Publ. RIMS, Kyoto Univ.
40 (2004), 507–536

The Equivariant Toda Lattice

By

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The Toda lattice is an infinite dimensional dynamical system of commuting flows

$$(\partial, \delta_n, \bar{\delta}_n \mid n > 0),$$

acting on functions $(q, a_k, \bar{a}_k \mid k > 0)$ defined on a one-dimensional lattice. In the limit of small lattice spacing ε, which is all that will concern us here (Takasaki and Takebe [12]), the functions $(q, a_k, \bar{a}_k)$ become functions of a real parameter x, and the role of translation by one unit of the lattice is taken by the operator $\mathsf{E} = e^{\varepsilon\partial}$, where ∂ is the infinitesimal generator of translations in x.

The derivations δ_1 and $\bar{\delta}_1$ act on the variables q, a_1 and $\bar{a}_1$ by the formulas

$$\delta_1 \bar{a}_1 = \bar{\delta}_1 a_1 = \nabla q, \qquad \delta_1 \log q = \nabla a_1, \qquad \bar{\delta}_1 \log q = \nabla \bar{a}_1, \tag{1}$$

where $\nabla : \mathcal{A} \to \mathcal{A}$ is the infinite-order differential operator

$$\begin{aligned} \nabla = \varepsilon^{-1}\big(\mathsf{E}^{1/2} - \mathsf{E}^{-1/2}\big) &= \sum_{k=0}^{\infty} \frac{\varepsilon^{2k}\partial^{2k+1}}{2^{2k}(2k+1)!} \\ &= \partial + \tfrac{1}{24}\,\varepsilon^2\,\partial^3 + O(\varepsilon^4). \end{aligned}$$

These formulas imply the **Toda equation**:

$$\delta_1 \bar{\delta}_1 \log q = \nabla^2 q. \tag{2}$$

The higher Toda flows are symmetries of this equation.

An abstract mathematical formulation of the Toda lattice is obtained by realizing the derivations $(\partial, \delta_n, \bar{\delta}_n)$ on the free differential algebra $\mathcal{A} =$

Communicated by K. Saito. Received September 18, 2002. Revised May 6, 2003.
2000 Mathematics Subject Classification(s): 37K10, 53D45.
The author was partially supported by the NSF under grant DMS-0072508, and, through the Institute for Advanced Study, under grant DMS-9729992.
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$\mathbb{Q}_\varepsilon\{q, a_k, \bar{a}_k \mid k > 0\}$, defined over the ring $\mathbb{Q}_\varepsilon = \mathbb{Q}[\![\varepsilon]\!]$. Reductions of the Toda lattice correspond to differential ideals $\mathcal{I}$ in $\mathcal{A}$ invariant under conjugation and closed under the derivations δ_n and $\bar{\delta}_n$. For example, the **Toda chain** is described by the differential ideal with generators

$$\mathcal{I} = \{a_1 - \bar{a}_1, a_2 - q, \bar{a}_2 - q, a_k, \bar{a}_k \mid k > 2\},$$

so that $\mathcal{A}/\mathcal{I} \cong \mathbb{Q}_\varepsilon\{q, v\}$, where v is the image of $a_1 \in \mathcal{A}$.

In this paper, we study a new reduction of the Toda lattice, which we call the equivariant Toda lattice. If ν is a formal parameter, this reduction is defined by the following constraint on the Lax operator:

$$(\delta_1 - \bar{\delta}_1)L = \nu\partial L. \tag{3}$$

Let $\mathcal{I}_\nu \subset \mathcal{A}[\nu]$ be the corresponding differential ideal. We prove that the differential algebra $\mathcal{A}[\nu]/\mathcal{I}_\nu$ is isomorphic to

$$\mathbb{Q}_{\varepsilon,\nu}\{q, v, \bar{v}\}/(\nu\partial q - \nabla(v - \bar{v})) \otimes_{\mathbb{Q}_{\varepsilon,\nu}} \mathbb{Q}_{\varepsilon,\nu}[z_k, \bar{z}_k \mid k > 0],$$

where $\mathbb{Q}_{\varepsilon,\nu} = \mathbb{Q}_\varepsilon[\nu]$, v and $\bar{v}$ are the images of a_1 and $\bar{a}_1 \in \mathcal{A}$, and z_k and $\bar{z}_k$ are constants of motion, which may be defined by the following equation:

$$\left(L - \nu + \sum_{k=1}^{\infty} z_k L^{-k}\right)\frac{\partial L}{\partial v} = L.$$

In an influential paper, Eguchi and Yang [3] conjectured that the Gromov-Witten invariants of $\mathbb{CP}^1$ are related to the Toda chain. (For more on this conjecture, see Eguchi, Hori and Yang [4], Pandharipande [10] and Getzler [5].) The equivariant Toda conjecture, as formulated in this paper in terms of the equivariant Toda lattice, provides a similar description of the equivariant Gromov-Witten invariants of $\mathbb{CP}^1$, which specializes to the conjecture of Eguchi and Yang in the non-equivariant limit.

Let $\mathbb{T}$ be the multiplicative group of $\mathbb{C}$, and let X be a topological space with an action of $\mathbb{T}$. The equivariant cohomology $H^*_{\mathbb{T}}(X, \mathbb{Z})$ of X is a module over the graded ring $H^\bullet_{\mathbb{T}} = H^\bullet(B\mathbb{T}, \mathbb{Z}) \cong \mathbb{Z}[\nu]$, where $\nu \in H^2_{\mathbb{T}}$. The equivariant cohomology $H^\bullet_{\mathbb{T}}(\mathbb{CP}^1, \mathbb{Z})$ of the projective line $\mathbb{CP}^1$ admits a presentation

$$H^\bullet_{\mathbb{T}}(\mathbb{CP}^1, \mathbb{Z}) \cong \mathbb{Z}[H, \nu]/(H(H - \nu)),$$

where H is the equivariant Chern class $c_1(\mathcal{O}(1)) \in H^2_{\mathbb{T}}(\mathbb{CP}^1, \mathbb{Z})$.

Denote the kth descendants of the cohomology classes 1 and H in Gromov-Witten theory by $\tau_{k,P}$ and $\tau_{k,Q}$ respectively; also, abbreviate $\tau_{0,P}$ and $\tau_{0,Q}$ to P and Q. The genus 0 equivariant Gromov-Witten invariants of $\mathbb{CP}^1$ are integrals over the moduli space $\overline{\mathcal{M}}_{0,n}(\mathbb{CP}^1)$ of stable maps of genus g with n marked points:

$$\langle \tau_{k_1,P} \dots \tau_{k_m,P} \tau_{\ell_1,Q} \dots \tau_{\ell_n,Q} \rangle_g \in H^\bullet_{\mathbb{T}} \otimes \mathbb{Q} \cong \mathbb{Q}[\nu].$$

The large phase space is the formal affine space with coordinates $\{s_k, t_k \mid k \geq 0\}$. The genus g Gromov-Witten potential $\mathcal{F}_g$ of $\mathbb{CP}^1$ is the generating function on the large phase space given by the formula

$$\mathcal{F}_g = \sum_{m,n=0}^{\infty} \frac{1}{m!\,n!} \sum_{\substack{k_1,\dots,k_m \\ \ell_1,\dots,\ell_n}} s_{k_1} \dots s_{k_m} t_{\ell_1} \dots t_{\ell_n} \langle \tau_{k_1,P} \dots \tau_{k_m,P} \tau_{\ell_1,Q} \dots \tau_{\ell_n,Q} \rangle_g .$$

We may combine the Gromov-Witten potentials into a single generating function by interpreting ε as a genus expansion parameter, and writing

$$\mathcal{F} = \sum_{g=0}^{\infty} \varepsilon^{2g} \mathcal{F}_g .$$

Based on explicit calculations using the topological recursion relations in genus 0 and 1, Pandharipande conjectured [11] that the following equation holds for the total Gromov-Witten potential:

$$\partial_0 \bar{\partial}_0 \mathcal{F} = \exp(\nabla^2 \mathcal{F}). \tag{4}$$

Here, $\nabla = \varepsilon^{-1}(\mathsf{E}^{1/2} - \mathsf{E}^{-1/2})$, where $\mathsf{E} = e^{\varepsilon\partial}$, and $\partial = \partial/\partial s_0$.

On applying the operator ∇^2 to both sides of (4) and identifying the vector fields ∂_0 and $\bar{\partial}_0$ with the Toda flows δ_1 and $\bar{\delta}_1$, we obtain the Toda equation (2) for $q = \exp(\nabla^2 \mathcal{F})$. Observe that $\partial_1 - \bar{\partial}_1 = \nu\partial$; this equation is formally identical to the constraint defining the equivariant Toda lattice.

The equivariant Toda lattice and the equivariant Gromov-Witten theory of $\mathbb{CP}^1$ each involve sequences $\{\delta_n, \bar{\delta}_n\}$ and $\{\partial_n = \partial/\partial t_n, \bar{\partial}_n = \partial/\partial t_n - \nu\partial/\partial s_n\}$ of commuting derivations, in the first case on the algebra $\mathcal{A}[\nu]/\mathcal{I}_\nu$, and in the second case on functions on the large phase space. These sequences of vector fields may be compared by means of a morphism

$$\mathcal{A}[\nu]/\mathcal{I}_\nu \longrightarrow \mathbb{Q}_{\varepsilon,\nu}[\![s_k, t_k \mid k \geq 0]\!]$$

of differential algebras which sends the generators q, v and $\bar{v}$ to $\exp(\nabla^2\mathcal{F})$, $\nabla\partial_0\mathcal{F}$ and $\nabla\bar{\partial}_0\mathcal{F}$, and the constants z_k and $\bar{z}_k$ to 0. In fact, the following relationship between these flows holds:

$$\sum_{k=0}^{\infty} z^{k+1}\partial_k = \sum_{n=1}^{\infty} \frac{z^n\delta_n}{(1+z\nu)(2+z\nu)\dots(n+z\nu)}, \tag{5a}$$

$$\sum_{k=0}^{\infty} z^{k+1}\bar{\partial}_k = \sum_{n=1}^{\infty} \frac{z^n\bar{\delta}_n}{(1-z\nu)(2-z\nu)\dots(n-z\nu)}. \tag{5b}$$

We conjectured this in a preprint of this paper, based on a proof of the result in genus 0 (see Section 4), together with calculations in genus 1 for small values of n; it has recently been proved by Okounkov and Pandharipande [9], along with the equivariant Toda equation (4). Thus, the equivariant Toda lattice yields a description of the equivariant Gromov-Witten invariants of $\mathbb{CP}^1$ in terms of a Lax operator whose coefficients are obtained by an explicit recursion. In particular, the descendent flows $\partial/\partial s_k$ of the puncture operator P are given in the non-equivariant limit by the formula

$$\frac{\partial}{\partial s_k} = \lim_{\nu\to 0}\left(\frac{\delta_{k+1}-\bar{\delta}_{k+1}}{\nu(k+1)!} - \frac{c_k(\delta_k+\bar{\delta}_k)}{k!}\right), \tag{6}$$

where c_k is the harmonic number $c_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$; this is exactly as conjectured by Eguchi and Yang [3].

