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Distributions of Exponential Growth with Support in a Proper Convex Cone

By

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Abstract

In this paper we will characterize the spaces of distributions of exponential growth with support in a proper convex cone by the heat kernel method. As application we can obtain the Paley-Wiener theorem for distributions of exponential growth supported by a proper convex cone and Edge-of-the-Wedge theorem for the space of the image by the Fourier-Laplace transform of them.

§1. Introduction

In this paper we shall study the space $H'(\mathbb{R}^n, K)$ of distributions of exponential growth. The spaces of distributions of exponential growth for the 1-dimensional case, direct product case or global case were investigated by many authors ([5], [7], [11], [15], [16], [18], [21], [24]). In [5] M. Hasumi studied the space $H(\mathbb{R}^n, \mathbb{R}^n)$ and the dual space $H'(\mathbb{R}^n, \mathbb{R}^n)$ (see Definition 3.2 and Definition 3.7). In [15] M. Morimoto studied the space $H(\mathbb{R}^n, K)$ and the dual space $H'(\mathbb{R}^n, K)$ (see Definition 3.2 and Definition 3.7). The purpose of this paper is to treat the space of distributions of exponential growth *supported by a proper convex cone* $\overline{\Gamma} \subset \mathbb{R}^n$, (denote by $H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$).

In §3 we introduce the base space $H(\mathbb{R}^n, K)$ and its dual space $H'(\mathbb{R}^n, K)$. The main purpose in this section is to obtain the structure theorem for $H'_{\overline{A}}(\mathbb{R}^n, K)$, the space of distributions of exponential growth *supported by a*

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set $\overline{A} \subset \mathbb{R}^n$ (Theorem 3.10). Therefore as corollary we obtain the structure theorem for $H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$, where $\overline{\Gamma} \subset \mathbb{R}^n$ is a proper convex cone, (Corollary 3.12), and the result which G. Lysik obtained for the case of direct product support of half lines ([11]). Furthermore we have the decomposition theorem for distributions of exponential growth with support in $\overline{\Gamma}_+ \cup \overline{\Gamma}_-$, (Corollary 3.14).

In §4 we shall characterize the space $H'(\mathbb{R}^n, K)$ by the heat kernel method, which T. Matsuzawa introduced for the spaces of distributions, ultradistributions and hyperfunctions [4], [12], [13], [14]. The main purpose in this section is to show that the convolution of the heat kernel and a distribution of exponential growth is a smooth solution of the heat equation with some exponential growth condition and conversely such a smooth solution can be represented by the convolution of the heat kernel and a distribution of exponential growth (Theorem 4.4).

In §5 we shall characterize the space $H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$ by the heat kernel method (Theorem 5.1).

In §6 we shall study the Paley-Wiener theorem for $H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$ by using the structure theorem given in §3 and the heat kernel method given in §4, §5. Then we shall show that the Fourier-Laplace transform of $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$ is a holomorphic function constructed by a finite sum of functions which are holomorphic on the domains whose imaginary parts are proper convex cones with vertex at the elements of K and with some polynomial growth conditions and conversely such a holomorphic function can be represented by the Fourier-Laplace transform of a distribution of exponential growth $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$. Then we can see that T is constructed by a finite sum of distributions of exponential growth supported by a proper convex cone $\overline{\Gamma}$ (Theorem 6.9). As corollary we have the result which M. Morimoto showed for the 1-dimensional case [15].

In §7 we shall study the space of the image by the Fourier-Laplace transform of $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$. Then by using the Paley-Wiener theorem given in §6, we can obtain the Edge-of-the-Wedge theorem for this space (Theorem 7.11). These results are generalizations of the work which M. Morimoto showed for the case of direct product ([16], Theorem 2).

§2. Preliminaries

Definition 2.1. We define some notations:

$$x = (x_1, \dots, x_n) \in \mathbb{R}^n,\ x^2 = x_1^2 + \cdots + x_n^2.$$

$$\langle x, \xi \rangle = \sum_{j=1}^{n} x_j \xi_j \quad \text{for } x, \xi \in \mathbb{R}^n.$$
$$z = (z_1, \ldots, z_n) \in \mathbb{C}^n, \quad z_j = x_j + \imath y_j, \quad j = 1, \ldots, n.$$
$$\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n, \quad \zeta_j = \xi_j + \imath \eta_j, \quad j = 1, \ldots, n.$$
$$B(x_0, \delta) = \{x \in \mathbb{R}^n; |x - x_0| < \delta, \ \delta > 0\}.$$
$$\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n.$$
$$\alpha! = \alpha_1! \ldots \alpha_n!.$$
$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}, \quad \triangle = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$
$$E(x,t) = (4\pi t)^{-\frac{n}{2}} \exp(-x^2/4t), \quad t > 0.$$

For $\zeta \in \mathbb{C}^n$, $\zeta = (\zeta_1, \ldots, \zeta_n)$, we put $|\zeta| = \sqrt{|\zeta_1|^2 + \cdots + |\zeta_n|^2}$.

Definition 2.2. Let K be a convex compact set in $\mathbb{R}^n$. Then we define supporting function of K by $h_K(x) = \sup_{\xi \in K} \langle x, \xi \rangle$.

Definition 2.3. Let Ω be an open set in $\mathbb{C}^n$. We denote by $\mathcal{H}(\Omega)$ the space of holomorphic functions on Ω and by $\mathcal{C}(\Omega)$ the space of continuous functions on Ω.

Definition 2.4. $\mathcal{D}(\mathbb{R}^n)$ is the space of $\mathcal{C}^\infty$ functions with compact support. $\mathcal{S}(\mathbb{R}^n)$ is the space of rapidly decreasing $\mathcal{C}^\infty$ functions and $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions.

Definition 2.5. For a function $\varphi(\xi) \in \mathcal{S}(\mathbb{R}^n)$, the Fourier transform $\mathcal{F}(\varphi)(x)$ is defined by

$$\mathcal{F}(\varphi)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \varphi(\xi) e^{\imath \xi x} d\xi$$

and the Fourier inverse transform $\mathcal{F}^{-1}(\varphi)(\xi)$ is defined by

$$\mathcal{F}^{-1}(\varphi)(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \varphi(x) e^{-\imath \xi x} dx.$$

Definition 2.6. For $\varphi(x) \in \mathcal{S}(\mathbb{R}^n)$ and $\phi(x) \in \mathcal{S}(\mathbb{R}^n)$, the convolution of $\varphi(x)$ and $\phi(x)$ is defined by

$$(\varphi * \phi)(x) = \int_{\mathbb{R}^n} \varphi(x - y) \phi(y) dy.$$

Definition 2.7. For a function $\varphi(x)$ on $\mathbb{R}^n$, if $\varphi(x)e^{\imath\zeta x} \in L^1(\mathbb{R}^n_x)$, then $\mathcal{LF}(\varphi)(\zeta)$ is defined by

$$\mathcal{LF}(\varphi)(\zeta) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \varphi(x)e^{\imath\zeta x}dx, \quad \zeta \in \mathbb{C}^n.$$

Definition 2.8. Let A be a set in $\mathbb{R}^n$. Then we denote by A° the interior of A, $\overline{A}$ the closure of A, for $\varepsilon > 0$, $A_\varepsilon = \{x \in \mathbb{R}^n; \mathrm{dis}(x, A) \le \varepsilon\}$ and by $\mathrm{ch}(A)$ convex hull of A.

Definition 2.9. Let Γ be a cone with vertex at 0. If $\overline{\mathrm{ch}\Gamma}$ contains no straight line, then we call Γ proper cone.

Definition 2.10 ([6], [22])**.** Let Γ be a cone. We put

$$\Gamma' := \{\xi \in \mathbb{R}^n; \langle y, \xi\rangle \ge 0 \quad \text{for all } y \in \Gamma\}.$$

Then we call Γ' dual cone of Γ.

Definition 2.11. Let Γ be a cone. Then we denote by $\mathrm{pr}\Gamma$ the intersection of Γ and the unit sphere. The cone Γ_1 is said to be a compact cone in the cone Γ_2 if $\mathrm{pr}\overline{\Gamma}_1 \subset \mathrm{pr}\Gamma_2$ and we write $\Gamma_1 \Subset \Gamma_2$.

Proposition 2.12 ([22], [23])**.** *Following conditions are equivalent*:

1. Γ *is proper cone.*
2. $(\Gamma')^\circ \ne \emptyset$.
3. *For any* $C \Subset (\Gamma')^\circ$, *there exists a number* $\sigma = \sigma(C) > 0$ *such that* $\langle \xi, x\rangle \ge \sigma|\xi||x|$, $\xi \in C$, $x \in \mathrm{ch}\overline{\Gamma}$.

Proposition 2.13 ([22])**.** $(\Gamma')' = \overline{\mathrm{ch}\Gamma}$ *and* $(\Gamma_1 \cap \Gamma_2)' = \mathrm{ch}({\Gamma'}_1 \cup {\Gamma'}_2)$. *Furthermore for a convex cone* Γ, *we have* $\Gamma = \Gamma + \Gamma$.

Definition 2.14. Let Γ_+ be a cone with vertex at 0. Then we put $\Gamma_- = -\Gamma_+$.

Definition 2.15. Let A be a set in $\mathbb{R}^n$. We put $\mathcal{S}'_{\overline{A}} := \{T \in \mathcal{S}'(\mathbb{R}^n);\ \mathrm{supp}\, T \subset \overline{A}\}$.

For the structure of $\mathcal{S}'_{\overline{\Gamma}}$, the following proposition is known:

Proposition 2.16 (Bros-Epstein-Glaser [1], [17])**.** *Let* Γ *be a proper open convex cone in* $\mathbb{R}^n$ *and let* $T \in \mathcal{S}'_{\overline{\Gamma}}$. *Then there exists a polynomially bounded continuous function* G *with support in* $\overline{\Gamma}$ *and a partial differential operator with finite order* $P(D)$ *so that* $T = P(D)G$.

Proposition 2.17 ([17]). *Let $\Gamma_+ \in \mathbb{R}^n$ be a proper open convex cone and $S \in \mathcal{S}'_{\overline{\Gamma}_+ \cup \overline{\Gamma}_-}$, $\overline{\Gamma}_- = -\overline{\Gamma}_+$. Then there exist $S_+ \in \mathcal{S}'_{\overline{\Gamma}_+}$ and $S_- \in \mathcal{S}'_{\overline{\Gamma}_-}$ such that*

$$S = S_+ + S_-.$$

§3. Distributions of Exponential Growth

In this section, we shall introduce $H'(\mathbb{R}^n, K)$, the space of distributions of exponential growth, and give the structure theorem of $H'_{\overline{A}}(\mathbb{R}^n, K)$.

Definition 3.1. Let K be a convex compact set in $\mathbb{R}^n$ and $\varepsilon > 0$. Then we define $H_b(\mathbb{R}^n, K_\varepsilon)$ as follows:

$$H_b(\mathbb{R}^n, K_\varepsilon) := \{\varphi \in C^\infty(\mathbb{R}^n); \sup_{x \in \mathbb{R}^n} |D^p \varphi(x) e^{h_K(x) + \varepsilon |x|}| < +\infty, \text{ for } \forall p \in \mathbb{N}^n\}.$$

Definition 3.2. We define the spaces $H(\mathbb{R}^n, \mathbb{R}^n)$ and $H(\mathbb{R}^n, K)$ as follows:

$$H(\mathbb{R}^n, \mathbb{R}^n) := \varprojlim_{\varepsilon > 0} H_b(\mathbb{R}^n, K_\varepsilon),$$
$$H(\mathbb{R}^n, K) := \varinjlim_{\varepsilon > 0} H_b(\mathbb{R}^n, K_\varepsilon),$$

where $\varprojlim_{\varepsilon > 0}$ means projective limit and $\varinjlim_{\varepsilon > 0}$ means inductive limit.

Remark 3.3. Now we give the relations of $H(\mathbb{R}^n, K)$ and the other function spaces:

(i) $\mathcal{D} \subset H(\mathbb{R}^n, K)$.

(ii) If $\{0\} \subset K$, then $H(\mathbb{R}^n, K) \subset \mathcal{S}$.

(iii) Let $r \geq 0$, $s \geq 0$, $\mathcal{S}^s_r(\mathbb{R}^n)$ be Gel'fand-Shilov space and $\mathcal{S}_r(\mathbb{R}^n) = \varinjlim_{s \to \infty} \mathcal{S}^s_r(\mathbb{R}^n)$. Then it is known that

$$\mathcal{S}_1(\mathbb{R}^n) = \{f \in \mathcal{C}^\infty(\mathbb{R}^n); \exists \delta > 0 \; \forall \alpha \sup_{x \in \mathbb{R}^n} |D^\alpha_x f(x)| e^{\delta |x|} < \infty\},$$

(for details we refer the reader [18]). Therefore

(a) If $K = \{0\}$, then $H(\mathbb{R}^n, K) = \mathcal{S}_1(\mathbb{R}^n)$.

(b) If $\{0\} \subset K$, then $H(\mathbb{R}^n, K) \subset \mathcal{S}_1(\mathbb{R}^n)$.

(iv) The space $H(\mathbb{R}^n, K)$ is slightly different from $\mathfrak{H}_E$ in [3]. In fact

$$\varphi(x) \in H(\mathbb{R}^n, K) \Leftrightarrow \exists \varepsilon > 0\ \forall p \in \mathbb{N}^n\ s.t. \sup_{x \in \mathbb{R}^n} |D^p \varphi(x) e^{h_K(x)+\varepsilon|x|}| < \infty.$$

$$\varphi(x) \in \mathfrak{H}_E \Leftrightarrow \forall p \in \mathbb{N}^n\ \exists k > 0\ s.t. \sup_{x \in \mathbb{R}^n} |D^p \varphi(x)| e^{k|x|} < \infty.$$

Therefore if $\{0\} \subset K$, then $H(\mathbb{R}^n, K) \subset \mathfrak{H}_E$.

Remark 3.4. L. Hörmander treated the base space $\mathcal{S}_f$ so that $\mathcal{D} \subset \mathcal{S}_f \subset H(\mathbb{R}^n, K)$ and the Fourier-Laplace transform of $\mathcal{S}_f$. For the details we refer the reader to [7].

For (i) of Remark 3.3, the following theorem is known:

Theorem 3.5 ([15]). *$\mathcal{D}(\mathbb{R}^n)$ is dense in $H(\mathbb{R}^n, K)$.*

Corollary 3.6. *If $\{0\} \subset K$, then $H(\mathbb{R}^n, K)$ is dense in $\mathcal{S}$.*

Definition 3.7. We denote by $H'(\mathbb{R}^n, \mathbb{R}^n)$ the dual space of $H(\mathbb{R}^n, \mathbb{R}^n)$ and by $H'(\mathbb{R}^n, K)$ the dual space of $H(\mathbb{R}^n, K)$. The elements of $H'(\mathbb{R}^n, \mathbb{R}^n)$ and $H'(\mathbb{R}^n, K)$ are called distributions of exponential growth.

