Publ. RIMS, Kyoto Univ.
40 (2004), 537–564

Relations for Multiple Zeta Values and Mellin Transforms of Multiple Polylogarithms

By

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Abstract

In this paper a relationship between the Ohno relation for multiple zeta values and multiple polylogarithms are discussed. First we introduce generating functions for the Ohno relation, and investigate their properties. We show that there exists a subfamily of the Ohno relation which recovers algebraically its totality. This is proved through analysis of Mellin transform of multiple polylogarithms. Furthermore, this subfamily is shown to be converted to the Landen connection formula for multiple polylogarithms by inverse Mellin transform.

§1. Introduction

§1.1. Definitions and examples

In this paper, we will consider the relationship between the Ohno relation for multiple zeta values (MZVs, for short) and the Landen connection formula for multiple polylogarithms (MPLs, for short) via Mellin transform and inverse Mellin transform.

Definition 1. For positive integers $k_1, \dots, k_n$ and $|z| < 1$, MPLs are defined by

$$\mathrm{Li}_{k_1,k_2,\dots,k_n}(z) := \sum_{m_1>m_2>\cdots>m_n>0} \frac{z^{m_1}}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}} \tag{1.1}$$

Communicated by A. Tamagawa. Received January 31, 2003. Revised July 24, 2003.
2000 Mathematics Subject Classification(s): Primary 11M06, 40B05; Secondary 16W25, 16W30, 16W50.
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and for the null sequence $\emptyset$, $\mathrm{Li}_{\emptyset}(z) := 1$. If $k_1 \geq 2$, MPLs also converge at $z = 1$ and define MZVs

$$\zeta(k_1, k_2, \ldots, k_n) := \sum_{m_1 > m_2 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}}, \tag{1.2}$$

and similarly $\zeta(\emptyset) := 1$. The weight and the depth of $\zeta(k_1, \ldots, k_n)$ are defined to be $k_1 + \cdots + k_n$ and n, respectively.

Through consideration on the dilogarithm $\mathrm{Li}_2(z)$ as an example, we explain an essential aspect of the relationship that will be considered in our papers.

The sum formula [G] for MZVs of depth 2

$$\zeta(3 + l) = \sum_{\substack{c_1 + c_2 = l \\ c_1, c_2 \geq 0}} \zeta(2 + c_1, 1 + c_2) \qquad (l \in \mathbb{Z}_{\geq 0})$$

is equivalent to the generating functional expression

$$\sum_{l=0}^{\infty} \zeta(3 + l)\, \lambda^l = \sum_{l=0}^{\infty} \left\{ \sum_{\substack{c_1 + c_2 = l \\ c_1, c_2 \geq 0}} \zeta(2 + c_1, 1 + c_2) \right\} \lambda^l.$$

Noting that, for a positive integer n,

$$\frac{1}{n - \lambda} = \sum_{l=0}^{\infty} \frac{\lambda^l}{n^{l+1}} \qquad \text{for} \quad |\lambda| < 1,$$

we see that the both sides in the above are meromorphic functions in λ,

$$\sum_{n=1}^{\infty} \frac{1}{n^2 (n - \lambda)} = \sum_{n_1 > n_2 > 0} \frac{1}{n_1 (n_1 - \lambda)(n_2 - \lambda)}.$$

Applying "inverse Mellin transform"

$$\widetilde{M}\left[f(\lambda)\right](z) = \frac{1}{2\pi\sqrt{-1}} \int_C f(\lambda) z^{\lambda}\, d\lambda \qquad (0 < z < 1)$$

to the left hand side (for details see Section 4), we have

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2} = \mathrm{Li}_2(z).$$

Note that

$$\begin{aligned}&\sum_{n_1>n_2>0}\frac{1}{n_1(n_1-\lambda)(n_2-\lambda)}\\&=\sum_{n_1>n_2>0}\frac{1}{n_1(n_1-n_2)(n_2-\lambda)}+\sum_{n_1>n_2>0}\frac{1}{n_1(n_2-n_1)(n_1-\lambda)}.\end{aligned}$$

Applying inverse Mellin transform to each term, we have

$$\begin{aligned}\sum_{n_1>n_2>0}\frac{z^{n_2}}{n_1(n_1-n_2)}&=-\mathrm{Li}_2\left(\frac{z}{z-1}\right),\\ \sum_{n_1>n_2>0}\frac{z^{n_1}}{n_1(n_2-n_1)}&=-\mathrm{Li}_{11}\left(\frac{z}{z-1}\right).\end{aligned}$$

Consequently we obtain the next functional equation for the dilogarithm:

$$\mathrm{Li}_2(z)=-\mathrm{Li}_2\left(\frac{z}{z-1}\right)-\mathrm{Li}_{11}\left(\frac{z}{z-1}\right), \tag{1.3}$$

which is known as the Landen connection formula for the dilogarithm [L]. This can be viewed as the connection formula for the dilogarithm between 1 and ∞.

§1.2. Main results and organization

Now we explain the Ohno relation [O], which is a generalization of the sum formula.

Any index $\boldsymbol{k}=(k_1,\dots,k_n)\in\mathbb{Z}^n_{\geq 1}$, $k_1\geq 2$ can be written uniquely as

$$\boldsymbol{k}=(a_1+1,\underbrace{1,\dots,1}_{b_1-1},\dots,a_s+1,\underbrace{1,\dots,1}_{b_s-1}), \tag{1.4}$$

with $s\in\mathbb{Z}_{\geq 1}$ and $a_i,b_i\in\mathbb{Z}_{\geq 1}$ $(i=1,\dots,s)$. The dual index $\boldsymbol{k}'=(k'_1,\dots,k'_{n'})$ of $\boldsymbol{k}$ is defined by

$$\boldsymbol{k}'=(b_s+1,\underbrace{1,\dots,1}_{a_s-1},\dots,b_1+1,\underbrace{1,\dots,1}_{a_1-1}), \tag{1.5}$$

and the dual of $\emptyset$ is itself.

Ohno relation. *Let $\boldsymbol{k}=(k_1,\dots,k_n)$ be any index and $\boldsymbol{k}'$ be its dual. For all $l\in\mathbb{Z}_{\geq 0}$, we have the following homogeneous (w.r.t. weight) relation,*

$$\sum_{\substack{c_1+\cdots+c_n=l\\ c_j\geq 0}}\zeta(k_1+c_1,\dots,k_n+c_n)=\sum_{\substack{c'_1+\cdots+c'_{n'}=l\\ c'_j\geq 0}}\zeta(k'_1+c'_1,\dots,k'_{n'}+c'_{n'}). \tag{1.6}$$

In particular, this contains such relations as the Hoffman relation [H1] $(l = 1)$, *the duality formula* [Z] $(l = 0)$ *and the sum formula* $(n = 1)$.

We introduce the generating functions of the both sides of the Ohno relation as follows:

$$f((a_i, b_i)_{i=1}^s; \lambda) := \sum_{l=0}^{\infty} \left\{ \sum_{\substack{c_1+\cdots+c_n=l \\ c_j \geq 0}} \zeta(k_1 + c_1, \ldots, k_n + c_n) \right\} \lambda^l, \tag{1.7}$$

$$g((a_i, b_i)_{i=1}^s; \lambda) := \sum_{l=0}^{\infty} \left\{ \sum_{\substack{c'_1+\cdots+c'_{n'}=l \\ c'_j \geq 0}} \zeta(k'_1 + c'_1, \ldots, k'_{n'} + c'_{n'}) \right\} \lambda^l. \tag{1.8}$$

The Ohno relation reads

$$f((a_i, b_i)_{i=1}^s; \lambda) = g((a_i, b_i)_{i=1}^s; \lambda). \tag{1.9}$$

We can show that f's and g's satisfy the same difference relations which play a fundamental role in our theory (Proposition 1, Section 2). Now we define the functions $F(\boldsymbol{k}; \lambda)$ and $G(\boldsymbol{k}; \lambda)$ by

$$F(\boldsymbol{k}; \lambda) := \sum_{\delta_i=0,1} (-\lambda)^{n-1-|\delta|} f((k_1, 1) \cup (k_i - \delta_i, 1)_{i=2}^n; \lambda),$$
$$G(\boldsymbol{k}; \lambda) := \sum_{\delta_i=0,1} (-\lambda)^{n-1-|\delta|} g((k_1, 1) \cup (k_i - \delta_i, 1)_{i=2}^n; \lambda).$$

Under the Ohno relation, we have

$$F(\boldsymbol{k}; \lambda) = G(\boldsymbol{k}; \lambda), \tag{1.10}$$

which we call the reduced Ohno relation. We can easily see that

$$F(\boldsymbol{k}; \lambda) = \sum_{m_1 > m_2 > \cdots > m_n > 0} \frac{1}{m_1^{k_1}(m_1 - \lambda) m_2^{k_2} \cdots m_n^{k_n}},$$

so that inverse Mellin transform of $F(\boldsymbol{k}; \lambda)$ is the multiple polylogarithm $\mathrm{Li}_{\boldsymbol{k}}(z)$. This fact gives us strong motivation to introduce these functions. The main theorem of this paper (Theorem 1) says that *the generating functions f's and g's are represented as MZVs-linear combinations of F's and G's, respectively. In other words, the reduced Ohno relation* (1.10) *recovers the totality of the Ohno relation* (1.9) (Section 3). This theorem is proved by virtue of the differential equations satisfied by MPLs and Mellin transform (Section 4 and 5).

Furthermore via inverse Mellin transform *the reduced Ohno relation is converted to the Landen connection formula for MPLs*:

$$\mathrm{Li}_{k_1,\dots,k_n}(z) = (-1)^n \sum_{\substack{c_1,\dots,c_n \\ |c_j|=k_j}} \mathrm{Li}_{c_1\cdots c_n}\left(\frac{z}{z-1}\right), \tag{1.11}$$

where c_i runs all compositions of k_i and the product of c_j's is given by concatenation. This can be viewed as the connection formula between 1 and ∞ for MPLs (Section 6). In Section 7 we give the proof of Proposition 1 and another proof of the Ohno relation.