In the second part of this paper (Sections 5–8), we relate the equivariant Toda lattice to the dressing operator formalism. Let $\log(L) = W\log(\Lambda)W^{-1}$ be the logarithm of the Lax operator L, related to the operator $\ell = \varepsilon(\partial W)W^{-1}$ by the formula

$$\log(L) = \log(\Lambda) - \ell.$$

Borrowing ideas of Carlet, Dubrovin and Zhang [1], we show that the equivariant Toda lattice may be characterized by the expansion

$$\Lambda + v + q\Lambda^{-1} = L + \nu\ell - \sum_{k=1}^{\infty}\frac{z_k}{k}L^{-k}.$$

In particular, the equation $(\delta_1 - \bar{\delta}_1)W = \nu\partial W$ is equivalent to the vanishing of the coefficients z_k. We also show that the equivariant Toda lattice has a Hamiltonian structure which is a deformation of the first Hamiltonian structure of the Toda chain. (We were however unable to find a bihamiltonian structure.) This gives a more direct relationship between the results of Okounkov and Pandharipande [9] and the original Toda conjecture.

This paper closes with an appendix in which the formulas (5a) and (5b) relating $\{\partial_k, \bar{\partial}_k\}$ and $\{\delta_n, \bar{\delta}_n\}$ are inverted.

§1. Difference Operators

In this section, we recall the mathematical structure underlying the Toda lattice; this material is adapted from the fundamental papers of Ueno and Takasaki [13] and Kupershmidt [8].

All of the commutative algebras which we consider in this paper carry an involution $p \mapsto \bar{p}$, and all ideals which we consider are closed under this involution. By a **differential algebra**, we mean a commutative algebra with derivation ∂ such that

$$\partial\bar{p} = \overline{\partial p}.$$

A **differential ideal** is an ideal closed under the action of the differential ∂. If S is a subset of a differential algebra $\mathcal{A}$, denote the differential ideal generated by $S \cup \bar{S}$ by (S), where $\bar{S} = \{\bar{x} \mid x \in S\}$ is the conjugate of S.

If $\mathcal{A}$ is a differential algebra and S is a set, the free differential algebra $\mathcal{A}\{S\}$ generated by S is the polynomial algebra

$$\mathcal{A}[\partial^n x, \partial^n \bar{x} \mid x \in S, n \geq 0],$$

with differential $\partial(\partial^n x) = \partial^{n+1} x$.

An **evolutionary derivation** δ of a differential algebra $\mathcal{A}$ is a derivation such that $[\partial, \delta] = 0$. The evolutionary derivations form a Lie subalgebra of the Lie algebra of derivations of $\mathcal{A}$, with involution

$$\bar{\delta}p = \overline{\delta\bar{p}}.$$

Let $\mathcal{A}$ be a differential algebra over $\mathbb{Q}_\varepsilon$, and let $q \in \mathcal{A}$ be a regular element (that is, having no zero-divisors) such that $\bar{q} = q$. The localization $q^{-1}\mathcal{A}$ of $\mathcal{A}$ is a filtered differential algebra, with differential $\partial(q^{-1}) = -q^{-2}\partial q$. Let $\Phi_\pm(\mathcal{A}, q)$ be the associative algebras of **difference operators**

$$\Phi_+(\mathcal{A}, q) = \left\{ \sum_{k=-\infty}^{\infty} p_k \Lambda^k \;\middle|\; p_k \in q^{-k}\mathcal{A},\, p_k = 0 \text{ for } k \ll 0 \right\},$$

$$\Phi_-(\mathcal{A}, q) = \left\{ \sum_{k=-\infty}^{\infty} p_k \Lambda^k \;\middle|\; p_k \in \mathcal{A},\, p_k = 0 \text{ for } k \gg 0 \right\},$$

with product

$$\sum_i a_i\Lambda^i \cdot \sum_j b_j\Lambda^j = \sum_k \Big(\sum_{i+j=k} \big(\mathsf{E}^{-j/2}a_i\big)\big(\mathsf{E}^{i/2}b_j\big)\Big)\Lambda^k.$$

Note that $\Phi_-(\mathcal{A},q)$ is in fact independent of q.

Let $A \mapsto A_\pm$ be the projections on $\Phi_\pm(\mathcal{A},q)$ defined by the formulas

$$\Big(\sum_{k=-\infty}^{\infty} p_k\Lambda^k\Big)_+ = \sum_{k=0}^{\infty} p_k\Lambda^k, \Big(\sum_{k=-\infty}^{\infty} p_k\Lambda^k\Big)_- = \sum_{k=-\infty}^{-1} p_k\Lambda^k.$$

We see that $A = A_- + A_+$. Define the residue $\operatorname{res} : \Phi_\pm(\mathcal{A},q) \to \mathcal{A}$ by the formula

$$\operatorname{res}\Big(\sum_{k=-\infty}^{\infty} p_k\Lambda^k\Big) = p_0.$$

For $k \in \mathbb{Z}$, let $[k]$ be the isomorphism of $\mathcal{A}$

$$[k] = \frac{\mathsf{E}^{k/2} - \mathsf{E}^{-k/2}}{\mathsf{E}^{1/2} - \mathsf{E}^{-1/2}} = \sum_{j=1}^{k} \mathsf{E}^{(k+1)/2-j} = k + O(\varepsilon^2).$$

Define $q^{[k]}$ by the recursion

$$q^{[k+1]} = \mathsf{E}^k q \cdot \mathsf{E}^{-1/2} q^{[k]},$$

with initial condition $q^{[0]} = 1$. The involution

$$A = \sum_{k=-\infty}^{\infty} p_k\Lambda^k \mapsto \bar{A} = \sum_{k=1}^{\infty} \bar{p}_k\, q^{[k]}\Lambda^{-k} + \bar{p}_0 + \sum_{k=1}^{\infty} \bar{p}_{-k}\, q^{-[k]}\Lambda^k,$$

defines an anti-isomorphism between the algebras $\Phi_+(\mathcal{A},q)$ and $\Phi_-(\mathcal{A},q)$.

§2. The Toda Lattice

To formulate the Toda lattice, we introduce the differential algebra

$$\mathcal{A} = \mathbb{Q}_\varepsilon\{q, a_k \mid k > 0\}/(q - \bar{q}).$$

It will be useful to define the symbol a_0 to equal 1.

The Lax operator of the Toda lattice is the difference operator

$$L = \Lambda + \sum_{k=1}^{\infty} a_k\Lambda^{-k+1} \in \Phi_-(\mathcal{A},q);$$

its conjugate $\bar{L}$ is given by the formula

$$\bar{L} = q\Lambda^{-1} + \sum_{k=1}^{\infty} \bar{a}_k q^{-[k-1]} \Lambda^{k-1} \in \Phi_+(\mathcal{A}, q).$$

Introduce elements $p_k(n) \in \mathcal{A}$, defined for all $n \geq 0$ and $k \in \mathbb{Z}$:

$$L^n = \sum_{k=-\infty}^{n} p_k(n)\Lambda^k.$$

To define the evolutionary derivation δ_n on the generators a_k of $\mathcal{A}$, introduce the difference operator $B_n = L^n_+$, and impose the Lax equation $\delta_n L = \varepsilon^{-1}[B_n, L]$. This equation means that

$$\varepsilon^{-1}[B_n, L] = \sum_{k=1}^{\infty} \delta_n a_k \, \Lambda^{-k+1}.$$

In order for this to be meaningful, it must be shown that the coefficient of Λ^k in $[B_n, L]$ vanishes for $k > 0$. This follows from the identity $[L^n, L] = 0$: we have

$$[B_n, L] = [B_n, L] - [L^n, L] = -[L^n_-, L],$$

and it is clear that the coefficient of Λ^k in $[L^n_-, L]$ vanishes if $k > 0$. We also see that $\delta_n a_k$ equals the coefficient of Λ^{-k+1} in

$$-\varepsilon^{-1} \sum_{j=0}^{k} [p_{j-k}(n)\Lambda^{j-k}, a_j\Lambda^{-j+1}],$$

hence that

$$\begin{aligned} (7) \quad & \delta_n a_k - \nabla p_{-k}(n) \\ &= \varepsilon^{-1} \sum_{j=1}^{k-1} \Big(\mathsf{E}^{(1-j)/2} p_{j-k}(n)\, \mathsf{E}^{(k-j)/2} a_j - \mathsf{E}^{(j-1)/2} p_{j-k}(n)\, \mathsf{E}^{(j-k)/2} a_j \Big) \\ &= \sum_{j=1}^{k-1} \Big(\mathsf{E}^{(1-j)/2} p_{j-k}(n)\, \nabla[k-j] a_j - \mathsf{E}^{(j-k)/2} a_j \nabla[j-1] p_{j-k}(n) \Big). \end{aligned}$$

In particular, $\delta_n a_1 = \nabla p_{-1}(n)$.

To define δ_n on the remaining generators q and $\bar{a}_k$ of $\mathcal{A}$, we impose the Lax equation $\delta_n \bar{L} = \varepsilon^{-1}[B_n, \bar{L}]$. In particular, we see that

$$\delta_n q = q\nabla p_0(n), \tag{8}$$

and hence that $\delta_n q^{[k]} = q^{[k]}\nabla[k]p_0(n)$. It also follows that

$$\delta_n(q^{-[k-1]}\bar{a}_k) = q^{-[k-1]}\big(\delta_n\bar{a}_k - \bar{a}_k\nabla[k-1]p_0(n)\big)$$

equals the coefficient of Λ^{k-1} in

$$\varepsilon^{-1}\sum_{j=0}^{k}[p_{k-j}(n)\Lambda^{k-j}, q^{-[j-1]}\bar{a}_j\Lambda^{j-1}],$$

hence that

$$\delta_n\bar{a}_k = \bar{a}_k\nabla[k-1]p_0(n) + \varepsilon^{-1}q^{[k-1]}\sum_{j=0}^{k}A_{k,j},$$

where

$$\begin{aligned} A_{k,j} &= \mathsf{E}^{(1-j)/2}p_{k-j}(n)\,\mathsf{E}^{(k-j)/2}q^{-[j-1]}\bar{a}_j - \mathsf{E}^{(j-1)/2}p_{k-j}(n)\,\mathsf{E}^{(j-k)/2}q^{-[j-1]}\bar{a}_j \\ &= \mathsf{E}^{(1-j)/2}\big(q^{[k-j]}p_{k-j}(n)\big)\,\mathsf{E}^{(k-j)/2}\bar{a}_j - \mathsf{E}^{(j-1)/2}\big(q^{[k-j]}p_{k-j}(n)\big)\,\mathsf{E}^{(j-k)/2}\bar{a}_j. \end{aligned}$$

Thus, we see that

$$\begin{aligned} \delta_n\bar{a}_k = \nabla\big(q^{[k]}p_k(n)\big) + \sum_{j=1}^{k-1}\Big(&\mathsf{E}^{(1-j)/2}\big(q^{[k-j]}p_{k-j}(n)\big)\,\nabla[k-j]\bar{a}_j \\ &-\mathsf{E}^{(j-k)/2}\bar{a}_j\nabla[j-1]\big(q^{[k-j]}p_{k-j}(n)\big)\Big). \end{aligned} \tag{9}$$

In particular, $\delta_n\bar{a}_1 = \nabla\big(qp_1(n)\big)$.