By Remark 3.3 and Theorem 3.5, we have $H'(\mathbb{R}^n, K) \subset \mathcal{D}'$.

For the space $H'(\mathbb{R}^n, K)$, the following proposition is known:

Proposition 3.8 ([15]). *A distribution T belongs to $H'(\mathbb{R}^n, K)$ if and only if for any $\varepsilon > 0$ there exist a partial differential operator $P_\varepsilon(D)$ and a bounded continuous function $F_\varepsilon(x)$ such that*

$$T = P_\varepsilon(D)\{e^{h_K(x)+\varepsilon|x|} F_\varepsilon(x)\}.$$

Definition 3.9. We put $H'_{\overline{A}}(\mathbb{R}^n, K) := \{T \in H'(\mathbb{R}^n, K); \operatorname{supp} T \subset \overline{A}\}$.

Now we have the structure theorem for distributions of exponential growth with support $\overline{A} \subset \mathbb{R}^n$:

Theorem 3.10. *Let A be a set in $\mathbb{R}^n$ and $T \in H'_{\overline{A}}(\mathbb{R}^n, K)$. Then for every $\varepsilon > 0$ there exist $S(x) \in \mathcal{S}'_{\overline{A}}$, $n_0 \in \mathbb{N}$ and $t_j \in K$, $j = 1, 2, \ldots, n_0$ such that*

$$T = S(x) e^{\varepsilon\sqrt{1+x^2}} \sum_{1 \le j \le n_0} e^{t_j x}.$$

Proof. Let $\varepsilon > 0$. By $K \subset \bigcup_{t\in K} B(t, \frac{\varepsilon}{2})$ and K is a compact set, there exists $n_0 \in \mathbb{N}$ such that $K \subset \bigcup_{1\le n\le n_0} B(t_n, \frac{\varepsilon}{2})$, $t_n \in K$.

Let $x_0 \in \mathbb{R}^n$. Then there exists $t'_{(x_0)} \in K$ and n_1, $1 \le n_1 \le n_0$ such that $h_K(x_0) = \langle t', x_0\rangle$ and $t' \in B(t_{n_1}, \frac{\varepsilon}{2})$. Furthermore

$$\begin{aligned} h_K(x_0) + \frac{\varepsilon}{2}|x_0| &\le h_{B(t_{n_1},\frac{\varepsilon}{2})}(x_0) + \frac{\varepsilon}{2}|x_0| \\ &\le t_{n_1}x_0 + h_{B(0,\frac{\varepsilon}{2})}(x_0) + \frac{\varepsilon}{2}|x_0| \\ &= t_{n_1}x_0 + \varepsilon|x_0| \\ &\le t_{n_1}x_0 + \varepsilon\sqrt{1+x_0^2}. \end{aligned}$$

Therefore, for any $x \in \mathbb{R}^n$,

$$e^{h_K(x)+\frac{\varepsilon}{2}|x|} \le e^{t_1x+\varepsilon\sqrt{1+x^2}} + \cdots + e^{t_{n_0}x+\varepsilon\sqrt{1+x^2}} = (e^{t_1x} + \cdots + e^{t_{n_0}x})e^{\varepsilon\sqrt{1+x^2}}.$$

Now we put

$$\begin{aligned} F_0(x) &:= \frac{1}{(e^{t_1x} + \cdots + e^{t_{n_0}x})e^{\varepsilon\sqrt{1+x^2}}}, \\ F_1(x) &:= \frac{1}{e^{t_1x} + \cdots + e^{t_{n_0}x}}, \\ F_2(x,t) &:= \frac{e^{tx}}{e^{t_1x} + \cdots + e^{t_{n_0}x}}, \quad t \in K. \end{aligned}$$

Then $F_0(x)$, $F_1(x)$, $F_2(x,t) \in \mathcal{C}^\infty(\mathbb{R}^n)$ and we have the following lemma:

Lemma 3.11. *Let $\alpha \in \mathbb{N}_0^n$. Then*

$$D^\alpha F_1(x) = P_\alpha(F_2(x,t_1), \dots, F_2(x,t_{n_0}))F_1(x), \tag{1}$$

where $P_\alpha(X_1, \dots, X_{n_0})$ is a polynomial.

Proof of Lemma. We use mathematical induction.

(i) Case of $|\alpha| = 0$. Then we obtain $P_\alpha = 1$.

(ii) Assume that when $|\alpha| = k$, (1) is true. Let $|\alpha| = k+1$. Then

$$\begin{aligned} D^\alpha F_1(x) &= \frac{\partial}{\partial x_j} D^\beta F_1(x) \quad (|\beta| = k) \\ &= \frac{\partial}{\partial x_j} P_\beta(F_2(x,t_1), \dots, F_2(x,t_{n_0}))F_1(x) \\ &\quad + P_\beta(F_2(x,t_1), \dots, F_2(x,t_{n_0}))\frac{\partial}{\partial x_j}F_1(x) \\ &= \Bigg\{\sum_{i=1}^{n_0} \frac{\partial}{\partial u_i} P_\beta(u_1, \dots, u_{n_0}) \times (t_{ij}F_2(x,t_i) - t_{1j}F_2(x,t_i)F_2(x,t_1) \end{aligned}$$

$$
\begin{aligned}
&-\cdots - t_{n_0 j}F_2(x,t_i)F_2(x,t_{n_0}))\Big\}F_1(x)\\
&- P_\beta(u_1,\dots,u_{n_0})\{t_{1j}F_2(x,t_1)+\cdots+t_{n_0 j}F_2(x,t_{n_0})\}F_1(x)\\
&\quad (u_1 = F_2(x,t_1),\dots,u_{n_0} = F_2(x,t_{n_0}))\\
&= P_\alpha(F_2(x,t_1),\dots,F_2(x,t_{n_0}))F_1(x).
\end{aligned}
$$
□

Since $0 < F_2(x,t_i) \le 1$, $\sup_{x\in\mathbb{R}^n}|P_\alpha| < \infty$. Therefore, for any $\varepsilon > 0$ there exists $\alpha \in \mathbb{N}_0^n$ such that

$$
\begin{aligned}
&|\langle F_0(x)T, \varphi(x)\rangle|\\
&\le \sup_{x\in\mathbb{R}^n} |D^\alpha(\varphi(x)F_0(x))e^{h_K(x)+\frac{\varepsilon}{2}|x|}|\\
&\le \sup_{x\in\mathbb{R}^n} \sum_{|m|=0}^{|\alpha|}\binom{\alpha}{m}\left|D^{\alpha-m}\varphi(x)\sum_{|l|=0}^{|m|}\binom{m}{l}D^{m-l}F_1(x)D^l(e^{-\varepsilon\sqrt{1+x^2}})e^{h_K(x)+\frac{\varepsilon}{2}|x|}\right|\\
&\le \sup_{x\in\mathbb{R}^n} \sum_{|m|=0}^{|\alpha|}\binom{\alpha}{m}\sum_{|l|=0}^{|m|}\binom{m}{l}|D^{\alpha-m}\varphi(x)||P_{m-l}||F_1(x)||Q_l|e^{-\varepsilon\sqrt{1+x^2}}e^{h_K(x)+\frac{\varepsilon}{2}|x|}\\
&\qquad\qquad\qquad\qquad \left(\sup_{x\in\mathbb{R}^n}|Q_l(x)| < \infty\right)\\
&\le \sup_{x\in\mathbb{R}^n} \sum_{|m|=0}^{|\alpha|}\binom{\alpha}{m}\sum_{|l|=0}^{|m|}\binom{m}{l}|D^{\alpha-m}\varphi(x)||P_{m-l}||Q_l||F_0(x)e^{h_K(x)+\frac{\varepsilon}{2}|x|}|\\
&\le C\sup_{x\in\mathbb{R}^n} \sum_{|m|=0}^{|\alpha|}\binom{\alpha}{m}\sum_{|l|=0}^{|m|}\binom{m}{l}|D^{\alpha-m}\varphi(x)|.
\end{aligned}
$$

This means that for $T \in H'(\mathbb{R}^n, K)$, $F_0(x)T \in \mathcal{S}'$ and if supp $T \subset \overline{A}$, then supp $F_0(x)T \subset \overline{A}$. For $\varphi(x) \in H(\mathbb{R}^n, K)$, we have

$$
\begin{aligned}
\langle T,\varphi\rangle &= \left\langle F_0(x)T \times \frac{1}{F_0(x)}, \varphi\right\rangle \\
&= \left\langle S(x)\frac{1}{F_0(x)}, \varphi\right\rangle.
\end{aligned}
\tag{2}
$$

So we obtain

$$
T = S(x)e^{\varepsilon\sqrt{1+x^2}}\sum_{1\le j\le n_0} e^{t_j x}, \quad S \in \mathcal{S}'_{\overline{A}}.
$$

□

For $H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$, we have the following corollary:

Corollary 3.12. *Let Γ be a proper open convex cone in $\mathbb{R}^n$ and let $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$. Then for any $\varepsilon > 0$ there exist $m_\varepsilon \in \mathbb{N}$ and bounded continuous functions $F_{\varepsilon,\alpha}(x)$, $|\alpha| \leq m_\varepsilon$, $\operatorname{supp}(F_{\varepsilon,\alpha}(x)) \subset \overline{\Gamma}$ such that*

$$T = \sum_{|\alpha|\leq m_\varepsilon} \left(\frac{\partial}{\partial x}\right)^\alpha \{e^{h_K(x)+\varepsilon|x|} F_{\varepsilon,\alpha}(x)\}.$$

Proof. By Lemma 2.16 and (2), for $\varphi(x) \in H(\mathbb{R}^n, K)$,

$$\begin{aligned}\langle T, \varphi\rangle &= \left\langle F_0(x)T, \frac{1}{F_0(x)}\varphi(x)\right\rangle \\ &= \left\langle P(D)G(x)\frac{1}{F_0(x)}, \varphi(x)\right\rangle.\end{aligned}$$

Therefore, for any $\varepsilon > 0$ there exist a partial differential operator $P(D)$ and a polynomially bounded continuous functions $G(x)$ with support in $\overline{\Gamma}$ such that

$$T = P(D)G(x) \times F^*(x), \quad F^*(x) = \frac{1}{F_0(x)} = e^{\varepsilon\sqrt{1+x^2}} \sum_{1\leq j\leq n_0} e^{t_j x}. \tag{3}$$

Let $\varepsilon_1 > 0$. For $\varphi(x) \in H(\mathbb{R}^n, K)$,

$$\begin{aligned}&\langle T, \varphi\rangle \\ &= \langle G(x), P(-D)(F^*(x)\varphi(x))\rangle \\ &= \langle G(x)e^{h_K(x)+\varepsilon_1|x|}, e^{-h_K(x)-\varepsilon_1|x|}P(-D)(F^*(x)\varphi(x))\rangle \\ &= \left\langle e^{h_K(x)+\varepsilon_1|x|}, G(x)e^{-h_K(x)-\varepsilon_1|x|} \sum_{|m_1|\leq m}\sum_{|\alpha|=0}^{|m_1|} \binom{m_1}{\alpha} D^{m_1-\alpha}F^*(x)D^\alpha\varphi(x)\right\rangle.\end{aligned}$$

Now we put

$$\begin{aligned}F_{3(\varepsilon,\varepsilon_1,m_1,\alpha)}(x) &:= G(x)e^{-h_K(x)-\varepsilon_1|x|}D^{m_1-\alpha}F^*(x) \\ &= G(x)e^{-h_K(x)-\varepsilon_1|x|}A_{m_1,\alpha}(t_1,\dots,t_{n_0},x,\varepsilon)F^*(x).\end{aligned}$$

Then $F_3(x) \in \mathcal{C}(\mathbb{R}^n)$, $\operatorname{supp} F_3(x) \subset \overline{\Gamma}$ and

$$\begin{aligned}&|F_3(x)| \leq C(1+|x|)^M e^{-h_K(x)-\varepsilon_1|x|}|A_{m_1,\alpha}(t_1,\dots,t_{n_0},x,\varepsilon)||F^*(x)|, \\ &\sup_{x\in\mathbb{R}^n} |A_{m_1,\alpha,}(t_1,\dots,t_{n_0},x,\varepsilon)| < \infty.\end{aligned}$$

Now we choose ε in (3) such that $0 < \varepsilon < \varepsilon_1$. Then

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^n}(1+|x|)^M e^{-h_K(x)-\varepsilon_1|x|}|F^*(x)| \\
&\le \sup_{x\in\mathbb{R}^n}(1+|x|)^M e^{-h_K(x)-\varepsilon_1|x|}(e^{h_K(x)}+\cdots+e^{h_K(x)})e^{\varepsilon(1+|x|)} \\
&\le e^{\varepsilon}\sup_{x\in\mathbb{R}^n}(1+|x|)^M e^{-(\varepsilon_1-\varepsilon)|x|} \\
&< \infty.
\end{aligned}$$

This means that $\sup_{x\in\mathbb{R}^n}|F_3(x)| < \infty$. Furthermore

$$\begin{aligned}
&\langle T, \varphi\rangle \\
&= \sum_{|m_1|\le m}\sum_{|\alpha|=0}^{|m_1|}\binom{m_1}{\alpha}\langle e^{h_K(x)+\varepsilon_1|x|}, F_3(x)D^\alpha\varphi(x)\rangle \\
&= \left\langle \sum_{|m_1|\le m}\sum_{|\alpha|=0}^{|m_1|}\binom{m_1}{\alpha}(-1)^{|\alpha|}D^\alpha(e^{h_K(x)+\varepsilon_1|x|}F_3(x)), \varphi(x)\right\rangle \\
&= \left\langle \sum_{|\alpha|\le m_\varepsilon}\left(\frac{\partial}{\partial x}\right)^\alpha(e^{h_K(x)+\varepsilon_1|x|}F_{\varepsilon_1,\alpha}(x)), \varphi(x)\right\rangle.
\end{aligned}$$

Since $\varepsilon_1 > 0$ is arbitrary, the proof is complete. □

By (3), we have the following corollary:

Corollary 3.13. *Let Γ be a proper open convex cone in $\mathbb{R}^n$ and let $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$. Then for any $\varepsilon > 0$ there exist n_0, a partial differential operator with finite order $P_\varepsilon(D)$ and a polynomially bounded continuous function $G_\varepsilon(x)$,* $\mathrm{supp}(G_\varepsilon(x)) \subset \overline{\Gamma}$ *such that*

$$T = P_\varepsilon(D)G_\varepsilon(x)\times F^*(x), \qquad F^*(x) = e^{\varepsilon\sqrt{1+x^2}}\sum_{1\le n\le n_0}e^{t_n x},$$

where $t_n \in K$, $(n = 1, \ldots, n_0)$.