§2. The Generating Functions and Their Properties

§2.1. Compositions

By a composition of a positive integer n, we mean an ordered sequence $\boldsymbol{c} = (c_1,\dots,c_l)$ of positive integers of which the sum is equal to n, and the composition of 0 is defined to be $\emptyset$. The "weight" $|\boldsymbol{c}|$ and the "length" $\mathrm{len}(\boldsymbol{c})$ of $\boldsymbol{c}$ are, by definition, n and l respectively. We allow 0's to appear in the middle elements of $\boldsymbol{c}$ and identify such compositions and normal compositions by removing 0's successively, i.e. we regard $(\dots,\underline{c_{i-1},0,c_{i+1},}\dots)$ to be the same composition as $(\dots,c_{i-1}+c_{i+1},\dots)$. For example,

$$(\underline{3,0,2},0,4) = (\underline{5,0,4}) = (9), \quad \text{or} \quad (3,0,\underline{2,0,4}) = (\underline{3,0,6}) = (9),$$
$$(3,0,2,1,\underline{0,0,4}) = (\underline{3,0,2},1,4) = (5,1,4).$$

We remark that the resulting composition dose not depend on the procedure of the identification. For compositions $\boldsymbol{c}$ and $\boldsymbol{c}'$ we define the partial order $\boldsymbol{c} \succ \boldsymbol{c}'$ if $\boldsymbol{c}'$ is obtained by decreasing some elements of $\boldsymbol{c}$. For example, $(5,1,4) \succ (4,1,3)$ and $(5,1,4) \succ (9) = (5,0,4)$. Between compositions of even length and compositions whose first element is greater than 1, we define the 1-1 correspondence κ as follows:

$$\kappa((a_i,b_i)_{i=1}^s) = (a_1+1,\underbrace{1,\dots,1}_{b_1-1},\dots,a_s+1,\underbrace{1,\dots,1}_{b_s-1}). \tag{2.1}$$

§2.2. Generating functions for the Ohno relation

Definition 2. For any composition $(a_i,b_i)_{i=1}^s = \kappa^{-1}(k_1,\dots,k_n)$, we set the generating functions of MZVs as

$$f((a_i,b_i)_{i=1}^s;\lambda) := \sum_{l=0}^{\infty}\left\{\sum_{\substack{c_1+\cdots+c_n=l\\ c_j\geq 0}} \zeta(k_1+c_1,\ldots,k_n+c_n)\right\}\lambda^l, \tag{2.2}$$

$$\begin{aligned} g((a_i,b_i)_{i=1}^s;\lambda) &:= \sum_{l=0}^{\infty}\left\{\sum_{\substack{c'_1+\cdots+c'_{n'}=l\\ c'_j\geq 0}} \zeta(k'_1+c'_1,\ldots,k'_{n'}+c'_{n'})\right\}\lambda^l \\ &= f((b_i,a_i)_{i=s}^1;\lambda), \end{aligned} \tag{2.3}$$

where $(k'_1,\ldots,k'_{n'})$ is the dual of $(k_1,\ldots,k_n)$. For convenience $f((a_i,b_i)_{i=1}^s;\lambda) := 0$ if a_1 or $b_s=0$. We set the weight of $f((a_i,b_i)_{i=1}^s;\lambda)$ and $g((a_i,b_i)_{i=1}^s;\lambda)$ to be $|(a_i,b_i)_{i=1}^s|$.

This power series absolutely converges for $|\lambda|<1$. The Ohno relation reads

$$f((a_i,b_i)_{i=1}^s;\lambda) = g((a_i,b_i)_{i=1}^s;\lambda), \tag{2.4}$$

for any compositions $(a_i,b_i)_{i=1}^s$.

Noting that for a positive integer n,

$$\frac{1}{n-\lambda} = \sum_{l=0}^{\infty}\frac{\lambda^l}{n^{l+1}} \qquad \text{for} \quad |\lambda|<1,$$

one can easily see that

(2.5)

$$f((a_i,b_i)_{i=1}^s;\lambda) = \sum_{m_1>\cdots>m_{B_s}>0}\prod_{i=1}^{s}\frac{1}{m_{B_{i-1}+1}^{a_i}\underbrace{(m_{B_{i-1}+1}-\lambda)\cdots(m_{B_i}-\lambda)}_{b_i}},$$

where $B_0=0$ and $B_i=b_1+\cdots+b_i$ for $i\geq 1$.

§2.3. Properties of the generating functions

The generating functions satisfy the following difference equations:

Proposition 1. *We set $\lambda' := \lambda-1$ and $I := \{(0,0),(1,0),(0,1)\}$, then for any composition $(a_i,b_i)_{i=1}^s$ the generating function f satisfies the following* *relations.*

$$\sum(-\lambda)^{s-|\delta|-|\epsilon|}f((a_i-\delta_i,b_i-\epsilon_i)_{i=1}^s;\lambda) = \sum{}'(-\lambda')^{s-|\delta'|-|\epsilon'|}f((a_i-\delta'_i,b_i-\epsilon'_{i+1})_{i=1}^s;\lambda'). \tag{2.6}$$

Here the sum $\sum$ *is taken over* $\{\delta_i,\epsilon_i\}\in I$, *the sum* $\sum'$ *taken over* $\delta'_1,\epsilon'_{m+1}\in\{0,1\}$ *and* $\{\delta'_i,\epsilon'_i\}\in I$ *for* $i=2,\dots,m$, *and* $|\delta^{(\prime)}|$ *(resp.* $|\epsilon^{(\prime)}|$*) is the sum of all* $\delta_i^{(\prime)}$ *(resp.* $\epsilon_i^{(\prime)}$*). The generating function* g *also satisfies the same relations. We define the weight of* λ *and* λ' *to be* -1 *and this relation is homogeneous of weight* $|(a_i,b_i)_{i=1}^s|-s$.

The proposition will play a crucial role in our theory. The proof is so long that it will be postponed until Section 7.

The generating functions are analytically continued to meromorphic functions with simple poles at positive integers.

Proposition 2. *The generating function* $f((a_i,b_i)_{i=1}^s;\lambda)$ *can be expanded to a partial fraction*

$$f((a_i,b_i)_{i=1}^s;\lambda)=\sum_{p=1}^{\infty}\left\{\sum_{j=1}^{B_s}\sum_{\substack{m_1>\cdots>m_{j-1}>p\\ p>m_{j+1}>\cdots>m_{B_s}}} C_p^{m_1\dots\overset{j}{\smile}{p}\dots m_{B_s}}\right\}\frac{1}{p-\lambda} \tag{2.7}$$

where

$$C_{m_j}^{m_1\dots m_{B_s}}=\frac{1}{m_1^{a_1}m_{B_1+1}^{a_2}\dots m_{B_{s-1}+1}^{a_s}}\prod_{i\neq j}\frac{1}{(m_i-m_j)}. \tag{2.8}$$

Proof. The generating function f can be written as follows:

$$f((a_i,b_i)_{i=1}^s;\lambda)=\sum_{m_1>\cdots>m_{B_s}>0}\sum_{j=1}^{B_s}\frac{C_{m_j}^{m_1\dots m_{B_s}}}{m_j-\lambda}.$$

For the proof, we have to show that it is possible to change the order of the summations. So it is sufficient to prove that for any j

$$\sum_{\substack{m_1>\cdots>m_{j-1}>m_j\\ m_j>m_{j+1}>\cdots>m_{B_s}}}\frac{C_{m_j}^{m_1\dots m_{B_s}}}{m_j-\lambda}$$

converges absolutely. Put $d_i=m_i-m_{i+1}$ for $i=1,\dots,B_s-1$ and $d_{B_s}=m_{B_s}$. Making use of the inequality

$$d_1+d_2+\cdots+d_{B_s}\geq B_s\sqrt[B_s]{d_1d_2\cdots d_{B_s}}$$

we have

$$\left|\frac{C_{m_j}^{m_1\ldots m_{B_s}}}{m_j-\lambda}\right| = \left|\left(\frac{1}{m_1^{a_1}\cdots m_{B_{s-1}+1}^{a_s}}\prod_{i\neq j}\frac{1}{(m_i-m_j)}\right)\frac{1}{m_j-\lambda}\right|$$
$$\leq \frac{1}{(d_1+d_2+\cdots+d_{B_s})^{a_1}}\left(\prod_{i=1}^{B_s-1}\frac{1}{d_i}\right)\frac{1}{|d_j+\cdots+d_{B_s}-\lambda|}$$
$$\leq \frac{1}{\left(B_s\sqrt[B_s]{d_1d_2\cdots d_{B_s}}\right)^{a_1}}\left(\prod_{i=1}^{B_s-1}\frac{1}{d_i}\right)\frac{1}{|d_j+\cdots+d_{B_s}-\lambda|}$$

Let λ be in a compact set which does not involve positive integers. Then there exists a positive constant A such that

$$\frac{1}{|d_j+\cdots+d_{B_s}-\lambda|}\leq\frac{A}{d_{B_s}}.$$

Hence

$$\sum_{m_1>\cdots>m_{B_s}>0}\left|\frac{C_{m_j}^{m_1\ldots m_{B_s}}}{m_j-\lambda}\right| \leq A\sum_{d_1,\ldots,d_{B_s}=1}^{\infty}\frac{1}{\left(B_s\sqrt[B_s]{d_1\ldots d_{B_s}}\right)^{a_1}}\left(\prod_{i=1}^{B_s}\frac{1}{d_i}\right)$$
$$< +\infty.$$

□

§3. Algebraic Reduction of the Ohno Relation

Definition 3. For any index $\boldsymbol{k}=(k_1,\ldots,k_n)$ we set the homogeneous functions of weight $|\boldsymbol{k}|+1$ as

$$F(\boldsymbol{k};\lambda) := \sum_{\delta_i=0,1}(-\lambda)^{n-1-|\delta|}f((k_1,1)\cup(k_i-\delta_i,1)_{i=2}^{n};\lambda), \tag{3.1}$$

$$G(\boldsymbol{k};\lambda) := \sum_{\delta_i=0,1}(-\lambda)^{n-1-|\delta|}g((k_1,1)\cup(k_i-\delta_i,1)_{i=2}^{n};\lambda). \tag{3.2}$$

It is easy to calculate $F(\boldsymbol{k};\lambda)$; we have

$$F(\boldsymbol{k};\lambda) = \sum_{m_1>m_2>\cdots>m_n>0}\frac{1}{m_1^{k_1}(m_1-\lambda)m_2^{k_2}\cdots m_n^{k_n}}. \tag{3.3}$$

On the other hand, it is difficult to write down the explicit form of $G(\boldsymbol{k};\lambda)$. These functions satisfy difference equations of the simple form:

Proposition 3. *For any index $\boldsymbol{k}=(k_1,\ldots,k_n)$, we have the relations homogeneous of weight $|\boldsymbol{k}|$:*

(i) *if* $k_1 \geq 2$
$\boldsymbol{k}'$ *to be the dual index of* $\boldsymbol{k}$ *defined by* (1.5). *Then*

$$\lambda F(k_1, k_2, \dots, k_n; \lambda) + \zeta(k_1, \dots, k_n) = F(k_1 - 1, k_2, \dots, k_n; \lambda), \tag{3.4}$$

$$\lambda G(k_1, k_2, \dots, k_n; \lambda) + \zeta(k'_1, \dots, k'_{n'}) = G(k_1 - 1, k_2, \dots, k_n; \lambda). \tag{3.5}$$

(ii) *if* $k_1 = 1$
$(k'_2 + 1, k_3, \dots, k'_{n'})$ *to be the dual index of* $(k_2 + 1, k_3, \dots, k_n)$. *Then*

$$\lambda F(1, k_2, \dots, k_n; \lambda) + \zeta(k_2 + 1, k_3, \dots, k_n) = \lambda' F(1, k_2, \dots, k_n; \lambda') + \lambda' F(k_2 + 1, \dots, k_n; \lambda'), \tag{3.6}$$

$$\lambda G(1, k_2, \dots, k_n; \lambda) + \zeta(k'_2 + 1, k'_3, \dots, k'_{n'}) = \lambda' G(1, k_2, \dots, k_n; \lambda') + \lambda' G(k_2 + 1, \dots, k_n; \lambda'). \tag{3.7}$$

Proof. Induction on compositions. Apply Proposition 1 to $(k_i, 1)_{i=1}^n$ and gather f's or g's whether $\epsilon_i = 0$ or not, then there are many cancel outs because of the identification $(\dots, k_i - 1, 0, k_{i+1}, \dots) = (\dots, k_i, 0, k_{i+1} - 1, \dots)$ and the induction hypothesis. □

It is easy to see that the inverse Mellin transform of $F(\boldsymbol{k}; \lambda)$ is the MPL $Li_{\boldsymbol{k}}(z)$. This is a motivation to introduce these functions. It is known that the Ohno relation is the largest systematic relation for MZVs, however there are many linear dependency among them. Actually we can prove

Theorem 1. *For any composition* $(a_i, b_i)_{i=1}^s$, *we have*

$$f((a_i, b_i)_{i=1}^s; \lambda) = \sum_{\boldsymbol{c}} \alpha_{\boldsymbol{c}}^{(a_i,b_i)} \zeta(\boldsymbol{k}_{\boldsymbol{c}}^{(a_i,b_i)}) F(\boldsymbol{c}; \lambda), \tag{3.8}$$

$$g((a_i, b_i)_{i=1}^s; \lambda) = \sum_{\boldsymbol{c}} \alpha_{\boldsymbol{c}}^{(a_i,b_i)} \zeta({\boldsymbol{k}'}_{\boldsymbol{c}}^{(a_i,b_i)}) G(\boldsymbol{c}; \lambda), \tag{3.9}$$

where the summation runs over some finite number of compositions $\boldsymbol{c}$, $\alpha_{\boldsymbol{c}}^{(a_i,b_i)} \in \mathbb{Q}$, $\kappa^{-1}(\boldsymbol{k}_{\boldsymbol{c}}^{(a_i,b_i)}) \prec (a_i, b_i)_{i=1}^s$, *and* ${\boldsymbol{k}'}_{\boldsymbol{c}}^{(a_i,b_i)}$ *is the dual index for* $\boldsymbol{k}_{\boldsymbol{c}}^{(a_i,b_i)}$. *Moreover the duality formula* $\zeta(\boldsymbol{k}_{\boldsymbol{c}}^{(a_i,b_i)}) = \zeta({\boldsymbol{k}'}_{\boldsymbol{c}}^{(a_i,b_i)})$ *comes from* $f(0) = g(0)$ *for compositions less than* $(a_i, b_i)_{i=1}^s$. *So the Ohno relation is reduced to*

$$F(\boldsymbol{k}; \lambda) = G(\boldsymbol{k}; \lambda) \tag{3.10}$$

as an algebraic relation.

We call (3.10) the reduced Ohno relation. For the proof of the theorem, we have to consider inverse Mellin transform of f's and g's.

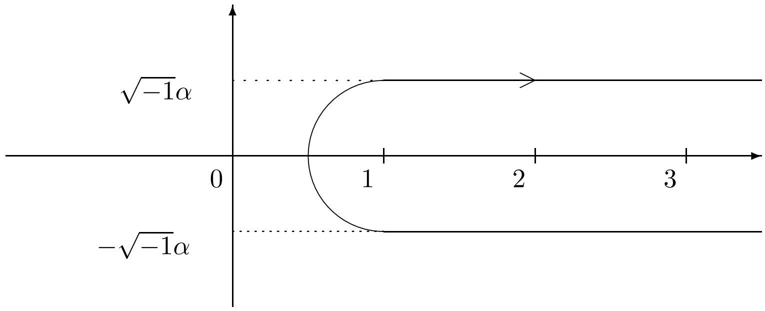
§4. Inverse Mellin Transform of the Generating Functions

§4.1. Integral transform of the generating functions

For any composition $(a_i, b_i)_{i=1}^s$ and any integer l we consider the integral transform

$$\widetilde{M}\left[\lambda^l f((a_i,b_i)_{i=1}^s;\lambda)\right](z) = \frac{1}{2\pi\sqrt{-1}}\int_C \lambda^l f((a_i,b_i)_{i=1}^s;\lambda) z^\lambda \, d\lambda, \tag{4.1}$$

where $0 < z < 1$ and the contour C for any $\alpha > 0$ is as follows:



Proposition 4. *The integral transform* (4.1) *absolutely converges and*

$$\widetilde{M}\left[\lambda^l f((a_i,b_i)_{i=1}^s;\lambda)\right](z) = \sum_{p=1}^{\infty} \operatorname*{Res}_{\lambda=p} \lambda^l f((a_i,b_i)_{i=1}^s;\lambda)\, z^p. \tag{4.2}$$

Proof. Since

$$\left|m_j - \left(t\sqrt{-1}\alpha\right)\right|^2 = (t^2+\alpha^2)\left(1 - \frac{m_j t}{t^2+\alpha^2}\right)^2 + \frac{m_j^2\alpha^2}{t^2+\alpha^2} \geq \frac{m_j^2\alpha^2}{t^2+\alpha^2},$$

we have

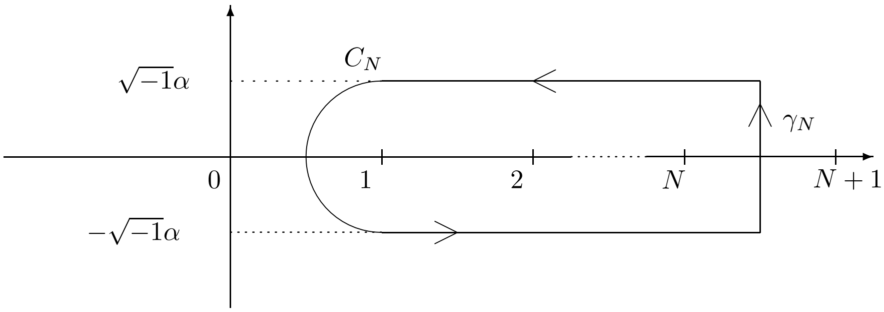
$$\begin{aligned} &\int_0^\infty \left|(\pm\sqrt{-1}\alpha+t)^l f((a_i,b_i)_{i=1}^s;\pm\sqrt{-1}\alpha+t) z^{\pm\sqrt{-1}\alpha+t}\right| dt \\ &\quad\leq \int_0^\infty (t^2+\alpha^2)^{\frac{l}{2}} \left|f((a_i,b_i)_{i=1}^s;\pm\sqrt{-1}\alpha+t)\right| z^t \, dt \\ &\quad\leq \zeta(k_1,k_2,\ldots,k_n)\int_0^\infty z^t \frac{(t^2+\alpha^2)^{(B_s+l)/2}}{\alpha^{B_s}} dt, \end{aligned} \tag{4.3}$$

where $(k_1,\ldots,k_n) = \kappa((a_i,b_i)_{i=1}^s)$ and $B_s = b_1 + \cdots + b_s$. Thus the integral absolutely converges.

Next, consider the integral

(4.4)
$$\frac{1}{2\pi\sqrt{-1}}\int_{C_N+\gamma_N}\lambda^l f((a_i,b_i)_{i=1}^s;\lambda)z^\lambda\,d\lambda=-\sum_{p=1}^N\operatorname*{Res}_{\lambda=p}\lambda^l f((a_i,b_i)_{i=1}^s;\lambda)\,z^p$$

where the contour is



and γ_N passes through $N+\frac{1}{2}$. We must prove that the integral on γ_N tends to 0 as $N\longrightarrow\infty$. Because of the inequality

$$\left|\frac{1}{m_j-\left(N+\frac{1}{2}+\sqrt{-1}t\right)}\right|\le\frac{1}{\left|m_j-\left(N+\frac{1}{2}\right)\right|}\le\frac{2(N+1)}{m_j},$$

the integral on γ_N is evaluated as

(4.5)
$$\left|\int_{-\alpha}^{\alpha}\left(N+\frac{1}{2}+\sqrt{-1}t\right)^l f((a_i,b_i)_{i=1}^s;N+\frac{1}{2}+\sqrt{-1}t)z^{N+\frac{1}{2}+\sqrt{-1}t}\,dt\right|$$
$$\le z^{N+\frac{1}{2}}(2(N+1))^{B_s}\zeta(k_1,\ldots,k_n)\int_{-\alpha}^{\alpha}\left(t^2+\left(N+\frac{1}{2}\right)^2\right)^{l/2}\,dt.$$

Since $0<z<1$, the right hand side converges to 0. □

We set

(4.6) $$\varphi((a_i,b_i)_{i=1}^s;z):=\widetilde{M}\left[f((a_i,b_i)_{i=1}^s;\lambda)\right](z),$$

(4.7) $$\psi((a_i,b_i)_{i=1}^s;z):=\widetilde{M}\left[g((a_i,b_i)_{i=1}^s;\lambda)\right](z).$$

Proposition 5.

(i) *The functions* $\varphi((a_i,b_i)_{i=1}^s;z)$ *and* $\psi((a_i,b_i)_{i=1}^s;z)$ *are holomorphic for* $|z|<1$.