We now recall the proof that the derivations δ_m and δ_n commute. The proof relies on the Zakharov-Shabat equation

$$\delta_m B_n - \delta_n B_m = \varepsilon^{-1}[B_m, B_n]. \tag{10}$$

To prove this equation, observe that

$$\begin{aligned} \delta_m B_n &= (\delta_m L^n)_+ = \varepsilon^{-1}[B_m, L^n]_+ = \varepsilon^{-1}[B_m, B_n + L^n_-]_+ \\ &= \varepsilon^{-1}[B_m, B_n] + \varepsilon^{-1}[B_m, L^n_-]_+. \end{aligned}$$

Since $[L^m, L^n] = 0$, we also see that

$$[B_m, L^n]_+ = [L^m - L^m_-, L^n]_+ = -[L^m_-, B_n]_+.$$

It follows that

$$\delta_m B_n - \delta_n B_m = \varepsilon^{-1}\big([B_m, B_n] + [B_m, L^n_-]_+\big) + \varepsilon^{-1}[L^n_-, B_m]_+ = \varepsilon^{-1}[B_m, B_n].$$

From (10), we easily see that the derivations δ_m and δ_n commute:

$$\begin{aligned}[\delta_m,\delta_n]L &= \varepsilon^{-1}\delta_m[B_n,L]-\varepsilon^{-1}\delta_n[B_m,L]\\ &= \varepsilon^{-1}[\delta_m B_n,L]+\varepsilon^{-2}[B_n,[B_m,L]]-\varepsilon^{-1}[\delta_n B_m,L]-\varepsilon^{-2}[B_m,[B_n,L]]\\ &= \varepsilon^{-1}[\delta_m B_n-\delta_n B_m,L]+\varepsilon^{-2}[B_n,[B_m,L]]-\varepsilon^{-2}[B_m,[B_n,L]]=0.\end{aligned}$$

The derivation $\bar\delta_n$ is defined to be the conjugate of δ_n, acting on the generators of $\mathcal{A}$ by the formulas

$$\bar\delta_n q=\overline{\delta_n q},\qquad \bar\delta_n a_k=\overline{\delta_n \bar a_k},\qquad \bar\delta_n \bar a_k=\overline{\delta_n a_k}.$$

The following proposition establishes the Lax equation for this derivation.

Proposition 2.1. *Let $C_n=-\bar L^n_-$; then $\bar\delta_n L=\varepsilon^{-1}[C_n,L]$ and $\bar\delta_n\bar L=\varepsilon^{-1}[C_n,\bar L]$.*

Proof. We have

$$\begin{aligned}\overline{\bar\delta_n\bar L} &= \overline{\bar\delta_n q}\Lambda^{-1}+\sum_{k=1}^{\infty}\overline{q^{-[k-1]}\big(\bar\delta_n\bar a_k-\bar a_k q^{-[k-1]}\bar\delta_n q^{[k-1]}\big)\Lambda^{k-1}}\\ &= \sum_{k=1}^{\infty}\overline{\bar\delta_n\bar a_k}\Lambda^{-k+1}+\nabla p_0(n)\Lambda-\sum_{k=1}^{\infty}a_k\nabla[k-1]p_0(n)\Lambda^{-k+1}\\ &= \sum_{k=1}^{\infty}\delta_n a_k\Lambda^{-k+1}-[p_0(n),L]=\varepsilon^{-1}[B_n,L]-[p_0(n),L]=\varepsilon^{-1}\overline{[C_n,\bar L]}.\end{aligned}$$

A similar proof shows that $\overline{\bar\delta_n L}=\varepsilon^{-1}\overline{[C_n,L]}$. □

It is automatic that the derivations $\bar\delta_m$ and $\bar\delta_n$ commute, since their conjugates do. To see that δ_m commutes with $\bar\delta_n$, we use the Zakharov-Shabat equation

$$\delta_m\bar B_n-\bar\delta_n B_m=\varepsilon^{-1}[B_m,\bar B_n]. \tag{11}$$

This is proved by combining the equations

$$\delta_m\bar B_n=(\delta_m\bar L^n)_-=\varepsilon^{-1}[B_m,\bar L^n]_-=\varepsilon^{-1}[B_m,\bar B_n]_-,$$

and

$$\bar\delta_m B_n=(\bar\delta_n L^m)_+=\varepsilon^{-1}[\bar B_n,L^m]_+=\varepsilon^{-1}[\bar B_n,B_m]_+.$$

It follows from (11) that δ_m and $\bar{\delta}_n$ commute:

$$\begin{aligned}[\delta_m, \bar{\delta}_n]L &= \varepsilon^{-1}\delta_m[\bar{B}_n, L] - \varepsilon^{-1}\bar{\delta}_n[B_m, L] \\ &= \varepsilon^{-1}[\delta_m\bar{B}_n, L] + \varepsilon^{-2}[\bar{B}_n, [B_m, L]] - \varepsilon^{-1}[\bar{\delta}_n B_m, L] - \varepsilon^{-2}[B_m, [\bar{B}_n, L]] \\ &= \varepsilon^{-1}[\delta_m\bar{B}_n - \bar{\delta}_n B_m, L] + \varepsilon^{-2}[\bar{B}_n, [B_m, L]] - \varepsilon^{-2}[B_m, [\bar{B}_n, L]] = 0.\end{aligned}$$

Denote by $\alpha : \mathcal{A} \to \mathbb{Q}_\varepsilon$ the homomorphism which sends the generators $\{q, a_k, \bar{a}_k\}$ of $\mathcal{A}$ to 0. By formulas (7), (8) and (9), we see that $\delta_n q$, $\delta_n a_k$ and $\delta_n \bar{a}_k$ all lie in the ideal $(\partial\mathcal{A})$ of $\mathcal{A}$, and hence

$$\alpha \cdot \delta_n = 0. \tag{12}$$

§3. The Equivariant Toda Lattice

Informally, a reduction of the Toda lattice is an invariant submanifold of the configuration space fixed by the involution; we formalize this as follows.

Definition 3.1. A **reduction** of the Toda lattice is a differential ideal $\mathcal{I} \subset \mathcal{A}$ invariant under conjugation and preserved by the action of the derivations δ_n and $\bar{\delta}_n$.

The simplest example of such a reduction is the **Toda chain**, defined by the constraint $L = \bar{L}$; the associated differential ideal

$$\mathcal{I} = (a_1 - \bar{a}_1, a_2 - q, a_k \mid k > 2)$$

is generated by the coefficients of $L - \bar{L}$. To see that $\mathcal{I}$ is closed under the action of the derivations δ_n, it suffices to observe that the operator $L - \bar{L}$ satisfies the Lax equation $\delta_n(L - \bar{L}) = \varepsilon^{-1}[B_n, L - \bar{L}]$, and that the coefficients of $\varepsilon^{-1}[B_n, L - \bar{L}]$ are contained in the differential ideal generated by the coefficients of $L - \bar{L}$.

The constraint $L = \bar{L}$ is equivalent to the relation $\delta_n = \bar{\delta}_n$ among the Toda flows, for all n; in particular, the Toda equation (2) becomes in this limit the equation $\delta_1^2 \log q = \nabla^2 q$.

In this paper, we study a reduction of the Toda lattice which is a deformation of the Toda chain. Let $\mathcal{A}[\nu]$ be the extension of the differential algebra $\mathcal{A}$ by a variable ν, such that $\partial\nu = 0$ and $\bar{\nu} = -\nu$, and consider families of reductions of the Toda lattice parametrized by $\mathcal{A}[\nu]$; that is, we consider differential ideals in $\mathcal{A}[\nu]$ satisfying the conditions of Definition 3.1.

Definition 3.2. The **equivariant Toda lattice** is the reduction of the Toda lattice defined over $\mathcal{A}[\nu]$ by the constraints

$$(\delta_1 - \bar{\delta}_1)L = \nu\partial L, \qquad (\delta_1 - \bar{\delta}_1)\bar{L} = \nu\partial\bar{L}. \tag{13}$$

Let $\mathcal{I}_\nu$ be the differential ideal determining this reduction.

Let K be the difference operator

$$K = B_1 - C_1 = \Lambda + a_1 + q\Lambda^{-1}. \tag{14}$$

Substituting the Lax equations into the constraints (13), we obtain an equivalent formulation of the equivariant Toda lattice: it is characterized by the pair of equations

$$\varepsilon^{-1}[K, L] = \nu\partial L, \qquad \varepsilon^{-1}[K, \bar{L}] = \nu\partial\bar{L}. \tag{15}$$

In other words, the differential ideal $\mathcal{I}_\nu$ defining the equivariant Toda lattice is generated by the coefficients of these equations.