Using Proposition 2.17, we have the following corollary:

Corollary 3.14. *Let $T \in H'_{\overline{\Gamma_+}\cup\overline{\Gamma_-}}(\mathbb{R}^n, K)$. Then there exist $T_+ \in H'_{\overline{\Gamma_+}}(\mathbb{R}^n, K)$ and $T_- \in H'_{\overline{\Gamma_-}}(\mathbb{R}^n, K)$ such that*

$$T = T_+ + T_-.$$

Proof. By Theorem 3.10, we have

$$T = \sum_{1 \le j \le n_0} S(x) e^{t_j x + \varepsilon\sqrt{1+x^2}}, \quad S \in \mathcal{S}'_{\overline{\Gamma_+ \cup \Gamma_-}}.$$

By Proposition 2.17, we have

$$\begin{aligned} T &= \sum_{1 \le j \le n_0} S_+(x) e^{t_j x + \varepsilon\sqrt{1+x^2}} + \sum_{1 \le j \le n_0} S_-(x) e^{t_j x + \varepsilon\sqrt{1+x^2}} \\ &\equiv T_+ + T_-. \end{aligned}$$

Since $S_+ \in \mathcal{S}'_{\overline{\Gamma_+}}$ and $S_- \in \mathcal{S}'_{\overline{\Gamma_-}}$, $T_+ \in H'_{\overline{\Gamma_+}}(\mathbb{R}^n, K)$ and $T_- \in H'_{\overline{\Gamma_-}}(\mathbb{R}^n, K)$. $\square$

Remark 3.15. M. Morimoto obtained this result for the 1-dimensional case in [15].

Example 3.16 (Example for Corollary 3.12)**.** Let $n = 2$, $K = \overline{B(0,1)}$ and $\Gamma := \{x = (x_1, x_2) \in \mathbb{R}^2; x_1^2 - x_2^2 > 0, x_1 > 0\}$. We define $T(x)$ by

$$T(x) = \begin{cases} \sqrt{x_1^2 - x_2^2} e^{|x|}, & x_1^2 - x_2^2 > 0,\ x_1 > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then $h_K(x) = |x|$, $T(x) \in H'_{\overline{\Gamma}}(\mathbb{R}^2, K)$ and for $\varepsilon > 0$,

$$T(x) = \sqrt{x_1^2 - x_2^2} e^{|x|} = \sqrt{x_1^2 - x_2^2} e^{-\varepsilon|x|} e^{|x|} e^{\varepsilon|x|} = F_\varepsilon(x) e^{h_K(x) + \varepsilon|x|},$$

where

$$F_\varepsilon(x) = \begin{cases} \sqrt{x_1^2 - x_2^2}\ e^{-\varepsilon|x|}, & x_1^2 - x_2^2 > 0,\ x_1 > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then $F_\varepsilon(x)$ is a bounded continuous function and $\mathrm{supp}(F_\varepsilon) \subset \overline{\Gamma}$.

§4. A Characterization for Distributions of Exponential Growth by the Heat Kernel Method

In this section, we shall characterize $H'(\mathbb{R}^n, K)$, the space of distributions of exponential growth, by the heat kernel method introduced by T. Matsuzawa in [12]. We notice that many authors make use of his idea ([2], [3], [9], [10], [20]).

Definition 4.1. For $\varphi(x) \in H(\mathbb{R}^n, K)$, we put $\varphi_t(x)$ by

$$\varphi_t(x) = \int_{\mathbb{R}^n} E(x - y, t) \varphi(y) dy, \quad t > 0.$$

We show the following lemma:

Lemma 4.2.

(i) $E(x,t) \in H(\mathbb{R}^n, K)$,

(ii) $\varphi_t(x) \in H(\mathbb{R}^n, K)$,

(iii) $\varphi_t(x) \to \varphi(x)$ *in* $H(\mathbb{R}^n, K)$, *as* $t \to 0_+$.

Proof. (i) Let $\varepsilon > 0$ and $K \subset [-R,R]^n$, $R > 0$. Then

$$\sup_{x\in\mathbb{R}^n} |D^\alpha E(x,t) e^{h_K(x)+\varepsilon|x|}| \le \sup_{x\in\mathbb{R}^n} |D^\alpha E(x,t) e^{(R+\varepsilon)|x|}|.$$

For the heat kernel we have the following estimate [14]:

$$|D^\alpha E(x,t)| \le \frac{\alpha!}{(4\pi t)^{\frac{n}{2}}} \left(\frac{en}{2t|\alpha|}\right)^{\frac{|\alpha|}{2}} e^{-\frac{x^2}{8t}}. \tag{4}$$

So we have

$$\begin{aligned}\sup_{x\in\mathbb{R}^n} |D^\alpha E(x,t) e^{h_K(x)+\varepsilon|x|}| &\le C \sup_{x\in\mathbb{R}^n} e^{-\frac{x^2}{8t}+(R+\varepsilon)|x|} \\ &< \infty.\end{aligned}$$

This means that $E(x,t) \in H(\mathbb{R}^n, K)$.

(ii) Let $\alpha \in \mathbb{N}_0^n$. For $\varphi(x) \in H(\mathbb{R}^n, K)$, we have by (4),

$$\begin{aligned}|D_x^\alpha E(x-y,t)\varphi(y)| &\le Ce^{-\frac{(x-y)^2}{8t}} e^{-h_K(y)-\varepsilon|y|} \\ &\le Ce^{-\frac{(x-y)^2}{8t}} e^{h_K(x-y)-h_K(x)+\varepsilon|x-y|-\varepsilon|x|} \\ &\le Ce^{-h_K(x)-\varepsilon|x|} e^{-\frac{(x-y)^2}{8t}+R|x-y|+\varepsilon|x-y|},\end{aligned} \tag{5}$$

where $K \subset [-R,R]^n$, $R > 0$. Since $e^{-\frac{(x-y)^2}{8t}+R|x-y|+\varepsilon|x-y|} \in L^1(\mathbb{R}_y^n)$,

$$D_x^\alpha \int_{\mathbb{R}^n} E(x-y,t)\varphi(y)dy = \int_{\mathbb{R}^n} D_x^\alpha E(x-y,t)\varphi(y)dy.$$

Since $E(x,t) \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$ is arbitrary, $\varphi_t(x) \in \mathcal{C}^\infty(\mathbb{R}^n)$. Furthermore by (5),

$$\begin{aligned}|D^\alpha \varphi_t(x) e^{h_K(x)+\varepsilon|x|}| &\le e^{h_K(x)+\varepsilon|x|} \int_{\mathbb{R}^n} |D^\alpha E(x-y,t)\varphi(y)|dy \\ &\le Ce^{h_K(x)+\varepsilon|x|} \int_{\mathbb{R}^n} e^{-h_K(x)-\varepsilon|x|} e^{-\frac{(x-y)^2}{8t}+R|x-y|+\varepsilon|x-y|} dy \\ &< \infty.\end{aligned}$$

Therefore, $\varphi_t(x) \in H(\mathbb{R}^n, K)$.

(iii) We notice that for $|y| \leq 1$,

$$
\begin{aligned}
(6)\qquad |D_x^\alpha \varphi(x-y) - D_x^\alpha \varphi(x)| &= \left| \int_0^1 D_u D_x^\alpha \varphi(u(x-y) + (1-u)x) du \right| \\
&= \left| \int_0^1 D_u F(v) du \right|, \\
&\quad (v = u(x-y) + (1-u)x, \quad F(v) = D^\alpha \varphi(v)) \\
&= \left| \int_0^1 \sum_{j=1}^n D_{v_j} F(v) \frac{\partial v_j}{\partial u} du \right| \\
&\leq \int_0^1 \sum_{j=1}^n |D^{\alpha+1} \varphi(u(x-y) + (1-u)x)||y_j| du \\
&\leq C \int_0^1 \sum_{j=1}^n e^{-h_K(x-uy) - \varepsilon|x-uy|} |y_j| du \\
&\leq C_1 e^{-h_K(x) - \varepsilon|x|} \sum_{j=1}^n |y_j|.
\end{aligned}
$$

Let $0 < \delta < 1$. Then

$$
\begin{aligned}
&D^\alpha(\varphi_t(x) - \varphi(x)) \\
&= D^\alpha \int_{\mathbb{R}^n} E(w,t) \varphi(x-w) dw - \int_{\mathbb{R}^n} E(w,t) D^\alpha \varphi(x) dw \\
&= \int_{\mathbb{R}^n} E(w,t)(D^\alpha \varphi(x-w) - D^\alpha \varphi(x)) dw \\
&= \int_{|w| \leq \delta} E(w,t)(D^\alpha \varphi(x-w) - D^\alpha \varphi(x)) dw \\
&\quad + \int_{|w| \geq \delta} E(w,t) D^\alpha \varphi(x-w) dw - \int_{|w| \geq \delta} E(w,t) D^\alpha \varphi(x) dw \\
&= I_1 + I_2 + I_3.
\end{aligned}
$$

By (6),

$$
\begin{aligned}
|I_1| e^{h_K(x) + \varepsilon|x|} &\leq \int_{|y| \leq \delta} E(y,t) |D^\alpha \varphi(x-y) - D^\alpha \varphi(x)| e^{h_K(x) + \varepsilon|x|} dy \\
&\leq C_3 \delta \int_{|y| \leq \delta} E(y,t) dy \\
&\leq C_3 \delta.
\end{aligned}
$$

$$\begin{aligned}|I_3|e^{h_K(x)+\varepsilon|x|} &\le \int_{|y|\ge\delta} E(y,t)dy \times |D^\alpha\varphi(x)|e^{h_K(x)+\varepsilon|x|}\\ &\le C'e^{-\frac{\delta^2}{8t}}\int_{|y|\ge\delta}\frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{y^2}{8t}}dy\\ &\le C''e^{-\frac{\delta^2}{8t}} \to 0 \quad \text{as} \ \ t\to 0_+.\end{aligned}$$

$$\begin{aligned}|I_2|e^{h_K(x)+\varepsilon|x|} &\le \int_{|y|\ge\delta} E(y,t)|D^\alpha\varphi(x-y)|dy \times e^{h_K(x)+\varepsilon|x|}\\ &\le C\int_{|y|\ge\delta} E(y,t)e^{-h_K(x-y)-\varepsilon|x-y|}dy \times e^{h_K(x)+\varepsilon|x|}\\ &\le C\int_{|y|\ge\delta} E(y,t)e^{h_K(y)-h_K(x)+\varepsilon|y|-\varepsilon|x|}dy \times e^{h_K(x)+\varepsilon|x|}\\ &\le C\int_{|y|\ge\delta} E(y,t)e^{R|y|+\varepsilon|y|}dy\\ &\le Ce^{-\frac{\delta^2}{8t}}\int_{\mathbb{R}^n}\frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\sum_{j=1}^n\{\frac{1}{8t}y_j^2-(R+\varepsilon)|y_j|\}}dy\\ &= C''e^{-\frac{\delta^2}{8t}}e^{2n(R+\varepsilon)^2t} \to 0, \quad \text{as } t\to 0_+.\end{aligned}$$

Since $0<\delta<1$ is arbitrary, this means that $\varphi_t(x)\to\varphi(x)$ in $H(\mathbb{R}^n,K)$, as $t\to 0_+$. □

Lemma 4.3. *Let $f(x)$ be a measurable function satisfying the following condition*:

$$\forall\varepsilon>0 \quad \exists C\ge 0 \quad \textit{such that} \quad |f(x)|\le Ce^{h_K(x)+\varepsilon|x|}.$$

Then $f(x)$ belongs to $H'(\mathbb{R}^n,K)$ in the following sense:

$$\langle T_f,\varphi\rangle = \int_{\mathbb{R}^n} f(x)\varphi(x)dx, \quad \varphi(x)\in H(\mathbb{R}^n,K).$$

Proof. We only prove the continuity. Let $\varepsilon'>0$, $\varphi(x)\in H_b(\mathbb{R}^n,K_{\varepsilon'})$ and $0<\varepsilon<\varepsilon'$. Since there exists a constant $C_1\ge 0$ such that $|f(x)|\le C_1e^{h_K(x)+\varepsilon|x|}$,

$$\begin{aligned}|\langle T_f,\varphi\rangle| &\le \int_{\mathbb{R}^n}|f(x)||\varphi(x)e^{h_K(x)+\varepsilon'|x|}|e^{-h_K(x)-\varepsilon'|x|}dx\\ &\le C_1\sup_{x\in\mathbb{R}^n}|\varphi(x)e^{h_K(x)+\varepsilon'|x|}|\int_{\mathbb{R}^n}e^{(\varepsilon-\varepsilon')|x|}dx\\ &\le C\sup_{x\in\mathbb{R}^n}|\varphi(x)e^{h_K(x)+\varepsilon'|x|}|.\end{aligned}$$

Since $\varepsilon'>0$ is arbitrary, the continuity is proved. □

Theorem 4.4. *Let $T \in H'(\mathbb{R}^n, K)$ and $U(x,t) = \langle T_y, E(x-y,t)\rangle$. Then $U(x,t) \in \mathcal{C}^\infty(\mathbb{R}^n \times (0,\infty))$ satisfying the following conditions*:

(7) $$\left(\frac{\partial}{\partial t} - \triangle\right) U(x,t) = 0,$$

(8) $$U(x,t) \to T,\ (t \to 0_+),\ \textit{in } H'(\mathbb{R}^n, K),$$

(9) $$\forall \varepsilon > 0\ \exists N_\varepsilon \geq 0\ \exists C_\varepsilon \geq 0$$
$$\textit{s.t. } |U(x,t)| \leq C_\varepsilon t^{-N_\varepsilon} e^{h_K(x)+\varepsilon|x|},\ 0 < t < 1,\ x \in \mathbb{R}^n.$$

Conversely, for a function $U(x,t) \in \mathcal{C}^\infty(\mathbb{R}^n \times (0,\infty))$ satisfying (7) *and* (9), *there exists a unique $T \in H'(\mathbb{R}^n, K)$ such that $\langle T_y, E(x-y,t)\rangle = U(x,t)$.*

Proof. By Proposition 3.8, we have

(10) $$\begin{aligned}\langle T_y, E(x-y,t)\rangle &= \langle e^{h_K(y)+\varepsilon|y|} F(y), P(-D)E(x-y,t)\rangle \\ &= \sum_{|\alpha|=0}^{m} \int_{\mathbb{R}^n} e^{h_K(y)+\varepsilon|y|} F(y) D^\alpha E(x-y,t)dy.\end{aligned}$$