(ii) *We have*

$$\widetilde{M}[\lambda^m f((a_i,b_i)_{i=1}^s;\lambda)](z) = \vartheta^m\big(\varphi((a_i,b_i)_{i=1}^s;z)\big), \tag{4.8}$$

$$\widetilde{M}[(\lambda-1)^m f((a_i,b_i)_{i=1}^s;\lambda-1)](z) = z\ \vartheta^m\big(\varphi((a_i,b_i)_{i=1}^s;z)\big), \tag{4.9}$$

and

$$\begin{aligned}\widetilde{M}\left[\frac{1}{\lambda-1}f((a_i,b_i)_{i=1}^s;\lambda-1)\right](z)\\ = z\left(-\zeta(k_1,\dots,k_n)+\int_0^z\frac{dz}{z}\varphi((a_i,b_i)_{i=1}^s;z)\right),\end{aligned}\tag{4.10}$$

where $\vartheta = zd/dz$ is the Euler derivation.

Proof.

(i) From Proposition 4 we obtain

$$\begin{aligned}\varphi((a_i,b_i)_{i=1}^s;z) &= \sum_{p=1}^{\infty}\operatorname*{Res}_{\lambda=p} f((a_i,b_i)_{i=1}^s;\lambda)\ z^p\\ &= \sum_{p=1}^{\infty}\left\{\sum_{j=1}^{B_s}\sum_{\substack{m_1>\cdots>m_{j-1}>p\\ p>m_{j+1}>\cdots>m_{B_s}}} C_p^{m_1\dots\overset{j}{\smile}{p}\dots m_{B_s}}\right\} z^p,\end{aligned}\tag{4.11}$$

where $C_p^{m_1\dots\overset{j}{\smile}{p}\dots m_{B_s}}$ is the same as in Proposition 2. The series

$$\sum_{p=1}^{\infty}\left\{\sum_{j=1}^{B_s}\sum_{\substack{m_1>\cdots>m_{j-1}>p\\ p>m_{j+1}>\cdots>m_{B_s}}} C_p^{m_1\dots\overset{j}{\smile}{p}\dots m_{B_s}}\right\}\frac{z^p}{p}$$

is convergent at $z=1$ because of Proposition 2, so the radius of convergence of (4.11) is at least 1.

(ii) The first equation can be shown by exchanging the derivation and the integration. For the second equation we shift the integral variable λ to $\lambda+1$ in the left hand side. For the last equation, we have

$$\begin{aligned}&\widetilde{M}\left[\frac{1}{\lambda-1}f((a_i,b_i)_{i=1}^s;\lambda-1)\right](z)\\ &= \frac{z}{2\pi\sqrt{-1}}\int_{\{\lambda-1|\lambda\in C\}}\frac{1}{\lambda}f((a_i,b_i)_{i=1}^s;\lambda)\ z^\lambda d\lambda\end{aligned}$$

$$= z\left(-\zeta(k_1,k_2,\ldots,k_n)+\frac{1}{2\pi\sqrt{-1}}\int_C \frac{1}{\lambda} f((a_i,b_i)_{i=1}^s;\lambda)\ z^\lambda d\lambda\right)$$
$$= z\left(-\zeta(k_1,k_2,\ldots,k_n)+\int_0^z \left\{\frac{1}{2\pi\sqrt{-1}}\int_C f((a_i,b_i)_{i=1}^s;\lambda)\ z^\lambda d\lambda\right\}\frac{dz}{z}\right).$$

In the last line above we have exchanged the order of the integrals.

□

Let us introduce the "inverse transform" of $\widetilde{M}$ by

$$M[\varphi(z)](\lambda)=\int_0^1 \varphi(z)z^{-\lambda-1}\ dz. \tag{4.12}$$

Proposition 6.

$$M[\varphi((a_i,b_i)_{i=1}^s;z)](\lambda)=f((a_i,b_i)_{i=1}^s;\lambda). \tag{4.13}$$

Proof. For $0<r<1$ and $\lambda<0$

$$\left|\int_0^r \left\{\sum_{p=1}^\infty \operatorname*{Res}_{\lambda=p} f((a_i,b_i)_{i=1}^s;\lambda)z^p\right\} z^{-\lambda-1}\ dz - f((a_i,b_i)_{i=1}^s;\lambda)\right|$$
$$=\left|\sum_{p=1}^\infty \operatorname*{Res}_{\lambda=p} f((a_i,b_i)_{i=1}^s;\lambda)\frac{r^{p-\lambda}}{p-\lambda}-\sum_{p=1}^\infty \operatorname*{Res}_{\lambda=p} f((a_i,b_i)_{i=1}^s;\lambda)\frac{1}{p-\lambda}\right|$$
$$\le \sum_{p=1}^\infty \left|\operatorname*{Res}_{\lambda=p} f((a_i,b_i)_{i=1}^s;\lambda)\frac{1}{p-\lambda}\right|(1-r^{p-\lambda})$$
$$\longrightarrow 0 \qquad (r\to 1)$$

by virtue of Abel's Theorem. □

§4.2. The differential-integral relations satisfied by φ's and ψ's

Proposition 7. *The functions φ's as well as ψ's satisfy the following relations*:

$$\sum(-\vartheta)^{s-|\delta|-|\epsilon|}\varphi((a_i-\delta_i,b_i-\epsilon_i)_{i=1}^s;z) = z\sum{}'(-\vartheta)^{s-|\delta|-|\epsilon|}\varphi((a_i-\delta_i,b_i-\epsilon_{i+1})_{i=1}^s;z). \tag{4.14}$$

Here $\sum$, $\sum'$, $|\delta^{(\prime)}|$, $|\epsilon^{(\prime)}|$ is the same for Proposition 1 and ϑ^{-1} is the integral operator

$$\begin{cases} \vartheta^{-1}\varphi\left((a_i,b_i)_{i=1}^s;z\right) = -\zeta(k_1,\ldots,k_n)+\displaystyle\int_0^z \frac{dz}{z}\varphi\left((a_i,b_i)_{i=1}^s;z\right), \\ \vartheta^{-1}\psi\left((a_i,b_i)_{i=1}^s;z\right) = -\zeta(k_1',\ldots,k_{n'}')+\displaystyle\int_0^z \frac{dz}{z}\psi\left((a_i,b_i)_{i=1}^s;z\right), \end{cases} \tag{4.15}$$

where $(k_1,\dots,k_n)=\kappa((a_i,b_i)_{i=1}^s)$ *and* $(k'_1,\dots,k'_{n'})$ *is its dual. We define the weight of* ϑ *to be* -1 *and this relation is homogeneous of weight* $|(a_i,b_i)_{i=1}^s|-s$.

Proof. This is a direct consequence from Proposition 1 and Proposition 5. □

Furthermore we set (see Definition 3 and Proposition 5)

$$\begin{aligned}\Phi(\boldsymbol{k};z) &:= \widetilde{M}[F(\boldsymbol{k};z)](z)\\ &= \sum_{\delta_i=0,1}(-\vartheta)^{n-1-|\delta|}\varphi((k_1,1)\cup(k_i-\delta_i,1)_{i=2}^n;z),\end{aligned}\tag{4.16}$$

$$\begin{aligned}\Psi(\boldsymbol{k};z) &:= \widetilde{M}[G(\boldsymbol{k};z)](z)\\ &= \sum_{\delta_i=0,1}(-\vartheta)^{n-1-|\delta|}\psi((k_1,1)\cup(k_i-\delta_i,1)_{i=2}^n;z).\end{aligned}\tag{4.17}$$

From (3.3) and (4.2) it follows that

$$\Phi(\boldsymbol{k};z)=\sum_{m_1>\cdots>m_n}\frac{z^{m_1}}{m_1^{k_1}\cdots m_n^{k_n}}=\mathrm{Li}_{\boldsymbol{k}}(z)\,.\tag{4.18}$$

Corollary 1. *The differential relations satisfied by* $\Phi(\boldsymbol{k};z)$ *and* $\Psi(\boldsymbol{k};z)$ *are the same as the differential relations for* $\mathrm{Li}_{\boldsymbol{k}}(z)$

$$\frac{d}{dz}\mathrm{Li}_{k_1,\dots,k_n}(z)=\begin{cases}\dfrac{1}{z}\mathrm{Li}_{k_1-1,k_2,\dots,k_n}(z) & \textit{if } k_1\geq 2,\\ \dfrac{1}{1-z}\mathrm{Li}_{k_2,\dots,k_n}(z) & \textit{if } k_1=1.\end{cases}\tag{4.19}$$

Proof. This is clear from Proposition 3 and Proposition 5. □

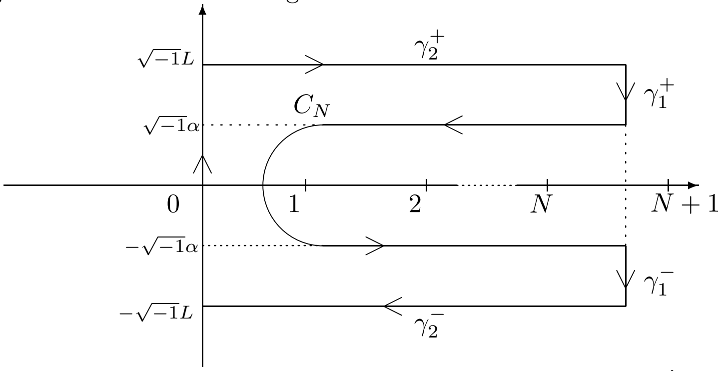
§4.3. Mellin transform and inverse Mellin transform

We call the integral transforms M and $\widetilde{M}$ Mellin transform and inverse Mellin transform respectively. In fact, for suitable compositions, $\widetilde{M}$ is actually inverse Mellin transform.

Proposition 8. *For any composition* $(a_i,b_i)_{i=1}^s$ *with* $a_{i_0}\geq 2$ *for some* i_0 *and* $0<c<1$

$$\widetilde{M}\left[f((a_i,b_i)_{i=1}^s;\lambda)\right](z)=\frac{1}{2\pi\sqrt{-1}}\int_{c-\sqrt{-1}\infty}^{c+\sqrt{-1}\infty}f((a_i,b_i)_{i=1}^s;\lambda)z^{\lambda}\,d\lambda.\tag{4.20}$$

Proof. Consider the following contour



Because the integrand has singular points only at positive integers the integral on this contour is 0. We check that the integrals on γ_1^+, γ_1^-, γ_2^+ and $\gamma_2^- \to 0$ if N, $L \to \infty$. First let N to ∞, then we can show that the integral on γ_1^+ tends to 0 in the same manner as in (4.5). Similarly the integral on $\gamma_1^- \longrightarrow 0$.