The following theorem collects the main properties of the differential ideal $\mathcal{I}_\nu$. Let $\mathsf{P} : \mathcal{A} \to \mathcal{A}$ be the infinite-order differential operator

$$\begin{aligned}\mathsf{P} &= \frac{\partial}{\nabla} = \sum_{g=0}^{\infty} \frac{\varepsilon^{2g}(2^{1-2g} - 1)B_{2g}}{(2g)!}\,\partial^{2g} \\ &= 1 - \tfrac{1}{24}\,\varepsilon^2\,\partial^2 + O(\varepsilon^4).\end{aligned}$$

Theorem 3.1. *The differential ideal $\mathcal{I}_\nu$ defining the equivariant Toda lattice equals $(\eta, \partial\zeta_k \mid k > 0)$, where*

$$\begin{aligned}\eta &= q\nabla(a_1 - \bar{a}_1) - \nu\partial q, \\ \zeta_k &= p_{-1}(k) - qp_1(k) - \nu\mathsf{P}p_0(k).\end{aligned}$$

The differential algebra $\mathcal{A}[\nu]/\mathcal{I}_\nu$ is isomorphic to

$$\widetilde{\mathcal{A}} = \mathbb{Q}_{\varepsilon,\nu}\{q, v, z_k \mid k > 0\}/(q - \bar{q}, y, \partial z_k \mid k > 0),$$

where v and z_k are identified with the images of a_1 and ζ_k in $\mathcal{A}[\nu]/\mathcal{I}_\nu$, and $y = q\nabla(v - \bar{v}) - \nu\partial q$.

The Toda flows δ_n and $\bar{\delta}_n$ map $\zeta_k \in \mathcal{A}[\nu]$ to $\mathcal{I}_\nu$; in particular, the variables $z_k \in \widetilde{\mathcal{A}}$ are constants of motion for the flows of the Toda lattice.

Proof. Let $\widetilde{\mathcal{I}}_\nu$ be the differential ideal $(\eta, \partial\zeta_k \mid k > 0)$. Define elements $f_k(n)$, $g_k(n) \in \mathcal{A}$ by the formulas

$$\varepsilon^{-1}[K, L^n] - \nu\partial L^n = \sum_{k=-\infty}^{\infty} f_k(n)\Lambda^{-k}, \varepsilon^{-1}[K, \bar{L}^n] - \nu\partial\bar{L}^n = \sum_{k=-\infty}^{\infty} g_k(n)\Lambda^k.$$

The differential ideal $\mathcal{I}_\nu$ is generated by the coefficients $f_k = f_k(1)$ and $g_k = g_k(1)$. The formulas

$$\begin{aligned} \varepsilon^{-1}[K, L^n] - \nu\partial L^n &= \sum_{i=1}^{n} L^{i-1}\big(\varepsilon^{-1}[K, L] - \nu\partial L\big)L^{n-i}, \\ \varepsilon^{-1}[K, \bar{L}^n] - \nu\partial\bar{L}^n &= \sum_{i=1}^{n} \bar{L}^{i-1}\big(\varepsilon^{-1}[K, \bar{L}] - \nu\partial\bar{L}\big)\bar{L}^{n-i}, \end{aligned}$$

show that the coefficients $f_k(n)$ and $g_k(n)$ lie in $\mathcal{I}_\nu$; hence $\partial\zeta_n = \mathsf{P}f_0(n)$ and $\partial\bar{\zeta}_n = \mathsf{P}g_0(n)$ do as well, showing that $\widetilde{\mathcal{I}}_\nu \subset \mathcal{I}_\nu$. We wish to prove the equality of these two differential ideals.

To do this, we show, by induction on k, that the coefficients f_k lie in $\widetilde{\mathcal{I}}_\nu$. We have $f_k = 0$ for $k < 0$. If $f_j \in \widetilde{\mathcal{I}}_\nu$ for $j < k$, we see that

$$\begin{aligned} \nabla z_{k+1} &= \operatorname{res}\big(\varepsilon^{-1}[K, L^{k+1}] - \nu\partial L^{k+1}\big) = \sum_{i=0}^{k} \operatorname{res}\big(L^i(\varepsilon^{-1}[K, L] - \nu\partial L)L^{k-i}\big) \\ &\equiv [k+1]f_k \pmod{\widetilde{\mathcal{I}}_\nu}, \end{aligned}$$

hence $f_k \in \widetilde{\mathcal{I}}_\nu$. A similar induction shows that $g_k \in \widetilde{\mathcal{I}}_\nu$; this induction starts with the fact that $g_{-1} = \eta$ lies in $\widetilde{\mathcal{I}}_\nu$.

Since $\zeta_k - [k]a_{k+1} \in (q, a_1, \dots, a_k)$, we see that the differential algebra $\mathcal{A}/\mathcal{I}_\nu$ is isomorphic to

$$\widetilde{\mathcal{A}} = \mathbb{Q}_{\varepsilon,\nu}\{q, v, z_k \mid k > 0\}/(q - \bar{q}, y, \partial z_k \mid k > 0).$$

It remains to prove that $\delta_n\zeta_k$ and $\bar{\delta}_n\zeta_k$ lie in $\mathcal{I}_\nu$. By the Zakharov-Shabat equations (10) and (11), we see that

$$\begin{aligned} \delta_n K &= \delta_n(B_1 - C_1) = (\delta_1 - \bar{\delta}_1)B_n + \varepsilon^{-1}[B_n, B_1 - C_1] \\ &= \nu\partial B_n + \varepsilon^{-1}[B_n, K]. \end{aligned}$$

It follows that

$$\begin{aligned}\nabla\delta_n\zeta_k &= \delta_n f_0(k) = \delta_n \operatorname{res}(\varepsilon^{-1}[K,L^k] - \nu\partial L^k)\\ &= \operatorname{res}(\varepsilon^{-1}[\delta_n K, L^k] + \varepsilon^{-1}[K,\delta_n L^k] - \nu\partial\delta_n L^k)\\ &= \varepsilon^{-1}\operatorname{res}([\nu\partial B_n + \varepsilon^{-1}[B_n,K],L^k] + \varepsilon^{-1}[K,[B_n,L^k]] - \nu\partial[B_n,L^k])\\ &= \varepsilon^{-1}\operatorname{res}([B_n,\varepsilon^{-1}[K,L^k] - \nu\partial L^k])\\ &= \nabla\sum_{j=1}^{n}[j]\big(p_j(n)f_j(k)\big).\end{aligned}$$

The extension of α to a homomorphism from $\mathcal{A}[\nu]$ to $\mathbb{Q}_{\varepsilon,\nu}$ continues to satisfy (12). It follows that $\alpha(\delta_n\zeta_k) = 0$, hence we obtain an explicit equation for $\delta_n\zeta_k$:

$$\delta_n\zeta_k = \sum_{j=1}^{n}[j]\big(p_j(n)f_j(k)\big) \in \mathcal{I}_\nu.$$

The proof that $\bar{\delta}_n\zeta_k \in \mathcal{I}_\nu$ follows along the same lines. □

Let us illustrate this theorem by calculating the coefficients a_2 and a_3 of the Lax operator L as elements of $\widetilde{\mathcal{A}}$. Applying $\operatorname{res} : \Phi_-(\mathcal{A},q) \to \mathcal{A}$ to the equation $\varepsilon^{-1}[K,L] = \nu\partial L$, we see that

$$a_2 = q + \nu\mathsf{P}v + z_1. \tag{16}$$

Taking the coefficient of Λ^{-1} in the equation $\varepsilon^{-1}[K,L] = \nu\partial L$, we see that

$$\nabla a_3 + (a_2 - q)\nabla v = \nu\partial a_2.$$

Lemma 3.1. $\nabla f\,\mathsf{P}g = \frac{1}{2}\nabla\big(f[2]\mathsf{P}g\big) - \frac{1}{2}[2]\big(f\partial g\big)$

Proof. We have

$$\nabla f\,\mathsf{P}g = \varepsilon^{-1}\mathsf{E}^{1/2}\Big(f\,\mathsf{E}^{-1/2}\mathsf{P}g\Big) - \varepsilon^{-1}\mathsf{E}^{-1/2}\big(f\,\mathsf{E}^{1/2}\mathsf{P}g\big).$$

The result follows, since $\mathsf{E}^{\pm1/2}\mathsf{P} = \frac{1}{2}[2]\mathsf{P} \pm \frac{1}{2}\varepsilon\partial$. □

By this lemma,

$$(a_2 - q)\nabla v = \nu\nabla v\,\mathsf{P}v + z_1\nabla v = \nu\nabla\big(\tfrac{1}{2}v[2]\mathsf{P}v - \tfrac{1}{4}\mathsf{P}[2]v^2\big) + z_1\nabla v.$$

It follows that

$$a_3 = \nu\big(\mathsf{P}\big(\tfrac{1}{4}[2]v^2 + q\big) - \tfrac{1}{2}v[2]\mathsf{P}v\big) + \nu^2\mathsf{P}v - z_1 v + \tfrac{1}{2}z_2. \tag{17}$$

This method of calculating the coefficients a_k becomes cumbersome for larger values of k: instead, it is better to use the recursion in $\widetilde{\mathcal{A}}$

$$p_{-1}(n) = qp_1(n) + \nu \mathsf{P} p_0(n) + z_n \tag{18}$$

which is a consequence of Theorem 3.1.

Let Ψ be the algebra of difference operators

$$\Psi = \{A \in \Phi_-(\widetilde{\mathcal{A}}, q) \mid \varepsilon^{-1}[K, A] = \nu\partial A\}.$$

Let $\mathbb{L} \in \Psi$ be the Lax operator defined by the recursion

$$p_{-1}(n) = qp_1(n) + \nu \mathsf{P} p_0(n). \tag{19}$$

This Lax operator plays a special role in the theory: the following lemma shows that the algebra Ψ may be identified with the commutative algebra $\widetilde{\mathcal{A}}_0(\!(\mathbb{L})\!)$, where

$$\widetilde{\mathcal{A}}_0 = \mathbb{Q}_{\varepsilon,\nu}[z_k, \bar{z}_k \mid k > 0]$$

is the kernel of the derivation $\partial : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}}$.

Lemma 3.2. *The homomorphism $\tilde{\alpha} : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}}_0$ which sends the generators $\partial^n q$, $\partial^n v$ and $\partial^n \bar{v}$ of $\widetilde{\mathcal{A}}$ to 0 induces an isomorphism between Ψ and $\widetilde{\mathcal{A}}_0(\!(\Lambda^{-1})\!)$.*

Proof. Since $\tilde{\alpha}(\mathbb{L}) = \Lambda$, the map $\tilde{\alpha} : \Psi \to \widetilde{\mathcal{A}}_0(\!(\Lambda^{-1})\!)$ is surjective. Suppose that $A \in \Psi$ lies in the kernel of $\tilde{\alpha}$, and let k be the smallest integer such that the coefficient $x \in \widetilde{\mathcal{A}}$ of Λ^{-k} in A is nonzero. We have

$$\varepsilon^{-1}[K, A] - \nu\partial A = \nabla x\, \Lambda^{1-k} + O(\Lambda^{-k}),$$

hence $x \in \widetilde{\mathcal{A}}_0$. In this way, we see that $\tilde{\alpha} : \Psi \to \widetilde{\mathcal{A}}_0(\!(\Lambda^{-1})\!)$ is injective. □

Theorem 3.2. *The evolutionary derivation $e = \partial_v + \partial_{\bar{v}}$ of $\widetilde{\mathcal{A}}$ preserves Ψ, and $e(L)$ satisfies the formula*

$$\left(L - \nu + \sum_{k=1}^{\infty} z_k L^{-k}\right) e(L) = L.$$

Proof. If $A \in \Psi$, we have

$$\varepsilon^{-1}[K, e(A)] - \nu\partial e(A) = e(\varepsilon^{-1}[K, A] - \nu\partial A) - [e(K), A] = 0,$$

since $e(K) = 1$. This shows that e preserves Ψ.