By $\sup_{y\in\mathbb{R}^n} |F(y)| < \infty$ and (4), for $\triangle$,

$$\begin{aligned}|e^{h_K(y)+\varepsilon|y|} F(y) \triangle \{D^\alpha E(x-y,t)\}| &\leq C e^{-\frac{(x-y)^2}{8t}} e^{h_K(y)+\varepsilon|y|} \\ &\leq C e^{-\frac{(x-y)^2}{8t}} e^{-h_K(x)-\varepsilon|x|+h_K(x-y)+\varepsilon|x-y|} \\ &\leq C e^{-h_K(x)-\varepsilon|x|} e^{-\frac{(x-y)^2}{8t}} e^{R|x-y|+\varepsilon|x-y|} \\ &= C_1 e^{2n(R+\varepsilon)^2 t} e^{-\sum_{j=1}^n \frac{\{|x_j-y_j|-4(R+\varepsilon)t\}^2}{8t}},\end{aligned}$$

where $K \subset [-R,R]^n$, $R > 0$. Since $e^{-\sum_{j=1}^n \frac{\{|x_j-y_j|-4(R+\varepsilon)t\}^2}{8t}} \in L^1(\mathbb{R}^n_y)$,

$$\begin{aligned}\triangle \langle T_y, E(x-y,t)\rangle &= \triangle \sum_{|\alpha|=0}^{m} \int_{\mathbb{R}^n} e^{h_K(y)+\varepsilon|y|} F(y) D^\alpha E(x-y,t)dy \\ &= \sum_{|\alpha|=0}^{m} \int_{\mathbb{R}^n} e^{h_K(y)+\varepsilon|y|} F(y) D^\alpha \{\triangle E(x-y,t)\}dy.\end{aligned}$$

Let $0 < a_0 < t < a_1$. Since

$$\left|\frac{\partial}{\partial t} E(x-y,t)\right| \leq \frac{(4\pi a_1)^{\frac{n}{2}} \frac{(x-y)^2}{4a_0{}^2} + 2\pi n (4\pi a_1)^{\frac{n}{2}-1}}{(4\pi a_0)^n} e^{-\frac{(x-y)^2}{4a_1}},$$

$$\begin{aligned}
&\left|e^{h_K(y)+\varepsilon|y|}\frac{\partial}{\partial t}E(x-y,t)\right| \\
&\le \frac{(4\pi a_1)^{\frac{n}{2}}\frac{(x-y)^2}{4{a_0}^2}+2\pi n(4\pi a_1)^{\frac{n}{2}-1}}{(4\pi a_0)^n}e^{-\frac{(x-y)^2}{4a_1}}e^{-h_K(x)-\varepsilon|x|+h_K(x-y)+\varepsilon|x-y|} \\
&\le Ce^{-\frac{(x-y)^2}{8t}}e^{R|x-y|+\varepsilon|x-y|} \\
&= C_1e^{2n(R+\varepsilon)^2t}e^{-\sum_{j=1}^n\frac{\{|x_j-y_j|-4(R+\varepsilon)t\}^2}{8t}},
\end{aligned}$$

where $K\subset[-R,R]^n$, $R>0$. Since $e^{-\sum_{j=1}^n\frac{\{|x_j-y_j|-4(R+\varepsilon)t\}^2}{8t}}\in L^1(\mathbb{R}^n_y)$ and $a_0>0$, $a_1>0$ are arbitrary, for $t>0$

$$\begin{aligned}
\frac{\partial}{\partial t}\langle T_y,E(x-y,t)\rangle &= \frac{\partial}{\partial t}\sum_{|\alpha|=0}^m\int_{\mathbb{R}^n}e^{h_K(y)+\varepsilon|y|}D^\alpha E(x-y,t)dy \\
&= \sum_{|\alpha|=0}^m\int_{\mathbb{R}^n}e^{h_K(y)+\varepsilon|y|}D^\alpha\left\{\frac{\partial}{\partial t}E(x-y,t)\right\}dy.
\end{aligned}$$

Since $\left(\frac{\partial}{\partial t}-\triangle\right)E(x,t)=0$, we have

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}-\triangle\right)\langle T_y,E(x-y,t)\rangle \\
&=\sum_{|\alpha|=0}^m\int_{\mathbb{R}^n}e^{h_K(y)+\varepsilon|y|}D^\alpha\left\{\left(\frac{\partial}{\partial t}-\triangle\right)E(x-y,t)\right\}dy=0.
\end{aligned}$$

Let $\varphi(x)\in H(\mathbb{R}^n,K)$, $\varphi(x)\in H_b(\mathbb{R}^n,K_{\varepsilon_1})$ and $0<\varepsilon<\varepsilon_1$. By Proposition 3.8, (4) and (10),

$$\begin{aligned}
&\int_{\mathbb{R}^n}|U(x,t)\varphi(x)|dx \\
&\le\sum_{|\alpha|=0}^m\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|e^{h_K(y)+\varepsilon|y|}F(y)D^\alpha E(x-y,t)\varphi(x)|dydx \\
&\le C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{h_K(y)+\varepsilon|y|}e^{-\frac{(x-y)^2}{8t}}e^{-h_K(x)-\varepsilon_1|x|}dydx \\
&\le C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{h_K(y-x)+h_K(x)+\varepsilon|y-x|+\varepsilon|x|}e^{-\frac{(y-x)^2}{8t}}e^{-h_K(x)-\varepsilon_1|x|}dydx \\
&\le Ce^{2n(R+\varepsilon)^2t}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{-\sum_{j=1}^n\frac{1}{8t}\{(y_j-x_j)-4(R+\varepsilon)t\}^2}e^{(\varepsilon-\varepsilon_1)|x|}dydx \\
&<\infty.
\end{aligned}$$

By Fubini's theorem,

$$\begin{aligned}\langle U(x,t),\varphi(x)\rangle &= \int_{\mathbb{R}^n}\langle T_y, E(x-y,t)\rangle\varphi(x)dx\\ &= \int_{\mathbb{R}^n} e^{h_K(y)+\varepsilon|y|}F(y)P(-D)\int_{\mathbb{R}^n}E(x-y,t)\varphi(x)dxdy\\ &= \langle T_y,\varphi_t(y)\rangle.\end{aligned}$$

By Lemma 4.2 (iii),

$$\begin{aligned}\langle U(x,t),\varphi(x)\rangle &= \langle T_y,\varphi_t(y)\rangle\\ &\to \langle T_y,\varphi(y)\rangle, \quad \text{as } t\to 0_+.\end{aligned}$$

This means that $U(x,t)\to T$ in $H'(\mathbb{R}^n,K)$.

Let $0<t<1$. By Proposition 3.8, (4) and (10),

$$\begin{aligned}&|\langle T_y, E(x-y,t)\rangle|\\ &\le C_1\sum_{|\alpha|=0}^{m}\int_{\mathbb{R}^n} e^{h_K(y)+\varepsilon|y|}\frac{\alpha!}{(4\pi t)^{\frac{n}{2}}}\left(\frac{en}{2t|\alpha|}\right)^{\frac{|\alpha|}{2}} e^{-\frac{(x-y)^2}{8t}}dy\\ &\le C_2 t^{-N}e^{h_K(x)+\varepsilon|x|}\int_{\mathbb{R}^n} e^{-\sum_{j=1}^n \frac{1}{8t}(y_j-x_j)^2+(R+\varepsilon)|y_j-x_j|}dy\\ &\le C_2 t^{-N}e^{2n(R+\varepsilon)^2t}e^{h_K(x)+\varepsilon|x|}\int_{\mathbb{R}^n} e^{-\sum_{j=1}^n\frac{1}{8t}\{|y_j-x_j|-4(R+\varepsilon)t\}^2}dy\\ &\le Ct^{-N}e^{h_K(x)+\varepsilon|x|}, \quad 0<t<1, \quad x\in\mathbb{R}^n.\end{aligned}$$

Now we shall prove the converse. For positive integer m, we put

$$f_m(t)=\begin{cases}\dfrac{1}{(m-1)!}t^{m-1} & (t\ge 0)\\[2ex] 0 & (t<0).\end{cases}$$

Let $u(t)$ be a $\mathcal{C}^\infty$ function such that

$$u(t)=\begin{cases}1\left(t\le\dfrac{t_1}{4}\right)\\[2ex] 0\left(t\ge\dfrac{t_1}{2}\right), \quad t_1>0,\end{cases}$$

and we set $v_m(t) = f_m(t)u(t)$. Then

$$v_m(t) = \begin{cases} f_m(t) & \left(t \leq \frac{t_1}{4}\right) \\ 0 & \left(t \geq \frac{t_1}{2}\right) \end{cases}$$

and

$$\left(\frac{d}{dt}\right)^m v_m(t) = \delta(t) + w(t), \tag{11}$$

where $w(t) \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $\delta(t)$ is Dirac's delta function, supp $w \subset \left[\frac{t_1}{4}, \frac{t_1}{2}\right]$. Now we put $m = N+2$, $v_{N+2}(t) = v(t)$ and

$$\tilde{U}(x,t) = \int_0^\infty U(x, t+s)v(s)ds.$$

By supp $v(s) \subset \left[0, \frac{t_1}{2}\right]$,

$$\begin{aligned} |\tilde{U}(x,t)| &\leq \int_0^{\frac{t_1}{2}} |U(x,t+s)||v(s)|ds \\ &\leq C_1 e^{h_K(x)+\varepsilon|x|} \int_0^{\frac{t_1}{2}} \frac{s^{N+1}}{(N+1)!(t+s)^N} ds \\ &\leq C e^{h_K(x)+\varepsilon|x|}. \end{aligned} \tag{12}$$

Since

$$U(x,t+s)v(s) = \begin{cases} U(x,t+s)v(s), (s>0) \\ 0, \qquad (s=0), \end{cases}$$

$\lim_{t \to 0_+} U(x,t+s)v(s) =: U(x,s)v(s)$ exists in $s \geq 0$. Therefore, by Lebesgue's dominated convergence theorem,

$$\begin{aligned} \lim_{t\to 0_+} \tilde{U}(x,t) &= \int_0^\infty \lim_{t \to 0_+} U(x,t+s)v(s)ds \\ &= \int_0^\infty U(x,s)v(s)ds. \end{aligned}$$

This means that $\tilde{U}(x,t)$ is a continuous function in $t \geq 0$. Now we put $g(x) = \tilde{U}(x,0)$. By (12),

$$|g(x)| \leq Ce^{h_K(x)+\varepsilon|x|}, \quad x \in \mathbb{R}^n. \tag{13}$$

Furthermore since $U(x,t) \in \mathcal{C}^\infty(\mathbb{R}^n \times (0,\infty))$, for any compact sets $K_1 \subset \mathbb{R}^n$ and $K_2 \subset (0,\infty)$, there exist constants M_1, $M_2 \geq 0$ such that

$$|\triangle U(x,t+s)v(s)| \leq M_1|s|,$$
$$\left|\frac{\partial}{\partial t}U(x,t+s)v(s)\right| \leq M_2|s|.$$

Since K_1, K_2 are arbitrary,

$$\triangle\tilde{U}(x,t) = \int_0^{\frac{t_1}{2}} \triangle U(x,t+s)v(s)ds,$$
$$\frac{\partial}{\partial t}\tilde{U}(x,t) = \int_0^{\frac{t_1}{2}} \frac{\partial}{\partial t}U(x,t+s)v(s)ds,$$

in $\mathbb{R}^n \times (0,\infty)$. Since

$$\left(\frac{\partial}{\partial t} - \triangle\right)U(x,t) = 0$$

in $\mathbb{R}^n \times (0,\infty)$, we have

$$\left(\frac{\partial}{\partial t} - \triangle\right)\tilde{U}(x,t) = \int_0^\infty \left(\frac{\partial}{\partial t} - \triangle\right)U(x,t+s)v(s)ds = 0 \tag{14}$$

in $\mathbb{R}^n \times (0,\infty)$.

By (11) and (14), for $t > 0$

$$\begin{aligned}(-\triangle)^{N+2}\tilde{U}(x,t) &= \left(-\frac{\partial}{\partial t}\right)^{N+2}\tilde{U}(x,t)\\ &= \int_0^\infty \left(-\frac{\partial}{\partial t}\right)^{N+2} U(x,t+s)v(s)ds\\ &= \left\langle U(x,t+s), \left(\frac{\partial}{\partial s}\right)^{N+2} v(s)\right\rangle\\ &= \langle U(x,t+s), \delta(s) + w(s)\rangle\\ &= U(x,t) + \int_0^\infty U(x,t+s)w(s)ds.\end{aligned}$$

Therefore, we have

$$U(x,t) = (-\triangle)^{N+2}\tilde{U}(x,t) - \int_0^\infty U(x,t+s)w(s)ds. \tag{15}$$

We put $H(x,t) = -\int_0^\infty U(x,t+s)w(s)ds$. Then by supp $w(s) \subset \left[\frac{t_1}{4}, \frac{t_1}{2}\right]$, $H(x,t)$ is $\mathcal{C}^\infty$-function in $t \geq 0$. Now we put $h(x) = H(x,0)$. Then

$$|h(x)| \leq Ce^{h_K(x)+\varepsilon|x|}. \tag{16}$$

We put $T_x = (-\triangle)^{N+2}g(x) + h(x)$. By Lemma 4.3, (13) and (16), we have $T \in H'(\mathbb{R}^n, K)$. Then

$$\begin{aligned}(17)\qquad &\langle T_y, E(x-y,t)\rangle \\ &= \langle (-\triangle_y)^{N+2}g(y) + h(y), E(x-y,t)\rangle \\ &= \int_{\mathbb{R}^n} g(y) \times (\triangle_y)^{N+2}E(x-y,t)dy + \int_{\mathbb{R}^n} h(y)E(x-y,t)dy \\ &= (-\triangle_x)^{N+2}\int_{\mathbb{R}^n} E(x-y,t)g(y)dy + \int_{\mathbb{R}^n} E(x-y,t)h(y)dy \\ &= (-\triangle)^{N+2}G_0(x,t) + H_0(x,t).\end{aligned}$$

For $G_0(x,t)$, we have

$$\left(\frac{\partial}{\partial t} - \triangle\right) G_0(x,t) = 0 \tag{18}$$

and

$$\begin{aligned}(19)\quad |G_0(x,t)| &\leq C\int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{(x-y)^2}{4t} + h_K(y)+\varepsilon|y|}dy \\ &\leq \frac{C}{(4\pi t)^{\frac{n}{2}}} e^{h_K(x)+\varepsilon|x|}\int_{\mathbb{R}^n} e^{-\frac{(y-x)^2}{4t} + h_K(y-x)+\varepsilon|y-x|}dy \\ &\leq \frac{C}{(4\pi t)^{\frac{n}{2}}} e^{h_K(x)+\varepsilon|x|}e^{n(R+\varepsilon)^2 t}\int_{\mathbb{R}^n} e^{-\frac{1}{4t}\sum_{j=1}^n \{|y_j-x_j|-2(R+\varepsilon)t\}^2}dy \\ &\leq C_1 e^{h_K(x)+\varepsilon|x|}, \quad 0 < t < T,\end{aligned}$$

where $K \subset [-R,R]^n$, $R > 0$.