Next let L to ∞. Assume that $i_0 = 1$. Noting that

$$\frac{1}{\left|m_1^{a_1} - \left(t + \sqrt{-1}L\right)\right|} \leq \frac{1}{m_1^{a_1} L}$$

we can show that in the same manner as in (4.3) the integral on γ_2^+ is

$$\begin{aligned}&\left|\int_0^\infty z^{\sqrt{-1}L+t} f((a_i, b_i)_{i=1}^s; \sqrt{-1}L + t)\ dt\right| \\ &\qquad \leq \frac{\zeta(k_1 - 1, k_2, \ldots, k_n)}{L} \int_0^\infty z^t \left(\frac{t^2 + L^2}{L^2}\right)^{B_s - 1} dt \\ &\qquad \longrightarrow 0 \qquad (L \longrightarrow \infty).\end{aligned}$$

The other cases of $i_0 \neq 1$ and that the integral on γ_2^- tends to 0 can be verified in the same way. □

§5. Proof and Examples of Theorem 1

§5.1. Proof of Theorem 1

From the equation (4.14) in Proposition 7 we obtain

$$\begin{aligned}&\left\{(1-z) \sum_{\substack{\epsilon_j \delta_{j+1} = 0 \\ \forall j}} + \sum_{\substack{\epsilon_j \delta_{j+1} = 1 \\ \exists j}}\right\} (-\vartheta)^{s - |\delta| - |\epsilon|} \varphi((a_i - \delta_i, b_i - \epsilon_i)_{i=1}^s; z) \\ &\qquad = z \sideset{}{'}\sum_{\substack{\delta_j \epsilon_{j+1} = 1 \\ \exists j}} (-\vartheta)^{s - |\delta| - |\epsilon|} \varphi((a_i - \delta_i, b_i - \epsilon_{i+1})_{i=1}^s; z).\end{aligned}$$

Here we note that $\varphi((a_i,b_i)_{i=1}^s;z)$ appears only in the first summation (it corresponds to the term with all $\delta_i=\epsilon_i=0$). Dividing by $\frac{1}{1-z}$ and applying $\int_0^z \frac{dz}{z}$ s times we have

$$\begin{aligned}
&(-)^s\varphi((a_i,b_i)_{i=1}^s;z)\\
&\quad-\sum_{\substack{\epsilon_j\delta_{j+1}=0\\ \forall j\\ 1\le|\delta|+|\epsilon|\le s}}\left(-\int_0^z\frac{dz}{z}\right)^{s-|\delta|-|\epsilon|}\varphi((a_i-\delta_i,b_i-\epsilon_i)_{i=1}^s;z)\\
&\quad+\sum_{\substack{\epsilon_j\delta_{j+1}=1\\ \exists j}}\left(\int_0^z\frac{dz}{z}\right)^{s}(-\vartheta)^{s-|\delta|-|\epsilon|}\varphi((a_i-\delta_i,b_i-\epsilon_i)_{i=1}^s;z)\\
&\quad+\sum_{\substack{\epsilon_j\delta_{j+1}=1\\ \exists j}}\left(\int_0^z\frac{dz}{z}\right)^{s-1}\int_0^z\frac{dz}{1-z}(-\vartheta)^{s-|\delta|-|\epsilon|}\varphi((a_i-\delta_i,b_i-\epsilon_i)_{i=1}^s;z)\\
&=\sum_{\substack{\delta_j\epsilon_{j+1}=1\\ \exists j}}{}'\left(\int_0^z\frac{dz}{z}\right)^{s-1}\int_0^z\frac{dz}{1-z}(-\vartheta)^{s-|\delta|-|\epsilon|}\varphi((a_i-\delta_i,b_i-\epsilon_{i+1})_{i=1}^s;z).
\end{aligned}$$

From the induction hypothesis φ's of less compositions than $(a_i,b_i)_{i=1}^s$ can be written by MZVs-linear combination of Φ's. Using Corollary 1 the second and third terms in the left hand side are expressed as MZVs-linear combination of Φ's. So we have

$$\begin{aligned}
\varphi((a_i,b_i)_{i=1}^s)&=\sum_{\boldsymbol{c}}\alpha_{\boldsymbol{c}}^{(a_i,b_i)}\zeta(\boldsymbol{k}_{\boldsymbol{c}}^{(a_i,b_i)})\Phi(\boldsymbol{c};z)\\
&\quad+\sum_{\boldsymbol{c}}\beta_{\boldsymbol{c}}^{(a_i,b_i)}\zeta(\boldsymbol{k}_{\boldsymbol{c}}^{(a_i,b_i)})\left(\int_0^z\frac{dz}{z}\right)^{s-1}\int_0^z\frac{dz}{1-z}\vartheta^{s-m_{\boldsymbol{c}}}\big(\Phi(\boldsymbol{c};z)\big),
\end{aligned}$$

where $2\le m_{\boldsymbol{c}}\le s+1$ and $\alpha_{\boldsymbol{c}}^{(a_i,b_i)}$, $\beta_{\boldsymbol{c}}^{(a_i,b_i)}\in\mathbb{Q}$. For the proof we must verify that the last term above is written in MZV-linear combination of Φ's. If $m_{\boldsymbol{c}}=s$, from Corollary 1

$$\left(\int_0^z\frac{dz}{z}\right)^{s-1}\int_0^z\frac{dz}{1-z}\vartheta^{s-m_{\boldsymbol{c}}}\big(\Phi(\boldsymbol{c};z)\big)=\left(\int_0^z\frac{dz}{z}\right)^{s-1}\int_0^z\frac{dz}{1-z}\big(\Phi(\boldsymbol{c};z)\big)$$

is Φ of the greater composition. If $m_{\boldsymbol{c}}=s+1$, one can show by virtue of (4.15),

$$\left(\int_0^z\frac{dz}{z}\right)^{s-1}\int_0^z\frac{dz}{1-z}\vartheta^{-1}\big(\Phi(\boldsymbol{c};z)\big).$$

is written in MZV-linear combination of $\Phi's$. For the Ψ's case the corresponding coefficients are MZVs of the dual indices. If $2\le m_{\boldsymbol{c}}\le s-1$, $\vartheta^{s-m_{\boldsymbol{c}}}\big(\Phi(\boldsymbol{c};z)\big)$

can be expressed as linear combination of the following

$$\vartheta^{s-m_{\boldsymbol{c}}}\big(\Phi(\boldsymbol{c};z)\big)=\sum_{\boldsymbol{c}',l,m}\alpha^{(a_i,b_i)}\Phi(\boldsymbol{c}',z)\frac{z^l}{(1-z)^m}, \tag{5.1}$$

where $\boldsymbol{c}'$ is less than $\boldsymbol{c}$, $l\le m\le s-m_{\boldsymbol{c}}$ and $\alpha^{(a_i,b_i)}\in\mathbb{Z}$. Next, we repeat the iterated integrals using the expansions

$$\frac{z^l}{(1-z)^m}=\sum_{j=0}^{n}\binom{n}{j}\frac{(-1)^{n-j}}{(1-z)^{m-n+j}},\qquad \frac{1}{(1-z)^m z}=\frac{1}{z}+\sum_{j=1}^{m}\frac{1}{(1-z)^j}.$$

Making once the iterated integral of the right hand side of (5.1) we have the following:

$$\begin{aligned}&\int_0^z\frac{dz}{(1-z)^m z}\Phi(\boldsymbol{k};z)\\&\quad=\int_0^z\frac{dz}{z}\Phi(\boldsymbol{k};z)+\int_0^z\frac{dz}{1-z}\Phi(\boldsymbol{k};z)+\sum_{j=2}^{m}\int_0^z\frac{dz}{(1-z)^j}\Phi(\boldsymbol{k};z),\end{aligned}$$

$$\begin{aligned}&\int_0^z\frac{dz}{(1-z)^m}\Phi(k_1,\dots,k_n;z)\\&=\frac{1}{m-1}\Bigg\{\frac{1}{(1-z)^{m-1}}\Phi(k_1,\dots,k_n;z)-\Phi(k_1,\dots,k_n;z)\\&\qquad-\Phi(1,k_1-1,\dots,k_n;z)-\sum_{l=2}^{m-1}\int_0^z\frac{dz}{(1-z)^l}\Phi(k_1-1,\dots,k_n;z)\Bigg\},\end{aligned}$$

$$\begin{aligned}&\int_0^z\frac{dz}{(1-z)^m}\Phi(1,k_2,\dots,k_n;z)\\&=\frac{1}{m-1}\left\{\frac{1}{(1-z)^{m-1}}\Phi(1,k_2,\dots,k_n;z)-\int_0^z\frac{dz}{(1-z)^m}\Phi(k_2,\dots,k_n;z)\right\},\end{aligned}$$

and

$$\int_0^z\frac{dz}{(1-z)^m}=-z\sum_{l=1}^{m-1}\frac{1}{l}\binom{m-1}{l-1}\frac{z^{l-1}}{(1-z)^{m-1}}.$$

Hence after making the iterated integral $s-1$ times we finally reach to the integral

$$\int_0^z\frac{dz}{1-z}=\Phi(1;z).$$

Thus $\varphi((a_i, b_i)_{i=1}^s)$ can be written by MZVs-linear combination of Φ's. The application of Mellin transform

$$M[\varphi(z)](\lambda) = \int_0^1 \varphi(z) z^{-\lambda-1} dz$$

gives us the theorem. □

§5.2. Examples of Theorem 1

We list examples of (3.8) up to weight 6. For simplicity we drop the variable λ.