For $n > 0$, we have by (19) that $\tilde{\alpha}(e(p_{-1}(n))) = \nu\tilde{\alpha}(e(p_0(n)))$, or equivalently,

$$\oint (\mathbb{L}-\nu)e(\mathbb{L}^n)\frac{d\mathbb{L}}{\mathbb{L}} = 0.$$

Since Ψ is a commutative algebra, $e(\mathbb{L})$ commutes with $\mathbb{L}$, hence $e(\mathbb{L}^n) = n\mathbb{L}^{n-1}e(\mathbb{L})$, and

$$\oint (\mathbb{L}-\nu)\mathbb{L}^{n-2}e(\mathbb{L})\, d\mathbb{L} = 0, \quad n > 0.$$

This shows that the coefficient of $\mathbb{L}^{-k}$ in $(\mathbb{L}-\nu)e(\mathbb{L})$ vanishes, hence $(\mathbb{L}-\nu)e(\mathbb{L}) = \mathbb{L}$.

Since $L/e(L)$ lies in Ψ, there is an expansion

$$\frac{L}{e(L)} = L - \nu + \frac{1}{2\pi i}\sum_{n=0}^{\infty} L^{-n-1}\oint L^n \frac{dL}{e(L)}.$$

(The constant term is determined by the fact that $e(L) = 1 + \nu\Lambda^{-1} + O(\Lambda^{-2})$.) We have

$$\frac{dL}{e(L)} = \frac{d\mathbb{L}}{e(\mathbb{L})} = (\mathbb{L}-\nu)\frac{d\mathbb{L}}{\mathbb{L}},$$

hence

$$\frac{1}{2\pi i}\oint L^n \frac{dL}{e(L)} = \frac{1}{2\pi i}\oint L^n\,(\mathbb{L}-\nu)\frac{d\mathbb{L}}{\mathbb{L}}.$$

It follows from the recursion (18) that

$$z_n = \tilde{\alpha}(p_{-1}(n)) - \nu\tilde{\alpha}(p_0(n)) = \frac{1}{2\pi i}\oint L^n\,(\mathbb{L}-\nu)\frac{d\mathbb{L}}{\mathbb{L}},$$

and the theorem follows. □

§4. The Dispersionless Limit of the Equivariant Toda Lattice

In this section, we consider the dispersionless limit of the equivariant Toda lattice, in which $\varepsilon \to 0$; we only consider the case in which the constants of motion z_k are set to 0. If $A \in \Phi_\pm(\widetilde{\mathcal{A}}, q)$, we write

$$A_0 = \lim_{\varepsilon\to 0} A \in \mathbb{Q}[\nu, q, v]((\Lambda^{-1})).$$

In the dispersionless limit, the algebra $\Phi_-(\widetilde{\mathcal{A}},q)$ degenerates to the commutative algebra $\widetilde{\mathcal{A}}((\Lambda^{-1}))$, and the leading order in the commutator is the Poisson bracket

$$\{A_0,B_0\}=\lim_{\varepsilon\to 0}\varepsilon^{-1}[A,B]=(\Lambda\partial_\Lambda A_0)\partial B_0-\partial A_0(\Lambda\partial_\Lambda B_0).$$

It is not hard to write down explicit formulas for the Lax operator $\mathbb{L}$ of the equivariant Toda lattice and its conjugate $\bar{\mathbb{L}}$ in the dispersionless limit.

Define the (unsigned) Stirling numbers (of the first kind) $\begin{bmatrix}n\\k\end{bmatrix}$ by the generating function

$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\nu^k=\prod_{j=0}^{n-1}(\nu+j).$$

We have the recursion

$$\begin{bmatrix}n\\k\end{bmatrix}=(n-1)\begin{bmatrix}n-1\\k\end{bmatrix}+\begin{bmatrix}n-1\\k-1\end{bmatrix}. \tag{20}$$

Theorem 4.1. *We have*

$$\mathbb{L}_0=K_0+\nu\sum_{n=0}^{\infty}\Big(\frac{\nu}{\Lambda}\Big)^n\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}n\\k\end{bmatrix}\frac{\log(K_0/\Lambda)^{n-k+1}}{(n-k+1)!}\Big(\frac{K_0}{\Lambda}\Big)^{-n},\ \textit{and}$$
$$\bar{\mathbb{L}}_0=K_0-\nu\sum_{n=0}^{\infty}\Big(\frac{-\nu\Lambda}{q}\Big)^n\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}n\\k\end{bmatrix}\frac{\log(\Lambda K_0)^{n-k+1}}{(n-k+1)!}\Big(\frac{\Lambda K_0}{q}\Big)^{-n}.$$

Proof. Denote by L_0 the expression which we wish to prove equals $\mathbb{L}_0$. It is clear that $\tilde{\alpha}(L)=\Lambda$, hence it suffices to prove the equation

$$\{K_0,L_0\}=\nu\partial K_0,$$

which is the dispersionless limit of the equation $\varepsilon^{-1}[K,\mathbb{L}]=\nu\partial\mathbb{L}$.

Since $\{K_0,\log(K_0/\Lambda)\}=\partial K_0$, we have

$$\{K_0,L_0\}=\nu\partial K_0\sum_{n=0}^{\infty}\Big(\frac{\nu}{\Lambda}\Big)^n\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}n\\k\end{bmatrix}\frac{\log(K_0/\Lambda)^{n-k}}{(n-k)!}\Big(\frac{K_0}{\Lambda}\Big)^{-n}.$$

Since $\partial\log(K_0/\Lambda)=K_0^{-1}\partial K_0$, we have

$$\partial L_0=\partial K_0\bigg(1+\sum_{n=0}^{\infty}\Big(\frac{\nu}{\Lambda}\Big)^n\sum_{k=0}^{n}(-1)^{n-k}A_{n,k}\begin{bmatrix}n\\k\end{bmatrix}\Big(\frac{K_0}{\Lambda}\Big)^{-n-1}\bigg),$$

where

$$A_{n,k} = \frac{\log(K_0/\Lambda)^{n-k}}{(n-k)!} - \frac{n\log(K_0/\Lambda)^{n-k+1}}{(n-k+1)!}.$$

The equation $\{K_0, L_0\} = \nu\partial L_0$ follows from (20).

Define $e^{\circ j}(L_0)$ by induction: $e^{\circ 0}(L_0) = L_0$ and $e^{\circ(j+1)}(L_0) = e(e^{\circ j}(L_0))$. Then

$$\begin{aligned} e^{\circ j}(L_0) &= \delta_{j,0}\, K_0 \\ &\quad + \nu^{1-j} \sum_{n=0}^{\infty} \Big(\frac{\nu}{\Lambda}\Big)^n \sum_{k=0}^{n-j+1} (-1)^{n-k-j} \begin{bmatrix} n \\ k \end{bmatrix} \frac{\log(K_0/\Lambda)^{n-k-j+1}}{(n-k-j+1)!} \Big(\frac{K_0}{\Lambda}\Big)^{-n}. \end{aligned}$$

This formula is proved by induction on j, using the formulas $e(K_0) = 1$ and $e(\log(K_0/\Lambda)) = K_0^{-1}$.

There is an embedding of the differential algebra $\widetilde{\mathcal{A}}$ in the differential algebra

$$\widetilde{\mathcal{A}}\{u\}/(\partial q - q\partial u) \cong \mathbb{Q}_{\varepsilon,\nu}[q]\{u, v\},$$

given by mapping $\bar{v}$ to $v - \nu \mathsf{P} u$. In the dispersionless limit, this embedding maps $\bar{v}$ to $v - \nu u$. We will prove the formula for $\bar{\mathbb{L}}_0$ by working with Laurent series in this larger algebra.

The Laurent series $\bar{\mathbb{L}}_0$ is obtained from $\mathbb{L}_0$ by replacing v by $v - \nu u$, Λ by q/Λ, and ν by $-\nu$. Let $\widetilde{\mathbb{L}}_0$ be the result of substituting $v - \nu u$ for v in $\mathbb{L}_0$; it is given by the formula

$$\begin{aligned} \widetilde{\mathbb{L}}_0 &= \sum_{j=0}^{\infty} \frac{(-\nu u)^j}{j!} e^{\circ j}(L) \\ &= K_0 + \nu \sum_{n=0}^{\infty} \Big(\frac{\nu}{\Lambda}\Big)^n \sum_{j=0}^{n+1} \sum_{k=0}^{n-j+1} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} \frac{u^j \log(K_0/\Lambda)^{n-k-j+1}}{j!(n-k-j+1)!} \Big(\frac{K_0}{\Lambda}\Big)^{-n} \\ &= K_0 + \nu \sum_{n=0}^{\infty} \Big(\frac{\nu}{\Lambda}\Big)^n \sum_{k=0}^{n+1} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} \frac{\log(qK_0/\Lambda)^{n-k+1}}{(n-k+1)!} \Big(\frac{K_0}{\Lambda}\Big)^{-n}. \end{aligned}$$

We obtain $\bar{\mathbb{L}}_0$ on substituting $-\nu$ for ν and Λ for q/Λ. □

We may now prove the formulas (5a) and (5b) relating the dispersionless limit of the equivariant Toda lattice to the equivariant genus 0 Gromov-Witten potential of $\mathbb{CP}^1$.

In the genus 0 limit, the functions $q = \exp(u)$ and v on the large phase space become $\exp(\partial^2\mathcal{F}_0)$ and $\partial\partial_0\mathcal{F}_0$, and $\bar{v}$ becomes $v - \nu u$. The proof of Theorem 4.2 of [5] extends to the equivariant case, and shows that

$$\partial^n u = s_n + O(|s|^2 + |t|^2), \qquad \partial^n v = \delta_{1,n} + t_n + O(|s|^2 + |t|^2). \tag{21}$$

Hence, we may identify the large phase space with the space of formal jets in an affine space with coordinates u and v.

The following lemma shows that the vector field e lifts to the puncture vector field on the large phase space.