Similarly, for $H_0(x,t)$ we have

$$\left(\frac{\partial}{\partial t} - \triangle\right) H_0(x,t) = 0 \tag{20}$$

and

$$|H_0(x,t)| \leq C_1 e^{h_K(x)+\varepsilon|x|}, \quad 0 < t < T. \tag{21}$$

Furthermore

$$\begin{aligned}G_0(x,t) - g(x) &= \int_{\mathbb{R}^n} E(u,t)\{g(x-u) - g(x)\}du \\ &= \frac{1}{\pi^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-s^2}\{g(x - \sqrt{4t}s) - g(x)\}ds.\end{aligned}$$

Since

$$\begin{aligned}
&|e^{-s^2}\{g(x-\sqrt{4t}s)-g(x)\}| \\
&\le Ce^{-s^2}(e^{-h_K(x-\sqrt{4t}s)-\varepsilon|x-\sqrt{4t}s|}+e^{-h_K(x)-\varepsilon|x|}) \\
&\le Ce^{-s^2}(e^{-h_K(x)-\varepsilon|x|+h_K(\sqrt{4t}s)+\sqrt{4t}\varepsilon|s|}+e^{-h_K(x)-\varepsilon|x|}) \\
&\le Ce^{-s^2-h_K(x)-\varepsilon|x|}(e^{\sqrt{4t}(R+\varepsilon)|s|}+1)
\end{aligned}$$

and $e^{-s^2}(e^{\sqrt{4t}(R+\varepsilon)|s|}+1)\in L^1(\mathbb{R}^n_s)$, by Lebesgue's dominated convergence theorem,

$$\lim_{t\to 0_+} G_0(x,t)-g(x)=\frac{1}{\pi^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-s^2}\left\{\lim_{t\to 0_+} g(x-\sqrt{4t}s)-g(x)\right\}ds=0,$$

because $g(x)$ is a continuous function. Therefore,

$$\lim_{t\to 0_+} G_0(x,t)=g(x). \tag{22}$$

Similarly,

$$\lim_{t\to 0_+} H_0(x,t)=h(x). \tag{23}$$

By (12), (14), (18), (19), (22) and uniqueness theorem of the heat equation [8], we have

$$G_0(x,t)=\tilde{U}(x,t). \tag{24}$$

Similarly,

$$H_0(x,t)=H(x,t). \tag{25}$$

By (15), (17), (24) and (25), we have

$$\begin{aligned}
\langle T_y, E(x-y,t)\rangle &= (-\triangle)^{N+2}G_0(x,t)+H_0(x,t) \\
&= (-\triangle)^{N+2}\tilde{U}(x,t)+H(x,t) \\
&= U(x,t).
\end{aligned}$$

□

Remark 4.5. C. Dong and T. Matsuzawa characterized Gel'fand-Shilov space $\mathcal{S}^s_r$ by the heat kernel method in [4]. But our result for the growth of t is better than their result. That is, they showed that the convolution of the heat kernel and a generalized function was $\mathcal{C}^\infty$-function with some *exponential* *growth for* t and conversely such a smooth solution could be represented by the convolution of the heat kernel and a generalized function u given by the following formula:

$$u = P(-\triangle)g_0(x) - h_0(x),$$

where $P(-\triangle)$ was the *infinite order* differential operator. In Theorem 4.4 we showed that the convolution was $\mathcal{C}^\infty$-function with some *polynomial growth for* t and conversely such a smooth solution could be represented by the convolution of the heat kernel and a generalized function T given by the following formula:

$$T = (-\triangle)^{N+2} g(x) + h(x),$$

of course, $(-\triangle)^{N+2}$ is a *finite order* differential operator. So we obtained the sharper result than them for the case of $H'(\mathbb{R}^n, K)$. For the details we refer the reader to [4].

§5. Distributions of Exponential Growth Supported by a Proper Convex Cone

In this section, we shall characterize $H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$, the space of distributions of exponential growth supported by a proper open convex cone $\Gamma \subset \mathbb{R}^n$.

Theorem 5.1. *Let* $\Gamma \subset \mathbb{R}^n$ *be a proper open convex cone,* $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$ *and* $U(x,t) = \langle T_y, E(x-y,t)\rangle$. *Then* $U(x,t) \in \mathcal{C}^\infty(\mathbb{R}^n \times (0,\infty))$ *satisfying the following conditions*:

$$\left(\frac{\partial}{\partial t} - \triangle\right) U(x,t) = 0, \tag{26}$$

$$U(x,t) \to T, \ (t \to 0_+), \ \text{in } H'(\mathbb{R}^n, K), \tag{27}$$

$$\forall \varepsilon > 0 \ \exists N_\varepsilon \geq 0 \ \exists C_\varepsilon \geq 0 \quad \text{s.t. } |U(x,t)| \leq C_\varepsilon t^{-N_\varepsilon} e^{-\frac{\mathrm{dis}(x,\overline{\Gamma})^2}{16t}} e^{h_K(x)+\varepsilon|x|}, \ 0 < t < 1, \ x \in \mathbb{R}^n. \tag{28}$$

Conversely, for a function $U(x,t) \in \mathcal{C}^\infty(\mathbb{R}^n \times (0,\infty))$ *satisfying* (26) *and* (28), *there exists a unique* $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$ *such that* $\langle T_y, E(x-y,t)\rangle = U(x,t)$.

Proof. By Theorem 4.4, (26) and (27) are obvious.

Let $0 < t < 1$. By Corollary 3.12 and (4),

$$\begin{aligned}
&|U(x,t)| \\
&\le \sum_{|\alpha|=0}^{m} \int_{\overline{\Gamma}} e^{h_K(y)+\varepsilon|y|} |F_{\varepsilon,\alpha}(y)||D^\alpha E(x-y,t)|dy \\
&\le C_1 t^{-N_1} \int_{\overline{\Gamma}} e^{h_K(y-x)+h_K(x)+\varepsilon|y-x|+\varepsilon|x|} e^{-\frac{|y-x|^2}{16t}} e^{-\frac{|y-x|^2}{16t}} dy \\
&\le C_1 t^{-N_1} e^{-\frac{\mathrm{dis}(x,\overline{\Gamma})^2}{16t}} e^{h_K(x)+\varepsilon|x|} \int_{\overline{\Gamma}} e^{h_K(y-x)+\varepsilon|y-x|} e^{-\frac{|y-x|^2}{16t}} dy \\
&\le C t^{-N} e^{-\frac{\mathrm{dis}(x,\overline{\Gamma})^2}{16t}} e^{h_K(x)+\varepsilon|x|}.
\end{aligned}$$

Therefore, we have (28).

Now we shall prove the converse. By (28),

$$\begin{aligned}
|U(x,t)| &\le C t^{-N} e^{-\frac{\mathrm{dis}(x,\overline{\Gamma})^2}{16t}} e^{h_K(x)+\varepsilon|x|} \\
&\le C t^{-N} e^{h_K(x)+\varepsilon|x|}, \quad 0 < t < 1.
\end{aligned}$$

By Theorem 4.4, there exists $T \in H'(\mathbb{R}^n, K)$ such that $\langle T_y, E(x-y,t)\rangle = U(x,t)$. Let $\varphi(x) \in \mathcal{D}$, $\mathrm{supp}(\varphi) \subset \mathbb{R}^n \backslash \overline{\Gamma}$, $K' = \mathrm{supp}(\varphi)$, $\delta = \mathrm{dis}(K', \overline{\Gamma}) > 0$. Then by Theorem 4.4,

$$\langle T, \varphi \rangle = \lim_{t \to 0_+} \int_{K'} U(x,t)\varphi(x)dx.$$

Therefore,

$$\begin{aligned}
|\langle T, \varphi\rangle| &\le C \lim_{t\to 0_+} t^{-N} \int_{K'} e^{-\frac{\mathrm{dis}(x,\overline{\Gamma})^2}{16t}} e^{h_K(x)+\varepsilon|x|} |\varphi(x)|dx \\
&\le C \lim_{t\to 0_+} t^{-N} e^{-\frac{\delta^2}{16t}} \int_{K'} e^{h_K(x)+\varepsilon|x|}|\varphi(x)|dx \\
&\le C_2 \lim_{t\to 0_+} t^{-N} e^{-\frac{\delta^2}{16t}} = 0.
\end{aligned}$$

This means that $\mathrm{supp}\, T \subset \overline{\Gamma}$. □

For (28) in Theorem 5.1, we have the following lemma:

Lemma 5.2. *Let* $U(x,t) \in \mathcal{C}^\infty(\mathbb{R}^n \times (0,\infty))$ *and satisfies* $(\frac{\partial}{\partial t} - \triangle)$ $U(x,t) = 0$. *Then* (28) *in Theorem* 5.1 *is equivalent to the following conditions*:

(29) $\forall \varepsilon > 0\ \exists N\ \exists C \ge 0$ *s.t.* $|U(x,t)| \le C t^{-N} e^{h_K(x)+\varepsilon|x|}$, $0 < t < 1$, $x \in \mathbb{R}^n$, *and* $U(x,t) \to 0, (t \to 0_+)$, *uniformly for all compact sets in* $\mathbb{R}^n \backslash \overline{\Gamma}$.

Proof. (28) $\Rightarrow$ (29) is obvious. Now we suppose (29). By the estimate in (29) and Theorem 4.4, there exists $T \in H'(\mathbb{R}^n, K)$ such that $U(x,t) = \langle T_y, E(x-y,t)\rangle$. Let $\varphi(x) \in \mathcal{D}(\mathbb{R}^n)$, $\mathrm{supp}(\varphi) \subset \mathbb{R}^n \backslash \overline{\Gamma}$. Then by (8) in Theorem 4.4 and the assumption in (29), we have

$$\langle T, \varphi\rangle = \lim_{t\to 0_+} \int_{\mathbb{R}^n} U(x,t)\varphi(x)dx = 0.$$

It means that $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$. By Theorem 5.1, we have (28). $\square$

By Lemma 5.2, we have the following corollary:

Corollary 5.3. *Let $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$ and $U(x,t) = \langle T_y, E(x-y,t)\rangle$. Then $U(x,t) \in \mathcal{C}^\infty(\mathbb{R}^n \times (0,\infty))$ satisfies the following conditions*:

(30) $$\left(\frac{\partial}{\partial t} - \triangle\right) U(x,t) = 0,$$

(31) $$U(x,t) \longrightarrow T, \ (t \to 0_+), \ \textit{in } H'(\mathbb{R}^n, K),$$

(32) $$\begin{aligned} &\forall\varepsilon > 0 \ \exists N \ \exists C \geq 0 \ \textit{s.t.} \ |U(x,t)| \leq Ct^{-N} e^{h_K(x)+\varepsilon|x|}, \\ &0 < t < 1, \ x \in \mathbb{R}^n \textit{and } U(x,t) \to 0, (t \to 0_+), \\ &\textit{uniformly for all compact sets in } \mathbb{R}^n \backslash \overline{\Gamma}. \end{aligned}$$

Conversely, for a function $U(x,t) \in \mathcal{C}^\infty(\mathbb{R}^n \times (0,\infty))$ satisfying (30) *and* (32), *there exists a unique $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$ such that $\langle T_y, E(x-y,t)\rangle = U(x,t)$.*

§6. Paley-Wiener Theorem for Distributions of Exponential Growth Supported by a Proper Convex Cone

In this section, we shall give the Paley-Wiener theorem for $H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$, the space of distributions of exponential growth supported by a proper open convex cone $\Gamma \subset \mathbb{R}^n$. For the 1-dimensional case, it is given in [15].

Definition 6.1. Let Γ be a proper open convex cone, K be a compact set and $\varepsilon' > 0$. Then we denote L and $L_{-\varepsilon'}$ by

$$L = \left\{ \bigcap_{u\in K} (\{u\} + (\overline{\Gamma}')^\circ) \right\}^\circ$$

$$L_{-\varepsilon'} = \mathbb{R}^n \backslash \left(\mathbb{R}^n \backslash L\right)_{\varepsilon'}.$$

Proposition 6.2. *For L and $L_{-\varepsilon'}$, we have the following properties*:

(i) $L \neq \emptyset$.

(ii) $L_{-\varepsilon'} \subset\subset L$.

Proof. (i) : Let $u_1,\ u_2 \in K \subset B(0, \frac{1}{2}R),\ R > 0,\ y \in (\overline{\Gamma}')^\circ,\ |y| = 1$. By Proposition 2.12, there exists $\delta > 0$ such that $B(y, \delta) \subset (\overline{\Gamma}')^\circ$, and $B(u_1+y, \delta) \subset \{u_1\} + (\overline{\Gamma}')^\circ$, $B(u_2 + y, \delta) \subset \{u_2\} + (\overline{\Gamma}')^\circ$. Let $x \in B(\frac{R}{\delta}y, R)$. Since

$$\left|\frac{R}{\delta}y - x\right| < R \Leftrightarrow \left|y - \frac{\delta}{R}x\right| < \delta,$$

we have $\frac{\delta}{R}x \in (\overline{\Gamma}')^\circ$. Therefore, $x \in (\overline{\Gamma}')^\circ$. This means that $B(\frac{R}{\delta}y, R) \subset (\overline{\Gamma}')^\circ$. Since

$$B\left(u_i + \frac{R}{\delta}y, R\right) \subset \{u_i\} + (\overline{\Gamma}')^\circ, \quad i = 1, 2,$$
$$\left|\left(\{u_1\} + \frac{R}{\delta}y\right) - \left(\{u_2\} + \frac{R}{\delta}y\right)\right| = |u_1 - u_2| < R$$

and $u_1,\ u_2 \in K$ are arbitrary, we have $\bigcap_{u\in K} (\{u\} + (\overline{\Gamma}')^\circ) \neq \emptyset$. Let $a \in \bigcap_{u\in K} (\{u\} + (\overline{\Gamma}')^\circ)$. By Proposition 2.13,

$$\begin{aligned} x \in a + (\overline{\Gamma}')^\circ \Rightarrow x &\in \bigcap_{u\in K} (\{u\} + (\overline{\Gamma}')^\circ) + (\overline{\Gamma}')^\circ \\ &\Leftrightarrow x \in u + (\overline{\Gamma}')^\circ + (\overline{\Gamma}')^\circ, \quad \text{for any } u \in K \\ &\Leftrightarrow x \in u + (\overline{\Gamma}')^\circ \\ &\Leftrightarrow x \in \bigcap_{u\in K} (\{u\} + (\overline{\Gamma}')^\circ). \end{aligned}$$

Therefore, $a + (\overline{\Gamma}')^\circ \subset \bigcap_{u\in K} (\{u\} + (\overline{\Gamma}')^\circ)$. By Proposition 2.12, $a + (\overline{\Gamma}')^\circ$ is an open set and not empty. Therefore, we have the condition (i). (ii) is obvious.