weight 2:

$$f(1,1) = F(1). \tag{5.2}$$

weight 3:

$$f(2,1) = f(1,2) = F(2). \tag{5.3}$$

weight 4:

$$f(3,1) = f(1,3) = F(3), \tag{5.4}$$

$$f(2,2) = 2F(3) + F(1,2) - \zeta(2)F(1), \tag{5.5}$$

$$f(1,1,1,1) = F(3) - F(2,1). \tag{5.6}$$

weight 5:

$$f(4,1) = f(1,4) = F(4), \tag{5.7}$$

$$f(3,2) = 3F(4) + F(1,3) + F(2,2) - \zeta(3)F(1) - \zeta(2)F(2), \tag{5.8}$$

$$f(2,3) = 3F(4) + F(1,3) + F(2,2) - \zeta(2,1)F(1) - \zeta(2)F(2), \tag{5.9}$$

$$f(2,1,1,1) = f(1,1,1,2) = 2F(4) - F(3,1) + F(2,2) - \zeta(2)F(2), \tag{5.10}$$

$$f(1,2,1,1) = f(1,1,2,1) = F(4) - F(3,1) - F(2,2). \tag{5.11}$$

weight 6:

$$f(5,1) = f(1,5) = F(5), \tag{5.12}$$

$$\begin{aligned} f(4,2) = 4F(5) &+ F(3,2) + F(2,3) + F(1,4) \\ &- \zeta(4)F(1) - \zeta(3)F(2) - \zeta(2)F(3), \end{aligned} \tag{5.13}$$

$$\begin{aligned} f(2,4) = 4F(5) &+ F(3,2) + F(2,3) + F(1,4) \\ &- \zeta(2,1,1)F(1) - \zeta(2,1)F(2) - \zeta(2)F(3), \end{aligned} \tag{5.14}$$

$$f(3,3) = 6F(5) + 2F(3,2) + 2F(2,3) + 2F(1,4) + F(1,2,2) - \zeta(3,1)F(1) - \zeta(3)F(2) - \zeta(2,1)F(2) - 2\zeta(2)F(3) - \zeta(2)F(1,2), \tag{5.15}$$

$$f(3,1,1,1) = 3F(5) - F(4,1) + F(2,3) + F(3,2) - \zeta(3)F(2) - \zeta(2)F(3), \tag{5.16}$$

$$f(1,1,1,3) = 3F(5) - F(4,1) + F(2,3) + F(3,2) - \zeta(2,1)F(2) - \zeta(2)F(3), \tag{5.17}$$

$$f(2,2,1,1) = 3F(5) - 2F(4,1) - F(2,2,1) - F(2,1,2) - \zeta(2)F(3) + \zeta(2)F(2,1) - \zeta(2,1)F(2), \tag{5.18}$$

$$f(1,1,2,2) = 3F(5) - 2F(4,1) - F(2,2,1) - F(2,1,2) - \zeta(2)F(3) + \zeta(2)F(2,1) - \zeta(3)F(2), \tag{5.19}$$

$$f(2,1,2,1) = f(1,2,1,2) = 2F(5) - F(4,1) - \zeta(2)F(3), \tag{5.20}$$

$$f(2,1,1,2) = 5F(5) - F(4,1) + 3F(3,2) + 2F(2,3) + F(2,1,2) - 3\zeta(2)F(3) - \zeta(2)F(2,1), \tag{5.21}$$

$$f(1,3,1,1) = f(1,1,3,1) = F(5) - F(4,1) - F(3,2) - F(2,3), \tag{5.22}$$

$$f(1,2,2,1) = F(5) - F(4,1) - 2F(3,2) - F(2,3) + F(2,2,1), \tag{5.23}$$

$$f(1,1,1,1,1,1) = 2F(5) - 2F(4,1) + F(3,1,1) - \zeta(2)F(3). \tag{5.24}$$

§6. Landen Formula and the Ohno Relation

The Landen connection formula for the dilogarithm (1.3) generalizes to the MPLs case. This formula is interpreted as the connection formula of MPLs between 1 and ∞.

Proposition 9.

$$\mathrm{Li}_{\boldsymbol{k}}(z) = (-1)^n \sum_{\boldsymbol{c}_1,\ldots,\boldsymbol{c}_n} \mathrm{Li}_{\boldsymbol{c}_1\cdots\boldsymbol{c}_n}\left(\frac{z}{z-1}\right), \tag{6.1}$$

where $\boldsymbol{c}_i$ runs all compositions of k_i.

Proof. For $\boldsymbol{k} = (1)$

$$\mathrm{Li}_1(z) = -\log(1-z) = \log\left(1 - \frac{z}{z-1}\right) = -\mathrm{Li}_1\left(\frac{z}{z-1}\right).$$

We assume that the proposition holds for $\boldsymbol{k} = (k_1, k_2, \ldots, k_n)$. Then using the differential relation

(6.2)
$$\frac{d}{dz}\mathrm{Li}_{k_1,\ldots,k_n}\left(\frac{z}{z-1}\right) = \begin{cases} \left(\frac{1}{z} + \frac{1}{1-z}\right)\mathrm{Li}_{k_1-1,k_2,\ldots,k_n}\left(\frac{z}{z-1}\right) & \text{if } k_1 \geq 2, \\ -\frac{1}{1-z}\mathrm{Li}_{k_2,\ldots,k_n}\left(\frac{z}{z-1}\right) & \text{if } k_1 = 1 \end{cases}$$

we have

$$\begin{aligned} &\mathrm{Li}_{k_1+1,k_2,\ldots,k_n}(z) = \int_0^z \frac{dz}{z}\mathrm{Li}_{k_1,k_2,\ldots,k_n}(z) \\ &= \int_0^z \left(\left(\frac{dz}{z} + \frac{dz}{1-z}\right)dz - \frac{dz}{1-z}\right)(-1)^n \sum_{\boldsymbol{c}_1,\ldots,\boldsymbol{c}_n} \mathrm{Li}_{\boldsymbol{c}_1\cdots\boldsymbol{c}_n}\left(\frac{z}{z-1}\right), \end{aligned}$$

and

$$\begin{aligned} \mathrm{Li}_{1,k_2,\ldots,k_n}(z) &= \int_0^z \frac{dz}{1-z}\mathrm{Li}_{k_1,k_2,\ldots,k_n}(z) \\ &= -\int_0^z -\frac{dz}{1-z}(-1)^n \sum_{\boldsymbol{c}_1,\ldots,\boldsymbol{c}_n} \mathrm{Li}_{\boldsymbol{c}_1\cdots\boldsymbol{c}_n}\left(\frac{z}{z-1}\right). \end{aligned}$$

□

One can obtain further information of the right hand side of (6.1) in the case of $\boldsymbol{k} = k$, a positive integer.

Lemma 1. *For any positive integers j and k with $j \leq k$, we have*

(6.3)
$$\sum_{m_1>\cdots>m_k} \frac{z^{m_j}}{m_1\prod_{i\neq j}(m_i - m_j)} = -\sum_{\substack{\boldsymbol{c} \\ |\boldsymbol{c}|=k \\ \mathrm{len}(\boldsymbol{c})=k-j+1}} \mathrm{Li}_{\boldsymbol{c}}\left(\frac{z}{z-1}\right).$$

Proof. We show by induction for k. For the case $k = 1$

$$\sum_{m=1}^{\infty} \frac{z^m}{m} = -\mathrm{Li}_1\left(\frac{z}{z-1}\right)$$

is obvious. We suppose that the proposition is correct for $k - 1$. Calculating the derivative of the series and applying the induction hypothesis we have

$$\begin{aligned} &\frac{d}{dz}\sum_{m_1>\cdots>m_k} \frac{z^{m_j}}{m_1\prod_{i\neq j}(m_i - m_j)} \\ &= \frac{1}{z(1-z)}\sum_{m_1>\cdots>m_{k-1}} \frac{z^{m_{j-1}}}{m_1\prod_{i\neq j}(m_i - m_{j-1})} \end{aligned}$$

$$-\frac{1}{1-z}\sum_{m_1>\cdots>m_{k-1}}\frac{z^{m_j}}{m_1\prod_{i\neq j}(m_i-m_j)}$$
$$=-\frac{1}{z(1-z)}\sum_{\substack{\boldsymbol{c}\\|\boldsymbol{c}|=k-1\\\mathrm{len}(\boldsymbol{c})=k-j+1}}\mathrm{Li}_{\boldsymbol{c}}\left(\frac{z}{z-1}\right)-\left(-\frac{1}{1-z}\right)\sum_{\substack{\boldsymbol{c}\\|\boldsymbol{c}|=k-1\\\mathrm{len}(\boldsymbol{c})=k-j}}\mathrm{Li}_{\boldsymbol{c}}\left(\frac{z}{z-1}\right).$$

Due to the differential relation (6.2) the lemma is proved for k. □

For any positive integer k, we have

$$G(k;\lambda)=\sum_{m_1>\cdots>m_k}\frac{1}{m_1(m_1-\lambda)\cdots(m_k-\lambda)},\tag{6.4}$$

hence

$$\Psi(k;z)=\sum_{m_1>\cdots>m_k}\sum_{j=1}^{k}\frac{z^{m_j}}{m_1\prod_{i\neq j}(m_i-m_j)}.\tag{6.5}$$

By virtue of Lemma 1 we have

Proposition 10.

$$\Psi(k;z)=-\sum_{j=1}^{k}\sum_{\substack{\boldsymbol{c}\\|\boldsymbol{c}|=k\\\mathrm{len}(\boldsymbol{c})=k-j+1}}\mathrm{Li}_{\boldsymbol{c}}\left(\frac{z}{z-1}\right).\tag{6.6}$$

Moreover using the differential relation for Li's and Ψ's and the equation above, we can see that

$$\Psi(k_1,\ldots,k_n;z)=(-1)^n\sum_{\boldsymbol{c}_1,\ldots,\boldsymbol{c}_n}\mathrm{Li}_{\boldsymbol{c}_1\cdots\boldsymbol{c}_n}\left(\frac{z}{z-1}\right).\tag{6.7}$$

Thus the relation $\Phi=\Psi$ can be interpreted as the Landen connection formula. We can think that the reduced Ohno relation is converted, via inverse Mellin transform, to the connection formula of MPLs between 1 and ∞.

Discussions. Proposition 9 says that the reduced Ohno relation is converted to the Landen connection formula for MPLs by inverse Mellin transform. In particular, since the explicit forms (6.4), (6.5) of $G(k;\lambda)$ and $\Psi(k;z)$ are revealed, the Landen connection formula for the polylogarithm $\mathrm{Li}_k(z)$ for any positive integer k is, via Mellin transform, converted to the relation $F(k;\lambda)=G(k;\lambda)$. This is nothing but the sum formula of depth k. Thus the sum formulas for MZVs are equivalent to the Landen connection formulas for polylogarithms. However, such equivalency is not achieved yet for the reduced Ohno relation of indices $\boldsymbol{k}$ of depth greater than 1. This is an important issue to be settled in future.