Lemma 4.1. *The puncture vector field*

$$e = \partial - \sum_{n=0}^{\infty} \left(s_{n+1} \frac{\partial}{\partial s_n} + t_{n+1} \frac{\partial}{\partial t_n} \right)$$

on the large phase space acts on elements of $\mathbb{Q}[\nu, q, v]$ *by the derivation* ∂_v.

Proof. Observe that the puncture vector field e commutes with $\partial 1$; this reflects the fact that $\mathbb{CP}^1$ is one-dimensional. The puncture (or string) equation says that

$$e(\mathcal{F}_0) = s_0 t_0 + \tfrac{1}{2}\nu t_0^2. \tag{22}$$

Applying the differential operators $\partial_0\partial$ and ∂^2 to this equation, we see that $e(v) = 1$ and $e(u) = 0$. □

Define generating functions $\pi_k(z) \in \mathbb{Q}[\nu, q, v]((z))$ by the formula

$$\mathbb{L}_0(z) = \sum_{n=-\infty}^{\infty} \frac{z^n}{[n]!} \mathbb{L}_0^n = \sum_{k=-\infty}^{\infty} \pi_k(z)\Lambda^k,$$

where $[n]!$ is the rational function

$$[n]! = \frac{\Gamma(\nu z + n + 1)}{\Gamma(\nu z + 1)} = (1 + z\nu)(2 + z\nu) \dots (n + z\nu).$$

It follows from the recursion (19) for the coefficients of $\mathbb{L}$ that

$$\pi_{-1}(z) = q\pi_1(z) + \nu\pi_0(z). \tag{23}$$

Lemma 4.2. $e(\pi_k(z)) = z\pi_k(z)$ *and* $\partial_q \pi_k(z) = z\pi_{k+1}(z)$.

Proof. We have

$$\begin{aligned}(\mathbb{L}_0-\nu)e(\mathbb{L}_0(z)) &= \sum_{n=-\infty}^{\infty}\frac{nz^n}{[n]!}\mathbb{L}_0^{n-1}(\mathbb{L}_0-\nu)e(\mathbb{L}_0) = \sum_{n=-\infty}^{\infty}\frac{nz^n}{[n]!}\mathbb{L}_0^n \\ &= \sum_{n=-\infty}^{\infty}\frac{(\nu z+n)z^n}{[n]!}\mathbb{L}_0^n - \sum_{n=-\infty}^{\infty}\frac{\nu z^{n+1}}{[n]!}\mathbb{L}_0^n = z(\mathbb{L}_0-\nu)\mathbb{L}_0(z).\end{aligned}$$

The equation for $e(\pi_k(z))$ follows on taking the coefficient of Λ^k. Since $\Lambda\partial_q K_0 = e(K_0)$, it follows from Theorem 4.1 that $\Lambda\partial_q\mathbb{L}_0(z) = e(\mathbb{L}_0(z))$. Taking the coefficient of Λ^{k+1}, we obtain the formula for $\partial_q\pi_k(z)$. □

Theorem 4.2. *The dispersionless limits of* (5a) *and* (5b) *hold.*

Proof. We will concentrate on the proof of (5a). The proof of the dispersionless limit of (5b) is the same, up to conjugation.

Let $\partial(z)$ be the generating function for vector fields

$$\partial(z) = \sum_{k=0}^{\infty} z^k\partial_k.$$

We must prove that

$$\partial(z)v = \sum_{n=1}^{\infty}\frac{z^{n-1}\partial p_0(n)}{[n]!}, \qquad \partial(z)u = \sum_{n=1}^{\infty}\frac{z^{n-1}\partial p_{-1}(n)}{[n]!}.$$

In terms of the generating functions

$$x = 1 + z\partial(z)\partial\mathcal{F}_0 - g_0(z), \qquad y = \nu + z\partial(z)\partial_0\mathcal{F}_0 - g_{-1}(z),$$

we wish to prove that $\partial x(z) = \partial y(z) = 0$. We will actually prove the stronger result, that $x(z) = y(z) = 0$: in other words, that

$$\sum_{k=0}^{\infty} z^k\partial_k\partial\mathcal{F}_0 = \sum_{n=1}^{\infty}\frac{z^{n-1}p_0(n)}{[n]!}, \qquad \sum_{k=0}^{\infty} z^k\partial_k\partial_0\mathcal{F}_0 = \sum_{n=1}^{\infty}\frac{z^{n-1}p_{-1}(n)}{[n]!}.$$

A theorem of Dijkgraaf and Witten [2] establishes that the Toda equation (2) holds in the dispersionless limit:

$$\partial_0^2\mathcal{F}_0 = q + \nu v.$$

Combining the topological recursion relations for equivariant Gromov-Witten invariants in genus 0 with Lemma 4.2, we see that

$$\begin{aligned}\partial x(z) &= z\big(x(z)\,\partial(v-\nu u)+y(z)\,\partial u\big)\\ \partial y(z) &= z\big(x(z)\,\partial q+y(z)\,\partial v\big).\end{aligned}$$

On the other hand, the string equation shows that $e(x(z)) = zx(z)$ and $e(y(z)) = zy(z)$.

Now apply the following principle (Proposition 4.1 of [5]):

> *A function f on the large phase space such that ∂f and $e(f)$ lie in $\mathbb{Q}[\nu]$ itself lies in $\mathbb{Q}[\nu]$.*

Arguing by induction, we see that the coefficients of z^k in $x(z)$ and $y(z)$ lie in $\mathbb{Q}[\nu]$; in other words, $x(z), y(z) \in \mathbb{Q}[\![\nu, z]\!]$. (In particular, we see that $\partial_k\partial\mathcal{F}_0$ and $\partial_k\partial_0\mathcal{F}_0$ lie in $\mathbb{Q}[\nu, q, v]$ for all $k \geq 0$.)

The proof is finished by observing that, by the divisor equation for Gromov-Witten invariants, the limits $\lim_{q\to 0} x(z)$ and $\lim_{q\to 0} y(z)$ are integrals over the degree 0 moduli space $\overline{\mathcal{M}}_{0,2}(\mathbb{CP}^1, 0)$; however, this moduli space is empty, hence $x(z) = y(z) = 0$. □

As mentioned in the introduction, the analogue of Theorem 4.2 is now known to hold in all genera (Okounkov and Pandharipande [9]).

§5. The Dressing Operator of the Toda Lattice

Let W be the universal **dressing operator** of the Toda lattice

$$W = 1 + \sum_{k=1}^{\infty} w_k \Lambda^{-k} \in \Phi_-(\mathcal{B}, q),$$

where $\mathcal{B}$ is the free differential algebra $\mathbb{Q}_\varepsilon\{q, w_k \mid k > 0\}/(q-\bar{q})$. The coefficients $w_k^* \in \mathcal{B}$ of

$$W^{-1} = 1 + \sum_{k=1}^{\infty} w_k^* \Lambda^{-k}$$

are characterized by the recursion obtained by extracting the coefficient of Λ^{-k} in the equation $WW^{-1} = 1$:

$$w_k^* = -w_k - \sum_{j=1}^{k-1} \big(\mathsf{E}^{(k-j)/2} w_j\big)\big(\mathsf{E}^{-j/2} w_{k-j}^*\big).$$

Lemma 5.1. *There is an embedding $\mathcal{A} \hookrightarrow \mathcal{B}$, characterized by the dressing equation $L = W\Lambda W^{-1}$, or equivalently, $LW = W\Lambda$.*

Proof. For all $k > 0$, $a_k + \varepsilon\nabla w_k$ lies in the differential ideal $(w_1, \ldots, w_{k-1})$. ☐

The conjugate Lax operator $\bar{L}$ is given by the dressing equation

$$\bar{L} = \overline{W}^{-1}(q\Lambda^{-1})\overline{W}.$$

Define evolutionary derivations $(\delta_n, \bar{\delta}_n \mid n > 0)$ of $\mathcal{B}$ by the formulas

$$\varepsilon\delta_n W + L_-^n W = \varepsilon\bar{\delta}_n W + \bar{L}_-^n W = 0. \tag{24}$$

Under the embedding $\mathcal{A} \hookrightarrow \mathcal{B}$, these derivations restrict to the flows of the Toda lattice on $\mathcal{A}$.

Let $\log(L) = W \log(\Lambda) W^{-1}$, where $\log(\Lambda)$ is a formal symbol for the operator $\varepsilon\partial$. Define ℓ to be the difference operator

$$\begin{aligned} \ell &= \log(\Lambda) - \log(L) = \varepsilon(\partial W)W^{-1} \\ &= \varepsilon\Bigg(\partial w_k + \sum_{j=1}^{k-1} \big(\mathsf{E}^{(k-j)/2}\partial w_j\big)\big(\mathsf{E}^{-j/2} w_{k-j}^*\big)\Bigg). \end{aligned}$$

The following is a result of Carlet, Dubrovin and Zhang [1]. (They work in the context of the Toda chain, so they assume that $a_1 = \bar{a}_1$ and $a_k = 0$, $k > 2$.)

Proposition 5.1. *The difference operator ℓ is an element of $\Phi_-(\mathcal{A}, q)$.*

Proof. Write

$$\ell = \sum_{k=1}^{\infty} b_k \Lambda^{-k} \in \Phi_-(\mathcal{B}, q).$$

We show that $b_k \in \mathcal{A}$ for all $k > 0$, by induction on k.

We have

$$\varepsilon\partial L = \varepsilon\partial(W\Lambda W^{-1}) = \varepsilon(\partial W)\Lambda W^{-1} - \varepsilon W\Lambda W^{-1}(\partial W)W^{-1} = [\ell, L],$$

hence for each $n > 0$, $\varepsilon\partial L^n = [\ell, L^n]$. Applying the linear map $\operatorname{res} : \Phi_-(\mathcal{A}, q) \to \mathcal{A}$, we obtain the equation

$$\nabla\Bigg([n]b_n + \sum_{k=1}^{n-1} [k](b_k p_k(n)) + \mathsf{P} p_0(n)\Bigg) = 0. \tag{25}$$

We see that $\alpha(\partial W) = 0$, and hence $\alpha(\ell) = 0$. Thus, the constant of integration in (25) vanishes, and we obtain the recursion

$$b_n = -\frac{1}{[n]}\left(\sum_{k=1}^{n-1}[k](b_k p_k(n)) + \mathsf{P}p_0(n)\right) \tag{26}$$

for the coefficients b_k, showing that they are elements of $\mathcal{A}$. □

§6. Fractional Powers of the Lax Operator

In this section, we study the fractional powers of the Lax operator L; this may be compared with the parallel construction for the KP hierarchy due to Khesin and Zakharevich [6]. The study of these fractional powers is closely related to the operator ℓ introduced in the last section.