Definition 6.3 ([15], [22])**.** For $T \in H'_{\Gamma}(\mathbb{R}^n, K)$, we define the Fourier-Laplace transform $\mathcal{LF}(T)$ of T by

$$\begin{aligned} \mathcal{LF}(T)(\xi + \imath\eta) &:= \mathcal{F}(e^{-\eta x}T)(\xi) \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\langle e^{-\eta x}T_x, e^{\imath\xi x}\rangle \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\langle T_x, e^{\imath\zeta x}\rangle. \end{aligned}$$

The last part means

$$\langle T_x, e^{\imath\zeta x}\rangle = \langle T_x, \chi(x)e^{\imath\zeta x}\rangle,$$

where $\chi(x) \in \mathcal{C}^\infty(\mathbb{R}^n)$ which satisfies

$$\chi(x) = \begin{cases} 1, & x \in \overline{\Gamma}_\varepsilon \\ 0, & x \notin \overline{\Gamma}_{2\varepsilon}, \quad \varepsilon > 0. \end{cases}$$

Definition 6.4. Let Γ be a proper open convex cone and K be a compact set. For $\varepsilon > 0$ and $u_j \in K$, $j = 1, \dots, j_0$, we set the following notations:

$$[\overline{\Gamma}]^j = (\{u_j\} + \overline{\Gamma})^\circ,$$
$$[\overline{\Gamma}]^j_{-\varepsilon} = \mathbb{R}^n \backslash (\mathbb{R}^n \backslash (\{u_j\} + \overline{\Gamma})^\circ)_\varepsilon.$$

Lemma 6.5. *Let Γ, Γ_C be proper open convex cones such that $\overline{\Gamma}_C \Subset (\overline{\Gamma}')^\circ$ and $\eta \in [\overline{\Gamma}_C]^j_{-2\varepsilon}$. Then for every $s \in \overline{B(u_j, \varepsilon)}$, $\eta - s \in (\overline{\Gamma}_C)^\circ$ and $|\eta - s| \geq \frac{1}{2}\varepsilon$.*

Proof. Let $\eta \in [\overline{\Gamma}_C]^j_{-2\varepsilon}$, namely $\eta \in \mathbb{R}^n \backslash (\mathbb{R}^n \backslash (\{u_j\} + \overline{\Gamma}_C)^\circ)_{2\varepsilon}$. By $\eta \notin (\mathbb{R}^n \backslash (\{u_j\} + \overline{\Gamma}_C)^\circ)_{2\varepsilon}$,

$$\begin{aligned} \eta + \overline{B\left(0, \frac{3}{2}\varepsilon\right)} \subset (\{u_j\} + \overline{\Gamma}_C)^\circ &\Leftrightarrow \eta - u_j + \overline{B\left(0, \frac{3}{2}\varepsilon\right)} \subset (\overline{\Gamma}_C)^\circ \\ &\Leftrightarrow \eta - u_j + \overline{B(0, \varepsilon)} + \overline{B\left(0, \frac{1}{2}\varepsilon\right)} \subset (\overline{\Gamma}_C)^\circ \\ &\Leftrightarrow \eta - \overline{B(u_j, \varepsilon)} + \overline{B\left(0, \frac{1}{2}\varepsilon\right)} \subset (\overline{\Gamma}_C)^\circ. \end{aligned}$$

For every $s \in \overline{B(u_j, \varepsilon)}$, since $\eta - s + \overline{B(0, \frac{1}{2}\varepsilon)} \subset (\overline{\Gamma}_C)^\circ$ and $\{0\} \in \partial\overline{\Gamma}_C$, we have $\eta - s \in (\overline{\Gamma}_C)^\circ$ and $|\eta - s| \geq \frac{1}{2}\varepsilon$. □

Proposition 6.6. *Let Γ be a proper open convex cone, K be a convex compact set, $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$ and $f(\zeta) = \mathcal{LF}(T)(\xi + \imath\eta)$. Then for every $\varepsilon > 0$ there exist $j_0 \in \mathbb{N}$, $l_\varepsilon \geq 0$ and the families $\{u_j\}_{j=1}^{j_0} \subset K$, $\{f_j(\zeta)\}_{j=1}^{j_0}$ satisfying the conditions* (33), (34), (35):

$$f_j(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath[\overline{\Gamma}']^j). \tag{33}$$

$\forall\overline{\Gamma}_C \Subset (\overline{\Gamma}')^\circ$ $\exists M_{\varepsilon,\overline{\Gamma}_C} \geq 0$ *such that*

$$|f_j(\zeta)| \leq M(1 + |\zeta|)^l, \quad \zeta \in \mathbb{R}^n + \imath[\overline{\Gamma}_C]^j_{-2\varepsilon}. \tag{34}$$

$$f(\zeta) = \sum_{1 \leq j \leq j_0} f_j(\zeta). \tag{35}$$

In particular, $f(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath L)$.

Proof. By Corollary 3.13,

(36)
$$\begin{aligned} f(\zeta) &= \langle T_x, e^{\imath\zeta x}\rangle \\ &= \sum_{|\alpha|=0}^{a} \sum_{|\beta|=0}^{|\alpha|} \binom{\alpha}{\beta} (\imath\zeta)^\beta \int_{\overline{\Gamma}} G(x) D^{\alpha-\beta} F^*(x) e^{\imath\zeta x} dx \\ &= \sum_{|\alpha|=0}^{a} \sum_{|\beta|=0}^{|\alpha|} \binom{\alpha}{\beta} (\imath\zeta)^\beta \sum_{1\le j\le j_0} \int_{\overline{\Gamma}} G(x) A_{\alpha,\beta}(u_1, \dots, u_{j_0}, x, \varepsilon) e^{u_j x + \frac{\varepsilon}{2}\sqrt{1+x^2}} e^{\imath\zeta x} dx, \end{aligned}$$

where $\sup_{x\in\mathbb{R}^n} |A_{\alpha,\beta}(u_1, \dots, u_{j_0}, x, \varepsilon)| < \infty$. Now we put

$$g_{j,\alpha,\beta}(\zeta) = \int_{\overline{\Gamma}} G(x) A_{\alpha,\beta}(u_1, \dots, u_{j_0}, x, \varepsilon) e^{u_j x + \frac{\varepsilon}{2}\sqrt{1+x^2} + \imath\zeta x} dx. \tag{37}$$

Then

$$\begin{aligned} |g_{j,\alpha,\beta}(\zeta)| &\le C_0 \int_{\overline{\Gamma}} (1+|x|)^M e^{u_j x + \frac{\varepsilon}{2}\sqrt{1+x^2}} e^{-\eta x} dx \\ &\le C_1 \int_{\overline{\Gamma}} e^{u_j x + \varepsilon|x|} e^{-\eta x} dx. \end{aligned} \tag{38}$$

Let $\eta \in [\overline{\Gamma}_C]^j_{-2\varepsilon}$. By Lemma 6.5 and Proposition 2.12, there exists $\sigma = \sigma(\overline{\Gamma}_C) > 0$ such that

$$\begin{aligned} (\eta - s)x &\ge \sigma|\eta - s||x| \\ &\ge \frac{1}{2}\sigma\varepsilon|x|, \quad \eta \in [\overline{\Gamma}_C]^j_{-2\varepsilon},\ x \in \overline{\Gamma},\ s \in \overline{B(u_j, \varepsilon)}. \end{aligned}$$

Therefore

$$\begin{aligned} |g_{j,\alpha,\beta}(\zeta)| &\le C_1 \int_{\overline{\Gamma}} e^{h_{\overline{B(u_j,\varepsilon)}}(x) - \eta x} dx \\ &\le C_1 \int_{\overline{\Gamma}} e^{sx - \eta x} dx, \quad s_{(x)} \in \overline{B(u_j, \varepsilon)}, \\ &\le C_1 \int_{\overline{\Gamma}} e^{-\sigma|\eta - s||x|} dx \\ &\le C_1 \int_{\overline{\Gamma}} e^{-\frac{1}{2}\varepsilon\sigma|x|} dx \\ &< \infty. \end{aligned} \tag{39}$$

If $\eta \in [\overline{\Gamma}_C]^j_{-2\varepsilon}$, then by the calculation from (38) to (39), we have

$$|g_{j,\alpha,\beta}(\zeta)| \leq \int_{\overline{\Gamma}} e^{-\frac{1}{2}\sigma\varepsilon|x|}dx,$$

and $e^{-\frac{1}{2}\sigma\varepsilon'|x|} \in L^1(\mathbb{R}^n_x)$. For $\zeta_0 \in \mathbb{R}^n + \imath[\overline{\Gamma}_C]^j_{-2\varepsilon}$ by (36) and Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}\lim_{\substack{\zeta\to\zeta_0\\ \eta\in[\overline{\Gamma}_C]^j_{-2\varepsilon}}} g_{j,\alpha,\beta}(\zeta) &= \lim_{\substack{\zeta\to\zeta_0\\ \eta\in[\overline{\Gamma}_C]^j_{-2\varepsilon}}} \int_{\overline{\Gamma}} G(x)A_{\alpha,\beta}(u_1,\dots,u_{j_0},x,\varepsilon)e^{u_jx+\frac{\varepsilon}{2}\sqrt{1+x^2}}e^{\imath\zeta x}dx,\\ &= \int_{\overline{\Gamma}} G(x)A_{\alpha,\beta}(u_1,\dots,u_{j_0},x,\varepsilon)e^{u_jx+\frac{\varepsilon}{2}\sqrt{1+x^2}}e^{\imath\zeta_0 x}dx,\\ &= g_{j,\alpha,\beta}(\zeta_0).\end{aligned}$$

Since $\varepsilon > 0$ and $\overline{\Gamma}_C \Subset (\overline{\Gamma}')^\circ$ are arbitrary, $g_{j,\alpha,\beta}(\zeta)$ is a continuous function in $\mathbb{R}^n + \imath[\overline{\Gamma}]^j$.

Let $\eta \in [\overline{\Gamma}_C]^j_{-2\varepsilon}$ and γ be a Jordan curve in $\{\mathbb{R}^n + \imath[\overline{\Gamma}_C]^j_{-2\varepsilon}\}_m$ which is mth component of $\mathbb{R}^n + \imath[\overline{\Gamma}_C]^j_{-2\varepsilon}$. By (39),

$$\int_\gamma |g_{j,\alpha,\beta}(\zeta)||d\zeta_m| < \infty.$$

By Fubini's theorem,

$$\begin{aligned}\int_\gamma g_{j,\alpha,\beta}(\zeta)d\zeta_m &= \int_{\overline{\Gamma}} G(x)A_{\alpha,\beta}(t_1,\dots,t_{j_0},x,\varepsilon)e^{u_jx+\frac{\varepsilon}{2}\sqrt{1+x^2}}\int_\gamma e^{\imath\zeta x}d\zeta_m dx\\ &= 0.\end{aligned}$$

By Morera's theorem, $g_{j,\alpha,\beta}(\zeta)$ is a holomorphic function of ζ_m. By Hartogs' theorem, $g_{j,\alpha,\beta}(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath[\overline{\Gamma}_C]^j_{-2\varepsilon})$. Since $\varepsilon > 0$ and $\overline{\Gamma}_C \Subset (\overline{\Gamma}')^\circ$ are arbitrary, we have $g_{j,\alpha,\beta}(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath[\overline{\Gamma}']^j)$. Now we put

$$f_j(\zeta) = \sum_{|\alpha|=0}^{a}\sum_{|\beta|=0}^{|\alpha|}\binom{\alpha}{\beta}(\imath\zeta)^\beta g_{j,\alpha,\beta}(\zeta).$$

Then we have

$$f(\zeta) = \sum_{1\leq j\leq j_0} f_j(\zeta),$$

$$f_j(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath[\overline{\Gamma}']^j), \quad j = 1,\dots,j_0,$$

$\exists l_\varepsilon \geq 0\ \forall \overline{\Gamma}_C \Subset (\overline{\Gamma}')^\circ\ \exists M_{\varepsilon, \overline{\Gamma}_C} \geq 0$ such that

$$|f_j(\zeta)| \leq M(1+|\zeta|)^l, \quad \zeta \in \mathbb{R}^n + \imath[\overline{\Gamma}_C]^j_{-2\varepsilon}, \quad j = 1, \ldots, j_0.$$

□

Proposition 6.7. *Assume that $f(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath L)$ and satisfies the conditions* (33), (34) *and* (35). *Then there exists a unique $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$ such that $f(\zeta) = \frac{1}{(2\pi)^{\frac{n}{2}}}\langle T_x, e^{\imath\zeta x}\rangle$.*

Proof. Let $\varepsilon > 0$, $0 < t < 1$ and $\zeta = \xi + \imath\eta$, $\eta \in \overline{\Gamma}_C$, $\overline{\Gamma}_C \Subset (\overline{\Gamma}')^\circ$ and $|\eta| = \varepsilon$. Now we put

$$U(x,t) = \frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{1 \leq j \leq j_0} \int_{\mathbb{R}^n} f_j(\zeta + \imath u_j) e^{-t(\zeta+\imath u_j)^2} e^{-\imath(\zeta+\imath u_j)x} d\xi,$$

$$U_j(x,t) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f_j(\zeta + \imath u_j) e^{-t(\zeta+\imath u_j)^2} e^{-\imath(\zeta+\imath u_j)x} d\xi.$$

We notice that $U_j(x,t)$ is independent of $\eta \in \overline{\Gamma}_C$ by Cauchy's integral theorem and satisfies

$$\left(\frac{\partial}{\partial t} - \triangle\right) U_j(x,t) = 0.$$

Furthermore

$$\begin{aligned} |U_j(x,t)| &\leq \int_{\mathbb{R}^n} |f_j(\zeta + \imath u_j)| e^{-t\xi^2 + t(\eta+u_j)^2} e^{(\eta+u_j)x} d\xi \\ &\leq M\{(1+|\eta_1 + u_{j_1}|)\cdots(1+|\eta_n + u_{j_n}|)\}^l e^{t(\eta+u_j)^2 + (\eta+u_j)x} \\ &\quad \times \int_{\mathbb{R}^n} \{(1+|\xi_1|)\cdots(1+|\xi_n|)\}^l e^{-t\xi^2} d\xi, \quad (u_j = (u_{j_1}, \ldots, u_{j_n})), \\ &\leq M_1 t^{-N} \{(1+|\eta_1 + u_{j_1}|)\cdots(1+|\eta_n + u_{j_n}|)\}^l e^{t(\eta+u_j)^2 + (\eta+u_j)x}. \end{aligned}$$

Then since $|\eta| = \varepsilon$,

$$\begin{aligned} |U_j(x,t)| &\leq M_2 t^{-N} (1 + \varepsilon + |u_j|)^{ln} e^{2t\eta u_j + tu_j^2 + \eta x + u_j x} \\ &\leq M_3 t^{-N} e^{2\varepsilon|u_j| + |u_j|^2 + \varepsilon|x| + u_j x} \\ &\leq M' t^{-N} e^{u_j x + \varepsilon|x|}, \quad 0 < t < 1, \quad x \in \mathbb{R}^n. \end{aligned}$$

By Theorem 4.4, there exists $T_j \in H'(\mathbb{R}^n, \{u_j\})$ such that $\langle T_{j_y}, E(x-y,t)\rangle = U_j(x,t)$.