We have another important issue: In Theorem 1, which is the main theorem in this paper, the indices $\boldsymbol{k}_{\boldsymbol{c}}^{(a_i,b_i)}$ and the rational numbers $\alpha_{\boldsymbol{c}}^{(a_i,b_i)}$ are not specified. They must be determined. Moreover, we conjecture that

$$f((a_i,b_i)_{i=1}^s;\lambda) = \sum_{\boldsymbol{c}} \alpha_{\boldsymbol{c}}^{(a_i,b_i)} \zeta(\boldsymbol{k}_{\boldsymbol{c}}^{(a_i,b_i)}) * F(\boldsymbol{c};\lambda), \tag{6.8}$$

$$g((a_i,b_i)_{i=1}^s;\lambda) = \sum_{\boldsymbol{c}} \alpha_{\boldsymbol{c}}^{(a_i,b_i)} \zeta(\boldsymbol{k}'^{(a_i,b_i)}_{\boldsymbol{c}}) * G(\boldsymbol{c};\lambda), \tag{6.9}$$

where $*$ is the harmonic product introduced in [H2].

§7. Proofs of the Difference Relations and the Ohno Relation

§7.1. Proof of Proposition 1

To prove Proposition 1, we need the following.

Lemma 2. *For any composition $(a_i,b_i)_{i=1}^s$, we set*

$$\begin{aligned}&[\{(a_i,d_i),b_i\}_{i=1}^s;\lambda] \\ &:= \sum_{m_1>\cdots>m_{B_s}>0} \prod_{i=1}^s \frac{1}{(m_{B_{i-1}+1}-d_i)^{a_i}\underbrace{(m_{B_{i-1}+1}-\lambda)\cdots(m_{B_i}-\lambda)}_{b_i}},\end{aligned} \tag{7.1}$$

where we interpret special cases with $a_i=0$ or $b_i=0$ for some i as follows:

$$[\{\dots,b_{i-1},(0,d_i),b_i,\dots\};\lambda] = [\{\dots,b_{i-1}+b_i,\dots\};\lambda], \tag{7.2}$$

$$[\{\dots,(a_{i-1},d),0,(a_i,d),\dots\};\lambda] = [\{\dots,(a_{i-1}+a_i,d),\dots\};\lambda]. \tag{7.3}$$

Then we have the following difference relations:

(i) (a) *If $a_1 \geq 2$,*

$$\begin{aligned}&\lambda\,[\{(a_1,0),b_1,\dots\};\lambda] \\ &\quad -[\{(a_1-1,0),b_1,\dots\};\lambda] - [\{(a_1,0),b_1-1,\dots\};\lambda] \\ &= \lambda'\,[\{(a_1,1),b_1,\dots\};\lambda] - [\{(a_1-1,1),b_1,\dots\};\lambda].\end{aligned} \tag{7.4}$$

(b) *If $a_1 = 1$,*

$$\begin{aligned}&\lambda\,[\{(1,0),b_1,\dots\};\lambda] - [\{(1,0),b_1-1,\dots\};\lambda] \\ &= \lambda'\,[\{(1,1),b_1,\dots\};\lambda].\end{aligned} \tag{7.5}$$

(ii) *If $i \neq 1$ and $i \neq s$, or $i = s$ and $b_s \neq 1$,*

$$(7.6)\quad \begin{aligned}&\lambda\,[\{\dots,(a_i,0),b_i,\dots\};\lambda]\\ &-[\{\dots,(a_i-1,0),b_i,\dots\};\lambda]-[\{\dots,(a_i,0),b_i-1,\dots\};\lambda]\\ =\;&\lambda'\,[\{\dots,b_{i-1},(a_i,1),\dots\};\lambda]\\ &-[\{\dots,b_{i-1},(a_i-1,1),\dots\};\lambda]-[\{\dots,b_{i-1}-1,(a_i,1),\dots\};\lambda].\end{aligned}$$

(iii) (a) *If $b_s \geq 2$,*

$$(7.7)\quad \begin{aligned}&[\{(a_1,1),b_1,(a_2,1),b_2,\dots,(a_s,1),b_s\};\lambda]\\ &=[\{(a_1,0),b_1,\dots,(a_s,0),b_s\};\lambda']\\ &\quad-\frac{1}{\lambda'}[\{(a_1,0),b_1,\dots,(a_s,0),b_s-1\};\lambda'].\end{aligned}$$

(b) *If $b_s = 1$,*

$$(7.8)\quad \begin{aligned}&\lambda\,[\{(a_1,1),b_1,(a_2,1),b_2,\dots,(a_{s-1},1),b_{s-1},(a_s,0),1\};\lambda]\\ &-[\{(a_1,1),b_1,(a_2,1),b_2,\dots,(a_{s-1},1),b_{s-1},(a_s-1,0),1\};\lambda]\\ &=\lambda'\,[\{(a_1,0),b_1,(a_2,0),b_2,\dots,(a_s,0),1\};\lambda']\\ &\quad-[\{(a_1,0),b_1,(a_2,0),b_2,\dots,(a_s-1),1\};\lambda']\\ &\quad-[\{(a_1,0),b_1,(a_2,0),b_2,\dots,b_{s-1}-1,(a_s,0),1\};\lambda'].\end{aligned}$$

Proof. We use the partial-fractions expansion:

$$(7.9)\quad \begin{aligned}&\frac{\lambda}{m^a(m-\lambda)}-\frac{1}{m^{a-1}(m-\lambda)}\\ &=\frac{\lambda'}{(m-1)^a(m-\lambda)}-\frac{1}{(m-1)^{a-1}(m-\lambda)}+\left(\frac{1}{(m-1)^a}-\frac{1}{m^a}\right),\end{aligned}$$

$$(7.10)\quad \frac{\lambda}{m(m-\lambda)}=\frac{\lambda'}{(m-1)(m-\lambda)}+\left(\frac{1}{m-1}-\frac{1}{m}\right).$$

(i) (a) We set B by

$$\prod_{j=1}^{s} m_{B_{j-1}+1}^{a_j}\underbrace{(m_{B_{j-1}+1}-\lambda)\cdots(m_{X_j}-\lambda)}_{b_j}=m_1^{a_1}(m_1-\lambda)\,B.$$

Then using (7.9) we have

$$\begin{aligned}&\lambda\,[\{(a_1,0),b_1,\dots\};\lambda]-[\{(a_1-1,0),b_1,\dots\};\lambda]\\ &=\sum_{m_1>\cdots>m_{B_s}>0}\left\{\frac{\lambda}{m_1^{a_1}(m_1-\lambda)}-\frac{1}{m_1^{a_1-1}(m_1-\lambda)}\right\}\frac{1}{B}\end{aligned}$$

$$= \sum_{m_1>\cdots>m_{B_s}>0} \left\{ \frac{\lambda'}{(m_1-1)^{a_1}(m_1-\lambda)} - \frac{1}{(m_1-1)^{a_1-1}(m_1-\lambda)} + \left(\frac{1}{(m_1-1)^{a_1}} - \frac{1}{m_1^{a_1}} \right) \right\} \frac{1}{B}$$

$$= \sum_{m_1>\cdots>m_{B_s}>0} \left\{ \frac{\lambda'}{(m_1-1)^{a_1}(m_1-\lambda)} - \frac{1}{(m_1-1)^{a_1-1}(m_1-\lambda)} \right\} \frac{1}{B} + \sum_{m_2>\cdots>m_{B_s}>0} \sum_{m_1=m_2+1}^{\infty} \left(\frac{1}{(m_1-1)^{a_1}} - \frac{1}{m_1^{a_1}} \right) \frac{1}{B}$$

$$= \lambda' [\{(a_1,1), b_1, \dots\}; \lambda] - [\{(a_1-1,1), b_1, \dots\}; \lambda] + [\{(a_1,0), b_1-1, \dots\}; \lambda].$$

(b) Using (7.10), it can be proved in the same manner as (ia).

(ii) We set A and B by

$$A := \prod_{j=1}^{i-1} m_{B_{j-1}+1}^{a_j} \overbrace{(m_{B_{j-1}+1}-\lambda)\cdots(m_{B_j}-\lambda)}^{b_j},$$

$$B := \frac{\prod_{j=i-1}^{s} m_{B_{j-1}+1}^{a_j} \overbrace{(m_{B_{j-1}+1}-\lambda)\cdots(m_{B_j}-\lambda)}^{b_j}}{m_{B_{i-1}+1}^{a_i}(m_{B_{i-1}+1}-\lambda)}.$$

Then

$$\lambda[\{\dots, b_{i-1}, (a_i,0), b_i, \dots\}; \lambda] - [\{\dots, b_{i-1}, (a_i-1,0), b_i, \dots\}; \lambda]$$
$$= \sum_{m_1>\cdots>m_{B_s}>0} \frac{1}{A} \left\{ \frac{\lambda}{m_{B_{i-1}+1}^{a_i}(m_{B_{i-1}+1}-\lambda)} - \frac{1}{m_{B_{i-1}+1}^{a_i-1}(m_{B_{i-1}+1}-\lambda)} \right\} \frac{1}{B}$$
$$= \sum_{m_1>\cdots>m_{B_s}>0} \frac{1}{A} \left\{ \frac{\lambda'}{(m_{B_{i-1}+1}-1)^{a_i}(m_{B_{i-1}+1}-\lambda)} - \frac{1}{(m_{B_{i-1}+1}-1)^{a_i-1}(m_{B_{i-1}+1}-\lambda)} \right.$$

$$
- \left(\frac{1}{(m_{B_{i-1}+1} - 1)^{a_i}} - \frac{1}{m_{B_{i-1}+1}^{a_i}} \right) \Bigg\} \frac{1}{B}
$$

$$
\begin{aligned}
&= \lambda'[\ldots, b_{i-1}, (a_i, 1), b_i, \ldots; \lambda] - [\ldots, b_{i-1}, (a_i - 1, 1), b_i, \ldots; \lambda] \\
&\quad + \sum_{\substack{m_1 > \cdots > m_{B_{i-1}} \\ m_{B_{i-1}} - 1 > m_{B_{i-1}+2} > \cdots > m_{B_s} > 0}} \frac{1}{A} \left(\frac{1}{(m_{B_{i-1}+2})^{a_i}} - \frac{1}{(m_{B_{i-1}} - 1)^{a_i}} \right) \frac{1}{B}.
\end{aligned}
$$