Let s be a complex number. The fractional power L^s of the Lax operator L is defined by means of the dressing operator:

$$L^s = W\Lambda^s W^{-1} = \Lambda^s + \sum_{k=1}^{\infty} a_k(s)\Lambda^{s-k} \in \Phi_-(\mathcal{B}, q). \tag{27}$$

The coefficient $a_k(s)$ is given by the explicit formula

$$a_k(s) = \mathsf{E}^{-s/2}w_k + \sum_{j=1}^{k-1}\big(\mathsf{E}^{(k-j-s)/2}w_j\big)\big(\mathsf{E}^{(s-j)/2}w^*_{k-j}\big) + \mathsf{E}^{s/2}w^*_k.$$

In particular, $a_k(0) = 0$ and $a_k(1) = a_k$. Differentiating the definition (27) of L^s with respect to s and setting $s = 0$, we obtain the formula

$$\left.\frac{dL^s}{ds}\right|_{s=0} = -\ell, \tag{28}$$

showing that $a'_k(0) = -b_k$. The following proposition is proved by extending this differential equation to all values s.

Proposition 6.1. *The coefficient $a_{k,i}(s)$ in the expansion*

$$a_k(s) = \sum_{i=0}^{\infty} \varepsilon^i a_{k,i}(s)$$

is a polynomial in s of degree $i + 1$ with coefficients in the differential algebra

$$\mathcal{A} \otimes_{\mathbb{Q}_\varepsilon} \mathbb{Q} \cong \mathbb{Q}\{q, a_k \mid k > 0\}/(q - \bar{q}).$$

Proof. By its definition, the fractional power L^s satisfies the differential equation

$$\frac{dL^s}{ds} = -\tfrac{1}{2}\big(L^s\,\ell + \ell\,L^s\big).$$

Taking the coefficient of Λ^{s-k} on both sides, we obtain the differential equation

$$\frac{da_k(s)}{ds} = -\frac{1}{2}\sum_{j=1}^{k-1}\big(\mathsf{E}^{(s-j)/2}b_{k-j}\,\mathsf{E}^{(k-j)/2}a_j(s) + \mathsf{E}^{(j-s)/2}b_{k-j}\,\mathsf{E}^{(j-k)/2}a_j(s)\big),$$

where we interpret $a_0(s)$ as 1. By an application of Proposition 5.1, the result follows. □

§7. Perturbation Theory for ℓ

Let $\Omega(\mathcal{A})$ be the vector space of Kähler differentials of the commutative $\mathbb{Q}_\varepsilon$-algebra $\mathcal{A}$; this is a free module over $\mathcal{A}$ with basis $\{dq, da_k, d\bar{a}_k \mid k > 0\}$. The differential $d : \mathcal{A} \to \Omega(\mathcal{A})$ extends to a morphism

$$d : \Phi_-(\mathcal{A}, q) \to \Phi_-(\mathcal{A}, q) \otimes_{\mathcal{A}} \Omega(\mathcal{A}).$$

We now calculate the differentials dL^s and $d\ell$ in terms of

$$dL = \sum_{k=1}^{\infty} da_k\,\Lambda^{-k+1}.$$

A basic formula of perturbation theory (Kumar [7]) says that for $f(z)$ an analytic function of z,

$$df(L) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!}\,\mathrm{ad}(L)^k(f^{(k+1)}(L)dL).$$

For $f(z) = z^s$, this becomes

$$dL^s = \sum_{k=0}^{\infty}(-1)^k\tbinom{s}{k+1}\,\mathrm{ad}(L)^k(L^{s-k-1}dL). \tag{29}$$

For completeness, we will now prove this formula directly in the context in which we need it.

For s a natural number n, the right-hand side of (29) is a finite sum, and the formula is then easily proved by induction on n: we have

$$\begin{aligned} d(L^{n+1}) = dL^n \cdot L + L^n \cdot dL &= \sum_{k=0}^{n-1} (-1)^k \tbinom{n}{k+1} \operatorname{ad}(L)^k (L^{n-k-1} dL) \cdot L + L^n \cdot dL \\ &= \sum_{k=0}^{n-1} (-1)^k \tbinom{n}{k+1} \big(\operatorname{ad}(L)^k (L^{n-k} dL) - \operatorname{ad}(L)^{k+1} (L^{n-k-1} dL) \big) + L^n \cdot dL \\ &= \sum_{k=0}^{n} (-1)^k \big(\tbinom{n}{k} + \tbinom{n}{k+1} \big) \operatorname{ad}(L)^k (L^{n-k} dL) \\ &= \sum_{k=0}^{n} (-1)^k \tbinom{n+1}{k+1} \operatorname{ad}(L)^k (L^{n-k} dL). \end{aligned}$$

By analytic continuation, (29) holds for all values of s. Indeed, the right-hand side is convergent in the ε-adic topology, since the operation $\operatorname{ad}(L)$ may be split into two terms: $\operatorname{ad}(\Lambda + a_1) = O(\varepsilon)$, and

$$\sum_{k=2}^{\infty} \operatorname{ad}(a_k \Lambda^{-k+1}) = O(\Lambda^{-1}).$$

It only remains to observe that by Proposition 6.1, the coefficient of ε^i in $da_{k,i}(s)$ is polynomial in s.

It is now straightforward to calculate $d\ell$: taking the derivative of (29) with respect to s and setting $s = 0$, we see that

$$d\ell = -\sum_{k=0}^{\infty} \tfrac{1}{k+1} \operatorname{ad}(L)^k (L^{-k-1}\, dL). \tag{30}$$

Theorem 7.1. *The constraint* (15) *defining the equivariant Toda lattice is equivalent to the identity*

$$K = L + \nu\ell - \sum_{k=1}^{\infty} \frac{z_k}{k} L^{-k}. \tag{31}$$

The vanishing of the constants z_k is equivalent to the constraint

$$(\delta_1 - \bar{\delta}_1) W = \nu \partial W, \tag{32}$$

or equivalently, the equation $(\delta_1 - \bar{\delta}_1) = \nu\partial$ on the differential algebra $\mathcal{B}$.

Proof. Written in terms of ℓ, (13) becomes

$$[K - \nu\ell, L] = 0.$$

This is equivalent to the statement that

$$K - \nu\ell \in \mathbb{Q}_{\varepsilon,\nu}(\!(L^{-1})\!).$$

It is not hard to see that for some constants ζ_k,

$$K - \nu\ell - L = \sum_{k=1}^{\infty} \zeta_k L^{-k} \in \mathbb{Q}_{\varepsilon,\nu}[\![L^{-1}]\!]; \tag{33}$$

the constant term vanishes since, by definition, $\operatorname{res}(K)$ and $\operatorname{res}(L)$ equal v, while $\operatorname{res}(\ell) = 0$.

It remains to identify the constants ζ_k. If δ is an evolutionary derivation of the differential algebra $\widetilde{\mathcal{A}}$, (30) implies that

$$\delta\ell = -\sum_{k=0}^{\infty} \tfrac{1}{k+1} \operatorname{ad}(L)^k (L^{-k-1}\delta L).$$

In particular, since L commutes with $e(L)$, we see that $e(\ell) = -L^{-1}e(L)$. Likewise, $e(L^{-k}) = -kL^{-k-1}$. Applying the derivation e to both sides of (33), we see that

$$1 = e(K) = e(L)\Big(1 - \nu L^{-1} - \sum_{k=1}^{\infty} k\zeta_k L^{-k-1}\Big).$$

It follows from Theorem 3.2 that $\zeta_k = -z_k/k$.

We have

$$\begin{aligned}(K - L - \nu\ell)W &= (L_+ + \bar{L}_-)W - LW - \varepsilon\nu\partial W \\ &= -L_-W + \bar{L}_-W - \varepsilon\nu\partial W = \varepsilon(\delta_1 - \bar{\delta}_1 - \nu\partial)W.\end{aligned}$$

Thus, the vanishing of the constants z_k in (31) is equivalent to the constraint (32). ☐

Theorem 7.1 implies that the equivariant Gromov-Witten invariants of $\mathbb{CP}^1$ are described by the equivariant Toda lattice with $z_k = 0$, $k > 0$. By the work of Okounkov and Pandharipande [9], the equivariant Gromov-Witten invariants of $\mathbb{CP}^1$ are associated with a τ-function of the Toda lattice which satisfies $(\delta_1 - \bar{\delta}_1)\tau = \nu\partial\tau$. The dressing operator W corresponding to this τ-function is given by the formula

$$W = \tau^{-1} \exp\Big(-\sum_{n=1}^{\infty} \frac{\delta_n}{n\Lambda^n}\Big)\tau;$$

it follows that W satisfies the equation $(\delta_1 - \bar{\delta}_1)W = \nu\partial W$. Thus, the Lax operator $\mathbb{L}$ defined by the recursion (19) governs the equivariant Gromov-Witten invariants of $\mathbb{CP}^1$.

§8. Hamiltonian Structure

In this section, we use Theorem 7.1 to show that the equivariant Toda lattice has a Hamiltonian structure.

Denote by $\mathcal{R}$ the quotient $\widetilde{\mathcal{A}}/\partial\widetilde{\mathcal{A}}$, and denote by $f \mapsto \int f\,dx$ the quotient map from $\widetilde{\mathcal{A}}$ to $\mathcal{R}$. The idea which this notation is intended to represent is that an element of $\widetilde{\mathcal{A}}$ is a density f, whose associated functional $\int f\,dx$ is obtained by integration with respect to the space variable x. In particular, $\int f\,dx$ vanishes on densities $f = \nabla g$.

Denote by Res the trace on $\Phi_-(\widetilde{\mathcal{A}}, q)$ with values in $\mathcal{R}$ given by the formula

$$\operatorname{Res}(f) = \int \operatorname{res}(f)\,dx.$$

Clearly, this map vanishes on total derivatives; it also vanishes on commutators, by the formula

$$\operatorname{Res}\Big[\sum_i a_i\Lambda^i, \sum_j b_j\Lambda^j\Big] = \nabla \sum_k [k](a_k b_{-k}).$$

There is a unique linear map

$$\operatorname{Res} : \Phi_-(\widetilde{\mathcal{A}}, q) \otimes_{\widetilde{\mathcal{A}}} \Omega(\widetilde{\mathcal{A}}) \to \Omega(\widetilde{\mathcal{A}})/\partial\Omega(\widetilde{\mathcal{A}})$$

such that $d\operatorname{Res}(A) = \operatorname{Res}(dA)$.

Associated to the equivariant Toda lattice, we have the basic sequence of functionals

$$h_n = \tfrac{1}{n+1}\operatorname{Res}(L^{n+1}), \quad n \geq 0,$$

with differentials $dh_n = \operatorname{Res}(L^n dL)$. In working with h_n, the following lemma is convenient.