Let $x_0 \notin \overline{\Gamma}$. Then there exists $\eta_0 \in (\overline{\Gamma}')^\circ$, $|\eta_0| = 1$ such that $\eta_0 x_0 = -2\delta < 0$. Then we have

$$\begin{aligned}\sup_{x\in B(x_0,\delta)} \eta_0 x &= \eta_0 x_0 + \delta \sup_{y\in B(0,1)} \eta_0 y \\ &< -2\delta + \delta \\ &= -\delta.\end{aligned}$$

Let $\eta' = \frac{\eta_0}{\sqrt{t}}$, $\varphi(x) \in \mathcal{D}$, supp$(\varphi)\subset B(x_0,\delta)$. Then we have

$$\begin{aligned}&|\langle U_j(x,t), \varphi(x)\rangle| \\ &= \left|\int_{B(x_0,\delta)} \int_{\mathbb{R}^n} f_j(\zeta + \imath u_j) e^{-t(\zeta+\imath u_j)^2} e^{-\imath(\zeta+\imath u_j)x} d\xi \varphi(x) dx\right| \\ &\le M\{(1+|\eta_1' + u_{j1}|)\cdots(1+|\eta_n' + u_{jn}|)\}^l e^{t(\eta'+u_j)^2} \\ &\quad \times \int_{B(x_0,\delta)} |\varphi(x)| e^{(\eta'+u_j)x} dx \int_{\mathbb{R}^n} \{(1+|\xi_1|)\cdots(1+|\xi_n|)\}^l e^{-t\xi^2} d\xi \\ &\le M_1 t^{-N} e^{t(u_j+\frac{\eta_0}{\sqrt{t}})^2} \int_{B(x_0,\delta)} |\varphi(x)| e^{\frac{1}{\sqrt{t}}\eta_0 x + u_j x} dx \\ &\le M_3 t^{-N} e^{tu_j^2 + 2\sqrt{t}\eta_0 u_j + |\eta_0|^2} \int_{B(x_0,\delta)} |\varphi(x)| e^{u_j x} e^{-\frac{\delta}{\sqrt{t}}} dx \\ &\le M_4 t^{-N} e^{-\frac{\delta}{\sqrt{t}}} \\ &\to 0, \quad t \to 0_+.\end{aligned}$$

Therefore, by Theorem 4.4, we have

$$\begin{aligned}\langle T_j, \varphi\rangle &= \lim_{t\to 0_+} \langle U_j(x,t), \varphi(x)\rangle \\ &= 0.\end{aligned}$$

Since $x_0 \notin \overline{\Gamma}$ is arbitrary, this means that supp $T_j \subset \overline{\Gamma}$.

Now for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$, let

$$\begin{aligned}V_\eta(x,t) &= e^{-t\eta^2 - \eta x}(E(x,t) * \varphi(-x)) \\ &= e^{-t\eta^2-\eta x} \int_{\mathbb{R}^n} E(x-y,t)\varphi(-y) dy, \quad 0 < t < 1.\end{aligned}$$

Then we have the following lemma:

Lemma 6.8 ([20]). $V_\eta(x,t) \to \varphi(-x)e^{-\eta x}$ *in* $\mathcal{S}(\mathbb{R}^n)$*, as* $t \to 0_+$.

For the details of the proof, we refer the reader to [20]. Now we resume the proof of Proposition 6.7.

Let $\varphi(x) \in \mathcal{D}$. Then

$$
\begin{aligned}
&\langle U_j(x,t), \varphi(x)\rangle \\
&= \left\langle \int_{\mathbb{R}^n} f_j(\zeta + \imath u_j) e^{-t(\zeta+\imath u_j)^2} e^{-\imath(\zeta+\imath u_j)x} d\xi, \varphi(x) \right\rangle \\
&= \left\langle f_j(\zeta + \imath u_j) e^{-t(\zeta+\imath u_j)^2}, \int_{\mathbb{R}^n} \varphi(x) e^{-\imath(\zeta+\imath u_j)x} dx \right\rangle \\
&= \langle f_j(\zeta + \imath u_j), e^{-t(\zeta+\imath u_j)^2} \mathcal{LF}(\varphi(-x)e^{-u_j x})(\zeta)\rangle \\
&= \langle f_j(\zeta + \imath u_j), \mathcal{LF}(E(x,t)e^{-u_j x})(\zeta) \times \mathcal{LF}(\varphi(-x)e^{-u_j x})(\zeta)\rangle \\
&= \langle f_j(\zeta + \imath u_j), \mathcal{LF}(e^{-u_j x}(E(x,t) * \varphi(-x)))(\zeta)\rangle \\
&= \langle \mathcal{F}(f_j(\zeta + \imath u_j)), e^{-u_j x - \eta x}(E(x,t) * \varphi(-x))\rangle.
\end{aligned}
$$

By Theorem 4.4 and Lemma 6.8, we have

$$
\begin{aligned}
&\langle T_j, \varphi\rangle \\
&= \lim_{t\to 0_+} e^{t(u_j+\eta)^2} \langle \mathcal{F}(f_j(\zeta + \imath u_j)), e^{-t(u_j+\eta)^2} e^{-u_j x - \eta x}(E(x,t) * \varphi(-x))\rangle \\
&= \langle \mathcal{F}(f_j(\zeta + \imath u_j)), \varphi(-x) e^{-u_j x - \eta x}\rangle \\
&= \langle f_j(\zeta + \imath u_j), \mathcal{F}^{-1}(\varphi(x) e^{u_j x + \eta x})\rangle \\
&\qquad\qquad \Leftrightarrow \langle T_j, \varphi(x) e^{-u_j x - \eta x}\rangle = \langle f_j(\zeta + \imath u_j), \mathcal{F}^{-1}(\varphi(x))\rangle \\
&\qquad\qquad \Leftrightarrow \langle e^{-u_j x - \eta x} T_j, \varphi(x)\rangle = \langle f_j(\zeta + \imath u_j), \mathcal{F}^{-1}(\varphi(x))\rangle \\
&\qquad \Leftrightarrow \langle \mathcal{F}(e^{-u_j x - \eta x} T_j), \varphi(x)\rangle = \langle f_j(\zeta + \imath u_j), \varphi(x)\rangle.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathcal{F}(e^{-u_j x - \eta x} T_j)(\xi) &= f_j(\zeta + \imath u_j) \\
\Leftrightarrow \frac{1}{(2\pi)^{\frac{n}{2}}} \langle T_{j_x}, e^{\imath(\zeta+\imath u_j)x}\rangle &= f_j(\zeta + \imath u_j) \\
\Leftrightarrow \frac{1}{(2\pi)^{\frac{n}{2}}} \langle T_{j_x}, e^{\imath\zeta x}\rangle &= f_j(\zeta).
\end{aligned}
$$

Now we put

$$
T = \sum_{1\le j\le j_0} T_j.
$$

Since $T_j \in H'_{\overline{\Gamma}}(\mathbb{R}^n, \{u_j\})$, $u_j \in K$, we have $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$ and

$$\begin{aligned} \frac{1}{(2\pi)^{\frac{n}{2}}}\langle T_x, e^{\imath\zeta x}\rangle &= \frac{1}{(2\pi)^{\frac{n}{2}}}\sum_{1\le j\le j_0}\langle T_{j\,x}, e^{\imath\zeta x}\rangle \\ &= \sum_{1\le j\le j_0} f_j(\zeta) \\ &= f(\zeta). \end{aligned}$$

Now we shall prove injective. Let T_1, $T_2 \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$ and assume that

$$\begin{aligned} \frac{1}{(2\pi)^{\frac{n}{2}}}\langle T_{1x}, e^{\imath\zeta x}\rangle &= \frac{1}{(2\pi)^{\frac{n}{2}}}\langle T_{2x}, e^{\imath\zeta x}\rangle \\ &= f(\zeta). \end{aligned}$$

For fixed $\eta_0 \in L_{-\varepsilon}$, we have

$$\begin{aligned} \frac{1}{(2\pi)^{\frac{n}{2}}}\langle e^{-\eta_0 x}T_{1x}, e^{\imath\xi x}\rangle &= \frac{1}{(2\pi)^{\frac{n}{2}}}\langle e^{-\eta_0 x}T_{2x}, e^{\imath\xi x}\rangle \\ &= f(\xi + \imath\eta_0). \end{aligned}$$

By (34) and Fourier transform in $\mathcal{S}'$ yields

$$e^{-\eta_0 x}T_1 = e^{-\eta_0 x}T_2, \quad \text{ in } \ \mathcal{S}'.$$

Let $\varphi(x) \in \mathcal{D}$. Then $\varphi(x)e^{\eta_0 x} \in \mathcal{D}$. Therefore,

$$\begin{aligned} \langle T_{1x}, \varphi(x)\rangle &= \langle e^{-\eta_0 x}T_{1x}, \varphi(x)e^{\eta_0 x}\rangle \\ &= \langle e^{-\eta_0 x}T_{2x}, \varphi(x)e^{\eta_0 x}\rangle \\ &= \langle T_{2x}, \varphi(x)\rangle. \end{aligned}$$

By Theorem 3.5, $T_{1x} = T_{2x}$ in $H'(\mathbb{R}^n, K)$. □

By Proposition 6.6 and Proposition 6.7, we have the following theorem:

Theorem 6.9. *Let Γ be a proper open convex cone, K be a convex compact set, $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$ and $f(\zeta) = \mathcal{LF}(T)(\xi + \imath\eta)$. Then for every $\varepsilon > 0$ there exist $j_0 \in \mathbb{N}$, $l_\varepsilon \ge 0$ and the families $\{u_j\}_{j=1}^{j_0} \subset K$, $\{f_j(\zeta)\}_{j=1}^{j_0}$ satisfying the conditions* (40), (41), (42):

$$f_j(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath[\overline{\Gamma}']^j). \tag{40}$$

$\forall\ \overline{\Gamma}_C \Subset (\overline{\Gamma}')^\circ\ \exists M_{\varepsilon,\overline{\Gamma}_C} \ge 0$ *such that*

$$|f_j(\zeta)| \le M(1 + |\zeta|)^l, \quad \zeta \in \mathbb{R}^n + \imath[\overline{\Gamma}_C]^j_{-2\varepsilon}. \tag{41}$$

$$f(\zeta) = \sum_{1 \le j \le j_0} f_j(\zeta). \tag{42}$$

In particular, $f(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath L)$.

Conversely if $f(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath L)$ *satisfies the conditions* (40), (41) *and* (42), *then there exists a unique* $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$ *such that* $f(\zeta) = \frac{1}{(2\pi)^{\frac{n}{2}}} \langle T_x, e^{\imath \zeta x} \rangle$. *Furthermore* T *is given by the following formula*:

$$T = \sum_{1 \le j \le j_0} T_j, \quad T_j \in H'_{\overline{\Gamma}}(\mathbb{R}^n, \{u_j\}), \tag{43}$$

$$f_j(\zeta) = \frac{1}{(2\pi)^{\frac{n}{2}}} \langle T_{j\,x}, e^{\imath \zeta x} \rangle. \tag{44}$$

Corollary 6.10. *Let* Γ *be a proper open convex cone*, $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, \{0\})$ *and* $f(\zeta) = \mathcal{LF}(T)(\xi + \imath\eta)$. *Then for* $\varepsilon > 0$ *there exists* $l_\varepsilon \ge 0$ *satisfying the conditions* (45), (46):

$$f(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath L). \tag{45}$$

$\forall \overline{\Gamma}_C \Subset (\overline{\Gamma}')^\circ\ \exists M_{\varepsilon, \overline{\Gamma}_C} \ge 0$ *such that*

$$|f(\zeta)| \le M(1 + |\zeta|)^l, \quad \zeta \in \mathbb{R}^n + \imath [\overline{\Gamma}_C]_{-2\varepsilon}. \tag{46}$$

Conversely if $f(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath L)$ *satisfies the conditions* (45) *and* (46), *then there exists a unique* $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, \{0\})$ *such that* $f(\zeta) = \frac{1}{(2\pi)^{\frac{n}{2}}} \langle T_x, e^{\imath \zeta x} \rangle$.

Remark 6.11 (Remark for Corollary 6.10). Now we consider more general Fourier-Laplace transforms. That is, if $T \in \mathcal{D}'$ and $e^{-\eta x} T \in \mathcal{S}'$, then we can define the Fourier-Laplace transform $\mathcal{LF}(T)(\zeta)$ of T. Furthermore it is known that we can obtain the Paley-Wiener theorem for $T \in \mathcal{D}'$ if Γ_T° is not empty where $\Gamma_T := \{\eta \in \mathbb{R}^n; e^{-\langle \cdot, \eta \rangle} T \in \mathcal{S}'\}$ (see Theorem 7.4.2 in [6]).

So we can assert that for the Paley-Wiener theorem for $T \in \mathcal{D}'$ (that is, for Theorem 7.4.2 in [6]) we can take the element of the space $H'_{\overline{\Gamma}}(\mathbb{R}^n, \{0\})$ as $T \in \mathcal{D}'$ if and only if the conditions of Corollary 6.10 are satisfied.