We divide the range of sum of the third term into two parts as

$$
\sum_{\substack{m_1 > \cdots > m_{B_{i-1}} \\ m_{M_{i-1}} - 1 > m_{B_{i-1}+2} > \cdots > m_{B_s} > 0}} = \sum_{n_1 > \cdots > m_{B_s} > 0} - \sum_{\substack{m_1 > \cdots > m_{B_{i-1}} \\ m_{B_{i-1}+2} = m_{B_{i-1}} - 1 \\ m_{B_{i-1}+2} > \cdots > m_{B_s} > 0}} .
$$

The later sum is equal to zero because of $m_{B_{i-1}+2} = m_{B_{i-1}} - 1$. Thus we have

$$
\begin{aligned}
&\lambda[\{\ldots, b_{i-1}, (a_i, 0), b_i, \ldots\}; \lambda] - [\{\ldots, b_{i-1}, (a_i - 1, 0), b_i, \ldots\}; \lambda] \\
&\quad = \lambda'[\{\ldots, b_{i-1}, (a_i, 1), b_i, \ldots\}; \lambda] - [\{\ldots, b_{i-1}, (a_i - 1, 1), b_i, \ldots\}; \lambda] \\
&\qquad + \sum_{m_1 > \cdots > m_{B_s} > 0} \frac{1}{A} \left\{ \frac{1}{(m_{B_{i-1}+2})^{a_i}} - \frac{1}{(m_{B_{i-1}} - 1)^{a_i}} \right\} \frac{1}{B} \\
&\quad = \lambda'[\{\ldots, b_{i-1}, (a_i, 1), b_i, \ldots\}; \lambda] - [\{\ldots, b_{i-1}, (a_i - 1, 1), b_i, \ldots\}; \lambda] \\
&\qquad - [\{\ldots, b_{i-1} - 1, (a_i, 1), b_i, \ldots\}; \lambda] + [\{\ldots, b_{i-1}, (a_i, 0), b_i - 1, \ldots\}; \lambda].
\end{aligned}
$$

(iii) (b) Repeating shift of $m_i \mapsto m_i + 1$, we have

$$
\begin{aligned}
&\lambda\left[\{(a_1, 1), b_1, \ldots, (a_{s-1}, 1), b_{s-1}, (a_s, 0), 1\}; \lambda\right] \\
&\quad - \lambda\left[\{(a_1, 1), b_1, \ldots, (a_{s-1}, 1), b_{s-1}, (a_s - 1, 0), 1\}; \lambda\right] \\
&\quad = - \sum_{m_1 > \cdots > m_{B_s} > 0} \frac{1}{(m_1 - 1)^{a_1} \cdots (m_{B_s - 1} - \lambda) m_{B_s}^{a_s}} \\
&\quad = - \sum_{m_1 > \cdots > m_{B_s} \geq 0} \frac{1}{m_1^{a_1} \cdots (m_{B_s - 1} - \lambda')(m_{B_s} + 1)^{a_s}} \\
&\qquad\quad \text{(by shift } m_i \longmapsto m_i + 1\text{)} \\
&\quad = - \sum_{m_1 > \cdots > m_{B_s - 1} \geq m_{B_s} > 0} \frac{1}{m_1^{a_1} \cdots (m_{B_s - 1} - \lambda') m_{B_s}^{a_s}} \\
&\qquad\quad \text{(by shift } m_{B_s} + 1 \longmapsto m_{B_s}\text{)} \\
&\quad = - \sum_{m_1 > \cdots > m_{B_s} > 0} \frac{m_{B_s} - \lambda'}{m_1^{a_1} \cdots (m_{B_s - 1} - \lambda') m_{B_s}^{a_s} (m_{B_s} - \lambda')}
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{m_1>\cdots>m_{B_s-1}>0}\frac{1}{m_1^{a_1}\cdots m_{B_s-1}^{a_s}(m_{B_s-1}-\lambda')}\\
&=\lambda'\,[\{(a_1,0),b_1,\ldots,b_{s-1}-1,(a_s,0),1\};\lambda']\\
&\quad-[\{(a_1,0),b_1,\ldots,b_{s-1},(a_s-1,0),1\};\lambda']\\
&\quad-[\{(a_1,0),b_1,\ldots,b_{s-1}-1,(a_s,0),1\};\lambda'].
\end{aligned}
$$

(a) Similarly as in the previous cases,

$$
\begin{aligned}
&[\{(a_i,1),b_i\}_{i=1}^{s};\lambda]\\
&=\sum_{m_1>\cdots>m_{B_s}>0}\frac{1}{(m_1-1)^{a_1}(m_1-\lambda)\ldots(m_{B_s-1}-\lambda)(m_{B_s}-\lambda)}\\
&=\sum_{m_1>\cdots>m_{B_s}\geq 0}\frac{1}{m_1^{a_1}(m_1-\lambda')\ldots(m_{B_s-1}-\lambda')(m_{B_s}-\lambda')}\\
&=(m_{B_s}>0\text{ part})+(b_{B_s}=0\text{ part})\\
&=[\{(a_i,0),b_i\}_{i=1}^{s};\lambda']-\frac{1}{\lambda'}[\{(a_i,0),b_i\}_{i=1}^{s-1}\cup\{(a_s,0),b_s-1\};\lambda'].
\end{aligned}
$$

□

Proof of Proposition 1. Using $[\{(a_i,d_i),b_i\}_{i=1}^{s}]$, the generating functions f and g are expressed as follows:

$$
\begin{cases}
f((a_i,b_i)_{i=1}^{s};\lambda)=[\{(a_i,0),b_i\}_{i=1}^{s};\lambda],\\
g((a_i,b_i)_{i=1}^{s};\lambda)=[\{(b_i,0),a_i\}_{i=s}^{1};\lambda].
\end{cases}
$$

If $a_1,b_s\geq 2$, applying Lemma 2 successively

$$
\begin{aligned}
&(\text{LHS})\\
&=\sum_{\{(\delta_i,\epsilon_i)\}_{i=1}^{s}\in I^s}(-\lambda)^{s-|\delta|-|\epsilon|}\,[\{(a_i-\delta_i,0),b_i-\epsilon_i\}_{i=1}^{s};\lambda]\\
&=\sum_{\{(\delta_i,\epsilon_i)\}_{i=2}^{s}\in I^{s-1}}\ \sum_{\delta_1'\in\{0,1\}}(-\lambda)^{s-1-|\delta|-|\epsilon|}(-\lambda')^{1-|\delta'|}\\
&\qquad\times[\{(a_1-\delta_1',1),b_1,(a_2-\delta_2,0),b_2-\epsilon_2,\ldots,(a_s-\delta_s,0),b_s-\epsilon_s\};\lambda]\\
&\qquad\qquad(\text{by Lemma 2 (ia)})\\
&=\sum_{\delta_1'\in\{0,1\}}\ \sum_{\{(\delta_j',\epsilon_j')\}_{j=2}^{s}\in I^{s-1}}(-\lambda')^{s-|\delta'|-|\epsilon'|}\\
&\quad\times[\{(a_1-\delta_1',1),b_1-\epsilon_2',(a_2-\delta_2',1),b_2-\delta_3',\ldots,(a_s-\delta_s',1),b_s\};\lambda]\\
&\qquad\qquad(\text{by Lemma 2 (ii) } s-1\text{ times})
\end{aligned}
$$

$$= \sum_{\delta'_1,\epsilon'_{s+1}\in\{0,1\}} \sum_{\{(\delta'_j,\epsilon'_j)\}_{j=2}^{s}\in I^{s-1}} (-\lambda')^{s-|\delta'|-|\epsilon'|}$$
$$\times[\{(a_1-\delta'_1,0),b_1-\epsilon'_2,(a_2-\delta'_2,0),b_2-\delta'_3,$$
$$\dots,(a_s-\delta'_s,0),b_s-\epsilon'_{s+1}\};\lambda']$$
(by Lemma 2 (iiia))
$$= (\text{RHS}).$$

Remaining relations and the relations of g's can be proved quite similarly. □

§7.2. Alternative proof of the Ohno relation

From the properties of the generating functions clarified in Section 2, we give an alternative proof for the Ohno relation

$$f((a_i,b_i)_{i=1}^s;\lambda) = g((a_i,b_i)_{i=1}^s;\lambda)$$

by induction on compositions.

If the composition is minimum i.e. $(a_i,b_i)_{i=1}^s = (1,1)$, it is obvious.

If the theorem is correct for compositions less than $(a_i,b_i)_{i=1}^s$, applying Proposition 1 to $(a_i,b_i)_{i=1}^s$ for f and g, we obtain two relations for f's and g's. Subtracting these two equations, we have

$$\sum \lambda^{s-|\delta|-|\epsilon|}\left\{f((a_i-\delta_i,b_i-\epsilon_i)_{i=1}^s;\lambda)-g((a_i-\delta_i,b_i-\epsilon_i)_{i=1}^s;\lambda)\right\}$$
$$=\sum {\lambda'}^{s-|\delta'|-|\epsilon'|}\left\{f((a_i-\delta'_i,b_i-\epsilon'_i)_{i=1}^s;\lambda')-g((a_i-\delta'_i,b_i-\epsilon'_i)_{i=1}^s;\lambda')\right\}.$$

But the terms whose compositions are less than $(a_i,b_i)_{i=1}^m$ are canceled out by the induction hypothesis. The remaining is

$$\lambda^s\left\{f((a_i,b_i)_{i=1}^s;\lambda)-g((a_i,b_i)_{i=1}^s;\lambda)\right\}$$
$$={\lambda'}^s\left\{f((a_i,b_i)_{i=1}^s;\lambda')-g((a_i,b_i)_{i=1}^s;\lambda')\right\}.$$

Hence $\lambda^s f((a_i,b_i)_{i=1}^s;\lambda)-\lambda^s g((a_i,b_i)_{i=1}^s;\lambda)$ is a periodic function in λ with period 1. Furthermore by Proposition 2 it is a meromorphic function such as

$$\lambda^s\sum_{p=1}^{\infty}\frac{C_k}{p-\lambda}.$$

Because of the periodicity, all C_k's must be zero. Thus we complete the proof.

Acknowledgements

The authors express their deep gratitude to Professor Masanobu Kaneko and Professor Shigeki Akiyama for their valuable suggestions.

The second author is partially supported by Grant-in-Aid Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology of Japan (12640046) and by Waseda University Grant for Special Research Project (2000A-124, 2001A-088 and 2002A-067).

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