Lemma 8.1.

$$p_0(n+1) = \sum_{k=0}^{n} [k+1]\big(a_{k+1}\, p_k(n)\big)$$

Proof. Applying the operator res to the equations $L^{n+1} = L \cdot L^n$ and $L^{n+1} = L^n \cdot L$, we see that

$$p_0(n+1) = \mathsf{E}^{1/2} p_{-1}(n) + \sum_{k=0}^{\infty} \mathsf{E}^{-k/2}\big(a_{k+1}\, p_k(n)\big),$$
$$p_0(n+1) = \mathsf{E}^{-1/2} p_{-1}(n) + \sum_{k=0}^{\infty} \mathsf{E}^{k/2}\big(a_{k+1}\, p_k(n)\big).$$

Taking $\mathsf{E}^{1/2}$ times the second of these equations minus $\mathsf{E}^{-1/2}$ times the first, we see that

$$\nabla p_0(n+1) = \nabla \sum_{k=0}^{n} [k+1]\big(a_{k+1}\, p_k(n)\big),$$

and hence, that

$$p_0(n+1) = \sum_{k=0}^{n} [k+1]\big(a_{k+1}\, p_k(n)\big) + \alpha(p_0(n+1)).$$

This proves the lemma, since $\alpha(p_0(n+1)) = 0$. □

Corollary 8.1.

$$h_n = \sum_{k=0}^{n} \frac{k+1}{n+1} \int \big(a_{k+1}\, p_k(n)\big)\, dx.$$

For example, using the formulas (16) and (17) for a_2 and a_3, we see that

$$\begin{aligned} h_0 &= \textstyle\int v\, dx, \\ h_1 &= \textstyle\int (\frac{1}{2}v^2 + a_2) dx = \int (\frac{1}{2}v^2 + q + \nu v + z_1) dx, \\ h_2 &= \textstyle\int (\frac{1}{3} v p_0(2) + \frac{2}{3}(a_2 p_1(2)) + a_3) dx \\ &= \textstyle\int (\frac{1}{3}v^3 + v[2]q + \nu\big(\frac{1}{2}v^2 + q + \frac{1}{2}v[2]\mathsf{P}v\big) + \nu^2 v + z_1 v + \frac{1}{2}z_2) dx. \end{aligned}$$

Proposition 8.1. *We have* $\operatorname{Res}(L^n\, dK) = dH_n$, *where*

$$H_n = h_n - \nu h_{n-1} + \sum_{k=1}^{n-1} z_k h_{n-k-1}.$$

Proof. From (31), (29) and (30), we see that

$$\begin{aligned} dK &= dL + \nu d\ell - \sum_{j=1}^{\infty} \frac{z_j}{j} dL^{-j} \\ &= dL + \sum_{k=0}^{\infty} (k+1)^{-1} \operatorname{ad}(L)^k \Big(\Big(-\nu + \sum_{j=1}^{\infty} \tbinom{j+k}{k} z_j L^{-j} \Big) L^{-k-1} dL \Big). \end{aligned}$$

Multiplying by L^n and applying Res, all of the terms with $k > 0$ drop out, and we obtain

$$\operatorname{Res}(L^n\,dK) = \operatorname{Res}\Big(\Big(L - \nu + \sum_{j=1}^{\infty} z_j L^{-j}\Big) L^{n-1} dL\Big),$$

which equals dH_n. □

Let δ_v and δ_u be the variational derivatives with respect to v and $u = \log(q)$.

Corollary 8.2. *We have* $\delta_v H_n = p_0(n)$, $\delta_u H_n = q p_1(n)$, $\delta_v \bar{H}_n = \bar{p}_0(n)$ *and* $\delta_u \bar{H}_n = q\bar{p}_1(n) - \nu \mathsf{P}\bar{p}_0(n)$.

Proof. The formulas for $\delta_v H_n$ and $\delta_u H_n$ follow since $dK = dv + q\,du\,\Lambda^{-1}$. The formulas for $\delta_v \bar{H}_n$ and $\delta_u \bar{H}_n$ now follow by taking conjugates, bearing in mind that $\bar{v} = v - \nu \mathsf{P} u$. □

For example, we have $H_0 = h_0 = \int v\,dx$,

$$H_1 = h_1 - \nu h_0 = \textstyle\int (\frac{1}{2}v^2 + q + z_1)dx, \quad \text{and}$$
$$H_2 = h_2 - \nu h_1 + z_1 h_0 = \textstyle\int (\frac{1}{3}v^3 + v[2]q + \frac{1}{2}\nu v[2]\mathsf{P}v + 2z_1 v - \nu z_1 + \frac{1}{2}z_2)dx.$$

It is now easy to show that the equivariant Toda lattice is Hamiltonian. Applying res to the equation $[K, L^n] = \nu \partial L^n$, we see that

$$\nabla p_{-1}(n) = \nabla(q p_1(n)) + \nu \partial p_0(n).$$

It follows that $\delta_n v = \nabla p_{-1}(n) = \nabla(q p_1(n)) + \nu \partial p_0(n)$. In conjunction with the formula $\delta_n u = \nabla p_0(n)$, we conclude that $\delta_n \left[\begin{smallmatrix} v \\ u \end{smallmatrix}\right] = \mathcal{H}\left[\begin{smallmatrix} \delta_v H_n \\ \delta_u H_n \end{smallmatrix}\right]$, where $\mathcal{H}$ is the Hamiltonian operator

$$\mathcal{H} = \begin{bmatrix} \nu\partial & \nabla \\ \nabla & 0 \end{bmatrix}.$$

Since $\bar{\delta}_n v = \nabla(q\bar{p}_1(n))$ and $\bar{\delta}_n u = \nabla \bar{p}_0(n)$, we also conclude that $\bar{\delta}_n \left[\begin{smallmatrix} v \\ u \end{smallmatrix}\right] = \mathcal{H}\left[\begin{smallmatrix} \delta_v \bar{H}_n \\ \delta_u \bar{H}_n \end{smallmatrix}\right]$. In other words, the equivariant Toda lattice is Hamiltonian with respect to the Hamiltonian structure

$$\{v(x), v(y)\} = \nu \partial \delta(x-y), \quad \{v(x), u(y)\} = \nabla_x \delta(x-y), \quad \{u(x), u(y)\} = 0.$$

It was proved by Getzler [5] and Zhang [14] that the flows $\partial_{k,P}$ in the Toda conjecture of Eguchi and Yang are Hamiltonian. Taking the limit $\nu \to 0$, we obtain the explicit formulas for these Hamiltonians due to Carlet, Dubrovin and Zhang [1]. By (6), the descendent flow $\partial_{k,P}$ has Hamiltonian

$$\lim_{\nu\to 0}\left(\frac{H_{k+1}-\bar{H}_{k+1}}{\nu(k+1)!}-\frac{c_k(H_k+\bar{H}_k)}{k!}\right).$$

Let ℓ_0 equal the limit as $\nu\to 0$ of ℓ. Since $L=K-\nu\ell=K-\nu\ell_0+O(\nu^2)$ and

$$\bar{L}=\bar{K}+\nu\bar{\ell}=K+\nu(\ell_0-\mathsf{P}u)+O(\nu^2),$$

we have

$$\begin{aligned}\nu^{-1}\big(H_{k+1}-\bar{H}_{k+1}\big)&=\nu^{-1}\tfrac{1}{k+2}\operatorname{Res}(L^{k+2}-\bar{L}^{k+2})-\tfrac{1}{k+1}\operatorname{Res}(L^{k+1}-\bar{L}^{k+1})\\&=\operatorname{Res}(K^{k+1}(\mathsf{P}u-2\ell_0)).\end{aligned}$$

It follows that $\partial_{k,P}$ has Hamiltonian $\frac{1}{(k+1)!}\operatorname{Res}(K^{k+1}(\mathsf{P}u-2\ell_0-2c_k))$.

Appendix. Another Formulation of the Equivariant Toda Conjecture

In this appendix, we prove that (5a) is equivalent to

$$\delta_n=n\sum_{k=1}^{n}\nu^{k-1}\begin{bmatrix}n\\k\end{bmatrix}\partial_{n-k}.$$

For example, $\frac{1}{2}\delta_2=\partial_1+\nu\partial_0$ and $\frac{1}{6}\delta_3=\partial_2+3\nu\partial_1+2\nu^2\partial_0$. A similar proof, which we omit, shows that (5b) is equivalent to

$$\bar{\delta}_n=n\sum_{k=1}^{n}(-\nu)^{k-1}\begin{bmatrix}n\\k\end{bmatrix}\bar{\partial}_{n-k}.$$

Equation (5a) may be restated as saying that

$$\partial_k=\sum_{n=1}^{k+1}(-\nu)^{k-n+1}h_{k-n+1}\big(1,\tfrac{1}{2},\dots,\tfrac{1}{n}\big)\frac{\delta_n}{n!},$$

where h_ℓ is the complete symmetric polynomial of degree ℓ. We wish to prove that

$$\frac{\delta_n}{n!}=\sum_{\ell=0}^{n-1}\nu^{n-\ell-1}e_{n-\ell-1}\big(1,\tfrac{1}{2},\dots,\tfrac{1}{n-1}\big)\partial_\ell,$$

where e_ℓ is the elementary symmetric polynomial of degree ℓ. In other words, we wish to prove that

$$\sum_{\ell=0}^{n-1}\nu^{n-\ell-1}(-\nu)^{\ell-m+1}e_{n-\ell-1}\big(1,\tfrac{1}{2},\dots,\tfrac{1}{n-1}\big)h_{\ell-m+1}\big(1,\tfrac{1}{2},\dots,\tfrac{1}{m}\big)=\delta_{n,m}.$$

This is clearly true if $n \le m$; thus, we have only to prove that the left-hand side vanishes when $n > m$. In this case, it equals ν^{n-m} times the coefficient of ν^{n-m} in the generating function

$$\prod_{j=1}^{n-1}(1+j\nu)\cdot\prod_{j=1}^{m}(1+j\nu)^{-1} = \prod_{j=m+1}^{n-1}(1+j\nu),$$

which is a polynomial of degree $n-m-1$; hence, the coefficient in question vanishes.

Acknowledgements

In writing this paper, I profited greatly from discussions with T. Eguchi and R. Pandharipande (at the Institute for Theoretical Physics of the University of California, Santa Barbara) and with B. Dubrovin and Y. Zhang (at the Institute for Advanced Study, Princeton, and the Max-Planck-Institut für Mathematik, Bonn). I thank these institutes for their hospitality and support.

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