Example 6.12 (Example for Theorem 6.9)**.** Let $n = 2$, $K = \{0\} \times [-1, 1]$ and $\Gamma := \{x = (x_1, x_2) \in \mathbb{R}^2; x_1^2 - x_2^2 > 0,\ x_1 > 0\} (= (\overline{\Gamma}')^\circ)$. We define $T(x)$ by

$$T(x) = \begin{cases} e^{|x_2|}, x_1^2 - x_2^2 > 0,\ x_1 > 0, \\ 0, \qquad \text{otherwise.} \end{cases}$$

We can see $T \in H'_{\overline{\Gamma}}(\mathbb{R}^2, K)$ and we have

$$\begin{aligned}\langle T_x, e^{\imath\zeta x}\rangle &= \int_{\overline{\Gamma}} e^{|x_2|} e^{\imath\zeta x} dx_1 dx_2 \\ &= \int_0^{\frac{\pi}{4}} \int_0^{\infty} e^{r(\imath\zeta_1 \cos\theta + (\imath\zeta_2+1)\sin\theta)} r dr d\theta \\ &\quad + \int_{-\frac{\pi}{4}}^{0} \int_0^{\infty} e^{r(\imath\zeta_1 \cos\theta + (\imath\zeta_2-1)\sin\theta)} r dr d\theta.\end{aligned}$$

If $\eta \in L := \{\eta = (\eta_1, \eta_2); \{(1,0)\} + (\overline{\Gamma}')^\circ\}$, then

$$\begin{aligned}&\langle T_x, e^{\imath\zeta x}\rangle \\ &= \int_0^{\frac{\pi}{4}} \frac{d\theta}{(\imath\zeta_1 \cos\theta + (\imath\zeta_2+1)\sin\theta)^2} + \int_{-\frac{\pi}{4}}^{0} \frac{d\theta}{(\imath\zeta_1 \cos\theta + (\imath\zeta_2-1)\sin\theta)^2} \\ &= \frac{1}{\imath\zeta_1(\imath\zeta_1 + \imath\zeta_2 + 1)} - \frac{1}{\imath\zeta_1(\imath\zeta_1 - \imath\zeta_2 + 1)} \\ &= f_1(\zeta) + f_2(\zeta).\end{aligned}$$

Then we can see $f_1(\zeta) \in \mathcal{H}(\mathbb{R}^2 + \imath L_1)$ and $f_2(\zeta) \in \mathcal{H}(\mathbb{R}^2 + \imath L_2)$, where

$$L_1 := \{\eta = (\eta_1, \eta_2); \{(0,1)\} + (\overline{\Gamma}')^\circ\}, \quad L_2 := \{\eta = (\eta_1, \eta_2); \{(0,-1)\} + (\overline{\Gamma}')^\circ\},$$

and $L = L_1 \cap L_2$. Now we define

$$T_1 = \begin{cases} e^{x_2}, & x_1 > x_2, \quad x_2 > 0, \\ 0, & \text{otherwise}, \end{cases}$$

$$T_2 = \begin{cases} e^{-x_2}, & x_1 > -x_2, \; x_2 < 0, \\ 0, & \text{otherwise}. \end{cases}$$

Then we have $T_1 \in H'_{\overline{\Gamma}}(\mathbb{R}^2, \{(0,1)\})$, $T_2 \in H'_{\overline{\Gamma}}(\mathbb{R}^2, \{(0,-1)\})$ and

$$\begin{aligned}&\langle T_{1_x}, e^{\imath\zeta x}\rangle = f_1(\zeta), \\ &\langle T_{2_x}, e^{\imath\zeta x}\rangle = f_2(\zeta), \\ &T = T_1 + T_2.\end{aligned}$$

§7. Edge-of-the-Wedge Theorem

In this section we give Edge-of-the-Wedge theorem for the space of the image by the Fourier-Laplace transform of $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$. First we introduce some spaces of holomorphic functions. For details we refer the reader to [15], [16].

Definition 7.1. For a subset A of $\mathbb{R}^n$, we define a set $\mathcal{T}(A)$ by $\mathcal{T}(A) = \mathbb{R}^n \times \imath A$.

Definition 7.2. For a convex compact set K of $\mathbb{R}^n$ and $\varepsilon > 0$,

$$\begin{aligned}&\mathcal{Q}_b(\mathcal{T}(K_\varepsilon))\\ &:= \{\varphi(\zeta) \in \mathcal{H}(\mathcal{T}(K_\varepsilon^\circ)) \cap \mathcal{C}(\mathcal{T}(K_\varepsilon));\ \sup_{\zeta\in\mathcal{T}(K_\varepsilon)} |\zeta^\alpha \varphi(\zeta)| < \infty \text{ for } \forall \alpha \in \mathbb{N}_0^n\},\end{aligned}$$

$$\mathcal{Q}(\mathcal{T}(K)) := \varinjlim_{\varepsilon>0} \mathcal{Q}_b(\mathcal{T}(K_\varepsilon)).$$

Definition 7.3. The dual space $\mathcal{Q}'(\mathcal{T}(K))$ of $\mathcal{Q}(\mathcal{T}(K))$ is called tempered ultrahyperfunctions.

Remark 7.4.

(i) A. U. Schmidt apply $\mathcal{Q}(\mathcal{T}(K))$ to study asymptotic expansions [18].

(ii) $\mathcal{Q}'(\mathcal{T}(K))$ is called tempered ultradistributions by S. e. Silva [19] and M. Hasumi [5], and called tempered ultrahyperfunctions by M. Morimoto [15], [16].

We have the following theorem for the spaces $H(\mathbb{R}^n, K)$ and $\mathcal{Q}(\mathcal{T}(K))$:

Theorem 7.5 ([15]). *Let $\varphi(x) \in H(\mathbb{R}^n, K)$. The Fourier inverse transform*

$$\mathcal{F}^{-1}(\varphi)(\zeta) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \varphi(x) e^{-\imath\zeta x} dx$$

establishes a topological isomorphism of $H(\mathbb{R}^n, K)$ onto $\mathcal{Q}(\mathcal{T}(K))$. The inverse mapping $\mathcal{F}$ is given by

$$\begin{aligned}\mathcal{F}(\psi)(x) &:= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \psi(\xi + \imath\eta) e^{\imath(\xi+\imath\eta)x} d\xi,\\ &\eta \in K_\varepsilon^\circ, \quad \psi \in \mathcal{Q}_b(\mathcal{T}(K_\varepsilon)).\end{aligned} \tag{47}$$

Remark 7.6. In (47), we notice that $\mathcal{F}(\psi)(x)$ is independent of $\eta \in K_\varepsilon^\circ$ by Cauchy's integral theorem.

Definition 7.7 ([15]). For $T \in H'(\mathbb{R}^n, K)$, we define the dual Fourier transform $\mathcal{F}(T)$ as a continuous linear functional on $\mathcal{Q}(\mathcal{T}(K))$ by the formula

$$\langle \mathcal{F}(T), \psi\rangle = \langle T, \mathcal{F}(\psi)\rangle, \quad \text{for } \psi \in \mathcal{Q}(\mathcal{T}(K)). \tag{48}$$

As a consequence of Theorem 7.5, we have the following theorem:

Theorem 7.8 ([15]). *The dual Fourier transform* (48) *gives topological isomorphisms*

$$\mathcal{F} : H'(\mathbb{R}^n, K) \to \mathcal{Q}'(\mathcal{T}(K)).$$

Definition 7.9. Let $K = \{u\}$, $\psi \in \mathcal{Q}_b(\mathcal{T}(K_{\varepsilon_1}))$ and assume that $f(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath L)$ satisfies

$$\begin{gathered} \forall\varepsilon > 0\ \exists l_\varepsilon \geq 0\ \forall\overline{\Gamma}_C \Subset (\overline{\Gamma}')^\circ\ \exists M_{\varepsilon,\overline{\Gamma}_C} \geq 0\ s.t. \\ |f(\zeta)| \leq M(1 + |\zeta|)^l, \quad \zeta \in \mathbb{R}^n + \imath[\overline{\Gamma}_C]_{-\varepsilon}. \end{gathered}$$

Then we define $\langle f(\zeta), \psi(\zeta)\rangle$ by

$$\begin{aligned} \langle f(\zeta), \psi(\zeta)\rangle &:= \langle f(\xi + \imath\eta_0), \psi(\xi + \imath\eta_0)\rangle \\ &= \int_{\mathbb{R}^n} f(\xi + \imath\eta_0)\psi(\xi + \imath\eta_0)d\xi, \end{aligned}$$

where $\eta_0 \in (\{u\} + (\overline{\Gamma}')^\circ) \cap (K_{\varepsilon_1}{}^\circ)$.

Definition 7.10. Let $K = \{u\}$, $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$ and $\psi \in \mathcal{Q}(\mathcal{T}(K))$, $\psi \in \mathcal{Q}_b(\mathcal{T}(K_{\varepsilon_1}))$. By Theorem 6.9 and Definition 7.9, we define $\langle\mathcal{LF}(T)(\zeta), \psi(\zeta)\rangle$ by

$$\langle\mathcal{LF}(T)(\zeta), \psi(\zeta)\rangle := \langle\mathcal{LF}(T)(\xi + \imath\eta_0), \psi(\xi + \imath\eta_0)\rangle, \tag{49}$$

where $\eta_0 \in (\{u\} + (\overline{\Gamma}')^\circ) \cap (K_{\varepsilon_1}{}^\circ)$.

Now we can show Edge-of-the-Wedge theorem. For the direct product case, it is given in [16].

Theorem 7.11 (Edge-of-the-Wedge Theorem). *Let* Γ_1, Γ_2 *be proper open convex cones in* $\mathbb{R}^n$,

$$L_m = \{u_m\} + (\overline{\Gamma}'_m)^\circ, \quad m = 1, 2.$$

Assume that $F_1(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath L_1)$ *and* $F_2(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath L_2)$ *satisfy*

$$\begin{gathered} \forall\varepsilon > 0\ \exists l_{m_\varepsilon} \geq 0\ \forall\overline{\Gamma}_{C_m} \Subset (\overline{\Gamma}'_m)^\circ\ \exists M_{\varepsilon,\overline{\Gamma}_{C_m}} \geq 0\ s.t. \\ |F_m(\zeta)| \leq M_{\varepsilon,\overline{\Gamma}_{C_m}}(1 + |\zeta|)^{l_{m_\varepsilon}}, \quad \zeta \in \mathbb{R}^n + \imath[\overline{\Gamma}_{C_m}]_{-2\varepsilon}, \quad m = 1, 2, \end{gathered} \tag{50}$$

where $[\overline{\Gamma}_{C_m}]_{-\varepsilon} = \mathbb{R}^n \backslash (\mathbb{R}^n \backslash (\{u_m\} + \overline{\Gamma}_{C_m})^\circ)_\varepsilon$.

Let K be a convex compact set which contains the segment with $\{u_1\}$ and $\{u_2\}$ as extremal point. Assume that

$$\langle F_1(\zeta), \psi(\zeta)\rangle = \langle F_2(\zeta), \psi(\zeta)\rangle \quad \forall \psi(\zeta) \in \mathcal{Q}(\mathcal{T}(K)). \tag{51}$$

Then there exists $F(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath(L_1' \cup L_2'))$ such that

$$F(\zeta)|_{(\mathbb{R}^n+\imath L_1)} = F_1(\zeta),$$
$$F(\zeta)|_{(\mathbb{R}^n+\imath L_2)} = F_2(\zeta),$$

where $L_1' = \{u_1\} + (\overline{\Gamma}_1' \cup \overline{\Gamma}_2')^\circ$ and $L_2' = \{u_2\} + (\overline{\Gamma}_1' \cup \overline{\Gamma}_2')^\circ$. Furthermore

(i) *if $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \{0\}$, then $F(\zeta)$ is polynomial,*

(ii) *if $\{u_1\} = \{u_2\}(=: \{u\})$, then we have*

$$F(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath(\{u\} + (\overline{\Gamma}'_1 \cup \overline{\Gamma}'_2)^\circ)) \tag{52}$$

and

$$\forall \varepsilon > 0\ \exists l_\varepsilon \geq 0\ \forall \overline{\Gamma}_C \Subset (\overline{\Gamma}'_1 \cup \overline{\Gamma}'_2)^\circ\ \exists M_{\varepsilon,\overline{\Gamma}_C} \geq 0 \tag{53}$$
$$|F(\zeta)| \leq M(1+|\zeta|)^l, \quad \zeta \in \mathbb{R}^n + \imath[\overline{\Gamma}_C]_{-\varepsilon},$$

where $[\overline{\Gamma}_C]_{-\varepsilon} = \mathbb{R}^n \backslash (\mathbb{R}^n \backslash (\{u\} + \overline{\Gamma}_C)^\circ)_\varepsilon$.

Proof. By (50) and Theorem 6.9, there exist $T_1 \in H'_{\overline{\Gamma}_1}(\mathbb{R}^n, \{u_1\})$ and $T_2 \in H'_{\overline{\Gamma}_2}(\mathbb{R}^n, \{u_2\})$ such that

$$\frac{1}{(2\pi)^{\frac{n}{2}}}\langle T_{1x}, e^{\imath\zeta x}\rangle = F_1(\zeta)$$
$$\frac{1}{(2\pi)^{\frac{n}{2}}}\langle T_{2x}, e^{\imath\zeta x}\rangle = F_2(\zeta).$$

Let $\varphi(x) \in H(\mathbb{R}^n, K)$. By Theorem 7.5, $\mathcal{F}^{-1}(\varphi)(\zeta) \in \mathcal{Q}(\mathcal{T}(K))$. By Definition 7.9, 7.10 and assumption (51), we have

$$\begin{aligned}\langle T_{1x}, \varphi(x)\rangle &= \langle \mathcal{LF}(T_1)(\zeta), \mathcal{F}^{-1}(\varphi)(\zeta)\rangle\\ &= \langle F_1(\zeta), \mathcal{F}^{-1}(\varphi)(\zeta)\rangle\\ &= \langle F_2(\zeta), \mathcal{F}^{-1}(\varphi)(\zeta)\rangle\\ &= \langle \mathcal{LF}(T_2)(\zeta), \mathcal{F}^{-1}(\varphi)(\zeta)\rangle\\ &= \langle T_{2x}, \varphi(x)\rangle.\end{aligned}$$

Therefore, $T_1 = T_2 =: T$ in $H'(\mathbb{R}^n, K)$ and supp $T \subset (\overline{\Gamma}_1 \cap \overline{\Gamma}_2)$. Now we put $F(\zeta) = \frac{1}{(2\pi)^{\frac{n}{2}}}\langle T_x, e^{\imath\zeta x}\rangle$. Then by the definition of T, $F(\zeta)|_{(\mathbb{R}^n+\imath L_1)} = F_1(\zeta)$, $F(\zeta)|_{(\mathbb{R}^n+\imath L_2)} = F_2(\zeta)$ and by Proposition 6.2 we have $F(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath(L_1' \cup L_2'))$.

If we have the assumption (i), then T is a distribution supported by $\{0\}$. By the structure theorem for distributions, $T = \sum_{|\alpha|\le m} c_\alpha D^\alpha \delta$. So $F(\zeta) = \mathcal{LF}(T)(\zeta)$ is polynomial.

If we have the assumption (ii), then by Proposition 2.13 and Theorem 6.9, we have (52) and (53). □

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