonic extension of g into Q. The zero extension of g into γ is denoted by $\hat{g}$. Moreover, for an arbitrary $\eta \in H_{00}^{1/2}(\gamma)$, let $\varphi \in K(Q)$ be the weak harmonic extension into Q. Then (A.2) implies

$$\int_\gamma \frac{\partial \hat{g}}{\partial y_1} \eta\ dy' = \int_Q \Big(\frac{\partial \varphi}{\partial y_1}\frac{\partial \hat{w}}{\partial y_3} - \frac{\partial \varphi}{\partial y_3}\frac{\partial \hat{w}}{\partial y_1}\Big) dy.$$

Now we define $\partial \hat{g}/\partial y_1 \in H^{-1/2}(\gamma)$ through this identity and then get

$$\left| \int_\gamma \frac{\partial \hat{g}}{\partial y_1} \eta \, dy' \right| \le \|\nabla w\|_{L^2(Q)} \|\nabla \varphi\|_{L^2(Q)} \quad (\forall \eta \in H_{00}^{1/2}(\gamma)).$$

The case of $i = 2$ is done in the similar way. □

Finally we state:

Proof of Lemma 4.2. Although the inequality (4.27) is essentially derived by González Burgos [16], we state another (and a somewhat simpler) proof. Taking the harmonic extension w of g into Q_{R_1} and using the same symbol w to indicate the zero extension of w into Q_R. Writing

$$D_h^i w(y) = \frac{1}{h} \int_0^h s_t^i \frac{\partial w}{\partial y_i}(y) \, dt, \quad (s_t^i : \text{ the shift operator defined by (4.15)})$$

we obtain, for any $\varphi \in K(Q_{R_2})$,

$$\begin{aligned}
\int_{S_R} (D_h^i g) \varphi(y', 0) \, dy' &= \frac{1}{h} \int_0^h \int_{S_R} \left(s_t^i \frac{\partial w}{\partial y_i}(y', 0) \right) \varphi(y', 0) \, dy' dt, \\
&= \frac{-1}{h} \int_0^h \int_{S_R} \left(\frac{\partial w}{\partial y_i}(y', 0) \right) s_{-t}^i \varphi(y', 0) \, dy' dt.
\end{aligned}$$

Since $s_{-t}^i \varphi(y', 0) \in H_{00}^{1/2}(S_R)$, by Lemma A.1, we have

$$\begin{aligned}
\int_{S_R} \left| (D_h^i g) \varphi(y', 0) \right| \, dy' &\le \frac{1}{h} \int_0^h \int_{S_R} \left| \frac{\partial g}{\partial y_i}(y') \right| \cdot \left| s_{-t}^i \varphi(y', 0) \right| \, dy' dt \\
&\le \left(\frac{1}{h} \int_0^h dt \right) \left\| \frac{\partial g}{\partial y_i} \right\|_{H^{-1/2}(S_R)} \| s_{-t}^i \varphi \|_{00, S_R} \\
&\le C(R) \|g\|_{00, S_{R_1}} \|\nabla \varphi\|_{L^2(Q_R)},
\end{aligned}$$

which completes the proof. □

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